Optimal investment decisions with minimum price guarantees under the constant elasticity of variance process

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Abstract

This paper offers an analytical representation for evaluating optimal investment decisions associated to a feed-in tariff (FIT) contract with a minimum price guarantee (i.e., a price-floor regime) under the constant elasticity of variance (CEV) model. The proposed analytic solutions can be used to optimally design FIT contractual schemes with both perpetual and finite maturity guarantees. We show that the argument that a perpetual guarantee only induces investment for prices below the price floor when offering a risk-free investment opportunity is still valid under the CEV process. We also demonstrate that the optimal price-floor level triggering immediate investment in the presence of a perpetual guarantee is independent of the elasticity parameter of the CEV model. By contrast, we show that such independence is not valid any more in the case of FIT contracts with a finite maturity guarantee. Our results provide evidence that care must be taken when a policymaker aims to design a given instrument to induce investment decisions with FIT contracts because the differences between trigger points under alternative modeling assumptions are quite significant and the excessive rents are usually paid at the expense of tax payers.

Keywords: Finance; real options; CEV model; feed-in tariff; price-floor regime

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1. Introduction

This paper proposes an analytical representation for analyzing optimal investment decisions associated to a feed-in tariff (henceforth, FIT) contract with a minimum price guarantee (i.e., a price-floor regime) under the constant elasticity of variance (hereafter, CEV) model, that is known to nest the limiting geometric Brownian motion (henceforward, GBM) process as a special case. Our analytic solutions are suitable to design FIT contractual schemes with both perpetual and finite-lived guarantees. Even though the project value of a FIT contract under the CEV process arises straightforwardly from the cap and floor solutions offered recently by Dias et al. (2023), the analytic formulae proposed in the present paper for studying investment decisions with minimum price guarantees constitutes a novel real options framework that should be useful for investors and policymakers in a wide variety of management decisions involving renewable energy projects.

The use of such support schemes has been considered as an important tool to design incentive policies related with renewable energy projects that might be relevant for mitigating many environmental problems and, hence, an abundant amount of research has emerged in the real options literature related with this topic, e.g., Shackleton and Wojakowski (2007), Couture and Gagnon (2010), Ritzenhofen et al. (2016), Ritzenhofen and Spinler (2016), Barbosa et al. (2018), Pineda et al. (2018), Adkins and Paxson (2019), Adkins et al. (2019), Barbosa et al. (2020), Barbosa et al. (2022), Paxson et al. (2022) and Li et al. (2023), just to name a few.

Building upon the work of Dias et al. (2023) and similarly to the case analyzed in Barbosa et al. (2018) under the GBM assumption, our real options model under the more general CEV process includes managerial flexibilities and calculates the optimal trigger level that induces a firm to invest in a FIT project with a minimum price guarantee. Within this policy, the producer receives a fixed amount if the market price is below the price floor or the market price, otherwise. Although the use of a simplistic real options approach based on a GBM diffusion is easier to be tackled analytically and numerically, it is well known that the CEV diffusion is better able to accommodate several stylized-facts that are commonly reported in the literature (including in a real options context), namely the so-called leverage and volatility smile effects.[1]

Our analytical formulae and the corresponding numerical analysis provide several interesting results that should be relevant for academics and researchers when studying such incentive policies. In particular, we show that the argument of Barbosa et al. (2018) that a perpetual guarantee only induces investment for prices below the price floor when offering a risk-free investment opportunity is still valid under the CEV process. This indicates that a perpetual guarantee is not economically sound even when considering a non-constant volatility setup such as the one considered in a CEV local volatility model. We also demonstrate that the optimal price-floor level L^* triggering immediate investment in the presence of a perpetual guarantee is independent of the elasticity parameter of the CEV model, which implies that it is equal to the one derived by Barbosa et al. (2018) under the GBM case. Nevertheless, for optimal threshold levels $\overline{P}$ above the optimal price-floor level L^* we observe that the leverage effect produces significant differences amongst models with different elasticity parameters.

By contrast, we show that such independence of L^* is not valid any more in the case of FIT contracts with a finite maturity guarantee. Using the European onshore wind farm with 25 wind turbines considered in Barbosa et al. (2018), we conclude that an annual difference higher than 1 million Euros of supporting rents (per farm) is required if the regulator negotiated the contract based on a GBM assumption when the price process was better described by an alternative CEV process. Hence, care must be taken when a policy-maker aims to design a given instrument to induce investment decisions with FIT contracts because the differences between trigger points under alternative modeling assumptions are quite significant and the excessive rents are usually paid at the expense of tax payers.

The remainder of the manuscript is organized as follows. Section 2 briefly summarizes the CEV model setup adopted in this paper. Section 3 presents the analytical formulae

[1]For additional theoretical background on the CEV process and applications in financial and real options contexts see, for instance, Cox (1975), Davydov and Linetsky (2001), Davydov and Linetsky (2003), Geman and Shih (2009), Nunes (2009), Dias and Nunes (2011), Larguinho et al. (2013), Ruas et al. (2013), Dias et al. (2015), Nunes et al. (2015) and Dias et al. (2020).

for evaluating optimal investment decisions associated to FIT contracts with a minimum price guarantee under the CEV model and considering both the cases with perpetual and finite maturity guarantees. Section 4 provides some numerical examples and Section 5 presents some concluding remarks. All the proofs are relegated to a supplementary file that is available as an internet appendix.

2. Model setup

Hereafter, we follow Dias et al. (2023) by assuming that the output price P is described via a CEV process represented by the stochastic differential equation

$$dP_t = (r - q)\, P_t dt + \sigma(P_t) P_t dW_t^{\mathbb{Q}}, \tag{1}$$

with the local volatility function—assumed to be continuous and strictly positive for all $P \in (0, \infty)$—given by

$$\sigma(P_t) := \delta P_t^{\frac{\beta}{2}-1}, \tag{2}$$

for $\delta \in \mathbb{R}^+$ and $\beta \in \mathbb{R}$, and where r and q are the (positive and constant) interest rate and dividend yield (or rate of return shortfall), respectively, while $\{W_t^{\mathbb{Q}} \in \mathbb{R} : t \geq 0\}$ is a standard Brownian motion under the risk-neutral measure $\mathbb{Q}$, initialized at zero and generating the augmented, right continuous and complete filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$.

The modeling specification given by equations (1) and (2) nests the lognormal assumption of Black and Scholes (1973) and Merton (1973) (if $\beta = 2$) as well as the absolute diffusion (if $\beta = 0$) and the square-root diffusion (if $\beta = 1$) processes of Cox and Ross (1976) as special cases. We recall also that if $\beta < 2$ (resp., $\beta > 2$), the local volatility function (2) is a decreasing (resp., increasing) function of the asset price, thus being able to generate downward-sloping (resp., upward-sloping) volatility skews that are observed in the market. As usual, the model parameter δ is interpreted as the scale parameter fixing the initial instantaneous volatility at time $t = 0$, $\sigma_P \equiv \sigma(P_0) = \delta P_0^{\beta/2-1}$. This calibration procedure is standard in the option pricing literature in order to ensure that CEV models with different values of β have the same variance at inception, thus permitting valid comparisons between different CEV diffusion processes.

3. Investment decisions with minimum price guarantees

This section considers a FIT contractual agreement with a minimum price guarantee—also known as a contract based on a price-floor regime—such as the one considered in Barbosa et al. (2018, Section 3) for the case of a renewable energy project and from a single investor's perspective, but now under the CEV modeling setup prescribed in Dias et al. (2023). Although the value of the project with a minimum price guarantee under the CEV process has been already derived in Dias et al. (2023, Section 4), the corresponding options to invest represent novel results to the real options literature.

3.1. Perpetual case

Let us start with a contractual scheme with a perpetual guarantee. This time-independent assumption implies that the investor may receive payments from the guarantee forever. Even though such perpetual guarantee is not used in practice, it is important to analyze it first in order to be compared with the more realistic case of a finite guarantee.

3.1.1. Value of the project

In the FIT contract under analysis the producer (or investor) is entitled to receive the price floor L for every unit of energy produced when the market price P is below the price floor; otherwise (i.e., when $P \geq L$), the producer receives the market price. Hence, the perpetual profit flow of such renewable energy project can be formally defined as $\Pi\left(P_t\right) = \max\left(P_t, L\right)$, with P_t being understood as the time-t revenue for one unit of energy produced and L being interpreted as the revenue from the guarantee for one unit of energy.

Assuming that the output price P follows the CEV process defined in equations (1) and (2), then the value of the (infinite horizon) project, $V_f(P_t, L, \infty)$, with a perpetual profit flow $\Pi\left(P_t\right)$ must satisfy the following ordinary differential equation (hereafter, ODE)

$$\frac{1}{2}\delta^2 P_t^{\beta}\frac{d^2V_f}{dP_t^2} + \left(r-q\right)P_t\frac{dV_f}{dP_t} - rV_f + \Pi\left(P_t\right) = 0, \tag{3}$$

whose analytical solution is dependent on the drift specification (i.e., whether $r = q$ or $r \neq q$) and on the elasticity parameter β, as already shown in Dias et al. (2023). For the sake of completeness and to introduce notation and explicit expressions that are required later, such results are highlighted in the next two propositions.

Proposition 1. *Assume the CEV process with $r = q$ and introduce the following notation:*

$$z_t \equiv z(P_t) := \frac{2\sqrt{2r}}{\delta|\beta - 2|} P_t^{1-\frac{\beta}{2}} \tag{4}$$

and

$$\nu := \frac{1}{|\beta - 2|}. \tag{5}$$

The time-t value of the project with a minimum price guarantee, $V_f(P_t, L, \infty)$, is given by

$$V_f(P_t, L, \infty) = \begin{cases} \left(A_0 P_t^{\frac{1}{2}} I_\nu(z_t) + \frac{L}{r}\right) \mathbb{1}_{\{P_t < L\}} & \\ + \left(B_0 P_t^{\frac{1}{2}} K_\nu(z_t) + \frac{P_t}{q}\right) \mathbb{1}_{\{P_t \geq L\}} & \Leftarrow \quad \beta < 2 \\ \left(B_0 P_t^{\frac{1}{2}} K_\nu(z_t) + \frac{L}{r}\right) \mathbb{1}_{\{P_t < L\}} & \\ + \left(A_0 P_t^{\frac{1}{2}} I_\nu(z_t) + \frac{P_t}{q}\right) \mathbb{1}_{\{P_t \geq L\}} - G_t^\infty(P_t) & \Leftarrow \quad \beta > 2 \end{cases}, \tag{6}$$

where $G_t^\infty(P_t)$ is the perpetual bubble value defined in Dias et al. (2023, equation 10), $I_\nu(z_t)$ and $K_\nu(z_t)$ are, respectively, the modified Bessel's functions of first and second kind with order ν, as given by Olver et al. (2010, equations 10.25.2 and 10.27.4), whereas $A_0 \equiv A_0(L)$ and $B_0 \equiv B_0(L)$ are constants (associated to the value of the project) that solve the following system of equations, for $\zeta := z(L)$ and for any $\beta \neq 2$:

$$\begin{cases} A_0 I_\nu(\zeta) - B_0 K_\nu(\zeta) = 0 \\ A_0 I_{\nu+1}(\zeta) + B_0 K_{\nu+1}(\zeta) = \frac{\delta}{q\sqrt{2r}} L^{\frac{\beta-1}{2}} \end{cases}. \tag{7}$$

Proof. Please see Dias et al. (2023, Proposition 8). ■

Remark 1. *Even though the constants A_0 and B_0 are easily determined by solving (numerically) the system of equations (7), it is still possible to calculate each one via a very* *simple closed-form solution after using some calculus and the Wronskian shown in Olver et al. (2010, equation 10.28.2), that is*

$$A_0 \equiv A_0(L) = \frac{2}{q|\beta-2|} L^{\frac{1}{2}} K_\nu(z(L)) \tag{8}$$

and

$$B_0 \equiv B_0(L) = \frac{2}{q|\beta-2|} L^{\frac{1}{2}} I_\nu(z(L)). \tag{9}$$

Proposition 2. *Assume the CEV process with $r \neq q$ and introduce the following notation:*

$$x_t \equiv x(P_t) := \frac{2|r-q|}{\delta^2|\beta-2|} P_t^{2-\beta}, \tag{10}$$

$$\epsilon := \operatorname{sign}\left[(r-q)(\beta-2)\right], \tag{11}$$

$$k := \epsilon\left(\frac{1}{2} + \frac{1}{2(\beta-2)}\right) - \frac{r}{|(r-q)(\beta-2)|}, \tag{12}$$

$$m := \frac{1}{2|\beta-2|}, \tag{13}$$

$$a := \frac{1}{2} + m - k, \tag{14}$$

$$b := 1 + 2m \tag{15}$$

and

$$Y := \left(\frac{2|r-q|}{\delta^2|\beta-2|}\right)^{\frac{b}{2}}. \tag{16}$$

The time-t value of the project with a minimum price guarantee, $V_f(P_t, L, \infty)$, is given by

$$V_f(P_t, L, \infty) = \begin{cases} \left(A_0 Y e^{(\epsilon-1)\frac{x_t}{2}} P_t M(a,b,x_t) + \frac{L}{r}\right) \mathbb{1}_{\{P_t < L\}} & \\ + \left(B_0 Y e^{(\epsilon-1)\frac{x_t}{2}} P_t U(a,b,x_t) + \frac{P_t}{q}\right) \mathbb{1}_{\{P_t \geq L\}} & \Leftarrow \quad \beta < 2 \\ \left(B_0 Y e^{(\epsilon-1)\frac{x_t}{2}} U(a,b,x_t) + \frac{L}{r}\right) \mathbb{1}_{\{P_t < L\}} & \\ + \left(A_0 Y e^{(\epsilon-1)\frac{x_t}{2}} M(a,b,x_t) + \frac{P_t}{q}\right) \mathbb{1}_{\{P_t \geq L\}} - G_t^\infty(P_t) & \Leftarrow \quad \beta > 2 \end{cases}, \tag{17}$$

where $M(a,b,x_t)$ and $U(a,b,x_t)$ are, respectively, the Kummer's and Tricomi's confluent hy-

pergeometric functions defined in Olver et al. (2010, equations 13.2.2 and 13.2.42), whereas $A_0 \equiv A_0(L)$ *and* $B_0 \equiv B_0(L)$ *are constants (associated to the value of the project) that solve the following system of equations, for* $\chi := x(L)$*:*

$$\begin{cases} A_0 M(a,b,\chi) - B_0 U(a,b,\chi) = \frac{r-q}{rq} Y^{-1} e^{(1-\epsilon)\frac{\chi}{2}} \\ A_0 \frac{a}{b} M(a+1,b+1,\chi) + B_0 a U(a+1,b+1,\chi) = e^{(1-\epsilon)\frac{\chi}{2}} \left(\frac{(1-\epsilon)(r-q)}{2rq} + \frac{\delta^2 L^{\beta-2}}{2r|r-q|} \right) Y^{-1} \end{cases}, \tag{18}$$

for $\beta < 2$*, and*

$$\begin{cases} A_0 M(a,b,\chi) - B_0 U(a,b,\chi) = \frac{q-r}{rq} L Y^{-1} e^{(1-\epsilon)\frac{\chi}{2}} \\ A_0 \frac{a}{b} M(a+1,b+1,\chi) + B_0 a U(a+1,b+1,\chi) = e^{(1-\epsilon)\frac{\chi}{2}} \left(\frac{(1-\epsilon)(q-r)}{2rq} + \frac{\delta^2 L^{\beta-2}}{2q|r-q|} \right) L Y^{-1} \end{cases}, \tag{19}$$

for $\beta > 2$*.*

Proof. Please see Dias et al. (2023, Proposition 9). ■

Remark 2. *Again, although the constants* A_0 *and* B_0 *can be easily determined by solving (numerically) the systems of equations* (18) *and* (19) *for* $\beta < 2$ *and* $\beta > 2$*, respectively, it is still possible to calculate each one via an analytic solution after using some calculus and combining the standard solution and the Wronskian presented in Olver et al. (2010, equations 13.2.4 and 13.2.34), that is*

$$\begin{aligned} A_0 \equiv A_0(L) \quad &= \quad \left[\frac{\delta^2 L^{\beta-2}}{2r|r-q|} + \frac{r-q}{rq} \left(\frac{1-\epsilon}{2} + a \, \frac{U(a+1,b+1,x(L))}{U(a,b,x(L))} \right) \right] \\ &\quad \times \frac{e^{-(1+\epsilon)\frac{x(L)}{2}}}{Y} \, \frac{\Gamma(a)}{\Gamma(b)} \, (x(L))^b \, U(a,b,x(L)) \end{aligned} \tag{20}$$

and

$$\begin{aligned} B_0 \equiv B_0(L) \quad &= \quad \left[\frac{\delta^2 L^{\beta-2}}{2r|r-q|} + \frac{r-q}{rq} \left(\frac{1-\epsilon}{2} - \frac{a}{b} \, \frac{M(a+1,b+1,x(L))}{M(a,b,x(L))} \right) \right] \\ &\quad \times \frac{e^{-(1+\epsilon)\frac{x(L)}{2}}}{Y} \, \frac{\Gamma(a)}{\Gamma(b)} \, (x(L))^b \, M(a,b,x(L)), \end{aligned} \tag{21}$$

for $\beta < 2$, and

$$\begin{aligned} A_0 \equiv A_0(L) &= \left[\frac{\delta^2 L^{\beta-2}}{2q|r-q|} + \frac{q-r}{rq}\left(\frac{1-\epsilon}{2} + a\,\frac{U(a+1,b+1,x(L))}{U(a,b,x(L))}\right)\right] \\ &\quad \times \frac{e^{-(1+\epsilon)\frac{x(L)}{2}}}{Y}\, L\, \frac{\Gamma(a)}{\Gamma(b)}\,(x(L))^b\, U(a,b,x(L)) \end{aligned} \tag{22}$$

and

$$\begin{aligned} B_0 \equiv B_0(L) &= \left[\frac{\delta^2 L^{\beta-2}}{2q|r-q|} + \frac{q-r}{rq}\left(\frac{1-\epsilon}{2} - \frac{a}{b}\,\frac{M(a+1,b+1,x(L))}{M(a,b,x(L))}\right)\right] \\ &\quad \times \frac{e^{-(1+\epsilon)\frac{x(L)}{2}}}{Y}\, L\, \frac{\Gamma(a)}{\Gamma(b)}\,(x(L))^b\, M(a,b,x(L)), \end{aligned} \tag{23}$$

for $\beta > 2$.

3.1.2. Value of the investment opportunity

Assuming that an investor has a perpetual investment opportunity to invest in a project with a sunk cost I, we can (simultaneously) determine the value of the (perpetual) option to invest in a project with a perpetual guarantee, $F_f(P_t, \overline{P}, \infty)$, and the optimal investment rule, with $\overline{P}$ being interpreted as the optimal output price trigger level that will induce the investment. Assuming that the output price P follows the CEV process defined in equations (1) and (2), then the value of the option to invest in a perpetual project must satisfy another ODE but with no profit flow, that is

$$\frac{1}{2}\delta^2 P_t^{\beta}\frac{d^2F_f}{dP_t^2} + (r-q)\,P_t\frac{dF_f}{dP_t} - rF_f = 0, \tag{24}$$

whose analytical solutions depend, again, on the drift level and β parameter and are determined subject to appropriate boundary conditions. The next two propositions offer such novel results for evaluating a perpetual option to invest in a project with an infinite-lived guarantee under the CEV process. We start, however, with a remark clarifying in which region the value-matching and smooth-pasting conditions can be applied.

Remark 3. *Similarly to what happens in the limiting GBM case considered in Barbosa et al. (2018), the value-matching and smooth-pasting conditions are not met for the first branch of equations* (6) *and* (17) *(i.e., when* $\overline{P} < L$*). Therefore, investment never occurs in this region, except when* $\frac{L}{r} > I$*, which would imply a counter-intuitive result of a project generating a positive net present value (hereafter, NPV) that is independent of* P*. Moreover, such risk-free payoff would be the same for any* $\beta \in \mathbb{R}$*, because the simultaneous presence of bubbles in the project value and in the option component would cancel out any bubble effect for* $\beta > 2$*. In summary, the optimal investment threshold for a FIT contract with a perpetual guarantee is restricted to the range* $\overline{P} \geq L$*.*[2]

Proposition 3. *Assume the CEV process with* $r = q$ *and the notation introduced in Proposition 1. The time-t value of the (perpetual American-style) option to invest in a project with a perpetual guarantee is given by*

$$\begin{aligned} & F_f(P_t, \overline{P}, \infty) \qquad (25) \\ = & \begin{cases} \left[V_{f,op}(\overline{P}, L, \infty) + G_t^{\infty}(\overline{P})\, \mathbb{1}_{\{\beta>2\}} - I\right] S_f(P_t, \overline{P}, \infty) - G_t^{\infty}(P_t)\, \mathbb{1}_{\{\beta>2\}} & \Leftarrow \quad P_t < \overline{P} \\ V_{f,op}(P_t, L, \infty) - I & \Leftarrow \quad P_t \geq \overline{P} \end{cases}, \end{aligned}$$

with $V_{f,op}(P_t, L, \infty)$ *being given by equation* (6) *but conditional on* $P_t \geq L$*, that is*

$$\begin{aligned} V_{f,op}(P_t, L, \infty) \quad := \quad & V_f(P_t, L, \infty) \mathbb{1}_{\{P_t \geq L\}} \\ = \quad & \begin{cases} B_0 P_t^{\frac{1}{2}} K_\nu(z_t) + \frac{P_t}{q} & \Leftarrow \quad \beta < 2 \\ A_0 P_t^{\frac{1}{2}} I_\nu(z_t) + \frac{P_t}{q} - G_t^{\infty}(P_t) & \Leftarrow \quad \beta > 2 \end{cases}, \end{aligned} \qquad (26)$$

and where the stochastic discount factor of the perpetual option to invest is given by

$$S_f(P_t, \overline{P}, \infty) = \begin{cases} \sqrt{\frac{P_t}{\overline{P}}} \frac{I_\nu(z(P_t))}{I_\nu(z(\overline{P}))} & \Leftarrow \quad \beta < 2 \\ \sqrt{\frac{P_t}{\overline{P}}} \frac{K_\nu(z(P_t))}{K_\nu(z(\overline{P}))} & \Leftarrow \quad \beta > 2 \end{cases}, \qquad (27)$$

which can be financially interpreted as the time-t price of a perpetual claim that pays \$1 the

[2] A formal proof of these arguments can be found in Appendix A.

first time P reaches $\overline{P}$ from below, with the optimal threshold $\overline{P}$ $(\geq L)$ and the constants $A_1 \equiv A_1(\overline{P})$ and $B_1 \equiv B_1(\overline{P})$ (associated to the value of the option to invest) being obtained as the solution of the following system of equations:

$$\begin{cases} A_1 \overline{P}^{\frac{1}{2}} I_\nu(z(\overline{P})) - B_0 \overline{P}^{\frac{1}{2}} K_\nu(z(\overline{P})) = \frac{\overline{P}}{q} - I \\ A_1 I_{\nu+1}(z(\overline{P})) + B_0 K_{\nu+1}(z(\overline{P})) = \frac{\delta I}{\sqrt{2r}} \overline{P}^{\frac{\beta-3}{2}} \end{cases}, \tag{28}$$

for $\beta < 2$, and

$$\begin{cases} B_1 \overline{P}^{\frac{1}{2}} K_\nu(z(\overline{P})) - A_0 \overline{P}^{\frac{1}{2}} I_\nu(z(\overline{P})) = \frac{\overline{P}}{q} - I \\ B_1 K_{\nu+1}(z(\overline{P})) + A_0 I_{\nu+1}(z(\overline{P})) = \frac{\delta}{q\sqrt{2r}} \overline{P}^{\frac{\beta-1}{2}} \end{cases}, \tag{29}$$

for $\beta > 2$, with the (already known) constants $A_0 \equiv A_0(L)$ and $B_0 \equiv B_0(L)$ (associated to the value of the project) being obtained as the solution of the system of equations (7) for any $\beta \neq 2$.

Proof. Please see Appendix B. ■

Proposition 4. *Assume the CEV process with $r \neq q$ and the notation introduced in Proposition 2. The time-t value of the (perpetual American-style) option to invest in a project with a perpetual guarantee is still given by equation (25), but now with $V_{f,op}(P_t, L, \infty)$ being given by equation (17) conditional on $P_t \geq L$, that is*

$$\begin{aligned} V_{f,op}(P_t, L, \infty) &:= V_f(P_t, L, \infty) 1\!\!1_{\{P_t \geq L\}} \\ &= \begin{cases} B_0 Y e^{(\epsilon-1)\frac{x_t}{2}} P_t U(a, b, x_t) + \frac{P_t}{q} & \Leftarrow \beta < 2 \\ A_0 Y e^{(\epsilon-1)\frac{x_t}{2}} M(a, b, x_t) + \frac{P_t}{q} - G_t^{\infty}(P_t) & \Leftarrow \beta > 2 \end{cases}, \end{aligned} \tag{30}$$

and where the stochastic discount factor of the perpetual option to invest being given by

$$S_f(P_t, \overline{P}, \infty) = \begin{cases} e^{(\epsilon-1)\frac{x(P_t)-x(\overline{P})}{2}} \frac{P_t}{\overline{P}} \frac{M(a,b,x(P_t))}{M(a,b,x(\overline{P}))} & \Leftarrow \beta < 2 \\ e^{(\epsilon-1)\frac{x(P_t)-x(\overline{P})}{2}} \frac{U(a,b,x(P_t))}{U(a,b,x(\overline{P}))} & \Leftarrow \beta > 2 \end{cases}, \tag{31}$$

with the optimal threshold $\overline{P}$ $(\geq L)$ and the constants $A_1 \equiv A_1(\overline{P})$ and $B_1 \equiv B_1(\overline{P})$ (as-

sociated to the value of the option to invest) being obtained as the solution of the following system of equations:

$$\begin{cases} A_1 M(a,b,x(\overline{P})) - B_0 U(a,b,x(\overline{P})) = \left(\frac{\overline{P}}{q} - I\right) Y^{-1} e^{(1-\epsilon)\frac{x(\overline{P})}{2}} \overline{P}^{-1} \\ A_1 \frac{a}{b} M(a+1,b+1,x(\overline{P})) + B_0 a U(a+1,b+1,x(\overline{P})) \\ \quad = \left[\frac{1}{q} - \left(\frac{\overline{P}}{q} - I\right)\left(\frac{\epsilon-1}{\delta^2}|r-q|\overline{P}^{1-\beta} + \overline{P}^{-1}\right)\right] Y^{-1} e^{(1-\epsilon)\frac{x(\overline{P})}{2}} \frac{\delta^2}{2|r-q|} \overline{P}^{\beta-2} \end{cases}, \tag{32}$$

for $\beta < 2$*, and*

$$\begin{cases} B_1 U(a,b,x(\overline{P})) - A_0 M(a,b,x(\overline{P})) = \left(\frac{\overline{P}}{q} - I\right) Y^{-1} e^{(1-\epsilon)\frac{x(\overline{P})}{2}} \\ B_1 a U(a+1,b+1,x(\overline{P})) + A_0 \frac{a}{b} M(a+1,b+1,x(\overline{P})) \\ \quad = \left[\frac{1}{q} + \left(\frac{\overline{P}}{q} - I\right)\frac{\epsilon-1}{\delta^2}|r-q|\overline{P}^{1-\beta}\right] Y^{-1} e^{(1-\epsilon)\frac{x(\overline{P})}{2}} \frac{\delta^2}{2|r-q|} \overline{P}^{\beta-1} \end{cases}, \tag{33}$$

for $\beta > 2$*, with the (already known) constants* $A_0 \equiv A_0(L)$ *and* $B_0 \equiv B_0(L)$ *(associated to the value of the project) being obtained as the solution of the system of equations* (18) *and* (19) *for* $\beta < 2$ *and* $\beta > 2$*, respectively.*

Proof. Please see Appendix C. ■

In summary, the optimal decision to invest in the case of a FIT contract with a perpetual guarantee can be analyzed via Propositions 3 and 4: exercise the (perpetual) option to invest only if $P_t \geq \overline{P}$; otherwise, i.e., if $P_t < \overline{P}$, it is better to wait and avoid loosing the value of waiting in the spirit of McDonald and Siegel (1986).

As highlighted in Remark 3, the perpetual case requires that $\overline{P} \geq L$ so that the problem is well behaved. In order to ensure that this condition is met, we determine the investment cost level I_L^* that separates the regions $\overline{P} \geq L$ and $\overline{P} < L$. This is accomplished by replacing $\overline{P}$ by the fixed price floor L (previously defined by the government) in the systems of equations (28), (29), (32) and (33) and solving each one with respect to $I \equiv I_L^*$. The obtained I_L^* value can be economically interpreted as the required investment cost that makes the fixed L the optimal trigger of investment. Notice that the constants A_0 and B_0 should be taken as functions of L and, hence, they are known in advance using the systems of equations (7), (18) or (19). Alternatively, it is also possible to use the novel analytic solutions offered in Remarks 1 and 2 to compute the value of such ingredients. Armed with the values of $A_0 \equiv A_0(L)$ and $B_0 \equiv B_0(L)$, the I_L^* level is then easily computed using the systems of equations (28), (29), (32) or (33) for the specific required case. If $I_L^* \leq I$, the optimal threshold $\overline{P}$ can be calculated since $\overline{P} \geq L$; otherwise, i.e., if $I_L^* > I$, the value-matching and smooth-pasting conditions are not met under the perpetual case and, hence, the investment problem falls in the counter-intuitive case exposed in Remark 3.

Several other interesting points can be also analyzed. For instance, for a given time-t price P_t (i.e., the current price) and a fixed price floor L (that is subject to the government policy), Barbosa et al. (2018, equation 16) allow us to determine the investment cost I^* that would induce the firm to exercise the option to invest immediately under the GBM process. In other words, this I^* value indicates the required level of investment that would make the time-t price P_t the trigger value to invest. In our case, this I^* value can be obtained by solving the systems of equations (28), (29), (32) or (33) with respect to $I \equiv I^*$ (and with $\overline{P}$ replaced by $P_t \geq L$) for the specific required case, which can then be compared with the initial investment cost I of the project. Again, the constants A_0 and B_0 are functions of L and, therefore, they are already known using the systems of equations (7), (18) and (19), or the analytical formulas shown in Remarks 1 and 2. As expected, if $I \leq I^*$ it would be better for the firm to invest immediately because $P_t = \overline{P}$ at the I^* level.

Furthermore, taking now the initial investment cost I of the project as given and substituting L for $\overline{P}$, Barbosa et al. (2018, equation 18) show how to compute (under the GBM process) the interesting point where the optimal threshold value $\overline{P}$ is equal to the price floor $L \equiv L^*$. The obtained L^* value can be economically interpreted as the required price floor level to be offered by the government to induce the firm to invest immediately (i.e., so that $\overline{P} = L^*$) for the fixed initial investment cost. In our case, this L^* value can be obtained by solving systems of four equations with the required constants being now functions of L^*. In particular, the following procedure is adopted to find the L^* value: (i) if $r = q$ and $\beta < 2$, use the systems of equations (7) with $L \equiv L^*$ and (28) with $\overline{P} \equiv L^*$; (ii) if $r = q$ and $\beta > 2$, use the systems of equations (7) with $L \equiv L^*$ and (29) with $\overline{P} \equiv L^*$; (iii) if $r \neq q$ and $\beta < 2$, use the systems of equations (18) with $L \equiv L^*$ and (32) with $\overline{P} \equiv L^*$; and (iv) if $r \neq q$ and $\beta > 2$, use the systems of equations (19) with $L \equiv L^*$ and (33) with $\overline{P} \equiv L^*$. The obtained L^* value can then be compared with the minimum price guarantee L that is offered by the government: if $L \geq L^*$ it would be better for the firm to invest right away.

Even though the levels of I_L^* and L^* can be calculated numerically using the aforementioned numerical procedures, it is possible to show analytically that $I_L^* = \frac{L}{r}$. This is accomplished using relatively simple calculus by appropriately combining the solutions offered in Remarks 1 and 2 with the systems of equations (28), (29), (32) and (33) and solving each particular case with respect to $I \equiv I_L^*$.[3] This also implies that $L^* = rI$. Therefore, I_L^* and L^* are both independent of β under the perpetual case and, hence, are the same triggers offered by Barbosa et al. (2018, equation 18) for the GBM case. The independence of I_L^* and L^* on the parameter β seems to be surprising at first glance, but is explained by the perpetual nature of the guarantee; as we will discuss in the next subsection, such independence is not valid any more in the case of FIT contracts with a finite maturity guarantee.

Nevertheless, such independence of the β parameter is only observed at the level L^*, because for the range of price floors $L \in [0, L^*[$ the optimal trigger value $\overline{P}$ decreases as we move further away to the left of $\beta < 2$ and increases as β rises for the case with $\beta > 2$. Only at $\overline{P} \equiv L^*$ the optimal threshold of investment is the same. Since alternative CEV models imply the existence of different optimal triggers or, equivalently, distinct optimal stopping times, a government assuming a GBM process to define a FIT policy might be paying a too high or a too low price floor to stimulate a firm to exercise the option to invest if the specific investment problem is better described by a different CEV model.

3.2. Finite case

Let us now analyze the FIT contractual agreement with a finite maturity guarantee. Similarly to the perpetual case, both the value of the project and the value of the option to invest will be analyzed.

[3]The full analytical proofs are available upon request.

3.2.1. Value of the project

Following Dias et al. (2023, Section 4.2), the value of the project with a finite guarantee, $V_f(P_t, L, T)$, must now satisfy the partial differential equation

$$\frac{1}{2}\delta^2 P_t^{\beta}\frac{\partial^2 V_f}{\partial P_t^2} + (r-q)\,P_t\frac{\partial V_f}{\partial P_t} - \frac{\partial V_f}{\partial \tau} - rV_f + \Pi\,(P_t) = 0, \tag{34}$$

with $\tau := T - t$, and whose solution can be expressed as the difference between the value of the project with a perpetual guarantee, $V_f(P_t, L, \infty)$, and the value of a forward start project with a perpetual guarantee, $V_f(P_T, L, \infty)$, that is

$$V_f(P_t, L, T) = V_f(P_t, L, \infty) - e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\left[V_f(P_T, L, \infty)|\mathcal{F}_t\right], \tag{35}$$

with $V_f(P_t, L, \infty)$ being computed by equations (6) and (17) for $r = q$ and $r \neq q$, respectively, whereas the risk-neutral expectation of the forward start perpetual FIT contract with a minimum price guarantee, $\mathbb{E}_{\mathbb{Q}}\left[V_f(P_T, L, \infty)|\mathcal{F}_t\right]$, is calculated using Dias et al. (2023, equations 70 and 71). These results are reported below to introduce notation and expressions explicitly dependent on the underlying spot P_t and price floor L that are required later.

For the no-drift case (i.e., for $r = q$),

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left[V_f(P_T, L, \infty)|\mathcal{F}_t\right] \qquad (36)\\
&= \begin{cases} A_0 I(-1, P_t, L) + X_r(-1, P_t, L) + B_0 K(1, P_t, L) + P_q(1, P_t, L) & \Leftarrow \quad \beta < 2 \\ B_0 K(-1, P_t, L) + X_r(-1, P_t, L) + A_0 I(1, P_t, L) + P_q(1, P_t, L) & \\ -G^{\infty}(t, T, P_t) & \Leftarrow \quad \beta > 2 \end{cases},
\end{aligned}$$

with the (already known) constants $A_0 \equiv A_0(L)$ and $B_0 \equiv B_0(L)$ (associated to the value of the project) being obtained as the solution of the system of equations (7) for any $\beta \neq 2$, the functions

$$P_q(1, P_t, L) := \begin{cases} e^{(r-q)\tau}\frac{P_t}{q}\,\Phi_{+1}\left(0, \frac{e^{-\alpha\tau}L^{2-\beta}}{f(\tau)}; 2+2\nu, \frac{P_t^{2-\beta}}{f(\tau)}\right) & \Leftarrow \quad \beta < 2 \\ \frac{1}{q}\left(f(\tau)\right)^{-\nu}\,\Phi_{-1}\left(-\nu, \frac{L^{2-\beta}}{f(\tau)}; 2+2\nu, \frac{e^{-\alpha\tau}P_t^{2-\beta}}{f(\tau)}\right) & \Leftarrow \quad \beta > 2 \end{cases} \tag{37}$$

and

$$X_r(-1,P_t,L) \tag{38}$$
$$:= \begin{cases} \frac{L}{r}P_t\left(f(\tau)\right)^{-\nu}\ \Phi_{-1}\left(-\nu,\frac{e^{-\alpha\tau}L^{2-\beta}}{f(\tau)};2+2\nu,\frac{P_t^{2-\beta}}{f(\tau)}\right)+\frac{L}{r}\frac{\Gamma\left(\nu,\frac{P_t^{2-\beta}}{2f(\tau)}\right)}{\Gamma(\nu)} & \Leftarrow\quad \beta<2 \\ \frac{L}{r}\,\Phi_{+1}\left(0,\frac{L^{2-\beta}}{f(\tau)};2+2\nu,\frac{e^{-\alpha\tau}P_t^{2-\beta}}{f(\tau)}\right) & \Leftarrow\quad \beta>2 \end{cases}$$

are borrowed from Dias et al. (2023, Proposition 4), with $\tau := T-t$,

$$\alpha := (r-q)|\beta-2|, \tag{39}$$

$$f(\tau) := \begin{cases} \frac{\delta^2|\beta-2|}{4(r-q)}\left(1-e^{-\alpha\tau}\right) & \Leftarrow\quad r\neq q \\ \frac{\delta^2(\beta-2)^2}{4}\tau & \Leftarrow\quad r=q \end{cases}, \tag{40}$$

and where $\Phi_\phi\left(p,x;v,\lambda\right)$ is interpreted as the p-th lower tail (if $\phi=-1$) or the p-th upper tail (if $\phi=+1$) truncated moment of a noncentral χ^2 random variable with v degrees of freedom and noncentrality parameter λ, as defined in Carr and Linetsky (2006, equations 5.12 and 5.11) and Dias and Nunes (2018, equations 21 and 24), $\frac{\Gamma(\nu,w)}{\Gamma(\nu)}$ is the incomplete gamma function ratio (or normalized function) shown in Olver et al. (2010, equation 8.2.4), with $\Gamma\left(\nu,w\right)$ and $\Gamma\left(\nu\right)$ being the upper incomplete gamma function and the Euler gamma function given in Olver et al. (2010, equations 8.2.2 and 5.2.1), respectively, for $\nu,w\in\mathbb{R}^+$, whereas the functions

$$I(\phi,P_t,L) := P_t^{\varphi}\sum_{j=0}^{\infty}\frac{\left(\frac{\sqrt{2r}}{\delta|\beta-2|}\right)^{2j+\nu}}{j!}\frac{(f(\tau))^j}{\Gamma(j+1+\nu)}\Phi_{+\tilde{\phi}}\left(j,\frac{L^{2-\beta}}{f(\tau)};2+2\nu,\frac{P_t^{2-\beta}}{f(\tau)}\right) \tag{41}$$

and

$$
\begin{aligned}
& K(\phi, P_t, L) && (42) \\
:= \quad & \frac{\pi}{2\sin(\nu\pi)} P_t^{\varphi} \sum_{j=0}^{\infty} \frac{\left(\frac{\sqrt{2r}}{\delta|\beta-2|}\right)^{2j-\nu}}{j!} \frac{(f(\tau))^{j-\nu}}{\Gamma(j+1-\nu)} \Phi_{+\tilde{\phi}} \left(j-\nu, \frac{L^{2-\beta}}{f(\tau)}; 2+2\nu, \frac{P_t^{2-\beta}}{f(\tau)} \right) \\
& + \mathbb{1}_{\{\beta<2, \phi=-1\}} \frac{1}{2} \Gamma\left(\nu, \frac{P_t^{2-\beta}}{2f(\tau)}\right) \left(\frac{\sqrt{2r}}{\delta|\beta-2|}\right)^{-\nu} \\
& - \frac{\pi}{2\sin(\nu\pi)} P_t^{\varphi} \sum_{j=0}^{\infty} \frac{\left(\frac{\sqrt{2r}}{\delta|\beta-2|}\right)^{2j+\nu}}{j!} \frac{(f(\tau))^{j}}{\Gamma(j+1+\nu)} \Phi_{+\tilde{\phi}} \left(j, \frac{L^{2-\beta}}{f(\tau)}; 2+2\nu, \frac{P_t^{2-\beta}}{f(\tau)} \right),
\end{aligned}
$$

with $\varphi := \mathbb{1}_{\{\beta<2\}}$ and $\tilde{\phi} := \phi \mathbb{1}_{\{\beta<2\}} - \phi \mathbb{1}_{\{\beta>2\}}$, are taken from Dias et al. (2023, Proposition 5).[4] Finally, the time-t expected value of the perpetual bubble, $G^{\infty}(t, T, P_t)$, is obtained from Dias et al. (2023, Proposition 7), that is

$$
G^{\infty}(t,T,P_t) = \frac{1}{\Gamma(\nu)} \int_T^{\infty} \left[\kappa_{T,u}^{\nu} e^{-r(u-T)} \int_1^{\infty} \frac{\exp\left(-\frac{s\tilde{f}A_{t,T,u}}{1+2sA_{t,T,u}}\right)}{(1+2sA_{t,T,u})^{1+\nu}} s^{\nu-1} ds \right] du, \tag{43}
$$

with

$$
\kappa_{T,u} := \begin{cases} \frac{2(r-q)}{\delta^2(2-\beta)\left[e^{(r-q)(2-\beta)(u-T)}-1\right]} & \Leftarrow \quad r \neq q \\ \frac{2}{\delta^2(2-\beta)^2(u-T)} & \Leftarrow \quad r = q \end{cases}, \tag{44}
$$

$$
\tilde{f} \equiv \tilde{f}(P_t) := \frac{e^{-\alpha\tau} P_t^{2-\beta}}{f(\tau)} \tag{45}
$$

and

$$
A_{t,T,u} := \begin{cases} \frac{1-e^{-\alpha(T-t)}}{2\left(e^{\alpha(u-T)}-1\right)} & \Leftarrow \quad r \neq q \\ \frac{T-t}{2(u-T)} & \Leftarrow \quad r = q \end{cases}. \tag{46}
$$

[4]Notice that the second line of Dias et al. (2023, equation 49) can be further simplified using the relation $\Gamma(\nu)\Gamma(1-\nu) = \pi/\sin(\nu\pi)$, for $\nu \neq 0, \pm 1, ...$, borrowed from Olver et al. (2010, equation 5.5.3), thus resulting in the second line of equation (42) of the present paper.

For $r \neq q$,

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left[V_f(P_T, L, \infty)|\mathcal{F}_t\right] \qquad (47)\\
=&\begin{cases} A_0 Y M_P(-1, P_t, L) + X_r(-1, P_t, L) + B_0 Y U_P(1, P_t, L) + P_q(1, P_t, L) & \Leftarrow \quad \beta < 2 \\ B_0 Y U_N(-1, P_t, L) + X_r(-1, P_t, L) + A_0 Y M_N(1, P_t, L) + P_q(1, P_t, L) & \\ -G^{\infty}(t, T, P_t) & \Leftarrow \quad \beta > 2 \end{cases},
\end{aligned}$$

with the (already known) constants $A_0 \equiv A_0(L)$ and $B_0 \equiv B_0(L)$ (associated to the value of the project) being obtained as the solution of the system of equations (18) and (19) for $\beta < 2$ and $\beta > 2$, respectively, and where the remaining required functions are borrowed from Dias et al. (2023, Proposition 6), that is

$$M_P(-1, P_t, L) := e^{(r-q)\tau} P_t \sum_{j=0}^{\infty} \frac{(c)_j}{(b)_j} \frac{1}{j!} \left(\frac{1 - e^{\alpha\tau}}{2}\right)^j \Phi_{-1}\left(j, \frac{e^{-\alpha\tau} L^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{P_t^{2-\beta}}{f(\tau)}\right), \quad (48)$$

with $c := a\mathbb{1}_{\{r<q,\beta<2\}} + (b-a)\mathbb{1}_{\{r>q,\beta<2\}}$ and $(c)_j$ is the Pochhammer's symbol as defined in Olver et al. (2010, equations 5.2.4 and 5.2.5),

$$M_N(1, P_t, L) := \sum_{j=0}^{\infty} \frac{(d)_j}{(b)_j} \frac{1}{j!} \left(\frac{1 - e^{-\alpha\tau}}{2}\right)^j \Phi_{-1}\left(j, \frac{L^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{e^{-\alpha\tau} P_t^{2-\beta}}{f(\tau)}\right), \quad (49)$$

with $d := (b-a)\mathbb{1}_{\{r<q,\beta>2\}} + a\mathbb{1}_{\{r>q,\beta>2\}}$,

$$\begin{aligned}
&U_P(1, P_t, L) \qquad (50)\\
=\;& \frac{\Gamma(-\nu)}{\Gamma(a-\nu)} e^{(r-q)\tau} P_t \sum_{j=0}^{\infty} \frac{(c)_j}{(b)_j} \frac{1}{j!} \left(\frac{1 - e^{\alpha\tau}}{2}\right)^j \Phi_{+1}\left(j, \frac{e^{-\alpha\tau} L^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{P_t^{2-\beta}}{f(\tau)}\right) \\
&+ \frac{\Gamma(\nu)}{\Gamma(a)} e^{(r-q)\tau} P_t \sum_{j=0}^{\infty} \frac{(c-\nu)_j}{(2-b)_j} \frac{\xi^j}{j!} \left(\xi \frac{1 - e^{\alpha\tau}}{2}\right)^{j-\nu} \Phi_{+1}\left(j - \nu, \frac{e^{-\alpha\tau} L^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{P_t^{2-\beta}}{f(\tau)}\right),
\end{aligned}$$

with $\xi := \mathbb{1}_{\{r<q\}} - \mathbb{1}_{\{r>q\}}$, and

$$
\begin{aligned}
& U_N(-1, P_t, L) && (51) \\
= \; & \frac{\Gamma(-\nu)}{\Gamma(a-\nu)} \sum_{j=0}^{\infty} \frac{(d)_j}{(b)_j} \frac{1}{j!} \left(\frac{1-e^{-\alpha\tau}}{2} \right)^j \Phi_{+1} \left(j, \frac{L^{2-\beta}}{f(\tau)}; 2+2\nu, \frac{e^{-\alpha\tau} P_t^{2-\beta}}{f(\tau)} \right) \\
& + \frac{\Gamma(\nu)}{\Gamma(a)} \sum_{j=0}^{\infty} \frac{(d-\nu)_j}{(2-b)_j} \frac{\tilde{\xi}^j}{j!} \left(\tilde{\xi} \frac{1-e^{-\alpha\tau}}{2} \right)^{j-\nu} \Phi_{+1} \left(j-\nu, \frac{L^{2-\beta}}{f(\tau)}; 2+2\nu, \frac{e^{-\alpha\tau} P_t^{2-\beta}}{f(\tau)} \right),
\end{aligned}
$$

with $\tilde{\xi} := \mathbb{1}_{\{r>q\}} - \mathbb{1}_{\{r<q\}} = -\xi$.

We recall that equation (35) allow us to compute the (time-t) value of the project until the maturity date of the FIT contract, i.e., only while the producer benefits from a contractual agreement guaranteing a minimum selling price of L. In practice, however, it might be reasonable to assume that the producer still runs the project after time T. Therefore, after the expiry date of the finite maturity FIT contract the producer is entitled to receive an unprotected cash flow equal to $e^{-r\tau}\frac{1}{q}\mathbb{E}_{\mathbb{Q}}\left[P_T \mid \mathcal{F}_t\right] - e^{-r\tau}\mathbb{E}_{\mathbb{Q}}\left[G_T^{\infty}(P_T) \mid \mathcal{F}_t\right] \mathbb{1}_{\{\beta>2\}}$ that should be economically interpreted as the cash amount that is obtained by selling energy at the prevailing market price for the remaining (perpetual) life of the project and with $G_T^{\infty}(P_T)$ being understood as a forward start perpetual bubble value in the spirit of Dias et al. (2023, equation 10).[5] Hence, the (time-t) full value of the project, $V_p(P_t, L, T)$, that contains both the protected and unprotected sources of value, is equal to[6]

$$
\begin{aligned}
& V_p(P_t, L, T) && (52) \\
= \; & \begin{cases} V_f(P_t, L, T) + \frac{P_t}{q} e^{-q(T-t)} & \Leftarrow \; \beta < 2 \\ V_f(P_t, L, T) + \frac{P_t}{q} e^{-q(T-t)} \frac{\gamma\left(\nu, \tilde{f}/2\right)}{\Gamma(\nu)} - e^{-r(T-t)} G^{\infty}(t, T, P_t) & \Leftarrow \; \beta > 2 \end{cases},
\end{aligned}
$$

where $\frac{\gamma(\nu,w)}{\Gamma(\nu)}$ is the incomplete gamma function ratio (or normalized function) shown in Olver et al. (2010, equation 8.2.4), with $\gamma(\nu, w)$ being the lower incomplete gamma function given

[5]This implies that we are implicitly assuming a perpetual concession with a finite maturity FIT. Nevertheless, the model can be also applied to the case of finite concessions having the same duration of the finite maturity FIT contract by simply ignoring the unprotected cash flow component.

[6]The proof of the present value of the unprotected cash flow component is presented in Appendix D.

in Olver et al. (2010, equations 8.2.1), for $\nu, w \in \mathbb{R}^+$, and with $G^\infty(t, T, P_t)$ being computed via equation (43).

Remark 4. *We notice that equation* (52) *nests immediately two special cases of interest: (i) if there is no guarantee—i.e., if* $L = 0$ *or, equivalently, if* $T \to t$ *so that* $\tau := T - t \to 0$*—, the present value of the project is simply equal to* $\lim_{T\to t} V_p(P_t, L, T) = \frac{P_t}{q} - G_t^\infty(P_t)\,\mathbb{1}_{\{\beta>2\}}$*, thus demonstrating that the value of the project is equal to a cash flow without a guarantee; (ii) in the case of a perpetual guarantee,* $\lim_{T\to\infty} V_p(P_t, L, T) = V_f(P_t, L, \infty)$*—i.e., the value of the project is obtained, as expected, via equations* (6) *and* (17) *for* $r = q$ *and* $r \neq q$*, respectively.*[7]

3.2.2. Value of the investment opportunity

Let us now consider that the investor has a perpetual investment opportunity to invest in a project with a sunk cost I. The goal is still to (simultaneously) determine the value of the (perpetual) option to invest in a project but now with a finite guarantee, $F_f(P_t, \overline{P}, T)$, and the optimal investment rule, with $\overline{P}$ being still interpreted as the optimal output price trigger level that will induce the investment.

Notice that we still have an infinite-horizon optimal stopping problem such as the one considered in Section 3.1.2. The only difference is that now once the (perpetual American-style) option to invest is exercised the owner is entitled to receive a project with a finite-lived guarantee (coupled with an unprotected cash flow component that is received after the expiry date of the finite maturity FIT contract), whereas in Section 3.1.2 the producer receives a project with an infinite-lived guarantee (hence, with no additional unprotected cash flow component).

Note also that, in contrast to the perpetual case, the value-matching and smooth-pasting conditions are now met in both regions $P_t < L$ and $P_t \geq L$ and, therefore, it is necessary to derive a system of nonlinear equations being dependent on the optimal threshold $\overline{P}$ in both regions and for $\beta < 2$ and $\beta > 2$. The next two propositions show how to evaluate a perpetual option to invest in a project with a finite-lived guarantee under the CEV process.

[7] The proof of such asymptotic analysis is presented in Appendix E.

Proposition 5. *Assume the CEV process with $r = q$ and the notation introduced in Proposition 1. The time-t value of the (perpetual American-style) option to invest in a project with a finite-lived guarantee is given by*

$$F_f(P_t, \overline{P}, T) \tag{53}$$
$$= \begin{cases} \left[V_p(\overline{P}, L, T) + G_t^{\infty}(\overline{P})\, \mathbb{1}_{\{\beta>2\}} - I\right] S_f(P_t, \overline{P}, \infty) - G_t^{\infty}(P_t)\, \mathbb{1}_{\{\beta>2\}} & \Leftarrow \quad P_t < \overline{P} \\ V_p(P_t, L, T) - I & \Leftarrow \quad P_t \geq \overline{P} \end{cases},$$

with the stochastic discount factor of the perpetual option to invest being still given by equation (27)*. Two cases must now be considered:*
(i) For $\overline{P} < L$, the optimal threshold $\overline{P}$ is obtained as the solution of the single nonlinear equation

$$\begin{aligned} &\left(\tfrac{L}{r} - e^{-r(T-t)} V_{f,nb}(\overline{P}, L, \infty) + \tfrac{\overline{P}}{q} e^{-q(T-t)} - I\right) \left(\overline{P}^{-1} + z'(\overline{P}) \tfrac{I_{\nu+1}(z(\overline{P}))}{I_\nu(z(\overline{P}))}\right) \\ &+ e^{-r(T-t)} \tfrac{\partial}{\partial \overline{P}} V_{f,nb}(\overline{P}, L, \infty) - \tfrac{1}{q} e^{-q(T-t)} = 0, \end{aligned} \tag{54}$$

for $\beta < 2$, and

$$\begin{aligned} &\left(\tfrac{L}{r} - e^{-r(T-t)} V_{f,nb}(\overline{P}, L, \infty) + g(\overline{P}) - I\right) \left(-z'(\overline{P}) \tfrac{K_{v+1}(z(\overline{P}))}{K_v(z(\overline{P}))}\right) \\ &+ e^{-r(T-t)} \tfrac{\partial}{\partial \overline{P}} V_{f,nb}(\overline{P}, L, \infty) - g'(\overline{P}) = 0, \end{aligned} \tag{55}$$

for $\beta > 2$, with

$$V_{f,nb}(\overline{P}, L, \infty) \tag{56}$$
$$:= \begin{cases} \left[A_0 I(-1, \overline{P}, L) + X_r(-1, \overline{P}, L) + B_0 K(1, \overline{P}, L) + P_q(1, \overline{P}, L)\right] & \Leftarrow \quad \beta < 2 \\ \left[B_0 K(-1, \overline{P}, L) + X_r(-1, \overline{P}, L) + A_0 I(1, \overline{P}, L) + P_q(1, \overline{P}, L)\right] & \Leftarrow \quad \beta > 2 \end{cases}$$

representing the time-t risk-neutral expectation (36) *but excluding the bubble effect,*

$$z'(\overline{P}) = \left(1 - \frac{\beta}{2}\right) \frac{2\sqrt{2r}}{\delta|\beta - 2|} \overline{P}^{-\frac{\beta}{2}}, \tag{57}$$

$$g(\overline{P}) = \frac{\overline{P}}{q} e^{-q(T-t)} \frac{\gamma\left(\nu, \tilde{f}(\overline{P})/2\right)}{\Gamma(\nu)} \tag{58}$$

and

$$g'(\overline{P}) = \frac{1}{q} e^{-q(T-t)} \left[\frac{\gamma\left(\nu, \tilde{f}(\overline{P})/2\right)}{\Gamma(\nu)} + \frac{2-\beta}{\Gamma(\nu)} \left[\tilde{f}(\overline{P})/2\right]^{\nu} e^{-\tilde{f}(\overline{P})/2} \right], \tag{59}$$

after employing Olver et al. (2010, equation 8.8.13) for tackling the derivative of the term involving the incomplete gamma function $\gamma(\nu, \tilde{f}(\overline{P})/2)$.

(ii) For $\overline{P} \geq L$, *the optimal threshold* $\overline{P}$ *and the constants* $A_1 \equiv A_1(\overline{P})$ *and* $B_1 \equiv B_1(\overline{P})$ *(associated to the value of the option to invest) are obtained as the solution of the system of equations*

$$\begin{cases} A_1 \overline{P}^{\frac{1}{2}} I_\nu(z(\overline{P})) - B_0 \overline{P}^{\frac{1}{2}} K_\nu(z(\overline{P})) - \frac{\overline{P}}{q}\left(1 + e^{-q(T-t)}\right) \\ \quad + e^{-r(T-t)} V_{f,nb}(\overline{P}, L, \infty) + I = 0 \\ A_1 I_{\nu+1}(z(\overline{P})) + B_0 K_{\nu+1}(z(\overline{P})) \\ \quad - e^{-r(T-t)} \frac{\delta}{\sqrt{2r}} \overline{P}^{\frac{\beta-3}{2}} \left[V_{f,nb}(\overline{P}, L, \infty) - \overline{P} \frac{\partial}{\partial \overline{P}} V_{f,nb}(\overline{P}, L, \infty)\right] - \frac{\delta I}{\sqrt{2r}} \overline{P}^{\frac{\beta-3}{2}} = 0 \end{cases}, \tag{60}$$

for $\beta < 2$, *and*

$$\begin{cases} B_1 \overline{P}^{\frac{1}{2}} K_\nu(z(\overline{P})) - A_0 \overline{P}^{\frac{1}{2}} I_\nu(z(\overline{P})) - \frac{\overline{P}}{q} - g(\overline{P}) \\ \quad + e^{-r(T-t)} V_{f,nb}(\overline{P}, L, \infty) + I = 0 \\ B_1 K_{\nu+1}(z(\overline{P})) + A_0 I_{\nu+1}(z(\overline{P})) - \frac{\delta}{\sqrt{2r}} \overline{P}^{\frac{\beta-1}{2}} \left[\frac{1}{q} + g'(\overline{P})\right] \\ \quad + e^{-r(T-t)} \frac{\delta}{\sqrt{2r}} \overline{P}^{\frac{\beta-1}{2}} \frac{\partial}{\partial \overline{P}} V_{f,nb}(\overline{P}, L, \infty) = 0 \end{cases}, \tag{61}$$

for $\beta > 2$, *with the (already known) constants* $A_0 \equiv A_0(L)$ *and* $B_0 \equiv B_0(L)$ *(associated to the value of the project) being obtained as the solution of the system of equations* (7) *for any* $\beta \neq 2$.

Proof. Please see Appendix F. ■

Proposition 6. *Assume the CEV process with $r \neq q$ and the notation introduced in Proposition 2. The time-t value of the (perpetual American-style) option to invest in a project with a finite-lived guarantee is still given by equation (53), but with the stochastic discount factor of the perpetual option to invest being given by equation (31). Again, two cases must be considered:*

(i) For $\overline{P} < L$, the optimal threshold $\overline{P}$ is obtained as the solution of the single nonlinear equation

$$\begin{aligned}&\left(\tfrac{L}{r} - e^{-r(T-t)}V_{f,nb}(\overline{P}, L, \infty) + \tfrac{\overline{P}}{q}e^{-q(T-t)} - I\right)\left(\overline{P}^{-1} + \left[\tfrac{\epsilon-1}{2} + \tfrac{\frac{a}{b}M(a+1,b+1,x(\overline{P}))}{M(a,b,x(\overline{P}))}\right]x'(\overline{P})\right)\\ &+e^{-r(T-t)}\tfrac{\partial}{\partial \overline{P}}V_{f,nb}(\overline{P}, L, \infty) - \tfrac{1}{q}e^{-q(T-t)} = 0,\end{aligned} \tag{62}$$

for $\beta < 2$, and

$$\begin{aligned}&\left(\tfrac{L}{r} - e^{-r(T-t)}V_{f,nb}(\overline{P}, L, \infty) + g(\overline{P}) - I\right)\left(\tfrac{\epsilon-1}{2} - \tfrac{aU(a+1,b+1,x(\overline{P}))}{U(a,b,x(\overline{P}))}\right)x'(\overline{P})\\ &+e^{-r(T-t)}\tfrac{\partial}{\partial \overline{P}}V_{f,nb}(\overline{P}, L, \infty) - g'(\overline{P}) = 0,\end{aligned} \tag{63}$$

for $\beta > 2$, with

$$\begin{aligned}&V_{f,nb}(\overline{P}, L, \infty) \qquad (64)\\ :=& \begin{cases}\left[A_0 Y M_P(-1, \overline{P}, L) + X_r(-1, \overline{P}, L) + B_0 Y U_P(1, \overline{P}, L) + P_q(1, \overline{P}, L)\right] & \Leftarrow \quad \beta < 2\\ \left[B_0 Y U_N(-1, \overline{P}, L) + X_r(-1, \overline{P}, L) + A_0 Y M_N(1, \overline{P}, L) + P_q(1, \overline{P}, L)\right] & \Leftarrow \quad \beta > 2\end{cases}\end{aligned}$$

representing the time-t risk-neutral expectation (47) but excluding the bubble effect and

$$x'(\overline{P}) = (2-\beta)\,\frac{2|r-q|}{\delta^2|\beta-2|}\overline{P}^{1-\beta}. \tag{65}$$

(ii) For $\overline{P} \geq L$, the optimal threshold $\overline{P}$ and the constants $A_1 \equiv A_1(\overline{P})$ and $B_1 \equiv B_1(\overline{P})$ (associated to the value of the option to invest) are obtained as the solution of the system of

equations

$$\begin{cases} A_1 e^{(\epsilon-1)\frac{x(\overline{P})}{2}} Y\overline{P} M(a,b,x(\overline{P})) - B_0 e^{(\epsilon-1)\frac{x(\overline{P})}{2}} Y\overline{P} U(a,b,x(\overline{P})) \\ \quad -\frac{\overline{P}}{q}\left(1+e^{-q(T-t)}\right) + e^{-r(T-t)} V_{f,nb}(\overline{P},L,\infty) + I = 0 \\ A_1 \frac{a}{b} M(a+1,b+1,x(\overline{P})) + B_0 a U(a+1,b+1,x(\overline{P})) \\ \quad - \left[I + e^{-r(T-t)} V_{f,nb}(\overline{P},L,\infty) - \frac{\overline{P}}{q}\left(1+e^{-q(T-t)}\right)\right] \\ \quad \times Y^{-1} e^{(1-\epsilon)\frac{x(\overline{P})}{2}} \overline{P}^{-1}\left(\frac{\epsilon-1}{2} + \frac{\delta^2}{2|r-q|}\overline{P}^{\beta-2}\right) \\ \quad -Y^{-1} e^{(1-\epsilon)\frac{x(\overline{P})}{2}} \frac{\delta^2}{2|r-q|}\overline{P}^{\beta-2}\left[\frac{1}{q}\left(1+e^{-q(T-t)}\right) - e^{-r(T-t)}\frac{\partial}{\partial \overline{P}} V_{f,nb}(\overline{P},L,\infty)\right] = 0 \end{cases}, \tag{66}$$

for $\beta < 2$*, and*

$$\begin{cases} B_1 e^{(\epsilon-1)\frac{x(\overline{P})}{2}} Y U(a,b,x(\overline{P})) - A_0 e^{(\epsilon-1)\frac{x(\overline{P})}{2}} Y M(a,b,x(\overline{P})) \\ \quad -\frac{\overline{P}}{q} - g(\overline{P}) + e^{-r(T-t)} V_{f,nb}(\overline{P},L,\infty) + I = 0 \\ B_1 a U(a+1,b+1,x(\overline{P})) + A_0 \frac{a}{b} M(a+1,b+1,x(\overline{P})) \\ \quad - \left[\frac{\overline{P}}{q} + g(\overline{P}) - e^{-r(T-t)} V_{f,nb}(\overline{P},L,\infty) - I\right]\frac{\epsilon-1}{2} Y^{-1} e^{(1-\epsilon)\frac{x(\overline{P})}{2}} \\ \quad -Y^{-1} e^{(1-\epsilon)\frac{x(\overline{P})}{2}} \frac{\delta^2}{2|r-q|}\overline{P}^{\beta-1}\left[\frac{1}{q} + g'(\overline{P}) - e^{-r(T-t)}\frac{\partial}{\partial \overline{P}} V_{f,nb}(\overline{P},L,\infty)\right] = 0 \end{cases}, \tag{67}$$

for $\beta > 2$*, with the (already known) constants* $A_0 \equiv A_0(L)$ *and* $B_0 \equiv B_0(L)$ *(associated to the value of the project) being obtained as the solution of the system of equations* (18) *and* (19) *for* $\beta < 2$ *and* $\beta > 2$*, respectively.*

Proof. Please see Appendix G. ■

To sum up, Propositions 5 and 6 can be used to analyze the optimal decision to invest in the case of a FIT contract with a finite maturity guarantee: exercise the (perpetual) option to invest only if $P_t \geq \overline{P}$; otherwise, i.e., if $P_t < \overline{P}$, it is better to wait and keep the option alive.

Notice that Propositions 5 and 6 are the CEV counterparts of the solution offered in Barbosa et al. (2018, Section 3.2) for the GBM process, whose numerical results can be obtained by taking $\beta \to 2$ in our solutions.[8] Fortunately, the optimal thresholds of the investment problem under analysis are independent of any bubble effect because both the perpetual option to invest and the value of the active project possess the same bubble. Therefore, equations (56) and (64) do not include any bubble effect for computing the trigger levels $\overline{P}$ in Propositions 5 and 6.

Note also that the value-matching and smooth-pasting conditions are now met in both regions $\overline{P} \geq L$ and $\overline{P} < L$. The case $\overline{P} < L$ in Propositions 5 and 6 requires only the use of a single nonlinear equation to determine the corresponding optimal trigger level $\overline{P}$ (in particular, equations (54) and (62) for $\beta < 2$ and equations (55) and (63) for $\beta > 2$). As expected, the associated constants of the perpetual option to invest ($A_1 \equiv A_1(\overline{P})$ for $\beta < 2$ and $B_1 \equiv B_1(\overline{P})$ for $\beta > 2$) can be calculated using equations (F.4) and (F.6), for $r = q$, and equations (G.3) and (G.5), for $r \neq q$, respectively.

To determine which region should be used to calculate the optimal trigger $\overline{P}$ it is necessary to find first the investment cost level I_L^* that separates the regions $\overline{P} \geq L$ and $\overline{P} < L$. This is accomplished by replacing $\overline{P}$ by the fixed price floor L (previously defined by the government) in the systems of equations (60), (61), (66) and (67) and solving each one with respect to $I \equiv I_L^*$. Notice that these are the equations that should be used since are the ones for which the condition $\overline{P} = L$ is valid. Similarly to the perpetual case, I_L^* is economically interpreted as the required investment cost that makes the current L the optimal trigger of investment, but now for the finite maturity case. Again, the constants A_0 and B_0 are functions of L and, hence, they are known in advance using the systems of equations (7), (18) and (19) or, alternatively, the analytic solutions offered in Remarks 1 and 2. Armed with the values of $A_0 \equiv A_0(L)$ and $B_0 \equiv B_0(L)$, the I_L^* level is then easily computed using the systems of equations (60) or (61), for the case $r = q$, and (66) or (67), for the case $r \neq q$. If $I_L^* \leq I$, the optimal threshold $\overline{P}$ is calculated using the region $\overline{P} \geq L$, i.e., $\overline{P}$ is obtained by solving the systems of nonlinear equations (60) or (61), for $r = q$, and (66) or (67), for $r \neq q$. If $I_L^* > I$, the optimal trigger $\overline{P}$ is computed in the region $\overline{P} < L$, i.e., $\overline{P}$ is calculated

[8] By the way, and for the sake of clarity, we notice that there is a very small typo in Barbosa et al. (2018, equation C.3). The correct formula is the one given in Barbosa et al. (2018, equation 26).

by solving the single nonlinear equations (54) or (55), for $r = q$, and (62) or (63), for $r \neq q$.

The knowledge of I_L^* under the finite guarantee case allow us to find an additional interesting point that is obtained in the appropriate region $\overline{P} < L$ or $\overline{P} \geq L$, depending on $I_L^* > I$ or $I_L^* \leq I$, respectively. For instance, suppose that $I_L^* \leq I$ for the case with $\beta < 2$ and $r = q$. By solving the system of equations (60) with respect to $I \equiv I^*$, and with $\overline{P}$ replaced by the given time-t price P_t and assuming a fixed price floor L (that is subject to the government policy), it is possible to find the required level of investment I^* that would induce the firm to exercise the option to invest immediately, which can then be compared with the initial investment cost I of the project. As before, this I^* value indicates the required level of investment that would make the time-t price P_t the trigger value to invest. Hence, it would be preferable for the firm to invest immediately if $I \leq I^*$ because $P_t = \overline{P}$ at the I^* level. A similar rational can be applied to the case of other combinations of β parameters and drift specifications.

Several other interesting points from the perspective of a policymaker can also be determined. For instance, taking now the initial investment cost I of the project as given and using the previous example, we can determine the point where the optimal threshold value $\overline{P}$ is equal to the price floor L, by solving the system of equations (60) with respect to $L \equiv L^*$, which can then be compared with the minimum price guarantee L that is offered by the government. Note that if the policymaker sets the minimum price guarantee $L \geq L^*$, the investor starts immediately the project and receives a revenue from the guarantee instead of a revenue from the market price. Again, a similar rational can be used in the case of other combinations of β parameters and drift specifications.

Even though the levels of I_L^* and L^* under the perpetual case have been shown to be independent of β, such independence is not valid in the case of FIT contracts with a finite maturity guarantee. In this case, in the range of price floors $L \in [0, L^*]$ both the price floor L^* and the optimal trigger value $\overline{P}$ decrease as we move further away to the left of $\beta < 2$ and increase as β rises for the case with $\beta > 2$. This contrasting behavior when compared with the perpetual case is explained by the finite nature of the guarantee, i.e., now the producer is not entitled to receive a fixed guarantee forever and, hence, the effect of β on I_L^* and L^* do not disappear. Again, since different CEV models imply the existence of different optimal triggers (including now distinct L^* levels), a policy scheme based on a GBM assumption might imply a too high or a too low price floor to encourage a producer to exercise the option to invest if the specific investment problem is better described with an alternative CEV specification.

A further interesting point from the policymaker perspective is the price floor level L_0 that turns the NPV of the project equal to zero, because any value of L above this point generates a positive NPV independently of the market price P. Notice that if $L \geq L_0$ the investment will be deployed immediately because there is no waiting option. Since under the CEV process with $\beta < 2$ the origin is attainable (and assumed to be absorbing), this point can be determined analytically by solving the equation NPV $:= V_p(P_t, L_0, T) - I = 0$ with respect to L_0 when $P_t = 0$ and with $V_p(0, L_0, T)$ being defined as the project value given in equation (52), which yields[9]

$$L_0 = \frac{Ir}{1 - e^{-r(T-t)}}. \tag{68}$$

However, since under the CEV process with $\beta > 2$ the origin is not attainable, it is not possible to obtain a closed-form solution as given in equation (68). Nevertheless, as shown in Appendix H, the level L_0 can still be determined numerically by solving the equation

$$\lim_{P_t \to 0^+} V_p(P_t, L_0, T) - I = 0, \tag{69}$$

$$\text{i.e.,} \quad \begin{cases} \lim_{P_t \to 0^+} \left[B_0(L_0) K(-1, P_t, L_0) \right] + \frac{L_0}{r}\left(1 - e^{-r(T-t)}\right) - I = 0 & \Leftarrow \quad r = q \\ \lim_{P_t \to 0^+} \left[B_0(L_0) YU_N(-1, P_t, L_0) \right] + \frac{L_0}{r}\left(1 - e^{-r(T-t)}\right) - I = 0 & \Leftarrow \quad r \neq q \end{cases},$$

with respect to L_0 when $P_t \to 0^+$ and with $B_0(L_0)$ being computed by equations (9) and (23) for $r = q$ and $r \neq q$, respectively. The differences found between equations (68) and (69) are clearly expected and can be explained by the fact that the process with $\beta > 2$ is

[9] The proof of such asymptotic analysis for both $\beta < 2$ and $\beta > 2$ is presented in Appendix H using results borrowed from Sankaran (1963), Temme (1993), Johnson et al. (1995, Chapter 29) and Ruas et al. (2013).

not trapped at the origin at time $t = 0$ and, therefore, the forward start perpetuity that is included in the project value (52) is influenced by a nonzero forward value P_T. By contrast, if $\beta < 2$ the process is absorbed at the origin and, hence, the forward start perpetuity that is included in the project value (52) does not depend on the forward value P_T because $P_T = 0$ (i.e., the process remains in a cemetery state forever).

Therefore, if the government sets a policy with a price floor $L \geq L_0$, the investment will be made immediately generating a risk-free profit. Moreover, by setting L in the range $L^* < L < L_0$, the policymaker gives the investor a waiting option and the project starts with a revenue from the guarantee.

Even though equation (68) that has been obtained using the CEV model with $\beta < 2$ is exactly the same to the one derived in Barbosa et al. (2018, equation 31) under the GBM assumption[10]—thus, implying that L_0 is, as expected, independent of β (for $\beta < 2$)—, the range of values $]L^*, L_0[$ is not the same across models with different skew patterns due to the influence of the β parameter on L^*. Furthermore, the limit of (68) when $T \to \infty$ is consistent with the observations that have been made in Remark 3, which also implies that the range $]L^*, L_0[$ vanishes as $T \to \infty$. This means that, under the perpetual case, there is no range of values in which the investor has an option to wait and a revenue from the guarantee when the project starts.

[10]A tiny technical detail arises here that deserves to be mentioned for the sake of completeness. Although Barbosa et al. (2018, equation 31) has been derived by simply imposing $P_t = 0$ in Barbosa et al. (2018, equation 19), this should not be formally made because, as argued, for instance, in Carr and Linetsky (2006, Remark 5.2) or Nunes et al. (2015, page 352), the limiting process of GBM never hits zero (i.e., zero is a natural boundary and not an exit boundary). Nevertheless, Barbosa et al. (2018, equation 31) is still obtained if we compute the limit of Barbosa et al. (2018, equation 19) when $P_t \to 0^+$.

3.3. Derivatives of the forward start perpetuity components

The implementation of Proposition 5 still requires the computation of the derivative of the risk-neutral expectation (56) with respect to $\overline{P}$, that is

$$\begin{aligned} & \frac{\partial}{\partial \overline{P}} V_{f,nb}(\overline{P}, L, \infty) \qquad (70) \\ = & \frac{\partial}{\partial \overline{P}} \begin{cases} \left[A_0 I(-1, \overline{P}, L) + X_r(-1, \overline{P}, L) + B_0 K(1, \overline{P}, L) + P_q(1, \overline{P}, L)\right] & \Leftarrow \quad \beta < 2 \\ \left[B_0 K(-1, \overline{P}, L) + X_r(-1, \overline{P}, L) + A_0 I(1, \overline{P}, L) + P_q(1, \overline{P}, L)\right] & \Leftarrow \quad \beta > 2 \end{cases}, \end{aligned}$$

with the (already known) constants A_0 and B_0 (associated to the value of the project) being obtained as the solution of the system of equations (7) for any $\beta \neq 2$, whereas the computation of Proposition 6 requires the derivative of the risk-neutral expectation (64) with respect to $\overline{P}$, that is

$$\begin{aligned} & \frac{\partial}{\partial \overline{P}} V_{f,nb}(\overline{P}, L, \infty) \qquad (71) \\ = & \frac{\partial}{\partial \overline{P}} \begin{cases} \left[A_0 Y M_P(-1, \overline{P}, L) + X_r(-1, \overline{P}, L) + B_0 Y U_P(1, \overline{P}, L)\right. & \\ \left. + P_q(1, \overline{P}, L)\right] & \Leftarrow \quad \beta < 2 \\ \left[B_0 Y U_N(-1, \overline{P}, L) + X_r(-1, \overline{P}, L) + A_0 Y M_N(1, \overline{P}, L)\right. & \\ \left. + P_q(1, \overline{P}, L)\right] & \Leftarrow \quad \beta > 2 \end{cases}, \end{aligned}$$

with the (already known) constants A_0 and B_0 (associated to the value of the project) being obtained as the solution of the system of equations (18) and (19) for $\beta < 2$ and $\beta > 2$, respectively.

Even though the derivatives (70) and (71) can be easily computed through a finite difference scheme, we prefer to use the analytical expressions shown in Appendix I that have been derived following the insights of Carr and Linetsky (2006), Ruas et al. (2013) and Dias and Nunes (2018).

Remark 5. *Clearly, the delta of a finite maturity FIT contract, i.e., $\partial V_f(P_t, L, T)/\partial P_t$, is straightforward to be evaluated via the appropriate combination of our analytical results. Moreover, Dias et al. (2023, equation 73) implies that the delta of a finite maturity FIT is* *equal to the delta of a finite-lived cap. Hence, we are able to extend the computation of the delta shown in Shackleton and Wojakowski (2007, equation 28) under the GBM diffusion to the more general CEV process. This should be helpful for the potential design of dynamic hedging strategies of such contracts.*

4. Numerical results

The goal of this section is to present some numerical experiments of the novel theoretical results proposed in this paper. To accomplish this purpose, we adopt the base case parameters configuration of Barbosa et al. (2018) that is associated to a typical European onshore wind farm. Even though we are adopting the case of an onshore wind farm in our numerical examples, our analytical formulae is suitable to be used in any other technology with FIT contracts.

Following Barbosa et al. (2018), we assume a riskless interest rate $r = 5\%$, a dividend yield (or rate of return shortfall) $q = 5\%$, a volatility $\sigma = 19\%$, a price floor $L = 25$ EUR/MWh and a finite duration of $T = 15$ years. It is also assumed that each wind turbine has a capacity of 2 MW and requires a total investment cost of 1.5 million EUR/MW, thus leading to a total investment cost per wind turbine of 3 million Euros, which implies an investment cost per MWh of $I = 3,000,000/(0.3 \times 2 \text{ MWh} \times 24 \text{ h} \times 365 \text{ days})$ for the case of a power plant with an assumed capacity factor of 30%. Although we analyze the investment threshold for a single wind turbine, the results can be easily calculated for a wind park with N turbines.

Figure 1 presents the optimal thresholds $\overline{P}$ under the case of a perpetual guarantee as a function of the price floor L for different CEV models with $\beta \in \{-6, -4, -2, 0, 1, 2, 3, 3.5\}$. The dashed line displays the price floor level L that separates the two regions in the graph, i.e., $\overline{P} \geq L$ and $\overline{P} < L$. Note that only the region of trigger values above or equal to the dashed line L represents the market conditions for which $\overline{P} \geq L$. Following the insights of Remark 3, trigger values below the dashed line L are not admissible for a FIT scheme with a perpetual guarantee since the value-matching and smooth-pasting conditions are not met and, hence, the investment problem is not well behaved in this case. The case with

$\beta \neq 2$ has been computed through Proposition 3, whereas the case with $\beta = 2$ has been implemented via Barbosa et al. (2018, equation 16).

We first notice that the triggers of all tested β parameters decrease as the price floor rises, thus inducing earlier investment decisions for higher price floor values. Moreover, for the range of price floors $L \in [0, L^*[$ the optimal trigger value $\overline{P}$ decreases as we move further away to the left of $\beta < 2$ and increases as β rises for the case with $\beta > 2$. This is explained by the different behavior of the leverage effects attached to CEV models for $\beta < 2$ and $\beta > 2$. Note also that the aforementioned independence of the β parameter is observed only at the optimal trigger level $\overline{P} \equiv L^* = 28.54$, which is exactly the L^* value that is obtained under the nested GBM assumption considered in Barbosa et al. (2018, equation 18). The perpetual nature of the guarantee makes the effect of β on L^* (and I_L^*) to disappear thus leading to this interesting finding.

In the case of a FIT contract with a perpetual guarantee under the CEV model, the value of $L^* = 28.54$ can be obtained numerically by solving the systems of four equations referred in Subsection 3.1.2 for each β value—with the relevant constants being functions of L^*, i.e., $A_0 \equiv A_0(L^*)$, $B_0 \equiv B_0(L^*)$, $A_1 \equiv A_1(L^*)$ and $B_1 \equiv B_1(L^*)$—or, alternatively, via the closed-form solution $L^* = rI$, which directly highlights the independence of L^* with respect to β. Nevertheless, different optimal triggers for distinct β values are obtained in the range $L \in [0, 28.54[$, which implies that a government that (wrongly) assumes a GBM process to define a FIT policy might be paying a too high or a too low price floor to stimulate a firm to exercise the option to invest if the specific investment problem is better described by a different CEV specification.

Figure 2 displays the optimal thresholds $\overline{P}$ under the case of a finite maturity guarantee as a function of the price floor L for different CEV models with $\beta \in \{-6, -4, -2, 0, 1, 2, 2.7, 4\}$. Again, the dashed line displays the price floor level L that separates the two regions in the graph, i.e., $\overline{P} \geq L$ and $\overline{P} < L$. As mentioned before, both regions are now valid because the value-matching and smooth-pasting conditions are always met in the presence of a finite maturity guarantee. The case with $\beta \neq 2$ has been computed through Proposition 5, whereas the case with $\beta = 2$ has been implemented via Barbosa et al. (2018, equation 26).

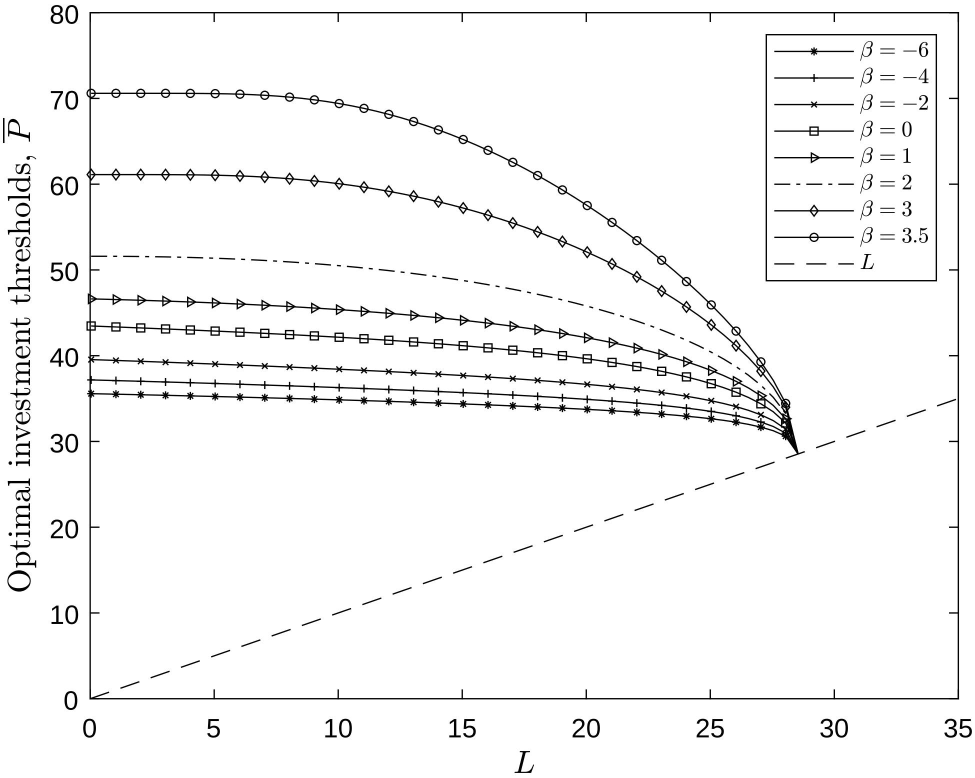

Figure 1: Perpetual optimal investment thresholds, $\overline{P}$, as a function of the price floor L for different β parameters with $P_0 = 25$.
Other parameters borrowed from Barbosa et al. (2018): $\sigma(P_0) = 0.19$, $r = 0.05$, $q = 0.05$ and total investment cost of 3 million Euros, which implies an investment cost per MWh of $I = 3,000,000/(0.3 \times 2 \text{ MWh} \times 24 \text{ h} \times 365 \text{ days})$. The case with $\beta \neq 2$ has been computed through Proposition 3, whereas the case with $\beta = 2$ has been implemented via Barbosa et al. (2018, equation 16).

Similarly to the perpetual case, the region $\overline{P} \geq L$ under the finite maturity case originates triggers that decrease as the price floor increases for the whole set of β parameters. As expected, this induces earlier investment decisions for higher price floor values. Furthermore, for the range of price floors $L \in [0, L^*]$ the optimal trigger value $\overline{P}$ still decreases as we move further away to the left of $\beta < 2$ and increases as β rises for the case with $\beta > 2$. The rational is again explained by the different behavior of the leverage effects associated to CEV models with $\beta < 2$ and $\beta > 2$. From a policymaker perspective, the region $\overline{P} \geq L$ creates scenarios for which the project starts receiving a revenue from the market price rather than a revenue from the guarantee. It is also possible to observe that for a wide range of price floors $L \in [0, L^*]$ the policy scheme is useless in practical terms because the

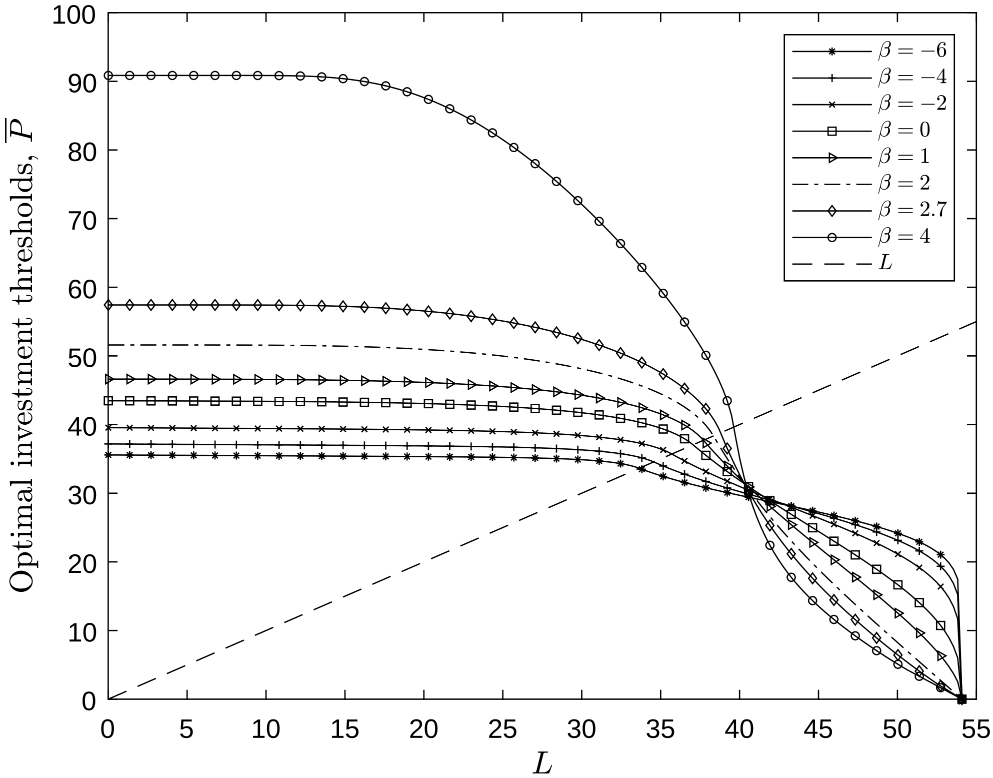

Figure 2: Finite maturity optimal investment thresholds, $\overline{P}$, as a function of the price floor L for different β parameters with $P_0 = 25$.
Other parameters borrowed from Barbosa et al. (2018): $T = 15$, $\sigma(P_0) = 0.19$, $r = 0.05$, $q = 0.05$ and total investment cost of 3 million Euros, which implies an investment cost per MWh of $I = 3,000,000/(0.3 \times 2 \text{ MWh} \times 24 \text{ h} \times 365 \text{ days})$. The case with $\beta \neq 2$ has been computed through Proposition 5, whereas the case with $\beta = 2$ has been implemented via Barbosa et al. (2018, equation 26).

lines representing the optimal thresholds are almost flat, at least for lower values of L and, in particular, for the case with $\beta < 2$.

In the case of a FIT contract with a finite maturity guarantee, however, the trigger point $\overline{P} \equiv L^*$ depends on the β parameter and needs to be determined numerically. Table 1 shows the optimal trigger points $\overline{P}$ that are equal to the price floor $L \equiv L^*$ of the finite maturity case displayed in Figure 2. We first note that the trigger point $L^* = 28.54$ determined for the perpetual guarantee case is always lower than the ones reported in Table 1 for the finite maturity case. This is expected since an investor requires a higher threshold to invest in a project that only ensures a finite guarantee instead of a perpetual one. We also conclude that the range $[0, L^*]$ widens as the β parameter rises. This should have important implications for a policymaking perspective. For instance, comparing the cases with $\beta = -6$ and $\beta = 2$, we observe that $(38.35 - 33.64) = 4.71$ EUR/MWh (per turbine) thus leading to an annual amount of $(38.35 - 33.64) \times 24$ h $\times$ 365 days $= 41,259.60$ EUR per turbine. For a typical European onshore wind farm with 25 wind turbines this would lead to a difference of 1,031,490 Euros. This would represent an amount that tax payers would be paying in excess (only for a single wind farm) if the regulator negotiated the contract based on a GBM assumption when the price process was better described by a CEV process with $\beta = -6$.

Table 1: Optimal investment thresholds $\overline{P}$ that are equal to the price floor $L \equiv L^*$ of the finite maturity case displayed in Figure 2.

β	-6	-4	-2	0	1	2	2.7	4
L^*	33.64	34.56	35.71	37.05	37.71	38.35	38.80	39.64

This table shows the optimal trigger points $\overline{P}$ that are equal to the price floor $L \equiv L^*$ of the finite maturity case displayed in Figure 2. For $r = q$ and $\beta < 2$, L^* is calculated by solving the systems of equations (7) with $L \equiv L^*$ and (60) with $\overline{P} \equiv L^*$. For $r = q$ and $\beta > 2$, L^* is computed by solving the systems of equations (7) with $L \equiv L^*$ and (61) with $\overline{P} \equiv L^*$. Notice that the relevant constants should be taken as functions of L^*, i.e., $A_0 \equiv A_0(L^*)$, $B_0 \equiv B_0(L^*)$, $A_1 \equiv A_1(L^*)$ and $B_1 \equiv B_1(L^*)$. Finally, for $\beta = 2$, L^* is determined via Barbosa et al. (2018, equation 30).

In contrast to the perpetual case, the region $\overline{P} < L$ allows the existence of threshold values that range in the interval $L \in [L^*, L_0]$, with $L_0 = 54.09$ EUR/MWh being determined through equations (68) and (69) for $\beta < 2$ and $\beta > 2$, respectively. Interestingly, this is the value that is obtained for the GBM process calculated via Barbosa et al. (2018, equation 31). This means that if the government sets a policy with a price floor $L \geq L_0$, then a positive NPV arises and the firm can undertake the project immediately. Finally, by setting the price floor in the range $L \in]L^*, L_0[$, the policymaker gives the investor a waiting option and the project starts with a revenue from the finite maturity guarantee. Notice also that in this region there is a price floor level L for which the optimal thresholds are the same for pairs of β parameters. Moreover, after this point lower β parameters lead to higher threshold values.

Even though the optimization problem involving the implementation of Propositions 3, 4, 5 and 6 is easily tackled in any computer language, the calculation and the convergence of the required optimal threshold values $\overline{P}$ depend on the combination of the used parameters. Nevertheless, for the cases where it might be hard to obtain convergent solutions it is possible to apply a linear interpolation scheme. For example, a simple linear interpolation between the values reported in Table 1 for $\beta = 1$ and $\beta = 2.7$ (both calculated through Proposition 5) allows us to find an interpolated value for the case with $\beta = 2$, which can then be compared with the one obtained via Barbosa et al. (2018, equation 30), that is: $37.71+(38.80-37.71)\times(2-1)/(2.7-1) = 38.35$, which is exactly the value that is obtained under the GBM process. Clearly, this is a further simple confirmation of the robustness of our theoretical results.

Table 2: Optimal investment thresholds of perpetual and finite maturity FIT contracts.

#	L	r	q	-4	-2	0	2	4	Maximum % difference	
Panel A: Optimal investment thresholds under the perpetual case										
1	10	0.07	0.03	41.03	42.63	46.16	54.88	106.44	25.24	93.93
2	15	0.07	0.03	41.03	42.61	46.10	54.75	105.99	25.06	93.61
3	20	0.07	0.03	41.03	42.59	46.01	54.41	102.64	24.59	88.66
4	10	0.05	0.05	36.28	38.42	42.16	50.51	89.16	28.16	76.53
5	15	0.05	0.05	35.70	37.69	41.17	48.78	81.15	26.82	66.36
6	20	0.05	0.05	34.91	36.66	39.65	45.80	67.03	23.77	46.34
7	10	0.03	0.07	32.44	33.88	36.79	43.59	61.56	25.58	41.22
8	15	0.03	0.07	27.13	27.37	28.40	31.14	34.37	12.88	10.36
9	20	0.03	0.07	-	-	-	-	-	-	-
Panel B: Optimal investment thresholds under the finite case with $T = 5$ years										
1	15	0.05	0.05	37.17	39.55	43.47	51.59	90.85	27.95	76.10
2	20	0.05	0.05	37.17	39.54	43.46	51.58	90.77	27.93	76.00
3	25	0.05	0.05	37.16	39.52	43.42	51.50	90.26	27.85	75.28
4	30	0.05	0.05	37.09	39.44	43.30	51.25	88.81	27.62	73.29
5	35	0.05	0.05	36.70	39.10	42.93	50.67	86.08	27.56	69.89
6	40	0.05	0.05	35.31	37.59	41.76	49.44	81.90	28.59	65.64
7	15	0.03	0.04	33.66	35.25	37.49	40.63	47.98	17.14	18.09
8	20	0.03	0.04	33.64	35.22	37.44	40.55	47.80	17.03	17.89
9	25	0.03	0.04	33.57	35.13	37.30	40.25	46.98	16.59	16.71
10	30	0.03	0.04	33.33	34.84	36.90	39.49	45.02	15.58	14.00
11	35	0.03	0.04	32.16	33.82	35.78	37.74	41.31	14.80	9.46
12	40	0.03	0.04	31.82	32.82	30.83	34.54	32.16	10.74	6.90

This table shows the values of optimal investment thresholds of perpetual and finite maturity FIT contracts under the CEV process with $\beta \in \{-4, -2, 0, 2, 4\}$ and $P_0 = 25$. The price floor L, the riskless interest rate r and the dividend yield (or rate of return shortfall) q vary as indicated in the second, third and fourth columns, respectively. Other parameters borrowed from Barbosa et al. (2018): $\sigma(P_0) = 0.19$ and total investment cost of 3 million Euros, which implies an investment cost per MWh of $I = 3,000,000/(0.3 \times 2 \text{ MWh} \times 24 \text{ h} \times 365 \text{ days})$. The two utmost right columns give the maximum absolute value of the percentage difference between the GBM trigger value and the CEV trigger values for $\beta < 2$ and $\beta > 2$, respectively, relative to the GBM trigger value. For Panel A, the case with $r = q$ and $\beta \neq 2$ has been computed through Proposition 3, the case with $r \neq q$ and $\beta \neq 2$ has been calculated using Proposition 4, whereas the case with $\beta = 2$ has been implemented via Barbosa et al. (2018, equation 16). For Panel B, the case with $r = q$ and $\beta \neq 2$ has been computed through Proposition 5, the case with $r \neq q$ and $\beta \neq 2$ has been calculated using Proposition 6, whereas the case with $\beta = 2$ has been implemented via Barbosa et al. (2018, equation 26).

Finally, even though we have adopted the base case parameters configuration of Barbosa et al. (2018), for the sake of completeness and future reproducibility of our results by other interested academics and practitioners, Table 2 provides investment thresholds for a wide variety of β parameters and different drift specifications. These results confirm the previous observations that care must be taken when a policymaker aims to design a given policy to induce investment decisions with the help of FIT contracts since the differences between trigger points under alternative modeling assumptions are quite significant and, in most of the cases, the excessive rents are paid at the expense of tax payers. Notice that contract #9 of Panel A is intended only to show an example of a contract where the value-matching and smooth-pasting conditions are not accommodated since it refers to the region $\overline{P} < L$ of the perpetual case.

5. Conclusions

This paper presents analytical formulae to analyze optimal investment decisions of FIT contracts with a minimum price guarantee under the CEV model considering both perpetual and finite-lived guarantees. Overall, we document that the argument that a perpetual guarantee only induces investment for prices below the price floor when offering a risk-free investment opportunity is still valid under the CEV process. We also show that the optimal price-floor level triggering immediate investment in the presence of a perpetual guarantee is independent of the elasticity parameter of the CEV model. Moreover, we show that such independence is not valid in the case of FIT contracts with a finite maturity guarantee. In summary, our results strongly highlight the importance of policymakers to move beyond the more simplistic real options analysis based on the GBM process to more realistic models incorporating leverage and volatility smile effects because the differences between trigger points originate excessive rents that are commonly paid at the expense of tax payers.

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