Compensation of high interest rates by tax holidays: A real options approach

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There is an important problem how to attract investments to the real sector of economics when credit risks are high. Our work is devoted to analysis of related tax mechanisms.

In order to compensate risks and other unfavorable factors for the investment attraction the following tax benefits are often used:

- tax holidays (exemption from tax during the certain period),
- reduction in tax rate,
- accelerated depreciation.

It is worth noting that increased credit risks imply increasing interest rates on credit.

In the paper, we study the possibility to apply the tax holidays mechanism (on corporate profit tax) for a compensation of high-level interest rates.

Tax holidays have been used until present days in South-eastern Asia, Eastern Europe and some Western European countries. One of the most successful applications was the tax holiday program in Puerto Rico which was initiated in 1949. Tax holidays granted for a period of 10-15 years have also been introduced for start-up corporations specialized in Hi-Tech products in Singapore. In Italy, beginning in 1986, 10 years tax holidays were established, for firms that were settling their activity in the southern part of the country. Many developed and developing countries (included China. India, Brazil, Malaysia et al.) provide tax holidays to attract foreign investment. In Russia, tax holidays were commonly used in the nineteen-nineties (a list of the regions where those mechanisms were introduced can be found in [4]). Tax

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holidays are one of the most used stimuli in order to attract investment into the so-called "Special Economic Zones" (as Skolkovo in Russia).

Various problems related to the influence of tax holidays on investment decisions, especially under risk and uncertainty, were studied in number of papers (see, e.g. [5], [10], [8]). Potential possibilities of tax holidays as a mechanism for maximization of the expected discounted tax payments from the created firm were explored in [4].

In economics with an increased risks (political, credit etc.) and other unfavorable factors the following question arises: can tax benefits provide investor with the same conditions (in some sense) for investment as he would have in "standard" economy without any risks and unfavorable factors.

Such a compensation problem was established and studied in [4], [3], where risk is modelled by an additive term to a discount rate ("risk premium"). As mechanisms for compensating were explored tax holidays, depreciation policy and reduction in profit tax rate.

In the present paper we formulate and solve (in real options framework) the similar problem with such an unfavorable (for investor) factor as high-level interest rates on credit.

1 The basic model

The model is related with an investment project directed to the creation of a new industrial firm (enterprise). The important feature of the considered model is the assumption that, at any moment, a decision-maker (investor) can either *accept* the project and proceed with the investment or *delay* the decision until he obtains new information regarding its environment (prices of the product and resources, demand etc.). Thus, the main goal of a decision-maker in this situation is to find, using the available information, a "good" time for investing the project (investment timing problem).

The real options theory is convenient and adequate tool for modelling the process of firm creation since it allows us to study the effects connected with a delay in investment (investment waiting). As in the real options literature, we model investment timing problem as an optimal stopping problem for present values of the created firm (see, e.g. [6], [9]).

A creation of an industrial enterprise is usually accompanied by certain tax benefits (in particular, the new firm is exempted from profit taxes during certain period). We represent the firm's present value as an integral of a profits flow. Such a consideration allows us to take into account in explicit form some peculiarities of a corporate profit taxation system, including tax exemption. Such an approach was applied by authors for a detailed model of investment project under taxation in [1], [3], [4].

Uncertainty in economic system is modelled by some probability space $(\Omega, \mathbb{F}, \mathbb{P})$ with filtration $\mathcal{F} = (\mathcal{F}_t,\ t \geq 0)$. $\mathcal{F}_t$ can be interpreted as the observable information about the system up to the time t. Time $t \geq 0$ is continuous. We denote by $\mathbf{E}$ the expectation operator associated to $\mathbb{P}$.

An infinitely-lived investor faces a problem to choose when to invest in a project directed to the creation of new firm.

The cost of investment required to create firm at time t is I_t. Investment are considered to be instantaneous and irreversible so that they cannot be withdrawn from the project any more and used for other purposes. We assume that $(I_t,\ t \geq 0)$ is $\mathcal{F}$-adapted random process.

Let us suppose that investment into creating a firm is made at time $\tau \geq 0$.

The result of such investment is the flow of profits from the firm (i.e. gross income minus production cost) with intensity $p^\tau_{\tau+t}$ at time $\tau + t$ $(t \geq 0)$. $p^\tau_{\tau+t}$ is assumed to be $\mathcal{F}_{\tau+t}$-measurable random variables $(t, \tau \geq 0)$.

The required investment I_τ is credited with repayment period L (years) and interest rate λ (annual).

The credit repayment $(D^\tau_{\tau+t}, t \geq 0)$ is started right after the creation of the firm τ. The principal repayment schedule (without interest rate repayment) is described by "flow of repayments" $D^\tau_{\tau+t} \geq 0,\ 0 \leq t \leq L : \int_0^L D^\tau_{\tau+t} dt = I_\tau$.

The total repayments (included interest) that the firm pays on credit, discounted to the investment time τ are :

$$K_\tau = K_\tau(\lambda) = \int_0^L (D^\tau_{\tau+t} + \lambda R^\tau_{\tau+t}) e^{-\rho t} dt = F_\tau + \frac{\lambda}{\rho}(I_\tau - F_\tau), \qquad (1)$$

where $R^\tau_t = \displaystyle\int_t^L D^\tau_{\tau+s} ds$ is an unpaid debt at the time $\tau + t$, ρ is the discount rate, and $F_\tau = \displaystyle\int_0^L D^\tau_{\tau+t} e^{-\rho t} dt$.

Further, we assume that total credit repayments $K_\tau(\lambda)$ increase in interest rate λ. It is a natural economic assumption which allows to avoid "bad" repayment schemes.

The created firm is granted with a tax holidays, during which it does not pay corporate profit tax. Let γ be a profit tax rate (tax burden), and ν be the duration of tax holidays.

Interest payments are included in profit tax base, but the maximal value of deductible interest rates is bounded by the limiting value λ_b (in Russia this value equals to the refinancing, or basic, rate of the Central Bank of Russia λ_{ref}, multiplied by 1.8, i.e. 13.95% at 2010).

The expected net present value (NPV) of the firm, discounted to the investment time τ is:

$$V_\tau = \mathbf{E}\left(\int_0^\nu p^\tau_{\tau+t} e^{-\rho t} dt + \int_\nu^{\max(\nu,L)} [p^\tau_{\tau+t} - \gamma(p^\tau_{\tau+t} - \bar{\lambda} R^\tau_{\tau+t})] e^{-\rho t} dt \right.$$

$$
+ \int\limits_{\max(\nu,L)}^{\infty} (1-\gamma) p^{\tau}_{\tau+t} e^{-\rho t} dt \Bigg| \mathcal{F}_{\tau} \Bigg), \tag{2}
$$

where $\bar{\lambda} = \min(\lambda, \lambda_b)$. Here we apply the unified formula for profit tax both for positive and negative tax base. It is in accordance with the principles of full-loss offset, or loss carry forward, i.e. transfer of the current year's losses to future years' profits that reduce profit tax base.

The starting point of this scheme is the known McDonald-Siegel model [9], which was the base for the real option theory (see, e.g.,[6],[14]). More complicated variants of this scheme, which take into account a detailed structure of cash-flows as well as a number of different taxes one can find in [3].

The behavior of the investor is assumed to be rational. It means that he solves *the investment timing problem*, namely, at any time τ prior to investment he chooses whether to invest and to earn present value V_τ, or to delay further his investment. So, the investor's decision problem is to find such a stopping time τ (investment rule), that maximizes the expected net present value from the future firm :

$$
\mathbf{E}\,(V_\tau - K_\tau)\, e^{-\rho\tau} \to \max_{\tau}, \tag{3}
$$

where the maximum is considered over all stopping times τ (w.r.t. filtration $\mathcal{F}$), and V_τ, K_τ are defined in (1)–(2).

2 Solution of the investment timing problem

Main assumptions

Let $(w^1_t,\ t \ge 0)$, $(w^2_t,\ t \ge 0)$ be two independent standard Wiener processes on the stochastic basis $(\Omega, \mathbb{F}, \mathcal{F}, \mathbb{P})$. These processes are thought as underlying processes modelling economic stochastics. So, we assume that σ-field $\mathcal{F}_t$ is generated by those processes up to t, i.e. $\mathcal{F}_t = \sigma\{(w^1_s, w^2_s),\ s \le t\}$.

The process of *profits* $p^{\tau}_{\tau+t}$ has finite expectations ($\mathbf{E}p^{\tau}_{\tau+t} < \infty$ for all $t, \tau \ge 0$), and is represented as follows:

$$
p^{\tau}_{\tau+t} = \pi_{\tau+t}\xi^{\tau}_{\tau+t}, \quad t, \tau \ge 0, \tag{4}
$$

where $(\pi_t,\ t \ge 0)$ is geometric Brownian motion, specified by the stochastic equation

$$
\pi_t = \pi_0 + \alpha_1 \int\limits_0^t \pi_s\, ds + \sigma_1 \int\limits_0^t \pi_s\, dw^1_s \qquad (\pi_0 > 0,\ \sigma_1 \ge 0), \quad t \ge 0, \tag{5}
$$

and $(\xi^\tau_{\tau+t},\ t \geq 0)$ is a family of non-negative diffusion processes, homogeneous in $\tau \geq 0$, defined by the stochastic equations

$$\xi^\tau_{\tau+t} = 1 + \int_\tau^{t+\tau} a(s-\tau, \xi^\tau_s)\, ds + \int_\tau^{t+\tau} b(s-\tau, \xi^\tau_s)\, dw^1_s, \quad t, \tau \geq 0, \tag{6}$$

with given functions $a(t,x),\ b(t,x)$ (these functions satisfy the standard conditions for the existence of the strongly unique solution in (6) – see, e.g. [11, Ch.5]).

The process π_t in representation (4) can be related to the (external) prices of produced goods and consumed resources (external uncertainty), whereas fluctuations $\xi^\tau_{\tau+t}$ can be generated by the firm created at time τ (firm's uncertainty). Obviously, $p^\tau_\tau = \pi_\tau$ for any $\tau \geq 0$.

The cost of the required investment I_t is also described by the geometric Brownian motion as follows

$$I_t = I_0 + \alpha_2 \int_0^t I_s\, ds + \int_0^t I_s(\sigma_{21}\, dw^1_s + \sigma_{22}\, dw^2_s), \quad (I_0 > 0) \quad t \geq 0, \tag{7}$$

where $\sigma_{21},\ \sigma_{22} \geq 0$. In order to avoid a degenerative case we assume that $\sigma_{22} > 0$. Then the linear combination $\sigma_{21} w^1_t + \sigma_{22} w^2_t$ has the same distribution as $(\sigma^2_{21} + \sigma^2_{22})^{1/2}\tilde{w}_t$, where $\tilde{w}_t$ is a Wiener process correlated with w^1_t, and the correlation coefficient is equal to $\sigma_{21}(\sigma^2_{21} + \sigma^2_{22})^{-1/2}$.

The flow of *principal repayment* at time $\tau + t$ (for the firm created at the time τ) will be represented as:

$$D^\tau_{\tau+t} = I_\tau d_t, \quad 0 \leq t \leq L, \tag{8}$$

where $(d_t,\ 0 \leq t \leq L)$ is the "repayment density" (per unit of investment), characterizing a repayment schedule, i.e. non-negative deterministic function such that $\int_0^L d_t\, dt = 1$. Formally, we will consider d_t for all $t \geq 0$ putting $d_t = 0$ for $t > L$. Note, that repayment density can depend, in general, on interest rate λ, i.e. $d_t = d_t(\lambda)$.

Such a scheme covers various schedules of credit repayment, accepted in practice (more exactly, their variants in continuous time). For example, fixed principal repayment can be described by the uniform density $d_t = 1/L$, while the well-known annuity scheme (fixed payments for a principal plus interest during the repayment period) corresponds to exponential density $d_t = \lambda e^{\lambda t}/(e^{\lambda L} - 1) \quad (0 \leq t \leq L)$.

Derivation of the present value. Optimal investment timing

The above assumptions allow us to obtain formulas for the present value of the future firm.

At first we need the following assertion.

Lemma 1. *Let τ be a stopping time. Then for all $t \ge 0$*

$$\mathbf{E}(p^{\tau}_{\tau+t}|\mathcal{F}_{\tau}) = \pi_{\tau} B_t, \quad \textit{where} \quad B_t = \mathbf{E}(\pi_t \xi^0_t)/\pi_0. \tag{9}$$

Proof. By the strong Markov property for a Wiener process the process $\widehat{w}_t = w^1_{t+\tau} - w^1_{\tau},\ t \ge 0$ will be Wiener process independent on $\mathcal{F}_{\tau}$. Using the explicit formula for geometric Brownian motion one can rewrite relation (4) as follows:

$$p^{\tau}_{\tau+t} = \pi_{\tau} \mathit{\Pi}^{\tau}_{t+\tau}, \quad \text{where } \mathit{\Pi}^{\tau}_{t+\tau} = \exp\{(\alpha_1 - \tfrac{1}{2}\sigma_1^2)t + \sigma_1 \widehat{w}_t\} \xi^{\tau}_{\tau+t}.$$

Homogeneity of the family (4) in τ implies that the process $\xi^{\tau}_{\tau+t}$ coincides (a.s.) with the unique (in the strong sense) solution of the stochastic equation

$$\xi_t = 1 + \int_0^t a(s, \xi_s)\, ds + \int_0^t b(s, \xi_s)\, d\widehat{w}_s.$$

Since $(\xi_t,\ t \ge 0)$ is independent on $\mathcal{F}_{\tau}$, the process $\mathit{\Pi}^{\tau}_{t+\tau}$ is independent also. Moreover, $\mathit{\Pi}^{\tau}_{t+\tau}$ has the same distribution as $\exp\{(\alpha_1 - \frac{1}{2}\sigma_1^2)t + \sigma_1 \widehat{w}_t\}\xi_t$, or as $(\pi_t/\pi_0)\xi^0_t$. Therefore, $\mathbf{E}(p^{\tau}_{\tau+t}|\mathcal{F}_{\tau}) = \pi_{\tau}\mathbf{E}\mathit{\Pi}^{\tau}_{t+\tau} = \pi_{\tau}\mathbf{E}(\pi_t \xi^0_t)/\pi_0$. $\square$

Let us define the following function :

$$B(t) = \int_t^{\infty} B_s e^{-\rho s}\, ds, \quad t \ge 0, \tag{10}$$

where B_s are defined in (9), and assume that $B(0) < \infty$.

We will denote a conditional expectation provided by $\mathcal{F}_{\tau}$ as $\mathbf{E}_{\tau}$.

The above Lemma 1 gives the following formulae for the present value (2):

$$V_{\tau} = \mathbf{E}_{\tau}\left(\int_0^{\nu} p^{\tau}_{\tau+t} e^{-\rho t} dt + (1-\gamma) \int_{\nu}^{\infty} p^{\tau}_{\tau+t} e^{-\rho t} dt + \gamma\bar{\lambda} \int_{\nu}^{\nu \vee L} R^{\tau}_{t+\tau} e^{-\rho t} dt \right)$$

$$= \pi_{\tau}[B(0) - \gamma B(\nu)] + \gamma\bar{\lambda} I_{\tau} D(\nu), \text{ where } D(\nu) = \int_{\nu \wedge L}^{L} \left(\int_t^{L} d_s ds \right) e^{-\rho t} dt. \tag{11}$$

Optimal investment timing

So, the above assumptions and formulas shows that investment timing problem (3) is reduced to optimal stopping problem for bivariate geometric Brownian motion and linear reward function (general theory of optimal stopping problems one can see, e.g., in [12], [13]). Indeed,

$$K_\tau = I_\tau \left[F + \lambda(1-F)/\rho\right], \quad \text{where } F = \int_0^L d_t e^{-\rho t} dt, \tag{12}$$

$$V_\tau - K_\tau = \pi_\tau[B(0) - \gamma B(\nu)] - I_\tau \left[F + \lambda(1-F)/\rho - \gamma\bar{\lambda}D(\nu)\right]. \tag{13}$$

Let β be a positive root of the quadratic equation

$$\frac{1}{2}\tilde{\sigma}^2\beta(\beta-1) + (\alpha_1 - \alpha_2)\beta - (\rho - \alpha_2) = 0, \tag{14}$$

where $\tilde{\sigma}^2 = (\sigma_1 - \sigma_{21})^2 + \sigma_{22}^2 > 0$ (since $\sigma_{22} > 0$) is a “total” volatility of investment project. It is easy to see that $\beta > 1$ whenever $\rho > \max(\alpha_1, \alpha_2)$.

The following theorem characterizes completely an optimal investment time.

Theorem 1. *Let the processes of profits and required investments be described by relations (4)–(7). Assume that $\rho > \max(\alpha_1, \alpha_2)$ and the following condition is satisfied:*

$$\alpha_1 - \frac{1}{2}\sigma_1^2 \geq \alpha_2 - \frac{1}{2}(\sigma_{21}^2 + \sigma_{22}^2).$$

Then the optimal investment time for the problem (3) is

$$\tau^* = \min\{t \geq 0 : \ \pi_t \geq \pi^* I_t\}, \tag{15}$$

where

$$\pi^* = \pi^*(\nu, \lambda) = \frac{\beta}{\beta - 1} \cdot \frac{F + \lambda(1-F)/\rho - \gamma\bar{\lambda}D(\nu)}{B(0) - \gamma B(\nu)}, \tag{16}$$

and $B(\cdot)$, $D(\cdot)$, F are defined at (10), (11), (12) respectively.

Formula of the type (15)–(16) for the difference of two geometric Brownian motions was first derived, probably, by McDonald and Siegel [9]. But rigorous proof and precise conditions for its validity appeared a decade later in [7]. It can also be immediately deduced from general results on optimal stopping for two-dimensional geometric Brownian motion and homogeneous reward function (e.g., [2]).

In order to avoid the trivial moment of investment $\tau^* = 0$, we will further suppose that the initial values of the processes satisfy the relation $\pi_0 < \pi^* I_0$.

The optimal investment level p^* characterizes the time when an investor accepts the project and makes investment. A decrease in p^* implies earlier investment time, and, on the contrary, an increase in p^* leads to a delayed investment.

Knowing the optimal investment rule, one can derive the expected net present value $N^* = \mathbf{E}\,(V_{\tau^*} - K_{\tau^*})\,e^{-\rho\tau^*}$ under the optimal behavior of potential investor. Using the standard technique for boundary value problems (Feynman-Kac formula – see, e.g., [11, Ch.9]), or results on homogeneous functionals of two-dimensional geometric Brownian motion ([2]), one can obtain the following formula.

Corollary 1. *Under assumptions of Theorem 1*

$$N^* = N^*(\nu,\lambda) = C[B(0) - \gamma B(\nu)]^{\beta}\left[F+\lambda(1-F)/\rho-\gamma\bar{\lambda}D(\nu)\right]^{1-\beta}, \qquad (17)$$

where $C = (\pi_0/\beta)^{\beta}[I_0/(\beta-1)]^{1-\beta}$.

3 Compensation of interest rates by tax holidays

Now we formulate a problem of compensation of high-level interest rates by tax exemptions.

The question is: can one choose such a duration of tax holidays ν that given index $\mathcal{M}$ (related to the investment project) under high interest credit rate λ will be greater (not less) than those index under "the reference" interest rate λ_0 and without tax holidays:

$$\mathcal{M}(\nu,\lambda) \geq \mathcal{M}(0,\lambda_0) \quad \text{for some } \nu \geq 0.$$

We consider the following indices :

1) optimal investment level π^*, that defines the time when an investor accepts the project and makes investment;
2) optimal NPV of the investor N^*.

As the reference interest rate we take the limit rate $\lambda_0 = \lambda_b$, which is deducted in profit tax base.

The assumption about an increasing (in interest rate) total payments on credit and explicit formulas (16)-(17) imply that the above indices are monotone in λ. Namely, the optimal investment level π^* increases, and the optimal NPV of the investor N^* decreases. Therefore, it makes sense to consider a compensation problem only for $\lambda > \lambda_0$.

Compensation in terms of optimal investment level

Let us begin with *optimal investment level* $\pi^* = \pi^*(\nu,\lambda)$.

We say that interest rate λ can be compensated (in terms of optimal investment level) by tax holidays, if $\pi^*(\nu,\lambda) \le \pi^*(0,\lambda_0)$ for some duration of tax holidays $\nu \ge 0$, i.e. in other words, if

$$\inf_{\nu \ge 0} \pi^*(\nu,\lambda) \le \pi^*(0,\lambda_0). \tag{18}$$

Since a decrease in π^* implies earlier investment time (for any random event), then a possibility to compensate in terms of optimal investment level can be interpreted as a possibility to increase investment activity in real sector. This situation is attractive for the State.

Let us denote $K = K(\lambda) = F + \lambda(1-F)/\rho$, where F is specified in (12) – the total discounted payments on credit per unit of investment (or, the same, per unit of credit).

Further, we assume that profits parameters B_t (defined in (9)) are such that B_t is differentiable increasing in t function. This assumption means that expected profit from the firm (after investment) grows in time, and allows to avoid some unessential technical difficulties.

The following result is the criterion for the compensation in terms of optimal investment level.

Theorem 2. *The interest rate λ can be compensated (in terms of optimal investment level) by tax holidays if and only if $\lambda \le \lambda_1$, where λ_1 is a unique root of the equation*

$$(1-\gamma)K(\lambda) = K(\lambda_0) - \gamma\lambda_0(1-F_0)/\rho, \tag{19}$$

and $F_0 = \displaystyle\int_0^L d_t(\lambda_0)e^{-\rho t}dt$ corresponds to the repayment schedule with interest rate λ_0.

In other words, there is a “critical” value of interest rate (λ_1) such that if interest rate is greater than this value, it can not be compensated (in terms of optimal investment level) by any tax holidays. Note that the “limiting” interest rate $\lambda = \lambda_1$ can be compensated only by tax holidays with infinite duration.

Proof. Let us denote $r_t = \displaystyle\int_t^L d_s ds$. From (16) we have

$$\begin{aligned}\frac{d\pi^*}{d\nu} &= \frac{\beta}{\beta-1}\cdot\frac{-\gamma\bar{\lambda}D'(\nu)[B(0)-\gamma B(\nu)] + [K-\gamma\bar{\lambda}D(\nu)]\gamma B'(\nu)}{[B(0)-\gamma B(\nu)]^2}\\ &= \gamma e^{-\rho\nu}\frac{\beta}{\beta-1}\cdot\frac{\bar{\lambda}r_\nu[B(0)-\gamma B(\nu)] - [K-\gamma\bar{\lambda}D(\nu)]B_\nu}{[B(0)-\gamma B(\nu)]^2}.\end{aligned} \tag{20}$$

As one can see from the latter formula, the optimal investment level is not in general monotone in ν. The sign of its derivative is completely defined by the function $Q(\nu) = \bar{\lambda} r_\nu [B(0) - \gamma B(\nu)] - [K - \gamma\bar{\lambda} D(\nu)]B_\nu$. Then

$$\begin{aligned} Q'(\nu) &= \bar{\lambda}\{r'_\nu [B(0)-\gamma B(\nu)] - r_\nu \gamma B'(\nu)\} - [K-\gamma\bar{\lambda}D(\nu)]B'_\nu + \gamma\bar{\lambda} B_\nu D'(\nu) \\ &= \bar{\lambda}\{-d_\nu[B(0)-\gamma B(\nu)] + r_\nu\gamma B_\nu e^{-\rho\nu}\} - [K-\gamma\bar{\lambda}D(\nu)]B'_\nu - \gamma\bar{\lambda}B_\nu r_\nu e^{-\rho\nu} \\ &= -\bar{\lambda} d_\nu [B(0)-\gamma B(\nu)] - B'_\nu [K-\gamma\bar{\lambda}D(\nu)] < 0 \end{aligned} \tag{21}$$

due to assumption $B'_t > 0$. Hence, if $\dfrac{d\pi^*}{d\nu} \le 0$ for some $\nu = \nu_0$ then $\dfrac{d\pi^*}{d\nu} < 0$ for all $\nu > \nu_0$. Since $r_\nu = 0$ for $\nu > L$ then (20) implies that π^* decreases in ν for enough large ν. So, the function π^* is either decreasing or having a unique maximum (in ν).

Therefore, applying formula (16) for optimal investment level and inequality $\pi^*(0,\lambda) > \pi^*(0,\lambda_0)$ for $\lambda > \lambda_0$, we have that relation (18) holds if and only if

$$\pi^*(\infty,\lambda) = \frac{\beta}{\beta-1}\cdot\frac{K(\lambda)}{B(0)} \le \pi^*(0,\lambda_0) = \frac{\beta}{\beta-1}\cdot\frac{K(\lambda_0)-\gamma\lambda_0 D(0)}{(1-\gamma)B(0)}, \tag{22}$$

where

$$D_0 = \int_0^L \left(\int_t^L d_s ds\right) e^{-\rho t} dt = \left(1 - \int_0^L d_t e^{-\rho t} dt\right)/\rho$$

and $d_t = d_t(\lambda_0)$ corresponds to repayment schedule with interest rate λ_0.

Now, the statement of Theorem 2 follows from (22).

In most cases the "critical" value λ_1 can be derived explicitly.

Corollary 2. *Let schedule of the principal repayments in credit do not depend on interest rate. Then the interest rate λ can be compensated (in terms of optimal investment level) by tax holidays if and only if $\lambda \le \lambda_1$, where*

$$\lambda_1 = \lambda_0 + \rho\frac{\gamma}{1-\gamma}\cdot\frac{F}{1-F}. \tag{23}$$

Proof. The corollary immediately follows from (19) and formula for $K(\lambda)$ (cf. (1)) :

$$K(\lambda) = F + \frac{\lambda}{\rho}(1-F).$$

Compensation in terms of optimal investor's NPV

Now let us consider an optimal investor's NPV $N^* = N^*(\nu,\lambda)$.

We say that interest rate λ can be compensated (in terms of optimal investor's NPV) by tax holidays, if $N^*(\nu,\lambda) \ge N^*(0,\lambda_0)$ for some duration of tax holidays $\nu \ge 0$, i.e.

$$\sup_{\nu \geq 0} N^*(\nu, \lambda) \geq N^*(0, \lambda_0). \tag{24}$$

An increase of N^* means a growth of expected investor's revenue, therefore a possibility to compensate in terms of optimal investor's NPV is linked to investment attraction for the investor.

The following result is similar to Theorem 2 above.

Theorem 3. *The interest rate λ can be compensated (in terms of optimal investor's NPV) by tax holidays if and only if $\lambda \leq \lambda_2$, where λ_2 is a unique root of the equation*

$$(1-\gamma)^{\beta/(\beta-1)} K(\lambda) = K(\lambda_0) - \gamma\lambda_0(1 - F_0)/\rho, \tag{25}$$

$F_0 = \displaystyle\int_0^L d_t(\lambda_0) e^{-\rho t} dt$ *corresponds to the repayment schedule with interest rate λ_0, and β is a positive root of the equation (14).*

Proof. The proof of Theorem 3 follows the general scheme of the proof of Theorem 2.

From formula (17) for the optimal investor's NPV we have

$$\begin{aligned}
\frac{dN^*}{d\nu} &= C\left\{-\gamma\beta[B(0){-}\gamma B(\nu)]^{\beta-1} B'(\nu)[K{-}\gamma\bar{\lambda}D(\nu)]^{1-\beta}\right.\\
&\quad \left. -\ (1-\beta)\gamma\bar{\lambda}[B(0){-}\gamma B(\nu)]^{\beta}[K{-}\gamma\bar{\lambda}D(\nu)]^{-\beta} D'(\nu)\right\}\\
&= \gamma e^{-\rho\nu}[B(0){-}\gamma B(\nu)]^{\beta-1}[K{-}\gamma\bar{\lambda}D(\nu)]^{-\beta} S(\nu)\\
&\quad \times \left\{\beta B_\nu[K{-}\gamma\bar{\lambda}D(\nu)] - (\beta-1)[B(0){-}\gamma B(\nu)]\bar{\lambda}r_\nu\right\}.
\end{aligned}$$

where $S(\nu) = -\beta Q(\nu) + [B(0){-}\gamma B(\nu)]\bar{\lambda}r_\nu$, and $Q(\nu)$ is defined in the proof of Theorem 2.

Using (21) we have

$$\begin{aligned}
S'(\nu) &= -\beta Q'(\nu) + \bar{\lambda}\left\{r'_\nu[B(0){-}\gamma B(\nu)] - \gamma r_\nu B'(\nu)\right\}\\
&= -\beta Q'(\nu) + \bar{\lambda}\left\{-d_\nu[B(0){-}\gamma B(\nu)] + \gamma r_\nu B_\nu e^{-\rho\nu}\right\}\\
&= (\beta-1)\bar{\lambda}d_\nu[B(0){-}\gamma B(\nu)] + \beta B'_\nu[K{-}\gamma\bar{\lambda}D(\nu)] + \gamma r_\nu B_\nu e^{-\rho\nu} > 0.
\end{aligned}$$

Using arguments, similar to those in the proof of Theorem 2, we get that the function N^* is either increasing or having a unique minimum (in ν).

Therefore, like above we can conclude that relation (24) holds if and only if $N^*(\infty, \lambda) \geq N^*(0, \lambda_0)$, or

$$C[B(0)]^\beta [K(\lambda)]^{1-\beta} \geq C(1-\gamma)^\beta [B(0)]^\beta [K(\lambda_0) - \gamma\lambda_0 D_0]^{1-\beta}, \tag{26}$$

where $D_0 = (1 - F_0)/\rho$ corresponds to the repayment schedule with interest rate λ_0. This implies the statement of Theorem.

Similar to the previous case of a compensation in terms of optimal investment level, the "critical" value λ_2 can be derived explicitly for the case when the principal repayments do not depend on interest rate.

Corollary 3. *Let schedule of the principal repayments does not depend on interest rate. Then the credit rate λ can be compensated (in terms of optimal investor's NPV) by tax holidays if and only if $\lambda \le \lambda_2$, where*

$$\lambda_2 = \lambda_0 (1-\gamma)^{-1/(\beta-1)} + \rho \frac{F}{1-F} \left[(1-\gamma)^{-\beta/(\beta-1)} - 1 \right]. \tag{27}$$

4 Concluding remarks

1. It will be interest to compare the obtained the "critical" interest rates λ_1 and λ_2 which give limits for the compensation (in relevant terms).

As Theorems 2 and 3 show, the bound λ_1 is a root of the equation $K(\lambda) = \tilde{K}/(1-\gamma)$, and λ_2 is a root of the equation $K(\lambda) = \tilde{K}/(1-\gamma)^{\beta/(\beta-1)}$, where $\tilde{K} = K(\lambda_0) - \gamma\lambda_0 D(0)$. Since the function $K(\lambda)$ increases, then $\lambda_2 > \lambda_1$.

This fact means that interest rates $\lambda < \lambda_1$ can be compensated by tax holidays both in terms of optimal investment level and in terms of investor's NPV. The opposite is not valid, in general, i.e. compensation in terms of NPV does not always imply compensation in terms of investment level, and therefore growth of investment activity.

2. Note, that the critical bound for compensation in terms of investor's NPV λ_2 depends (in contrast to the bound λ_1) on the parameters of the project but only through the value β (a positive root of the equation $0.5\tilde{\sigma}^2\beta(\beta-1)+(\alpha_1-\alpha_2)\beta-(\rho-\alpha_2)=0$). As a consequence, if volatility of the project σ increases then the bound of compensation (in terms of NPV) λ_2 will be increase also.

3. Usually, it is assumed the reduction in the refinancing (basic) rate λ_{ref} is a positive factor for a revival of investment activity in real sector. But this differs from the conclusions of our model.

Indeed, if tax holidays are trivial ($\nu = 0$), then optimal investment level

$$\pi^* = \pi^*(\lambda_{\text{ref}}) = \frac{\beta}{\beta-1} \cdot \frac{F+\lambda(1-F)\rho^{-1}-\gamma 1.8\lambda_{\text{ref}} D(0)}{B(0)(1-\gamma)}$$

decreases in λ_{ref}. So, π^* raises and, hence, investment activity (earlier investor entry) falls when λ_{ref} diminishes.

Similarly, optimal investor's NPV increases in λ_{ref}, and therefore decreasing refinancing rate λ_{ref} de-stimulates investor.

As calculations show when the refinancing rate λ_{ref} falls to 2 times (from the current value of 8%) the optimal investment level grows and optimal investor's NPV declines up to 20%.

4. We performed a number of calculations with "reasonable" (for Russian economy) data. Namely, we put tax burden $\gamma \approx 40\%$, discount rate $\rho \approx 10\%$, credits with period $L \approx 10$ (years) and fixed-principal repayment schedule, $\lambda_0 = 1.8 \times$ (refinancing rate CB of Russia) $= 13.95\%$. Typical characteristics of profits and investment cost gave that "aggregated" parameter β lies in the interval between 3 and 8.

Then, the received estimations for "critical" compensation bounds were the following: $\lambda_1 \approx$ 25–30%, $\lambda_2 \approx$ 30–40%, that seems to be not extremely high (especially, for the current economic situation in Russia).

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