The effect of both sides operational flexibility in capacity investment under model uncertainty*

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Abstract

This paper analyzes a firm's capacity expansion and reduction problem under uncertainty. The firm expands (resp., contracts) the capacity if the output price is sufficiently large (resp., low). Consequently, the firm has two types of operational flexibility. The firm faces uncertainty about the output price and can not uniquely identify its distribution. The firm treats the price dynamics as an approximation of its actual dynamics. Then, the firm decides its managerial strategy under model uncertainty. To deal with the model uncertainty, we employ the robust control approach. We reveal the effect of both sides' operational flexibility on the firm's decision-making under model uncertainty.

Keywords: real options; operational flexibility; capacity expansion and reduction; model uncertainty

1 Introduction

Uncertainty in business environments makes the prospect of business activities difficult. The real options approach provides a framework for strategic decision-making in business management under uncertainty (Trigeorgis and Reuer, 2017). Many researches of the real options approach treat uncertainty as risk in the context of Knight's. However, the current business environment is highly uncertain and analyses of management strategies under such uncertainty are required.

The analyses on real options approach under Knightian uncertainty has started to grow only in the last 20 decades (Nishimura and Ozaki, 2007; Trojanowska and Kort, 2010; Wang, 2010; Miao and Wang, 2011; Flor and Hesel, 2015; Viviani et al., 2018; Delaney, 2022; Luo and Tian, 2022). Nishimura and Ozaki (2007) investigate the irreversible investment under Knightian uncertainty by adopting the continuous-time multiple-priors utility model developed by Chen and Epstein (2002). The approach is called κ-ignorance approach. Many other researches (Trojanowska and Kort, 2010; Wang, 2010; Viviani et al., 2018; Delaney, 2022) also employ κ-ignorance approach to investigate the irreversible investment under Knightian uncertainty.

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Another approach to analyzing firms' investment problems under Knightian uncertainty is the robust control approach developed by Hansen, Sargent, and coauthors (Hansen and Sargent, 2001; Hansen et al., 2006). In the robust control approach, a decision-maker considers the reference model as an approximation of the true model due to the existence of Knightian uncertainty. The decision-maker considers a set of approximation models by disturbing the reference model. The disturbances depict the set of possible probability measures. Consequently, the term model uncertainty is used in the robust control approach. A fictitious decision-maker is introduced to deal with uncertainty in robust control approach. The fictitious decision-maker chooses the worst possible alternative measure with considering the cost of taking the alternative measure. Hansen and Sargent (2001) and Hansen et al. (2006) use the relative entropy to evaluate the difference between the reference measure and the alternative measure. Then, a decision-making problem is formulated as a two-player zero-sum game.

Hansen and Sargent (2001) and Hansen et al. (2006) investigate the utility maximization problem under model uncertainty. Other applications of the robust control approach include the analysis of environmental problems (Roseta-Palma and Xepapadeas, 2004; Athanassoglou and Xepapadeas, 2012; Yoshioka and Tsujimura, 2022), financial problems (Uppal and Wang, 2003; Ghaoui et al., 2003; Liu, 2010; Zawisza, 2015; Balter and Pelsser, 2020), and firms' management strategies (Flor and Hesel, 2015; Miao and Rivera, 2006; Imai and Tsujimura, 2022; Luo and Tian, 2022; Zhao, 2022) under model uncertainty.

This paper analyzes a firm's managerial decision-making under model uncertainty. The firm considers changing its capacity in response to the output demand, i.e., the output price. The firm expands the capacity if the demand is sufficiently large, while the firm reduces the capacity if the demand is sufficiently low. In sum, the firm has operational flexibility regarding the capacity size and has two managerial options: capacity expansion and reduction options. In this paper, we consider the case in which the firm can either expand or reduce the capacity only once. In making decisions, the firm faces uncertainty on the output price and can not uniquely identify its distribution. The firm treats the price dynamics as an approximation of its true dynamics. Then, the firm decides its capacity management strategy under model uncertainty. To deal with the model uncertainty, we employ the robust control approach developed by Hansen, Sargent, and their coauthors (Hansen and Sargent, 2001; Hansen et al., 2006) as mentioned above. Then, the firm's problem is a maxmin problem: the firm optimally chooses the timing of capacity expansion/reduction to maximize the present value of net profit, while the fictitious decision-maker optimally chooses the distortion of probability measure to minimize the present value of profit with the cost of taking distortion. In this paper, we numerically investigate the firm's optimal capacity management strategy, which is characterized by two thresholds for capacity expansion and reduction. Further, we examine the impact of uncertainty on both managerial options values.

2 Firm's Problem

Suppose that a firm produces an output Q in a competitive market. We assume that one unit output is produced by per unit capacity. This means that the change in Q is equivalent to the change in capacity. The firm changes the level of its capacity depending on the demand. The inverse demand function, i.e., the output price P, is exogenously given by the assumption of a competitive market. Then, the firm changes the level of capacity depending on the output price.

We consider the case in which the firm has two managerial options: the option to expand and to contract the capacity. The firm expands the capacity from Q_0 to $Q_H := Q_0 + \Delta Q$ with $\Delta Q \in (0, Q_0)$ when the output price is larger than or equal to p_H, while the firm contracts the capacity from $Q_L := Q_0 - \Delta Q$, when the output price is less than or equal to p_L. Then, the output Q_t at time $t \geq 0$ is given by:

$$Q_t = \begin{cases} Q_H, & P_t \geq p_H, \\ Q_0, & p_L < P_t < p_H, \\ Q_L, & P_t \leq p_L. \end{cases} \tag{2.1}$$

Expanding the capacity by ΔQ costs $q_H \Delta Q$, while reducing the capacity by ΔQ yields sales gains of $q_L \Delta Q$. Here, q_H is the purchasing price of capacity and q_L is the selling price of capacity with $q_L < q_H$.

The firm's operating profit π at time t is given by:

$$\pi(P_t, Q_t) = (P_t - c)Q_t, \tag{2.2}$$

where P_t is the output price and $c > 0$ is the constant operating cost. The dynamics of the output price P_t is governed by the following geometric Brownian motion:

$$\mathrm{d}P_t = \mu P_t \mathrm{d}t + \sigma P_t \mathrm{d}W_t, \quad P_0 = p > 0, \tag{2.3}$$

where $\mu, \sigma > 0$ and W_t is a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$, where $\mathcal{F}_t$ is generated by W_t. The firm's manager feels the possibility of errors in identifying the reference probabilities and treats (2.3) as an approximation. That is, the firm's manager faces model uncertainty. To express the model uncertainty, we introduce a set of equivalent probability measures, $\mathcal{P}$, on $(\Omega, \mathcal{F})$. Then, the reference probability measure $\mathbb{P}$ replaced by another equivalent probability measure $\mathbb{Q} \in \mathcal{P}$. We replace W_t in (2.3) by $W_t^{\mathbb{Q}} + \int_0^t h_s \mathrm{d}s$, where h_t is progressively measurable and $W_t^{\mathbb{Q}}$ is a Brownian motion under the measure $\mathbb{Q}$. Then, equation (2.3) is rewritten as:

$$\mathrm{d}P_t = (\mu + \sigma h_t) P_t \mathrm{d}t + \sigma P_t \mathrm{d}W_t^{\mathbb{Q}}, \quad P_0 = p > 0. \tag{2.4}$$

We use the relative entropy to measure the distance between two probability measures and introduce the discounted relative entropy as in Hansen and Sargent (2001) and Hansen et al. (2006):

$$\begin{aligned} R(\mathbb{Q}) &= r \int_0^\infty \mathrm{e}^{-rt} \left(\int \log \left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \right) \mathrm{d}\mathbb{Q} \right) \mathrm{d}t \\ &= \mathbb{E}_{\mathbb{Q}} \left[\int_0^\infty \mathrm{e}^{-rt} \frac{h_t^2}{2} \mathrm{d}t \right], \end{aligned} \tag{2.5}$$

where $r \in (0, 1)$ is the discount rate. Further, we set the instantaneous relative entropy constraint for all t as in Vardas and Xepapadeas (2010):

$$R_t(\mathbb{Q}) = \mathbb{E}_{\mathbb{Q}} \left[\frac{h_t^2}{2} \right] \leq \frac{\zeta^2}{2}, \tag{2.6}$$

where ζ represents the acceptable degree of model misspecification.

The firm's problem is to maximize the expected net present value of profit under model uncertainty. To this end, the firm chooses whether capacity expansion or reduction timing over the set of timing $\mathcal{T}$. Here, we employ the robust control approach to investigate the firm's decision-making under model uncertainty. A fictitious decision-maker chooses the worst possible alternative measure to deal with the model uncertainty with considering the cost of taking the alternative probability measure. The relative entropy represents the cost. Then, the firm's problem is formulated as:

$$V(p) = \sup_{\tau \in \mathcal{T}} \inf_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} \left[\int_0^\infty \mathrm{e}^{-rt} \pi(P_t, Q_t) \mathrm{d}t + \theta R(\mathbb{Q}) - \mathrm{e}^{-r\tau} \left(q_H \Delta Q \mathbf{1}_{\{\tau = \tau_H\}} - q_L \Delta Q \mathbf{1}_{\{\tau = \tau_L\}} \right) \right], \tag{2.7}$$

where V is the value function of the firm's problem and $\theta > 0$ is the robustness parameter. τ is the capacity expansion/reduction time given by:

$$\tau := \min\{\tau_H, \tau_L\}, \tag{2.8}$$

where τ_H and τ_L represent capacity expansion and reduction time, respectively:

$$\tau_H := \inf\{t \geq 0; P_t \geq p_H\} \text{ and } \tau_L := \inf\{t \geq 0; P_t \leq p_L\}. \tag{2.9}$$

$\mathbf{1}_S$ is the indicator function such that $\mathbf{1}_S = 1$ if $\mathbf{1}_S$ is true and $\mathbf{1}_S = 0$ otherwise. The second term of the right-hand side, $\theta R(\mathbb{Q})$, represents the cost of taking the alternative probability measure.

We assume the expected present value of the operating profit π is finite so that the firm's problem is meaningful:

$$\mathbb{E}_{\mathbb{Q}} \left[\int_0^\infty \mathrm{e}^{-rt} \pi(P_t, Q_t) \mathrm{d}t \right] = \frac{pQ_0}{r - (\mu + \sigma h_0)} - \frac{cQ_0}{r} < \infty. \tag{2.10}$$

3 Variational Inequalities of the Firm's Problem

The firm's problem (2.7) is formulated as an optimal stopping problem. The optimal stopping problem is solved via the variational inequalities (Bensoussan and Lions, 1982; Øksendal and Reikvam, 1998).

The variational inequalities of the firm's problem (2.7) is the followings:

$$\begin{aligned} \max \Bigg\{ & \inf_h \left\{ \mathcal{L}V(p) + \pi(p, Q_0) + \theta \frac{h^2}{2} \right\}, \inf_h \{ (G_H(p) - q_H \Delta Q) - V(p) \}, \\ & \inf_h \{ (G_L(p) + q_L \Delta Q) - V(p) \} \Bigg\} = 0, \end{aligned} \tag{3.1}$$

where $\mathcal{L}$ is the infinitesimal operator:

$$\frac{1}{2}\sigma^2 p^2 V''(p) + (\mu + \sigma h) p V'(p) - rV(p), \tag{3.2}$$

and G_i ($i = \{0, H, L\}$) the value of operating profit with the output level Q_i and the cost of taking the distortion:

$$
\begin{aligned}
G_i(P_t) &:= \mathbb{E}_{\mathbb{Q}}\left[\int_t^{\infty} \mathrm{e}^{-r(t-s)}\pi(P_s;Q_i)\mathrm{d}s + \theta R(\mathbb{Q})|\mathcal{F}_t\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\int_t^{\infty} \mathrm{e}^{-r(t-s)}\pi(P_s;Q_i)\mathrm{d}s + \theta\int_t^{\infty} \mathrm{e}^{-r(t-s)}\frac{h_s^2}{2}\mathrm{d}s|\mathcal{F}_t\right] \\
&= \frac{P_tQ_i}{r-(\mu+\sigma h_t)} - \frac{cQ_i}{r} + \theta\frac{h_t^2}{2r}.
\end{aligned} \tag{3.3}
$$

From the variational inqqualities (3.1), the capacity expansion region $\mathcal{E}$ and the capacity reduction region $\mathcal{R}$ are respectively given by:

$$
\mathcal{E} := \left\{p; \inf_h\{V(p) \le G_H(p) - q_H\Delta Q\}\right\} \text{ and } \mathcal{R} := \left\{p; \inf_h\{V(p) \le G_L(p) + q_L\Delta Q\}\right\}. \tag{3.4}
$$

The continuation region $\mathcal{C}$, where the firm does not expand/reduce the capacity, is given by:

$$
\mathcal{C} := \{p; \mathbb{R}_{++} \setminus \mathcal{E}\cup\mathcal{R}\} = \left\{p; \inf_h\{V(p) > G_H(p) - q_H\Delta Q\} \text{ and } \inf_h\{V(p) > G_L(p) + q_L\Delta Q\}\right\}. \tag{3.5}
$$

For $p \in \mathcal{C}$, the variational inequalities (3.1) leads to

$$
\inf_h\left\{\mathcal{L}V(p) + \pi(p,Q_0) + \theta\frac{h^2}{2}\right\} = 0. \tag{3.6}
$$

Equation (3.6) yields the the optimal distortion h^* as:

$$
h^* = -\frac{\sigma p V'(p)}{\theta}. \tag{3.7}
$$

From (3.7), h^* goes to 0 as $\theta \to \infty$. This means that the firm's problem (2.7) goes to the problem without model uncertainty. Then, the parameter θ represents the degree of concern for model uncertainty. Replacing h by h^* in (3.6), we obtain a general solution to (3.6):

$$
V(p) = A_1p^{\beta_1} + A_2p^{\beta_2} + \frac{pQ_0}{r-(\mu+\sigma h^*)} - \frac{cQ_0}{r} + \theta\frac{(h^*)^2}{r}, \tag{3.8}
$$

where $A_1, A_2 > 0$ are the constants to be determined and $\beta_1 > 1$, $\beta_2 < 0$ are the roots of the particular equation:

$$
\frac{1}{2}\sigma^2\beta(\beta-1) + (\mu+\sigma h)\beta - r\beta = 0. \tag{3.9}
$$

We consider that the capacity expansion and reduction region are characterized by thresholds as in (2.1). If the output price is higher (resp., lower) than or equal to p_H (resp., p_L), the firm expands (resp., reduces) the capacity. Then, the continuation, expansion, and reduction regions are rewritten as

$$
\mathcal{C} = \{\{p > p_L\} \cup \{p < p_H\}\},\ \mathcal{E} = \{p \ge p_H\},\ \text{and } \mathcal{R} = \{p \le p_L\}. \tag{3.10}
$$

If h^* is calculated, four unknown parameters A_1, A_2, p_H, and p_L, are derived by the following value-matching conditions and smooth-pasting conditions:

$$
V(p_H; h^*) = G_H(p_H; h_H^*) - q_H\Delta Q, \tag{3.11}
$$

$$V(p_L;h^*) = G_L(p_L;h^*_L) + q_L\Delta Q, \tag{3.12}$$

$$V'(p_H;h^*) = G'_H(p_H;h^*_H), \tag{3.13}$$

$$V'(p_L;h^*) = G'_L(p_L;h^*_L), \tag{3.14}$$

where h^*_i $(i = \{H, L\})$ is derived from the fictitious decision-maker's problem when either of two options is exercised:

$$\inf_h \{G_H(p_H) - q_H\Delta Q\} \text{ or } \inf_h \{G_L(p_L) + q_L\Delta Q\}. \tag{3.15}$$

From (3.3) and (3.15), h^*_i is the real solution to the following cubic equation:

$$\frac{\theta\sigma^2}{r}h^3 - \frac{2\theta(r-\mu)}{r}h^2 + \frac{\theta(r-\mu)^2}{r}h + \sigma p Q_i = 0. \tag{3.16}$$

Notice also that when the output price is equal to the thresholds, $P_t = p_i$, we have $h_t = h^* = h^*_i$ $(i = \{H, L\})$.

4 Numerical Analysis

We will present the results of numerical analysis on the conference.

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