A Non-Censored Binomial Model for Mean Reverting Stochastic Processes

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Abstract

Binomial trees are widely used for both financial and real option pricing due to their ease of use, versatility and precision. However, the classic approach developed by Cox, Ross, and Rubinstein (1979) applies only to a Geometric Brownian Motion diffusion processes, limiting the modeling choices. Nelson and Ramaswamy (1990) provided a general method to construct recombining binomial lattices which was used by Hahn and Dyer (2008) to develop a censored recombinant Mean Reverting model. These models, although more computationally complex in programming than the Cox et. al. (1979) binomial model, are fundamentally simpler than alternative approaches such as trinomial trees or simulation methods for American options. In this paper we extend the mean reverting model of Hahn and Dyer (2008) and propose a non-censored model that is more precise and has some other distinct advantages. We compare these two approaches and present the results of applying these models to evaluate a hypothetical real option.

Keywords

Real Options, Mean Reversion, Binomial Lattice, Brazilian Sugar-Ethanol industry.

1. Introduction: Recombinant binomial trees for real options valuation

The mathematical complexity associated with the real options theory derives from the need for a probabilistic solution for the optimal investment decision throughout the life to the option. The solution to this dynamic optimization problem, as described by Dixit and Pindyck (1994), is to model the uncertainty of the underlying asset as a stochastic process where the optimum decision value of investment is obtained by solving a differential equation with the appropriate boundary conditions. In many cases, however, this differential equation has no analytical solution or the simplified assumptions concerning the boundary conditions do not reflect the actual complexity of the problem. In these cases, a discrete approximation to the underlying stochastic process can be used in order to obtain a solution that is computationally efficient for the dynamic valuation problem at hand.

One of these alternatives is the binomial lattice, which is a robust, precise and intuitively appealing tool for option valuation models. The discrete recombinant binomial model developed by Cox, Ross and Rubinstein (1979) to evaluate real options is widely accepted as an efficient approximation to the Black, Scholes and Merton's (1973) model due to its ease of use, flexibility and the fact that if converges weakly to a Geometric Brownian Motion (GBM) as the time step (Δt) decreases. Furthermore, as opposed to the Black, Scholes and Merton model, this approach provides the solution to the early exercise of American type options. The approach used by Cox, et al (1979), where the branch nodes recombine due to the fact that the upward movement (u) is the inverse of downward movement (d), means that at each step N, one obtains $N + 1$ node, and not 2^N as in the case of a non-recombining tree. The recombinant lattice is simple and practical to implement in spreadsheet such as Excel or even in decision tree programs. In the approach developed by Brandão, Hahn, and Dyer (2005), for example, the payoffs in each branch correspond to cash flows of each state of the underlying asset.

Often, however, the relevant uncertainty is poorly modeled by a GBM stochastic diffusion process. This occurs when the value of a variable is a function of a long-term mean level, as is usually the case of non-financial commodities or interest rates. Several authors, such as Bessimbinder, Coughenour, Sequin and Smoller (1995), Schwartz (1997, 1988), Laughton and Jacoby (1993) among others, suggest that this type of variable often exhibits auto-regressive behavior and point to the fact that modeling such variable with a GBM can exaggerate the range of values depicted and, as a result, overstate the value of options written on the variable.

This paper is organized as follows. After this introduction, in section 2 we review the censored model of Hahn and Dyer (2008), develop a non-censored version . In section 3 we apply these two models to value a hypothetical real option and compare the results of these two approaches. In section 4 we conclude.

2. Binomial approximation for mean reverting models

A mean reverting (MR) stochastic process model is a Markov process in which the direction and intensity of deviation are a function of the long term average to which the current price must revert. The logic behind a Mean Reverting Model derives from microeconomics: when prices are depressed (or below their long term mean level), the demand for this product tends to increase while the production tends to decrease. This is due to the fact that consumption of a commodity increases as prices decrease, while low returns to producers will lead to the decision to postpone investment to close less efficient units, thereby reducing the supply of the product. The opposite will occur if prices are high (or above the long term mean). As an example, empirical studies (Pindyck & Rubinfeld, 1991) have shown that these microeconomic forces do indeed cause oil prices to exhibit mean reverting stochastic behavior.

The simplest form of MR process is the single factor Ornstein-Uhlenbeck process, also called Arithmetic MR process, which is defined by Eq. (1):

$$dx_t = \eta\left(\bar{x} - x_t\right)dt + \sigma dz_t \qquad (1)$$

where x_t is the natural log of the variable S_t, η the mean reversion speed, $\bar{x}$ is the long term average to which x_t reverts, σ the volatility of process and dz is the standard Wiener process. The natural logarithm of the variable is used since in the case of commodities it is generally assumed that these prices have a lognormal distribution. This is convenient because since $x=ln(S)$, S cannot be negative. In this case we are assuming that S_t follows a Geometric Orstein-Uhlenbeck process, where $S_t = \exp(x_t)$. Therefore, the expected value and variance of the Orstein-Uhlenbeck process are given by Dixit and Pindyck (1994):

$$E\left[x_t\right]=\bar{x}+\left(x_0-\bar{x}\right)e^{-\eta t} \tag{2}$$

$$Var\left[x_t\right]=\frac{\sigma^2}{2\eta}\left(1-e^{-2\eta t}\right) \tag{3}$$

We can see that when $t \rightarrow \infty$, $Var[x_t] \rightarrow \sigma^2/2\eta$ and not to infinity as is the case of a GBM.

The use of binomial lattices similar to the classic GBM model of Cox, et al (1979) to model MR processes has been essentially nonexistent due to the fact that such models often produce transition probabilities greater than 1 or less than zero when the influence of mean reversion is particularly strong. Consequently, Monte Carlo simulation or discrete trinomial and multi-nomial trees (Hull, 1999) are have been the primary methods used to model MR processes. Unfortunately, trinomial trees, such as those suggested by Tseng and Lin (2007), Clewlow and Strickland (1999), Hull and White (1994[a], 1994[b]) and Hull (1999), require more involved methodologies for specifying valid branching probabilities and lattice cell sized to ensure convergence of the stochastic process. This requires more sophisticated programming and results in difficulty in applying trinomial trees to a wide range of specific projects and cases.

As an alternative, Monte Carlos simulation approaches such as the Least Squares method (LS) of Longstaff and Schwartz (2001) are able to accommodate almost any stochastic process, including a combination of various processes, thereby eliminating the so-called “curse of dimensionality and modeling”. However, the shortcoming of these models is in modeling decisions, which can pose problems in the modeling of compound options, for example.

We propose an approach similar to the one described by Hahn and Dyer (2008) involving censuring of transition probabilities. While both binomial approaches provide results that are sufficiently precise for the use in real options applications and are sufficiently robust for use for modeling two factor bi-variate processes, this new method does not require the censoring step. This approach assumes that the stochastic behavior of modeled variable is homoscedastic, but heteroskedastic behaviour can also be modeled with some adjustments.

2.1. Censured Mean Reversion Binomial Model (Nelson and Ramaswamy, 1990)

Nelson and Ramaswamy (1990) proposed an approach that can be used in a wide range of conditions, and which is appropriate for the Ornstein-Uhlenbeck process. Their model is a simple binomial sequence of *n* periods of duration Δt, with a time horizon T: $T = n\,\Delta t$, which then allows a recombinant binomial tree to be built.
The general form for the differential equation of a stochastic process is given by: $dx = \alpha(x,t)dt + \sigma(x,t)dz$, and the proposed model is given by the following equations:

$$x_t^+ \equiv x + \sqrt{\Delta t}\sigma(x,t) \qquad \text{(up movement)}$$

$$x_t^- \equiv x - \sqrt{\Delta t}\sigma(x,t) \qquad \text{(down movement)}$$

$$p_t \equiv 1/2 + 1/2\sqrt{\Delta t}\,\frac{\alpha(x,t)}{\sigma(x,t)} \qquad \text{(up probability)} \qquad (4)$$

$$1\text{-}p_t \qquad \text{(down probability)}$$

However, in this model, the probability p_t can assume values or values greater than 1. This condition is remedied by censoring the probabilities p_t (and therefore: 1- p_t), to the range of 0 to 1 in the following manner:

$$p \equiv \begin{cases} \dfrac{1}{2} + \dfrac{1}{2}\dfrac{\alpha(x,t)}{\sigma(x,t)}\sqrt{\Delta t} & \text{if } p \geq 0 \text{ and } p_t \leq 1 \\ 0 & \text{if } p_t < 0 \text{ , } p_t \text{ is censored} \\ 1 & \text{if } p_t > 1 \text{ , } p_t \text{ is censored} \end{cases}$$

For the process shown in Eq. (1), the terms in Eq. (4) are:

$\alpha(x,t) = \eta(\bar{x} - x_t)$, and

$\sigma(x,t) = \sigma$

However, in this case we can obtain negative values or values greater than 1 in the following cases:

If $(\bar{x} - x_t)\sqrt{\Delta t} > \sigma$, then $p_{x_t} > 1$

If $(\bar{x} - x_t)\sqrt{\Delta t} < -\sigma$, then $p_{x_t} < 0$

In these cases the value of p_t can be censured according to scheme shown below:

$$p \equiv \begin{cases} \dfrac{1}{2} + \dfrac{\eta(\bar{x} - x)\sqrt{\Delta t}}{2\sigma} & \text{if } p \geq 0 \text{ and } p_t \leq 1 \\ 0 & \text{if } p_t < 0 \text{ , } p_t \text{ is censored} \\ 1 & \text{if } p_t > 1 \text{ , } p_t \text{ is censored} \end{cases}$$

These conditions are shown in Eq. (5):

$$p_{x_t} = \max\left(0, \min\left(1, \frac{1}{2} + \frac{1}{2}\frac{\eta(\bar{x} - x_t)}{\sigma}\sqrt{\Delta t}\right)\right) \tag{5}$$

where:

$\Delta x^+ = \sigma\sqrt{\Delta t}$; $\Delta x^- = -\sigma\sqrt{\Delta t}$

As x_t is the ln of price S, then $\Delta S^+ = e^{\sigma\sqrt{\Delta t}}$ and $\Delta S^- = e^{-\sigma\sqrt{\Delta t}}$. These expressions are identical to those used in the recombinant tree for a GBM, therefore the result is a recombinant binomial tree similar to that obtained with the Cox et al. (1979) approach. The probability calculations and their censoring will produce a model that converges weakly to a MR process, as shown by Hahn (2005). Note that at each node in the lattice the probability of an up movement (p_t) will depend of x_t, generating, according to the equation, a second up probability lattice $p_{x,t.}$

The adjustment required to transform a MR process into a risk neutral process for use in option pricing is done in the long term average $\bar{x}$, which is penalized by the normalized risk premium of the process: $\bar{x} - \lambda_x/\eta$ (Dixit and Pindyck, 1994, Bastian-Pinto and Brandão, 2007). For the risk neutral censored binomial tree, the adjustment is made in Eq. (6):

$$p_{x_t} = \max\left(0, \min\left(1, \frac{1}{2} + \frac{1}{2}\frac{\eta\left[\left(\overline{x} - \lambda_x/\eta\right) - x\right]}{\sigma}\sqrt{\Delta t}\right)\right) \tag{6}$$

2.2. Non Censored Mean Reverting Lattice Model

To develop a binomial model the 1st and 2nd moments (expected value and variance) of the stochastic process must match the corresponding moments of the binomial lattice. The problem is finding a binomial sequence that converges to a stochastic differential equation (SDE) in the form:

$$dx_t = \alpha(x,t)\,dt + \sigma(x,t)\,dz$$

where $\alpha(x,t)$ and $\sigma(x,t)$ are respectively functions of the continuous instantaneous growth rate (*drift*) and volatility, and dz is a standard Weiner increment. The conditions for a binomial sequence of x_t converges to the SDE above is that $x_t = x_0 + \int_0^t \alpha(x_s,s)ds + \int_0^t \sigma(x_s,s)dz$, exists in $0 < t < \infty$, and that $\left|x_{\Delta t}^{\pm}(x,t) - x\right|$, $\left|\alpha_{\Delta t}(x,t) - \alpha(x,t)\right|$, e $\left|\sigma_{\Delta t}^2(x,t) - \sigma^2(x,t)\right| \to 0,\ when\ \Delta t \to 0$.

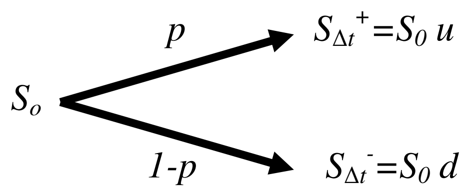
Using the discretization: $\Delta t = t - t_0$ we can write Eq. (2) and Eq. (3) as:

$$E[x_t] = \overline{x} + \left(x_{t-1} - \overline{x}\right)e^{-\eta\Delta t} \tag{7}$$

$$Var[x_t] = \frac{\sigma^2}{2\eta}\left(1 - e^{-2\eta\Delta t}\right) \tag{8}$$

The objective is to match Eq. (7) and Eq. (8) to the analogous terms for a one period binomial process of price <u>S</u>, as shown in **Figure 1**:

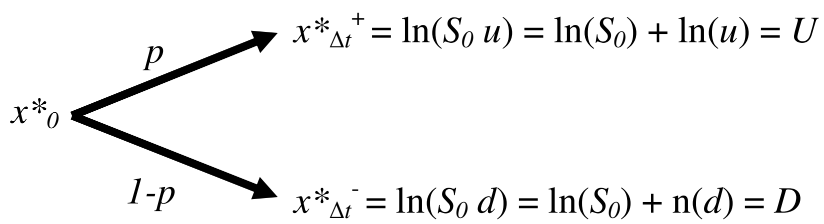
Figure 1. Binomial node



For our model we used the approach suggested by Hull and White (1994[a], 1994[b]) as described in Clewlow and Strickland (1999) and in Hull (1999), for the case of a trinomial tree model of a MR process. First, we define an additive tree, which models an Ornstein-Uhlenbeck arithmetic process with a long term mean equal to zero: $\bar{x}^* = 0$, and initial value of zero: $x_0^* = 0$. In this lattice the nodes will have a value of x_t^*. The expected values of the Ornstein-Uhlenbeck model are added to the value of the nodes in each period from Eq. (7) using the real long term average of the process: $\bar{x}$, and the real beginning value of: x_0. Hence, this tree of values x_t is used to obtain the tree of a price process S_t with lognormal distribution defined by $S_t = e^{x_t}$

Since we are considering $x_t = \ln(S_t)$, to study the dynamics of the effect of the binomial node we can consider S_0 as a unit value, i.e.: $S_0 = 1$ in such a way that the relative magnitudes in the binomial process remain unaltered. Since $x_o^* = \bar{x}^* = 0$ we can write the binomial relationship of the process, which is now arithmetic, as x_0^* (**Figure 2**):

Figure 2. Binomial node for Ornstein-Uhlenbeck process



(9)

To approximate this binomial process with the Eq. (7) and Eq. (8) of the Ornstein-Uhlenbeck process we obtain the following relationships:

$$x^{*+} = x^* + \sigma\sqrt{\Delta t} \tag{10}$$

$$x^{*-} = x^* - \sigma\sqrt{\Delta t} \tag{11}$$

$$p_{x_t} = \frac{1}{2} + \frac{1}{2}\frac{\eta\left(-x_t^*\right)\sqrt{\Delta t}}{\sqrt{\eta^2\left(-x_t^*\right)^2 \Delta t + \sigma^2}} \tag{12}$$

The derivation of Eq. (10), Eq. (11) and Eq. (12) is shown in Appendix 1. With these we can model the additive recombinant binomial tree of mean 0 and initial value 0 for an Arithmetic MR process of x_t^*. As in Clewlow and Strickland (1999) and in Hull (1999), to these values of nodes we should then add the expected values obtained by Eq. (7), considering now x_o and $\bar{x}$ (both no longer equal to 0, but with real parameter values of a MR process). The x value after i up movements, and j down movements will be:

$$t = (i + j)\Delta t$$

$$x_{(i,j)} = \bar{x} + \left(x_0 - \bar{x}\right) e^{-\eta(i+j)\Delta t} + \underbrace{\left(i - j\right)\sigma\sqrt{\Delta t}}_{x^*} \text{, or:}$$

$$x_{(i,j)} = \bar{x}\left(1 - e^{-\eta(i+j)\Delta t}\right) + x_0 e^{-\eta(i+j)\Delta t} + \underbrace{\left(i - j\right)\sigma\sqrt{\Delta t}}_{x^*} \tag{13}$$

The non censured binomial recombinant tree for the geometric MR process, defined by: $S_t = e^{x_t}$, is obtained by directly transforming $x_{(i,j)}$ values in $S_{(i,j)}$. This yields a recombinant Geometric MR binomial tree. The relationship between the non-censored model and the Nelson and Ramaswamy (1990) censored is shown in Appendix 2. We note that in this non-censored model, the adjustment for risk neutrality is given in the equation of expected value of the process, altering the value of x given in Eq. (13) to:

$$x_{(i,j)} = \left(\bar{x} - \lambda_x/\eta\right)\left(1 - e^{-\eta(i+j)\Delta t}\right) + x_0 e^{-\eta(i+j)\Delta t} + \underbrace{\left(i - j\right)\sigma\sqrt{\Delta t}}_{x^*} \tag{14}$$

In the following section we will apply these approaches to the valuation of a hypothetical real option, based on historical data, and will compared the results of both methods.

3. Ethanol Industry Expansion Option Valuation using MR Lattices

The bio-fuels sector, especially in Brazil, is well known for having several managerial flexibilities which must be valued as real options, as shown by Brandão, Penedo & Bastian-Pinto (2009), Bastian-Pinto & Brandão (2007) and Goncalves, Neto & Brasil (2006), among other authors. To illustrate, in this section we value an option to sugar refining plants to produce ethanol from the same base input, sugarcane. For this option to be available an operator must pay the exercise price, which in this case is the investment cost of the ethanol plant.

Investments in a sugar refining plant are substantially higher than those of an equivalent ethanol plant. A sugarcane processing plant can either be a sugar only refinery, an ethanol distillery or a flexible plant capable of producing any mix of each product. In order to value option to expand a sugar refinery into a flexible plant (sugar and ethanol) we consider the possible cash flows from each mode: production of sugar and a small amount of ethanol as byproduct, production of ethanol only, and production of both from the same sugarcane processing plant.

As the industrial investment in a sugar refinery is higher than that of an equivalent ethanol plant, it is reasonable to assume that a sugar plant that is already in operation might want to consider the opportunity to invest in an ethanol distillery. We model this embedded flexibility as an American real option where the exercise price is the cost of the ethanol distillery unit.

3.1. Modeling the Option to Expand

The free cash flows from the sugar refinery and from the ethanol distillery plant are proportional to the prices of the respective commodities paid to producers. The series of prices used are available online at CEPEA (2009). Both series are historical prices and were converted into monthly averages from May 1998 to January 2010, deflated so as to represent prices of January 2010. The series used for ethanol are a mix of anhydrous ethanol (70%) and hydrated ethanol (30%), reflecting the ratio produced in the distillery (GONÇAVES et al, 2006; EPE, 2008).

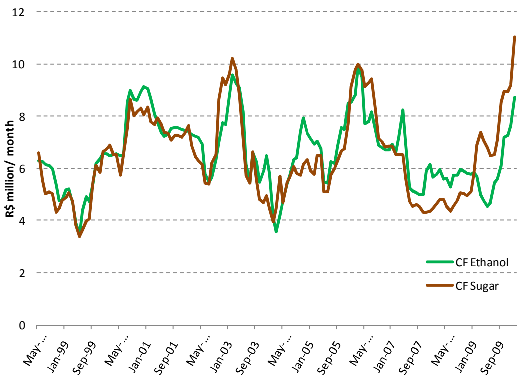
For each ton of sugarcane that is processed, the sugar refining plant produces 107 kg of sugar and 12 liters of ethanol, whereas an ethanol distillery plant will produce 80 liters from the same amount of sugarcane. Direct taxes are assumed to be 16% for sugar income and 4% for ethanol, but production of sugar also involves higher variable costs than those of ethanol production, due among other factors, to higher energy consumption by this process. The case considers a typical high capacity plant capable of processing 2.6 million tons of sugarcane yearly. For this plant size variable costs for sugar were considered to be R$ 4 million/ year greater than that of ethanol. Income tax is assumed to be 34% and the necessary investment for the expansion to the flex plant is R$ 83.2 million.

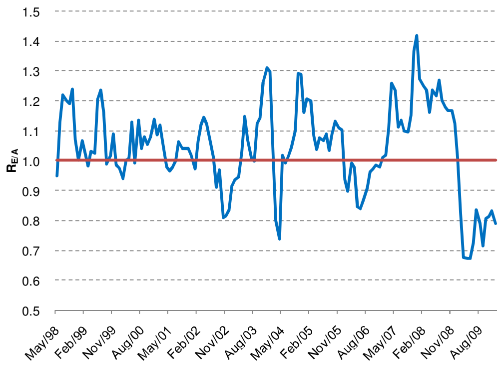
The uncertain variable modeled for valuation of the real option is the ratio of free cash flows of both modes of production:

$$R_{E/A} = \frac{\textit{Free cash flow of ethanol production}}{\textit{Free cash flow of sugar production}}$$

This approach allows the reduction of uncertainties from two to only one stochastic variable. The historical behavior of $R_{E/A}$ can be seen in Figure 3, together with the historical values of the free cash flows used in the calculation of $R_{E/A}$. Free cash flows were estimated from the prices series available, as mentioned.

Figure 3. Monthly free cash flows for sugar and ethanol plants and Ratio of cash flows (*R*$_{EA}$)





The option studied is valued as follows: we assume a maturity time of ten years in quarter periods (Δt = 0.25), after which the higher cash flow (ethanol or sugar) is perpetuated without additional growth. This consideration suggests that the option is no longer available after 10 years, which is a limitation of this example.

Nevertheless, it is a reasonable assumption, since after this time the value of the option should be significantly lower than during the time frame considered.

The base cash flow of sugar production is modeled as the expected value of a geometric mean reversion, and the parameters were obtained from the historical series (**Figure 3**). The starting value for cash flow of quarterly production of sugar, is: CFs_0 = R$ 33.057 million, and the long term level to which it reverts: $\overline{CFs}$ = R$ 22.633 million. The yearly discount rate used, in real terms, is: K = 11.87% and the risk free rate Rf = 6.18%. This yields a base case present value of R$ 795.8 million which is consistent with the values of acquisitions and mergers presently occurring in Brazil.

At each node of the binomial lattice, we have a value of the multiple $R_{E/A}$. In cases where it is lower than 1, the cash flow of the production of sugar is higher than that of ethanol production from the same quantity of processed sugarcane. In cases where it is higher than 1, the cash flow of ethanol production is that of the production of sugar, multiplied by the ratio: $R_{E/A}$. The initial value (at time $t_0 = 0$) of the variable $R_{E/A}$ is 0.78969. This indicates that at the start of the projection, the free cash flow of the production of ethanol is lower than that of sugar. This value is one of the lowest of the whole series analyzed and should be an inhibitor of the investment in the expansion for ethanol production. Nevertheless the high volatility of $R_{E/A}$ should almost certainly attribute some value to the expansion option.

Initially a lattice is constructed with the values of $R_{E/A}$ according to the stochastic process chosen for modeling this variable, with 40 quarterly steps. With the values of $R_{E/A}$ modeled in a MR lattice, the values of the free cash flow of ethanol production is calculated by multiplying the values of $R_{E/A}$ by the deterministic value of the Sugar production cash flows at that step. With the values of the ethanol production cash flow, a second lattice is calculated for the present values of the expansion project, starting from the end of the 10^{th} year (40 quarters). At this point at each nod we calculate the present value of the production of ethanol from its perpetuated cash flows discounted four quarters at the risk free rate (time necessary for starting of operation after decision) minus the cost of expansion, or that of the perpetuated sugar production cash flows, whichever is higher.

We then proceed backwards to step 39, where the value at each node is calculated as follows: the values from the previous step (step 40), weighted by the risk neutral probabilities of the $R_{E/A}$ lattice, and discounted by the risk free rate to which is added the ethanol cash flow at the node, or the sugar cash flow, whichever is higher, and the value of expansion: the present value of the production of ethanol from its perpetuated cash flows discounted four quarters minus the cost of expansion.
We proceed backwards up to step 0. At step 0 we finally have the present value of the project with the expansion option.

3.2. Comparison of the Modeling of $R_{E/A}$ with the both MR Lattices

The stochastic variable in this option is the rate of cash flows of ethanol production to that of sugar production: $R_{E/A}$. We modeled it as a geometric mean reversion using the binomial lattice models presented in the section 2. To verify the precision of both the censured and non-censored models, we will show the results of the modeling of $R_{E/A}$ as a geometric MR process according to the model 1 of Schwartz (1997) as defined by Eq. .

$$dR = \eta\left(\bar{x} + \frac{\sigma^2}{2\eta} - \ln[R]\right)Rdt + \sigma Rdz \qquad (15)$$

where:

$x_t = \ln(R_t)$, and

$dx = \eta(\bar{x} - x)dt + \sigma dz$

For this model, it is necessary to determine the values of the following parameters:

R_o – initial value (in $t = 0$) of stochastic variable $R_{E/A\ t}$
$x_0 = \ln(R_o)$
$\bar{x}$ – long term mean level to which $x_t = \ln(R_t)$ converges
η – mean reversion speed
σ – volatility of the process

Δt – discreet time interval

The expected value for $R_{E/A}$ behaving according to Equ. (15) is:

$$E[R_t] = \exp\left\{ \ln\left(R_{t_0}\right) e^{-\eta(t-t_0)} + \left[\ln\left(\bar{R}\right) - \frac{\sigma^2}{2\eta} \right]\left(1 - e^{-\eta(t-t_0)}\right) + \frac{\sigma^2}{4\eta}\left(1 - e^{-2\eta(t-t_0)}\right) \right\} \quad (16)$$

It is important to note that Eq. (16) converges to a long term value $\bar{R}^* = \exp\left(\bar{x} + \sigma^2/4\eta\right)$ (see Schwartz, 1997). Geometric binomial models with $R_t = \exp(x_t)$ must converge to a value: $\bar{R} = \exp\left(\bar{x} + \sigma^2/2\eta\right)$. Therefore if we have $\bar{R}^* = \exp\left(\bar{x} + \sigma^2/2\eta\right)$ as the long term mean, this transforms the process defined by Eq. (15) to: $dR = \eta\left(\ln\left[\bar{R}^*\right] - \ln[R]\right) R dt + \sigma R dz$

which means that $\bar{R} = \bar{R}^* \exp\left(-\sigma^2/4\eta\right)$

From the historical data available of $R_{E/A\ t}$ the parameters required to model the stochastic variable as e geometric mean reversion were derived by running the following regression: $\log[R_t] - \log[R_{t-1}] = \beta_0 + \beta_1 \log[R_{t-1}] + \varepsilon$

The mean reversion coefficient η is obtained from the regression output as $\eta = \dfrac{-\log(\beta_1 + 1)}{\Delta t}$, the volatility is given by $\sigma = \sigma_\varepsilon \sqrt{\dfrac{2\log(\beta_1 + 1)}{\Delta t\left[(\beta_1 + 1)^2 - 1\right]}}$ where σ_ε^2 is the variance of the regression's errors, and the long term mean is given by $\bar{R} = \exp\left[-\dfrac{\beta_0}{\beta_1} + \dfrac{\sigma^2}{2\eta}\right]$. For a more detailed discussion of the parameter definition in mean reverting models we refer the reader to Bastian-Pinto, Brandão & Hahn (2009).

Parameters for equation (15) were found to be:

η = 1.97 (per year)

σ = 30.30 % (per year)

$\bar{R}_{EA}$ = 1.0388

In **Figure 4** and **Figure 5**, both censored and non-censored MR lattices for $R_{E/A}$ are shown compared with indication of probability of occurrence of each nod (size of blue dots), censored nods (red dots) in the case of censored lattice, expected value from the lattice (solid blue line), expected value from analytic expression (solid red line), together with 95% certainty analytical interval (dotted red line) and equilibrium level (solid green line).

Figure 4 – Censored Lattice for $R_{E/A}$

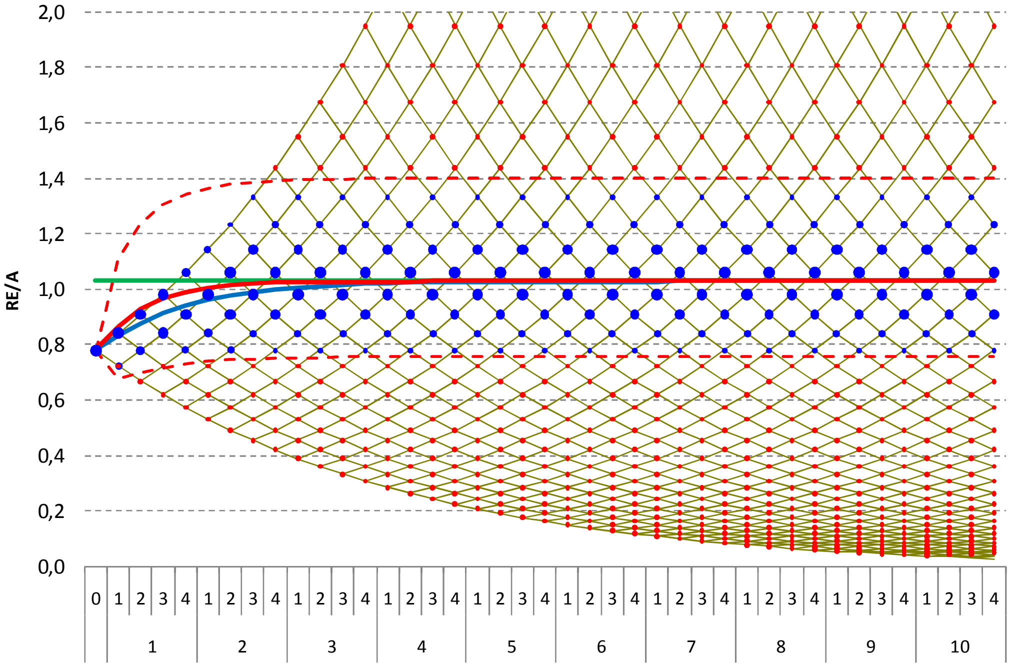
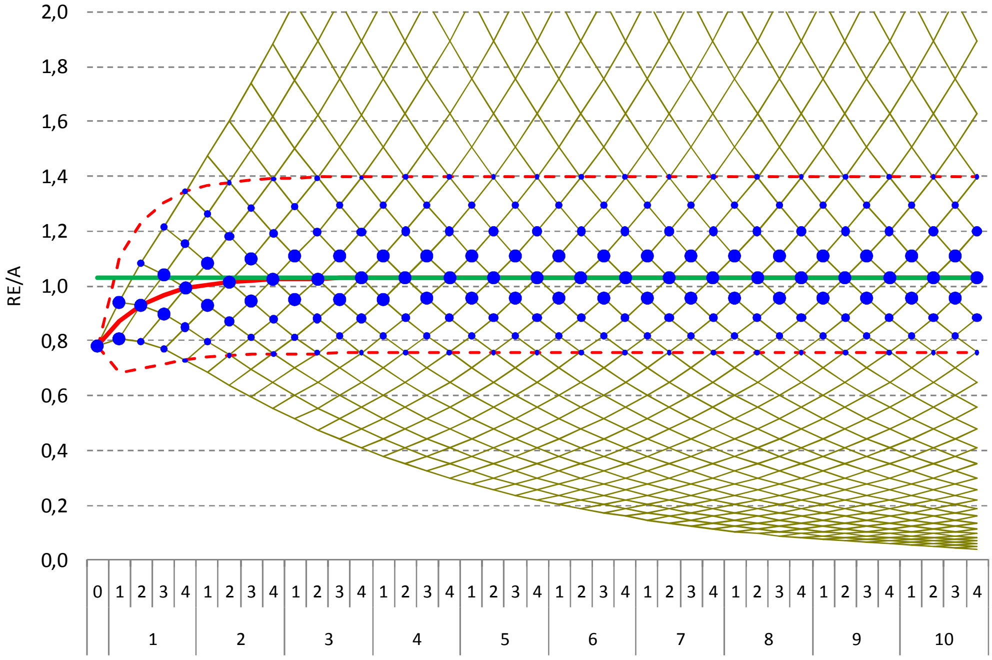


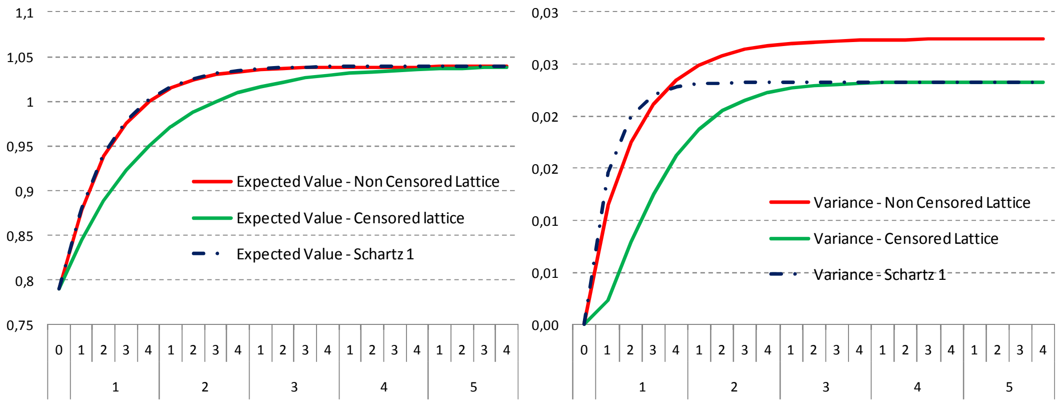
Figure 5 – Non-censored MR Lattice for $R_{E/A}$



: probability of occurrence, between 0,5 % and 5%
: probability of occurrence, between 5 % and 20%
: probability of occurrence, between 20 % and 35%
: probability of occurrence, above 35 %
: censored nods (probability =0)
: expected value from lattice
: expected value analytical
: 95% certainty analytical

One difference that is apparent in the lattices is that the expected value for the censored lattice differs from the analytical value obtained from equation (16), whereas the expected value from the non-censored lattice closely approximates it. We then checked the values obtained analytically and from each lattice for the 1st and 2nd moment (expected value and variance) of the process described, for the first five years (20 periods of $\Delta t = 0.25$). These are plotted in **Figure 6**.

Figure 6 – Expected Value and Variance from both Lattices and Analytical form



We can note that the non-censored lattice closely matches the expected value of the Schwartz model 1 (1997) process, but its variance estimate although in the first periods in very close to the analytical form, stabilizes at a higher level. With regard to the censored lattice, the opposite seems to happen: its expected value initially diverges from that of the analytical solution, due to the construction characteristics of this particular lattice, and then converges to the long term mean. Its variance, although at first also diverging from that of the analytical form, then catches up with it with much precision.

From the results above we can infer that the non-censored lattice might slightly overestimate the value of an option modeled with it, due to the slightly inflated variance, whereas the censored lattice will on the contrary slightly undervalue it due to the divergence in the expected value. But it is important to remember that these observations will only apply to this specific case, and that other parameters for mean reversion might yield different results.

3.3. Results for the Real Option Estimation

In order to value derivatives, options and real options, a risk neutral price process is required since it allows the use of the risk-free interest rate as the adjusted price appreciation rate. We then use the Capital Asset Pricing Model to estimate the risk premium λ of the stochastic variable $R_{E/A}$. This was done by regressing the log return of $R_{E/A}$ against the return of the Ibovespa stock index of Brazil stock market, in order to find the β factors (the beta coefficient of the CAPM), for $R_{E/A}$. It was found to be not statistically different from 0, indicating that the risk premium for the stochastic variable can also be considered zero. This is reasonable since the variable $R_{E/A}$ is a rate of two cash flows and should have no correlation with market risk. Therefore for the stochastic processes considered in this paper, the risk premium is zero, and the parameters found for these are already risk adjusted for the real option calculation.

Valuing the expansion option of the sugar processing plant into a flexible ethanol/sugar plant with the two lattices developed in this paper yields the results shown in Table 1:

Table 1 – Expansion Option Value

	Total value $	Option Value $ (%)
Base case	R$ 795.8 million	
With expansion option Censored lattice method	R$ 847.6 million	R$ 51.8 million (+6.51%)
With expansion option Non-Censored lattice method	R$ 858.8 million	R$ 63.0 million (+7.91%)

As expected, the option has value and it differs slightly while using each different model of lattice. In this particular case, the non-censored lattice seems to approach more closely the behavior of the stochastic uncertainty, as its expected value replicates more precisely the values of the analytic estimation. Therefore the option value calculated by the non-censored lattice probably is more precise than with the censored lattice. In this example the initial value of the stochastic variable differs significantly (-24%) from the equilibrium level to which it converges at a considerably high reversion speed (η = 1.97). Because of the construction characteristics of the censored lattice, the expected value calculated from it diverges initially from the true analytic result, therefore yielding a lower value for the option than what is the correct one.

This particular example of the value estimation of a hypothetical real option, although based on real data, has the purpose of demonstrating the applicability of the non-censored MR process lattice developed in this article, and to show how different stochastic modeling may alter the value of a real option. We note that this simplified case does not consider any restrictions that may apply in sugar-ethanol plants such as supply contracts and logistic limitations.

4. Conclusions

Binomial lattices are an accurate, robust and intuitively appealing approach for option valuation. In this article we have developed a non censored lattice model for MR processes which has some distinct advantages over currently available models. We compare this model with the censored lattice model of Ramaswamy (1994) as extended by Hahn and Dyer (2008) and show some of its advantages. We then apply both models to the valuation of American options by an mean reverting stochastic process, which are typically only modeled with much more computationally complex methods such as trinomial lattices or least square simulation approaches.

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Appendix 1: Derivation of up and down values and up probabilities for the non-censored binomial lattice for MR process

Writing $e^{-\eta\Delta t}$ as a Taylor series, we have:

$$e^{-\eta\Delta t} = \sum_{n=0}^{\infty}\frac{(-\eta\Delta t)^n}{n!} = 1 + (-\eta\Delta t) + \frac{(-\eta\Delta t)^2}{2!} + \frac{(-\eta\Delta t)^3}{3!} + ... \qquad (17)$$

Since a binomial lattice approach implies that we are using short time intervals, we can consider all Δt with an exponent greater than 2 to converge to 0. This yields:

$$e^{-\eta\Delta t} \approx 1 - \eta\Delta t \qquad (18)$$

With this relation, we can write Eq. (7) and Eq. (8) as:

$$E[x_t] \approx \bar{x} + (x_0 - \bar{x})(1 - \eta\Delta t) = x_0 + (\bar{x} - x_0)\eta\Delta t \qquad (19)$$

$$Var[x_t] \approx \frac{\sigma^2}{2\eta}(1 - 1 + 2\eta\Delta t) = \sigma^2\Delta t \qquad (20)$$

With $\Delta t = t - t_0$, from the binomial process (9), we have:

$$E[x_t] = pU + (1-p)D \qquad (21)$$

For the variance, we also have:

$$Var[x_t] = E\left[x_t^2\right] + E[x_t]^2 \qquad (22)$$

$$Var[x_t] = pU^2(1-p)D^2 - (pU + (1-p)D)^2$$

$$Var[x_t] = pU^2 + (1-p)D^2 - p^2U^2 - (1-p)^2 D^2 - 2p(1-p)UD$$

$$Var[x_t] = pU^2 + D^2 - pD^2 - p^2U^2 - D^2 - p^2D^2 + 2pD^2 - 2pUD + 2p^2UD$$

$$Var[x_t] = pU^2 - p^2U^2 - p^2D^2 + pD^2 - 2pUD + 2p^2UD$$

$$Var[x_t] = p(U^2 + D^2 - 2UD) - p^2(U^2 + D^2 - 2UD)$$

$$Var[x_t] = p(1-p)(U-D)^2 \qquad (23)$$

Then have the moment matching equations:

Eq. (19) $\equiv$ Eq. (21), Eq. (20) $\equiv$ Eq. (23)

From the first relation above, with $x_o^* = \bar{x}^* = 0$, we have:

$$(-x_t^*)\eta\Delta t \equiv pU + (1-p)D$$

From the second relation:

$$\sigma^2\Delta t \equiv p(1-p)(U-D)^2$$

Therefore we have two equations with three unknowns: *p*, *U* and *D*. In order to stay with only two unknowns, we make a consideration that turns the binomial tree into a recombining lattice: *D* = - *U*. So these equations are now:

$$\eta(-x_t^*)\Delta t \equiv (2p-1)U \qquad (24)$$

$$\sigma^2\Delta t \equiv 4p(1-p)U^2 \qquad (25)$$

Writing Eq. (24) ^{2} + Eq. (25):

$$\eta^2\left(-x_t^*\right)^2\Delta t^2 + \sigma^2\Delta t = U^2$$

and substituting in Eq. (24)^2, we obtain:

$$\eta^2\left(-x_t^*\right)^2\Delta t^2 = (2p-1)^2\left(\eta^2\left(-x_t^*\right)^2\Delta t^2 + \sigma^2\Delta t\right)$$

$$2p-1 = \frac{\eta\left(-x_t^*\right)\Delta t}{\sqrt{\eta^2\left(-x_t^*\right)^2\Delta t^2 + \sigma^2\Delta t}}$$

$$p_{x_t} = \frac{1}{2} + \frac{1}{2}\frac{\eta\left(-x_t^*\right)\sqrt{\Delta t}}{\sqrt{\eta^2\left(-x_t^*\right)^2\Delta t + \sigma^2}} \qquad (26)$$

This yields an expression for *p* as a function of the process parameters (η, σ) and the time step Δ*t*, as well as the value x_t^*. It is easy to verify that this expression for *p* will always be in the range from 0 to 1, and therefore there is no need for censoring as in the Nelson and Ramaswamy (1990) model. With

$\theta = -x_t^*\eta\sqrt{\Delta t}$ we can write *p* as:

$$p = 0,5\left(1 + \frac{\theta}{\sqrt{\theta^2 + \sigma^2}}\right)$$

Thus, for *p* > 1 to happen it is necessary that: $\theta > \sqrt{\theta^2 + \sigma^2}$, which is impossible, independently of the sign of θ.

Furthermore, for *p* < 0, to happen it is necessary that $-\theta > \sqrt{\theta^2 + \sigma^2}$, which is also impossible.

Although it is not necessary to censor this lattice, we still have a lattice of up movements probabilities $p_{(i,j)}$ which is dependent of the values of $x_{(i,j)}^*$. The subscripts *i* and *j* represent the number of up movements (*i*) and down movements (*j*) in the trajectory leading to node $x^*_{(i,j)}$ from the starting point: $x^*_0 = 0$.

To derive the magnitude of the up and down movements, from Eq. (24) we have:

$$U = \frac{-x_t^* \eta \Delta t}{(2p-1)}$$

with:

$$U = \frac{\eta(-x_t^*)\Delta t \sqrt{\eta^2 \left(-x_t^*\right)^2 \Delta t^2 + \sigma^2 \Delta t}}{\eta\left(-x_t^*\right)\Delta t}$$

$$U = \sqrt{\eta^2 \left(-x_t^*\right)^2 \Delta t^2 + \sigma^2 \Delta t} = -D$$

However, the expression $\eta^2 \left(-x_t^*\right)^2 \Delta t^2$ prevents the tree of recombining because even with $U = -D$, it is still a function of x_t and therefore does not allow that the result from a superior node be equal to that from a low nod. Nonetheless, we can consider that $\Delta t^2 \rightarrow 0$, if we are considering small time intervals. We thus use:

$$U = -D \equiv \sigma\sqrt{\Delta t}$$

So, for this non-censored mean reversion model, of mean 0 and starting point 0:

$$x^{*+} = x^* + \sigma\sqrt{\Delta t}$$

$$x^{*-} = x^* - \sigma\sqrt{\Delta t}$$

$$p_{x_t} = \frac{1}{2} + \frac{1}{2}\frac{\eta\left(-x_t^*\right)\sqrt{\Delta t}}{\sqrt{\eta^2\left(-x_t^*\right)^2 \Delta t + \sigma^2}}$$

Appendix 2: Derivation of the censored mean reversion model of Nelson and Ramaswamy (1990)

In order to get to Nelson & Ramaswamy (1990) model, from Eq. (26), we first consider that in this model there is a long term mean $\bar{x}$:

$$p = \frac{1}{2} + \frac{1}{2}\frac{1}{\sqrt{\dfrac{\eta^2\left(\bar{x}-x_t\right)^2\Delta t^2 + \sigma^2\Delta t}{\eta^2\left(\bar{x}-x_t\right)^2\Delta t^2}}}$$

$$p = \frac{1}{2} + \frac{1}{2}\frac{1}{\sqrt{1 + \dfrac{\sigma^2}{\eta^2\left(\bar{x}-x_t\right)^2\Delta t}}}$$

We then consider that with small values of Δt, the expression $\dfrac{\sigma^2}{\eta^2\left(\bar{x}-x_t\right)^2\Delta t}$ is large when compared to 1, and we can therefore simplify the above equation by taking out the 1 from the denominator:

$$p \equiv \frac{1}{2} + \frac{\eta\left(\bar{x}-x_t\right)\sqrt{\Delta t}}{2\sigma}$$

Comparing this equation to Eq. (4) we have:

$\alpha(x,t) = \eta\left(\bar{x}-x_t\right)$, and

$\sigma(x,t) = \sigma$

After the simplification above, note that we might get negative values or values above 1:

If $\left(\bar{x}-x_t\right)\sqrt{\Delta t} > \sigma$, then $p_{x_t} > 1$

If $\left(\bar{x}-x_t\right)\sqrt{\Delta t} < -\sigma$, then $p_{x_t} < 0$

In these cases, the value of p_t needs censoring as shown:

$$p \equiv \begin{cases} \dfrac{1}{2} + \dfrac{\eta\left(\bar{x}-x\right)\sqrt{\Delta t}}{2\sigma} & \text{if } p \geq 0 \text{ and } p_t \leq 1 \\ 0 & \text{if } p_t < 0\text{ , } p_t \text{ is censored} \\ 1 & \text{if } p_t > 1\text{ , } p_t \text{ is censored} \end{cases}$$

These conditions match those already shown by Eq. (6).