Real Options under Ambiguity

David Roubaud[a*], André Lapied[a] and Robert Kast[b]

[a] *Université Paul Cézanne, GREQAM ,15-19 Allée Claude Forbin, 13627 Aix-en-Provence*

[b] *CNRS, LAMETA, 2 Place Viala, 34060 Montpellier, France*

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[*] Corresponding Author: david_roubaud@hotmail.com

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ABSTRACT

A new class of real options models has recently emerged, characterized by the presence of "ambiguity" (or "Knightian uncertainty") and based on recursive multiple-priors preferences. Ambiguity is typically described through a range of Geometric Brownian motions and solved by application of a maxmin expected utility criterion among them (worst case), but this reduces individual preferences to an extreme form of pessimism. In contrast, by relying on dynamically consistent "Choquet-Brownian" motions, we show that a much broader spectrum of attitudes towards ambiguity may be accounted for, improving the explanatory and application potentials of these appealing expanded real options models. In the case of a perpetual real option to invest, it is shown that ambiguity aversion may delay the moment of exercise of the option, while the opposite holds true for an ambiguity seeking decision maker. Furthermore, an intricate relationship between risk and ambiguity appears strikingly in our model.

Keywords: Real Options; Ambiguity; Irreversible investment; Optimal stopping*;* Knightian uncertainty; Choquet-Brownian motions.

JEL classification codes: C61; D81; D92; G31

1. INTRODUCTION

Dealing better with uncertainty may contribute to gaining competitive edge and foster value creation. Such ambition is at the very core of the real options approach to capital budgeting. But what sort of uncertainty entrepreneurs truly face? Is it *objective* or rather a matter of *beliefs* and/or *tastes*? How may a real option model be affected after introducing uncertainty? Over the last two decades, breakthroughs in decision theory certainly improved the understanding of uncertainty and its multiple forms. Still, giving axiomatic foundations to what derives from various psychological biases remains challenging. Transfering theoretical advances into practical recommendations is even more delicate, especially in dynamic settings.

Recently, a new class of real option models was built, characterized by the presence of *Knightian uncertainty* or *ambiguity* (Nishimura and Ozaki, 2007; Asano, 2005; Trojanowska and Kort, 2010; Miao and Wang, 2009). Solving uncertainty by application of a *maxmin* criterion over the potential outcomes of decision, these recursive multiple-priors models identify a significant impact of ambiguity on real option valuations and timing of exercise, but only in the case of extreme aversion towards uncertainty. In this paper, real options models under uncertainty are expanded to account for the variety of preferences towards ambiguity. Indeed, many experiments and studies confirmed that if aversion may be a prevalent reaction to uncertainty, excluding ambiguity seeking *a priori* may often be unjustified.

To account for a wider range of preferences towards ambiguity, we suggest the use of dynamically consistent *Choquet-Brownian* processes to model uncertainty (see section 2.2). This is the key originality of our model. In our Choquet expected utility framework, the impact of perceived ambiguity on the expected cash flows from a project is summarized by the value of a parameter c: it expresses the nature and intensity of the psychological bias revealed by decision makers under ambiguity, that we call *c-ignorance*. The probabilistic case is a special case in our generalized real option model to invest, as well as the multiple-priors. Section 1 discusses the rationale behind the construction of such models, identify some limits, as well as underline how the real options theory is deeply concerned over uncertainty and contributes to its improved representation. Key notions and methodological choices are clarified.

1.1 Key definitions:

As Real options[1] (Myers, 1977) may appear quite frequently, getting a quantitative estimate of the cost of opportunity of acting *now* rather than later may be relevant in many situations. The core of the real options theory applies powerful techniques from financial options theory (derived from Black and Scholes, 1973, and Merton, 1973), in conjunction with decision analysis concepts, to achieve valuation of capital investments (Brennan and Schwartz, 1985; McDonald and Siegel, 1986; Pindyck, 1991; Ingersoll and Ross, 1992; Dixit and Pindyck, 1994). Valuing the flexibility inherent to most decision-making processes is especially crucial in front of large capital budget decisions bearing high uncertainty, such as R&D projects, M&A or intangible asset valuations[2]. Some limits of the standard approach of investment decision, net present value and/or discounted cash flow methods (Fisher, 1930; Williams, 1938) are improved. It has indeed been shown on many occasions (Dean, 1951; Hayes and Garvin, 1982 and others) that discounted cash flows can lead to non-optimal decisions, such as investing too early in projects while waiting would allow to create more value, or conversely to wrongly reject projects, for instance by ignorance of "growth options" (Myers, 1977). Not always easy to apply in practice, real options models may improve risk analyses by giving management the incentive and ability to actively manage sources of risks rather than passively following DCF threshold methods[3].

Next, if we turn to decision theory, many forms of *uncertainty*, not reducible to the usual concept of risk, have been described and shown to be meaningful. Moreover, preferences *revealed* by decision makers show repeatedly that they are typically not neutral towards uncertainty. Such preferences may be due to various psychological biases, whose multiple origins will not be debated specifically here[4].

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[1] A project combining the following three key characteristics is a standard real option: *irreversibility* (due to the presence of at least partially sunk costs), *uncertainty* (at least as regards to future payoffs) and *managerial flexibility* (regarding the timing of option exercise).

[2] Mining and petroleum projects often benefit from real option approach. Other industries, such as pharmaceuticals, are now increasingly applying these models in their project analyses and management decisions. Hence the strong interest expressed towards real options by many theorists and practitioners alike.

[3] Still, the success of real options theory should not be overestimated yet, as real options are often seen as too complex to apply and difficult to put into practice. Real option theory has also suffered from being misused to justify unrealistic valuation levels, especially during the internet bubble (for instance with a fraudulent use by Enron).

[4] See for instance Heath and Tversky (1991) on the striking influence of the *self perceived* competence level on a decision-maker attitude towards ambiguity and subsequent decisions.

Let's underline that we are not to discuss here the normative status of such attitudes towards ambiguity; rather, formalization is given to commonly observed revealed preferences in front of uncertain outcomes, challenging the well established expected utility framework. Our approach is axiomatic and subjective (the measure derives from the decision maker's preferences), without reference to an objective probability distribution that would be subjectively distorted (although it could be an interpretation).

If uncertainty remains a somewhat vague notion in the litterature, as many different interpretations coexist, a clear definition is needed to avoid confusion or misunderstanding[5]. The label "uncertainty" may be better used as a generic term, while a specific description is given to its different forms, such as risk or ambiguity. Even if the very distinction between *risk* and *uncertainty* remains often unfamiliar, or perceived as overly theoretical and/or unpractical, it should not be ignored in many settings. Dissociating risk and ambiguity was initiated by Knight (1921) and Keynes (1921). A *risky* situation is defined through the existence of a unique probabilistic model, known from the decision maker: it is well aware of the random nature of some elements at least, but remains perfectly confident that no model misspecification needs to be considered. This is equivalent to adopting a rational expectations framework. To the contrary, *ambiguity* appears when the decision maker is not fully confident that his beliefs on the possible states of the world are perfect, when uncertainty cannot be reduced to a single Kolmogorov type of probability measure. It may be typically necessary to rely on a *range* of probability measures.

Another formulation would be to emphasize that under Knightian uncertainty the information is too limited to allow the use of *precise* probabilities. Experimental studies showed repeatedly that decision makers usually prefer to deal with known probabilities rather than imprecise ones and that applying the subjective utility model when confronted with ambiguity was consequently often misleading. Most typically, decision makers prefer more *transparent* settings if they are averse to ambiguity (see how the knowledge of urn composition dictates preferences in Ellsberg, 1961).

[5] Lets note that such difficulties once also prevailed immediately after the impact of attitudes towards risk was first identified in the literature. Following pioneer works (Pratt, 1964: Arrow, 1965) the need to incorporate psychological biases into dominant expected utility models has been gradually recognized, for instance in the financial asset pricing theory (see portfolio diversification in Kelsey and Milne, 1995) or in the insurance literature.

Accounting for preferences towards uncertainty may generate additional complexity in real options models. Minimizing this likely drawback is fundamental as such models without uncertainty are already often criticized as overly complex! The trade-off between empirical realism vs. tractability of the model needs to be recognized and assumed. Furthermore, modeling uncertainty still remains challenging, defining *ambiguity* and/or *aversion to ambiguity* remains controversial. We follow Schmeidler's definition of aversion to ambiguity in relation to the convexity of capacities[6]. Finally, uncertainty frequently remains a somewhat confusing term (Smithson, 1988). Very often, especially in the finance literature, uncertainty is reduced to risk only. In this paper, we leave the Bayesian expected utility and its updating issues aside to look at the impact of *"ambiguity" or* "*Knightian uncertainty"*[7]. We exclude situations where *fundamental uncertainty* prevails[8]. In our framework, ambiguity is generated by missing information that *could* be known (Camerer and Weber, 1992). Moreover, to isolate the effect of uncertainty, the decision maker is assumed to be *risk-neutral*.

1.2 Real option models under ambiguity: a recent proposal

The introduction of uncertainty inside economic decision-making models confirmed that the effect of ambiguity is different from that of risk alone and is well documented. The real option literature so far almost only discusses the impact of increasing *risk* on valuation of real options and exercise timing[9]. Nevertheless, a small number of articles have recently suggested to expand real options models to ambiguity. Indeed, as illustrated by Miao and Wang (2009), a wide range of economic decisions may indeed be reduced to option exercise choices or *optimal stopping problems under uncertainty*. Furthermore, real options models were precisely developed to allow for a more efficient decision-making in presence of flexibility and irreversibility. Finally, as Montesano (2008) points out, ambiguity aversion

[6] Other definitions coexist: Ghirardato and Marinacci (1997) refer to «the neutrality towards uncertainty *a priori* » and Epstein (1999) suggests the use of *sophisticated* probabilities. Several competing notions coexists such as « the aversion towards uncertainty » (Châteauneuf, 1994), « pessimism » (Arrow et Hurwicz, 1972) or the preference for « *randomization* » (Eichberger et Kelsey, 1996).

[7] These expressions may be used indifferently. Refering to Knightian uncertainty is frequent in the literature, but may seem a little farfetched if Knight's original proposal is replaced in its context.

[8] Even subjective probabilities should not be used if uncertainty is fundamental (Davidson , 1991; Vickers , 1994).

[9] Contrary to the standard conclusion of neoclassical theory on investment (Markowitz, 1952; Tobin, 1958), an increase in risk (volatility) is almost always shown to have a positive impact on the valuation of real options (see McDonald and Siegel, 1986). Renewed controversy may be found in Sarkar (2000) and Lund (2005).

seem important in financial markets, where agents are concerned over the level of *transparency* (i.e the reliability of the probability distribution of outcomes they refer to). Real options decision makers should just be as concerned over it. This apparent lack of interest may be due to the relative distance between these fields of research and expertise or may result from the persistent fragmentation of the decision-making research itself when dealing with ambiguity.

Indeed, ambiguity representations generally translate into non-linearity of probabilities (Feynman, 1963), while many non additive models coexist (Cohen and Tallon, 2001). Moreover, several technical and theoretical issues persist, especially the difficulty in insuring *dynamic consistency*. Dynamic consistency implies that decision makers, once committed to a contingent plan, are not changing plans later on during the process. This apparently limits freedom of choices at successive stages, but is a condition for rational inter-temporal behavior, preserving decision makers from irrational erratic behaviors (such as *money pumps* or *Dutch books*). Dynamic consistency remains a key prerequisite to using non-additive models in a dynamic setting, one that often remains highly problematic.

Adressing it, among many, two models have taken the lead: on the one hand, the *multiple-priors* preferences (Gilboa and Schmeidler, 1989) approach which is based on the *maxmin* criterion (optimization under a worst case scenario), and on the other one, the *Choquet expected utility* (CEU) models (Gilboa, 1987; Schmeidler, 1989; Sarin and Wakker, 1992). None of them truly managed to gain full acceptance but they offer far more potential than most of their competitors, especially to deal with practical issues[10]. Another stream of literature has been associated with the representation of "Knightian ambiguity", that of *robust control*[11]. Klibanoff et al. (2005) propose a promising *smooth* version of ambiguity, by adjusting subjective probabilities through a smooth functional. But so far, the most established model in the decision-making under uncertainty literature remains the recursive multiple-priors model, that was adapted to a dynamic setting by Chen and Epstein (2002).

[10] Let's underline that when CEU applications are based on convex non-additive measure, then multiple priors and CEU may correspond (through the *core*), at least in a static framework.
[11] Hansen et al. (2001) shed light on the relation between robust control and aversion to ambiguity, and relate it to the solving of a *maxmin* problem with multiplier preferences. Chen and Epstein (2002) underline some key differences.

This model uses properties from recursive utility functions[12] and multiple-priors (Wald, 1950; Dreze, 1961) to allow for dynamic consistency[13], even in the presence of uncertainty (Epstein and Zin, 1989; Duffie and Epstein, 1992; Epstein and Wang, 1994; Epstein and Schneider, 2003). Most recently, Riedel (2009) develops a martingale theory for multiple-priors, generalizing existing optimal stopping theory under multiple-priors uncertainty. Overall, multiple-priors models helped re-interpret several apparent paradoxes in finance, especially when observations repeatedly contradicted predictions from standard expected utility models, such as the "equity premium" puzzle (as identified by Merha and Prescott, 1985) or the "home bias" puzzle (Epstein and Miao, 2003). Other articles have presented remarkable contributions, expanding applications to portfolio choices (Epstein and Wang, 1994), contract theory (Mukerji, 1998)[14] or to explain the own-equity effect (Boyle et al., 2003). On the other hand, Choquet rationality[15] may appear less intuitive than the *maxmin* optimization. But a few articles[16] used it to describe the potentially striking impact of ambiguity in the financial markets. Montesano (2008) showed recently that uncertainty aversion with Choquet expected utility leads to decreasing trading volume on the call options market as ambiguity increases. These pioneer articles confirmed the explanatory potential and specificities of Choquet ambiguity representations.

1.3. Real option models under multiple-priors :

Looking at a real option to invest, Nishimura and Ozaki (2007) show that ambiguity impacts the value of irreversible investments in a « drastically » different manner from that of "traditional uncertainty" (risk). In their model, increasing uncertainty affects negatively the investment value, while an increase in risk increases it. But as regards the timing of exercise, in both cases the value of waiting increases, thereby delaying option exercise. This notable conclusion illustrates the power of introducing ambiguity in real options models. Nevertheless, other articles have been less conclusive regarding the impact of ambiguity

[12] Koopmans (1960) and Lucas and Stokey (1984) paved the way for recursive utility framework.
[13] We refer to Sarin and Wakker (1988), Machina (1989) and Eichberger and Kelsey (1996) for detailed discussion of dynamic consistency issues within such models.
[14] Uncertainty aversion may lead to incompleteness of financial markets (Mukerji and Tallon, 2001), or to unstable portfolio preferences with no equilibrium identifiable (Dow and Verlang, 1992; Epstein and Wang, 1994).
[15] The article on Choquet rationality by Ghirardato and Le Breton (1999) describes how the usual definition of rationality is expanded to englobe a larger set of beliefs, including non additive beliefs (or *capacities*).
[16] See for instance the impact of Choquet preferences on portfolio allocation in Basset et al. (2004).

(early exercise or no, increased option value or not). Nishimura and Ozaki (2004) themselves showed that in a job search real option model, more ambiguity may lead to earlier option exercise.

Other models at least converge in demonstrating that introducing ambiguity is not trivial. In Trojanowska and Kort (2010), ambiguity aversion has an equivocal impact on the value of waiting, accelerating investment only in certain situations. Asano (2005) show that an increase in uncertainty may delay adoption of environmental policies. Miao and Wang (2009) suggest to reconcile some contradictions in these results by considering *the moment of resolution* (or not) of ambiguity: the prospect of a persisting ambiguity after option exercise may possibly delay option exercise rather than accelerate it. Overall, these models confirm that ambiguity impacts real options valuation and timing of exercise. But by construction only the worst-case scenario is considered, which reduces the behavioral bias to pessimism[17]. Furthermore, ambiguity and attitude towards ambiguity are mixed and impossible to distinguish[18]. It is arguable that through the size of the set of priors a belief towards the level of ambiguity may be expressed (the larger the set, the more ambiguity is introduced) but that is not sufficient.

The remainder of the paper is organized as follows. In section 2, our alternative proposal to the recursive multiple-priors models is described, as dynamically consistent *Choquet-Brownian* motions[19] (CBM) are used to model uncertainty. Section 3 applies this approach to a *real option to invest* and identifies the threshold project values. Section 4 provides with a sensitivity analysis illustrating the characteristics of our new optimal investment rule. Some analytical results are established, then complemented by simulations, as standard in the literature on real options models with multiple-priors, in order to get clear comparison of results. Section 5 discusses main results in relation to previous works and presents our concluding remarks.

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[17] To avoid ignoring the existence of a whole range of probability measures, several other criteria have been proposed. Ghirardato and Marinacci (2004) following Arrow and Hurwicz (1972) have for instance proposed to combine worst case scenario with best case in a convex combination. See also Chateauneuf, Eichberger and Grant (2006) on *neo-additive* capacities, or Schroder's non dynamically consistent proposal (2007).

[18] This is much weaker than the dichotomy established for risk: *risk* is determined by the shape of the probability distribution of outcomes, while *risk aversion* results from the curvature of the decision maker's utility function.

[19] A Choquet-Brownian motion (Kast and Lapied, 2008) is a distorted Wiener process, where the distorsion derives from preferences towards ambiguity. It was shown to be the continuous time limit of a specific kind of random walk, the Choquet Random Walk (CRW). A Choquet Random Walk may be described as a binomial lattice (Bernoulli model) with equal capacities (instead of additive probabilities) on the two states at each node. (See 2.2.2)

2. RECURSIVE MULTIPLE-PRIORS PREFERENCES VS. CHOQUET-BROWNIAN PROCESSES

2.1. Recursive multiple-priors:

Suppose the profit flow of a project (patent, etc) at the disposal of a decision maker is described as following a geometric Brownian motion (GBM) with a drift[20]. Launching the project has a defined cost (construction, etc), which is sunk (irreversibility). So far, this set up may be assimilated to that of a classical financial call option, allowing to rely on financial options pricing techniques. But suppose the decision maker is not perfectly confident about the extent to which the GBM actually model properly the expected profit flow dynamics. Ambiguity is consequently introduced and takes the form of a distortion from the original GBM. Chen and Epstein (2001) proposed the use of a set of *density generators* to build a range of probability measures representing small deviations from the original probability measure. Small deviations only are allowed as the subjective beliefs are constrained by adopting an additional boundary condition. A constant κ is used to limit the scope of the accepted deviations in a range $\left[-\kappa,\kappa\right]$. Chen and Epstein refer to the concept of κ*-ignorance*, where constant κ derives from a fundamental hypothesis, *rectangularity,* in order to guarantee dynamic consistency. Such construction is possible in application of Girsanov's theorem[21] on equivalent probability measures (applying a density generator to a Brownian motion results indeed in another Brownian motion). Nishimura and Ozaki (2007) offer a detailed presentation of such a construction, including clear explanations of the crucial hypothesis in the Epstein model, that of rectangularity, which insures dynamic consistency[22]. To sum it up, ambiguity is introduced in a limited way inside an optimal stopping model, through a set of geometric Brownian motions which differ only by their *drift*. The subsequent optimization is solved by dynamic programming and use of Ito's

[20] This special connection between stochastic processes and finance roots back to Bachelier's dissertation (1900). Success of such processes have to be balanced with some limits, especially due to the basic reduction of real life phenomena (such as share prices moves for instance) to an overly simple geometric Brownian motion. Controversy is further discussed in many articles and recent financial crises led to strong criticism of models based on GBM.

[21] Girsanov's theorem applies to finite time intervals only (Karatzas and Shreve, 1991). But it is a common approximation in the literature on financial options when referring to infinite horizon. We adopt it here as well.

[22] On rectangularity: beliefs are constrained to a set of "one-step-ahead" conditional probabilities in Chen an Epstein (2002), Epstein (2003) and Roorda (2004). The rectangularity property allows for recursivity, which in turn insures dynamic consistency (Sarin and Wakker, 1998). This property is sometimes referred to as *time-consistency* or *stability under pasting*. We refer to Kast and Lapied (2008) for a discussion of the rectangularity hypothesis in the context of Choquet-Brownian motions.

Lemma in order to identify the continuation region (values V of the project for which it is not yet optimal to invest so no action is recommended) and the critical value V^*, such that it is optimal to invest only once $V \geq V^*$. It is typical to consider a profit flow following a geometric Brownian motion $(\pi_t)_{0 \leq t \leq T}$, where T is the expiration date, $\pi_0 > 0$, B_t is the standard Brownian motion with respect to the original probability measure (towards which the decision maker is perfectly confident) and μ and σ are some real numbers, with $\sigma > 0$ and $\mu < \rho$, where $\rho > 0$, is the firm discount rate. In the *absence of uncertainty,* the profit flow is then represented by the following expression: $d\pi_t = \mu\pi_t dt + \sigma\pi_t dB_t$ (1)

This contrasts with the result obtained under *Knightian uncertainty* and recursive *multiple- priors* model, as a set of stochastic differential equations is now to be used. By construction, $dB_t^\theta = dB_t + \theta_t dt$, so now we obtain the following modified expression for the profit flow:

$$d\pi_t = (\mu - \sigma\theta)\pi_t dt + \sigma\pi_t dB^\theta{}_t \tag{2}$$

The difference between adopting ambiguity (2) or not (1) is then resumed in a single modification, as μ is simply replaced by $(\mu - \sigma\theta)$. The absence of ambiguity is included as a special case, when $\theta = 0$. this is sufficient to demonstrate that uncertainty has an impact different from risk alone. But as the decision maker considers only the worst case, ambiguity aversion leads to a unique case: only the lowest possible value of the project cash flow growth rate is considered.

2.2 Uncertainty through Choquet-Brownian processes

2.2.1. Foundations

In this paper, we adopt another approach to model uncertainty, in order to avoid some limits inherent to the *maxmin* criterion. As usual, the decision maker expresses preferences relative to the uncertain payoffs generated by a real option project at various dates. But this time they are taken into account in a different way, by referring to *capacities* (instead of additive probabilities) to weight likelihood of events and by relying on *discounted Choquet integrals* to compute payoffs value[23]. Let's first clarify these key notions before showing how the dynamics of the real option cash flows will consequently be represented by a distorted kind of Brownian motions (that we call Choquet-Brownian motions) rather than by a standard geometric brownian.

A capacity on a finite set of states of nature S is a real-valued function υ on the subsets of S such that: $\nu(\varnothing)=0, \nu(S)=1;\ A\subseteq B \Rightarrow \nu(A)\leq \nu(B).$ So one of the key characteristic of a capacity is to be nonadditive, which can be used to explain preferences for known probabilities (Ellsberg) and represent a wide range of attitudes towards ambiguity. Why capacities rather than probabilities? Schmeidler (1989) linked the convexity of capacities with a representation of ambiguity aversion[24]. This behavioral interpretation of capacities is at the basis of our construction. Let's note that a capacity is convex if: $\nu(A)+\nu(B)\leq \nu(A\cup B)+\nu(A\cap B).$

When beliefs are represented by capacities, the resulting expected utility cannot be computed through Lebesgue integrals for several reasons, such as the violation of monotonicity. A specific notion of integration is required, which in particular will take into account the rank of outcomes (see Rank Dependant Expected Utility in risk models). To compute the decision maker preferences, which take the form of cash valuations regarding future uncertain payoffs, we need to use a criterion allowing computation of a certainty equivalent when integrals are non-linear. Using Choquet integrals allows just that in our setting. We adopt the framework described in Kast and Lapied (2007), including axiomatization of dynamic consistency[25] and discussion of conditioning. In this setting, preferences of the decision maker for a process of payoffs $X = (X_0,\ldots, X_T)$ are represented by the discounted Choquet expectations, at rate *r*, with respect to a capacity ν. The certainty equivalent of the process is then:

$$DE(X)=\sum_{t=0}^{T} r(t)E_{\nu}(X_t)\text{, where: } E_{\nu}(X_t)=\sum_{s_t\in S_t} X_t(s_t)\Delta\nu(s_t), \tag{3}$$

[23] The « expected value » of an outcome on a given capacity may be computed through the use of Choquet integral. Applying Choquet integrals and capacities was suggested in modern decision theory by Schmeidler (1982, 1986).
[24] Epstein (1999) or Ghirardato and Marinacci(2002) discuss the equivalence convexity/ ambiguity aversion.
[25] One of the key axioms used being the property of additivity of Choquet integrals for co-monotone functions.

with the usual notation for a Choquet integral for which, if, for instance, $X_t(s_1) \leq \ldots \leq X_t(s_N)$, $\Delta\nu(s_n) = \nu(\{s_n,\ldots, s_N\}) - \nu(\{s_{n+1}, \ldots, s_N\})$, with $\{s_{N+1}\} = \varnothing$, for notational convenience. It is then possible to compute the Choquet expected value over the Choquet-Brownian motions (CBM) that are used to describe the dynamics of uncertain future cash flows in our real option to invest model.

CBM may be better understood as dynamically consistent continuous-time limits of Choquet Random Walks[26] (CRW). CRW are defined in discrete time by refering to a binomial lattice representing uncertainty with equal *capacities* (rather than probabilities) on the two states at each node. In order to characterize a CRW, suppose that for any node s_t at date t ($0 \leq t < T$), if s^u_{t+1} and s^d_{t+1} are the two possible successors of s_t at date t+1 (for, respectively, an "up" or a "down" movement in the binomial tree), the conditional capacity is a constant c. Suppose we consider a *symmetrical* random walk (when "up" and "down" movements are of the same magnitude[27]), such that: $\nu(s^u_{t+1}/s_t) = \nu(s^d_{t+1}/s_t) = c$, with $0 < c < 1$.

The constant conditional capacity c plays a key role in such setting: it summarizes the decision makers'attitude towards ambiguity. Indeed dynamics is now described by a discrete time motion in which probability ½ is replaced by this constant c : it represents the ambiguous weight that the decision maker is putting both on the event « up » and the event « down » instead of the unambiguous ½. Just like κ is used to describe κ - ignorance[28], we may use the expression *c-ignorance* in relation to the value of c.

When the decision maker is ambiguity averse, the capacity is sub-linear: this is the case if and only if parameter c < 1/2. This relates to Yaari's definition of aversion to risk as a result of sub-linearity (1969, 1987)[29]. Obviously if c=1/2 then we get back to the probabilistic framework, as a special case. Let's underline that an *increase* in perceived ambiguity in our setting means that the value of parameter c is going further away from its central key anchor ½: the capacity becomes more convex (increasing ambiguity for an ambiguity averse) or more concave (increasing ambiguity for an ambiguity seeker).

[26] Just like a standard binomial tree converges to a Brownian motions in continuous time.
[27] Expansion to non symmetrical random walks would be possible in this setting, at least in discrete time.
[28] As a continuous time counterpart, in a different context, to ε-dissemination, where ε represents the degree of "contamination" of confidence in the probability measure (Chen and Epstein, 2002). See also the relation with *i.d.d* uncertainty ("independently and indistinguishably distributed").
[29] See Montesano (1990, 1991) for discussion of competing definitions of aversion to risk (*mean preserving spreads* versus *risk premium*) and impact of adopting non-expected utility models.

2.2.2 Convergence towards Choquet-Brownian motions:

A symmetrical CRW was shown to converge in continuous time to a general Wiener processes with distorted mean $m = 2c - 1$ and variance $s^2 = 4c(1-c)$. This allows to solve basic optimal investment problems, such as real option models under uncertainty. Overall, not only is taken into account the impact of the intrinsic randomness of trajectories due to the stochastic nature of profit flows and project value (which is already typically achieved by using geometric Brownian motions), but also simultaneously the level of c-ignorance, hence the attitude towards ambiguity. What are the consequences of adopting this framework? The profit flow is modified as follows: $d\pi_t = \mu\, \pi_t\, dt + \sigma\, \pi_t\, dW_t$, (4)

with $dW_t = m\, dt + s\, dB_t$, where W_t is a general Wiener process with mean $m = 2c–1$ and variance $s^2 = 4c(1–c)$. So that we derive the following modified equation: $d\pi_t = (\mu + m\, \sigma)\, \pi_t\, dt + s\, \sigma\, \pi_t\, dB_t$ (5)

This relation is naturally of the same type as the one obtained in the no ambiguity case (1) or with the *maxmin* ambiguity (2). In both cases, parameter uncertainty replaces model uncertainty after a change of measure. Only this time parameters m and s, directly deriving from c, are introduced to represent the decision maker's attitude towards ambiguity. Some implications appear clearly: if for instance the decision maker is ambiguity averse, then parameter c<1/2. Consequently, $0< c < ½$ implies $– 1 < m < 0$ and $0 < s < 1$, and then $\mu + \mathrm{m}\sigma < \mu$ and $0 < \mathrm{s}\sigma < \sigma$. We already get an insight into the potential impact of *Choquet-Brownian* uncertainty, at least on the profit flow: it introduces a reduction of the instantaneous mean, but also of the volatility in the case of aversion to ambiguity.

The last result was not necessarily expected. Overall, (5) leads to different results from the case for risk only, as well as the *maxmin* recursive model, for which the profit flow is also modified but only its drift[30] (Epstein and Schneider, 2003). With *Choquet-Brownian* uncertainty, to the contrary, the effect of the Choquet distortion on the standard profit flow is equivocal, reducing both the instantaneous mean and the volatility for an ambiguity averse decision maker. Introducing ambiguity with CBM is not neutral, but consequences remain unclear at first.

[30] Ambiguity into a multiple-priors model is reduced to risk, which may be questionable.

3. MODEL: A REAL OPTION TO INVEST UNDER AMBIGUITY

Suppose a decision maker enjoying an option to invest into a new project (for instance a patent). This project presents the essential characteristics of a real option: it is irreversible (once decided, investment is instantaneous while its cost, noted I is sunk); it is only affected by time decay; its exercise can be delayed and the choice of timing belongs exclusively to the decision maker. Decision will be taken based exclusively on observed information about stochastic cash flows. In the absence of uncertainty, McDonald and Siegel (1986) presented the seminal version of such irreversible investment decision. Notice that we adopt a continuous time horizon[31], time being indexed by $t \geq 0$. If the project has finite life, T is the expiration date of the project. All information available at each t is represented by an increasing filtered probability space (Ω, F_T, P). The decision maker is risk neutral and cash flows are discounted at rate $\rho > 0$. Cash flows for the project have to be estimated. They move over time at least partially in a random way, so we need to rely on some sort of stochastic processes, combining dynamics with uncertainty. Over a given sequence of possible stochastic payoffs, an optimal stopping time has to be identified, maximising the expected overall result[32].

Proposition 1. *Suppose a decision maker considering a real option to invest in a project at sunk cost I and facing Knightian uncertainty. Suppose that this uncertainty affects the profit flow $(\pi_s)_{s\geq t}$, expected from exercising the option, and that this state variable follows a Choquet-Brownian motion, as characterized earlier in section 2. Then, the project value W_t at time t, with expiration time T, is equal, once exercised, to the expected value E^P of the discounted[33] cash flows with respect to the probability measure P conditional on the filtration F_t defined previously, such that:*

$$W(\pi_t, t) = E^P\left[\int_t^T e^{-\rho(s-t)}\pi_s ds / F_t\right] \tag{6}$$

[31] Continuous time models leads to more explicit computations, but sometimes by using numerical methods.

[32] Optimal stopping problem grew in the 1960s (see Chow and Robbins (1961, 1963, 1971) following original generalization of sequential analysis by Snell (1952). In general, stopping rule problems do not have closed form solutions and methods of finding approximate solutions must be used.

[33] At exogenous discount rate ρ, such that $\rho > \mu$ in order to avoid triviality.

Proof: Simply derived and slightly adapted from standard demonstration in the literature since McDonald and Siegel (1986). For more progressive treatment, we refer to Dixit and Pindyck (1994, chapter 4 to 6). Trojanowska and Kort (2010) offer clear and detailed proofs in the context of real options under ambiguity with multiple-priors. The same holds true for proposition 2, 3 and 4.

The decision maker has to determine the optimal moment t', $t' \in [t, T]$ when to exercise the option to invest. This F_t-optimal stopping time is the one which maximises the value in $t = 0$ of the project, over the whole period considered (*principle of optimality*), taking into account the discounted cost of investing, at discount rate ρ. The stopping time is a random variable that described the exercise date of the option. We rely on dynamic programming[34] to identify optimal sequential decision under uncertainty[35].

Proposition 2. *Option value V_t at time t, while still not exercised, is the following*:

$$V_t = \max_{t' \in (t,T)} E^P \left[\int_{t'}^{T} e^{-\rho(s-t)} \pi_s ds - e^{-\rho(t'-t)} I \,/\, F_t \right] \tag{7}$$

Proof: As justified earlier.

As proved many times, (see for instance Asano, 2005), we obtain from (6):

$$W(\pi_t, t) = \int_t^T \pi_t \exp(-(\rho - \mu)(s - t)) ds = \frac{\pi_t}{\rho - \mu}(1 - e^{-(\rho-\mu)(T-t)}) \tag{8}$$

If the project is *perpetual*, then computation is much eased: it is indeed common assumption to adopt an infinite planning horizon and a never expiring project (cf. Dixit and Pindyck, 1994, or Trigeorgis, 1996). We will then proceed by adopting a stationary model.

Proposition 3. *Under stationary hypothesis, the value for the project is the standard expected value of a perpetual profit flow, which can be simplified as such*:

$$W(\pi_t) = \frac{\pi_t}{\rho - \mu} \tag{9}$$

Proof: See for instance Dixit and Pindyck (1994, p72).

[34] See Stokey, Lucas and Prescott (1989) for deep treatment of dynamic programming in economic settings.
[35] See Markov stopping rule problems in Chow, Robbins and Siegmund (1972) and Shirayev (1973).

It is not possible to apply ordinary rules of derivation to Ito processes. But the use of Ito's lemma allows differentiation and integration of functions of stochastic processes. In the absence of ambiguity, we obtain the following expression by applying Ito's lemma to (9): $dW_t = \mu W_t dt + \sigma W_t dB_t$ (10)

Under Choquet-Brownian ambiguity, this relation is naturally of the same type as for the cases of no ambiguity or *maxmin* ambiguity, only this time $\mu + m\sigma$ and $s\sigma$ in place of μ and σ. Hence, we rewrite the formula (10), as described earlier in previous section: $dW_t = (\mu + m\sigma)W_t dt + s\sigma W t dB_t$ (11)

Project value (9) is consequently rewritten to take into account the presence of ambiguity:

$$W(\pi_t) = \frac{\pi_t}{\rho - (\mu + m\sigma)} \tag{12}$$

Proposition 4. *If the project value W is now technically supposed to be independent from physical time* t, *then the option value* V_t *only depends on* W_t. *Consequently, it is a solution of the following 'Hamilton-Jacobi-Bellman' type of function, which will be solved by dynamic programming:*

$$V(W_t) = \max\left\{W_t - I, E^P\left[dV_t \middle| F_t\right] + V(W_t) - \rho V(W_t)dt\right\} \tag{13}$$

Proof: As justified earlier. See for instance Nishimura and Ozaki (2007), p681.

We now clearly identify characteristics of the optimal investment strategy: existence of a (unique) critical value W^* such that option is exercised if and only if $W_t \geq W^*$; if not, option is kept moving forward, defining a continuation region where $W_t < W^*$. In the right side part of (13), the first term $W_t - I$ represents the value of *investing now* by exercising the option, while the second term corresponds to the *value of waiting*. Notice that both terms on the right hand side of (13) must be equal in the continuation region. Hence, in this continuation region: $E^P\left[dV_t \middle| F_t\right] = \rho V(W_t)dt$ (14)

Applying Ito's lemma to expand $dV(W_t)$, supposing that V is twice differentiable in the continuation region and $V' > 0$, if we now combine (14) and (11) we obtain:

$$dV_t = V'(W_t)((\mu + m\sigma)W_t dt + s\sigma W_t dB_t) + \frac{1}{2}(s\sigma)^2 W_t^2 V''(W_t)dt \tag{15}$$

From (15), the relation obtained is satisfied for every dt, then we derive the following second-order ordinary differential equation for V, as:

$$\frac{1}{2}(s\sigma)^2 W_t^2 V^{''}(W_t) + (\mu + m\sigma) W_t V^{'}(W_t) - \rho V(W_t) = 0 \tag{16}$$

Further assumptions are necessary in order to solve equation (16), holding in the continuation region. We adopt the following standard boundary conditions: value matching, smooth pasting and absorbing barrier. If we note W^* the critical value (or reservation value) triggering the option exercise, then:

$$\begin{cases} V(W^*) = W^* - I & \text{"value matching condition"}^{36} \quad (17) \\ V^{'}(W^*) = 1 & \text{"smooth pasting condition"} \quad (18) \\ F(0) = 0 & \text{"absorbing barrier condition"} \quad (19) \end{cases}$$

We explicitly solve (16) under conditions (17) to (19) in order to get the option value in the continuation region, $V(W_t)$ as well as the critical value (or free boundary) W^*, so we obtain[37]:

$$\begin{cases} V(W_t) = \left(\frac{I}{\alpha - 1}\right)^{1-\alpha} \alpha^{-\alpha} W_t^{\alpha} \qquad W_t < W^* & (20) \\ \text{and } W^* = \frac{\alpha}{\alpha - 1} I & (21) \end{cases}$$

In the exercise region, $W \geq_t W^*$, $V(W_t) = W_t - I$. (22)

and α is a constant[38], whose value depends on parameters μ, ρ, σ, c, so defined:

$$\alpha = \frac{-((\mu + m\sigma) - \frac{1}{2}(s\sigma)^2) + \sqrt{\left\{(\mu + m\sigma) - \frac{1}{2}(s\sigma)^2\right\}^2 + 2\rho(s\sigma)^2}}{(s\sigma)^2} \tag{23}$$

with $m = 2c-1$ and $s^2 = 4c(1-c)$ (see section 2).

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[36] (17) implies that investing in the project gains a net payoff equal to $V^* - I$. (18) is derived from the first-order condition when maximizing project value. (19) if the investment value has no value, then the option is worthless.

[37] See for instance Dixit and Pindyck for simple treatment (1994, p142-143), getting solution through dynamic programming (using linear combination), as well as description of fundamental quadratic's intuition.

[38] $\alpha > 1$ so that W* and V are well defined, which in the multiple-priors also holds as $\rho > \mu$ and $\kappa > 0$ (see Nishimura and Ozaki, 2007, Annex A.5.).

Let's summarize our optimal stopping problem in the context of ambiguity represented by CBM:

Proposition 5. *Assuming a real option to invest under Choquet-Brownian ambiguity as defined in propositions 1-4, optimal strategy and value of investment are summarized in (20), (21), (22) and (23).*

This is of course close from what is obtained in the case of absence of uncertainty, which becomes a special case. We observe the introduction in key formulas of parameters *m* and *s*, directly deriving from c, which resumes the attitude of the decision maker towards ambiguity.

4. COMPARATIVE STATICS

After identifying the optimal investment rule, what happens if parameters change? We compare the effect of a change in risk (that is an increase in volatility) with that of a change in Knightian uncertainty (either represented by κ in the multiple-priors or by c with Choquet-Brownian ambiguity). We obtain several analytical results, more specifically regarding the impacts of risk and ambiguity on project values in the stopping region. Regarding the critical reservation value and the timing of option exercise in the presence of ambiguity, we use some simulations[39] results, as standard in the real option under ambiguity literature, to show the critical impact of changing parameters and examine the characteristics of the optimal investment rule when analytical results are not easily computable.

4.1. An increase in risk

In the absence of ambiguity, the well know result of an increase in risk consists in the increase of the value of the project in the continuation region and in the reservation value, while the value of the project once the option has been exercised does not change. Exercise of the option is delayed. If now we introduce ambiguity, we show that a change in risk also impacts the exercised project value. Differences in original attitude towards ambiguity may explain why the same variation in risk may be looked over differently by decision makers revealing different attitudes towards perceived uncertainty, with potentially drastic consequences in terms of valuation.

[39] Simulation results have often to be taken with a pinch of salt. Different parameters may likely influence each other and interpreting can be hazardous at time. Nevertheless, simulations are an important tool in practice when dealing with real options. From collecting adequate data on past demand for instance, are generated prospective future demand trajectories. Stochastic dynamic programming in the context of real options rely on quality of information used, that is sound data collection, analysis and industry expertise.

Proposition 6. *In the presence of ambiguity, a change in risk levels impacts project value in the stopping region. An increase in risk leads to an increase in the value of the project once the option has been exercised if and only if the decision maker is ambiguity lover (c>1/2). The opposite holds true if the decision maker is averse to ambiguity (c<1/2).*

Proof: See appendix A

4.2. An increase in ambiguity (c-ignorance):

In this subsection we now wish to analyze the effect of a change in c-ignorance on project value, continuation value, reservation value and timing of exercise[40].

4.2.1. Project value in the stopping region $W_T \geq W^*$ *:*

This time let's recall that in the presence of Choquet-Brownian uncertainty $(\mu + m\sigma)$ replaces μ , so:

$$W(\pi_t) = \frac{\pi_t}{\rho - (\mu + m\sigma)} \tag{12}$$

The impact of an increase in c-ignorance in the stopping region is the following: if the decision maker is ambiguity averse, $\mu + m\sigma \searrow$, thus project value W in the stopping region decreases. The opposite holds true for an ambiguity lover. This result generalizes the multiple-priors model, in which an increase in κ-ignorance always translates into a decreased value for the project (indeed the decision maker remaining averse to ambiguity and only considers the lower born).

4.2.2. Project value in the continuation region $W_T < W^*$ *:*

From now on, computation is not trivial and we cannot derive analytical solutions through simplifying derivatives. Using simulation, here we find that if we fix μ, σ, ρ : if the decision maker is ambiguity averse, an increase in ambiguity leads to a decrease in project value in the continuation region. The opposite holds true for an ambiguity lover.

[40] Let's recall first that an *increase* in perceived ambiguity in our setting means that the value of parameter *c* is going further away from its central key anchor ½ (corresponding to the limit probabilistic case, that of an absence of ambiguity). Possible deviations are confined in a range and c represents the index of the intensity and nature of perceived ambiguity (or c-ignorance).

4.2.3. Reservation value: $W^* = \dfrac{\alpha}{\alpha - 1} I$

Let's note that when $\alpha \nearrow, \alpha > 1,$ then the hysteresis factor $\dfrac{\alpha}{\alpha - 1} \searrow$[41]; this means the reservation value decreases. As shown in the previous section, α depends on 4 parameters, the degree of c-ignorance c (which in turn determines the values for *s* and *m*), the growth rate μ, the discount rate ρ, as well as the volatility σ.

We may summarize briefly some side results concerning growth rates μ and discount rates ρ, identifying how they also impact reservation values and timing of exercise of option:

- If c, σ, ρ are fixed, then according to our simulation: $\nearrow \mu \Rightarrow \searrow \alpha$, that is an increase in the growth rate decreases α, which in turn means the reservation value W^* increases (see fig.1 in appendix B). The attitude towards ambiguity (lover, averse, neutral) does not change the direction of the trend, but an ambiguity lover's reservation value is always higher than that of a neutral or averse one.
- Now if c, σ, μ are fixed, then according to our simulation: $\nearrow \rho \Rightarrow \nearrow \alpha$, that is an increase in the discount rate increases α, meaning the reservation value W^* decreases (see fig.2 in appendix B). The attitude towards ambiguity does not affect the trend, but an ambiguity lover reservation value is once more always higher than that a neutral or averse one.

Let's now turn to our main point of discussion, the impact of a change in c when all other parameters are fixed: if μ, σ, ρ are fixed, then if the decision maker is ambiguity lover, an increase in c-ignorance will lead to a decrease in α. Reservation value increases. The opposite holds true for an ambiguity averse (see fig.3 in appendix B). In the case of aversion towards ambiguity, the observed decrease in reservation value is similar to that in Nishimura and Ozaki (2007). But we also establish the opposite result for an ambiguity seeker.

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[41] Let's note that as long as $\alpha > 1, \dfrac{\alpha}{\alpha - 1} > 1$. Hence $W^* > 1$, which is sufficient to rule out as incorrect the traditional static NPV criterion.

4.3.4. Value of waiting

Next, we explore the connection between a change in reservation value and the subsequent impact on timing of option exercise. If we reinterpret the reservation value W^* in terms of a reservation profit flow π^*, then from adapting (12), we get: $W^* = \dfrac{\pi^*}{\rho - (\mu + m\sigma)}$ that can be rewritten: $\pi^* = \{\rho - (\mu + m\sigma)\} W^*$. In our model[42], simulations on reservation profit flow π* show two distinct areas depending on the nature of c-ignorance (seeker or averse): π* is increasing with the degree of ambiguity for an ambiguity averse (that is the value of waiting increases), while decreasing for an ambiguity seeker. This leads to the adoption of opposite behaviors, with an accelerated (ambiguity seeker) or a delayed (ambiguity averse) option exercise (See fig.4 in appendix B).

It does not come as a huge surprise at this stage that preferences towards perceived ambiguity play such a defining role when deciding over the optimal moment of exercise of our real option. Just like project and option valuations are affected by individual preferences, the timing of exercise is modified according to the nature of the attitude of the decision maker towards ambiguity. For an ambiguity averse, the present value effect (decrease in the net present value of the project) dominates the option value effect (the cost of opportunity of acting is reduced), and exercise is delayed. The opposite holds true for an ambiguity seeker. Let's summarize our findings:

Proposition 7. *A change in the level of perceived ambiguity has an impact on project value as well as on reservation value, consequently impacting the timing of exercise of real options with Choquet-Brownian motions. While an ambiguity averse decision maker will delay option exercise, an ambiguity seeker will exercise it earlier than if he was neutral towards ambiguity.*

[42] See Nishimura and Ozaki (2007) and Trojanowska and Kort (2010) for discussion of this relation in the context of multiple-priors. In the latter especially, the impact of Knightian uncertainty on triggers W^* and π^* appears equivocal for *finite life* projects: an increase in reservation value may not lead to delayed investment (for instance, larger life-times finite projects are negatively associated with investment enhancing). For *perpetual* projects, to the contrary, monotonicity is demonstrated, with π^* increasing with ambiguity.

5. CONCLUSION

Few articles within the real options literature have so far explored the impact of ambiguity. Moreover, the few pioneer real options models under Knightian uncertainty are based on the multiple-priors model. They have given great insight into the importance of addressing the existence of preferences towards perceived ambiguity. Unfortunately such models are also very restrictive by definition, as they rely exclusively on a *maxmin* criterion.

By introducing a wider spectrum of attitudes towards ambiguity represented through Choquet-Brownian motions, we show that individual preferences matter and lead to significant and contrasted impacts on option valuations and subsequent timing of exercise. Indeed, aversion towards ambiguity will increase the value of waiting and delay exercise, while ambiguity loving preferences will encourage an earlier exercise of a real options to invest. So far in the literature only the result for an averse ambiguity had been established in the case of a perpetual option (Nishimura and Ozaki, 2007).

Real options models under ambiguity so far concur in underlining that ambiguity should not be purely and simply ignored. Considering the limited number of papers on this subject, that some points of debate remain open, such as the moment of resolution of ambiguity (Miao and Wang, 2009), is hardly surprising and actually bodes well for further research. Let's notice that the very effect of risk itself on options may still be subject of debate (see Sarkar's controversial stance, 2000; discussed in Lund, 2005).

Furthermore, in our model a complex inter-relation between risk and ambiguity, often completely ignored in the literature, appears and raises new questions. In the combined presence of ambiguity and risk, individual revealed preferences towards such forms of uncertainty may deeply impact real option valuations and subsequent actions: as we pointed out, even a risk neutral decision maker will react to changes in risk if he is not neutral towards ambiguity. This relation may appear more strikingly here than in the multiple priors approach where ambiguity is largely "assimilated" to risk.

To conclude, we wish to underline that more research will be necessary to deepen the understanding of ambiguity revealed preferences, to compare their various representations and ponder their respective interest and limits. Even the interpretation of ambiguity itself remains somehow controversial and a better distinction between beliefs and tastes may be desirable.

Recently axiomatized Choquet-Brownian motions are tractable enough to be adapted to more complex real options settings, including compound options or multiple sources of ambiguity such as typical stochastic costs (see Kast, Lapied and Roubaud, 2010). Besides, expanding our model to finite life projects would allow comparisons with research on finite life projects in the context of multiple-prior (Trojanowska and Kort, 2010; Gryglewicz, Huisman and Kort, 2008).

As decision makers' preferences towards ambiguity matter, they should often be taken into consideration when examining the timing and valuation of real options projects. Obviously it may be helpful when assessing decisions *ex-post*, to better understand why in practice real options may be exercised much later (or sooner) than predicted in the expected utility framework. But it may also contribute to a better framing *a priori* of the impact of ambiguous key drivers on a real option project, if only through additional sensitivity analyses including ambiguity parameters.

The multiple-priors approach corresponds to a cautious attitude in front of potential model misspecification, a "robust decision rule" for an investment. From a managerial point of view, ambiguity should not necessarily be feared though: embracing it when strategically justified may prove wise and source of competitive advantage, when caution would prevent from undertaking potentially profitable investments. Obviously the intuition of managers will not be replaced by quantitative estimates, but those who adopted the real options approach often underline that it contributes to better thinking, planning and conducting of projects under uncertainty. Adopting some sort of ambiguity parameter should just help them in doing that in a more explicit way!

Finally, at this early stage of real options models under ambiguity, it may be argued that they already contribute to the idea that dominant models in finance should maybe more often than not take ambiguity preferences into consideration… if only by making the hypothesis of "ambiguity neutrality" at least as explicit than its omnipresent "risk neutral" counterpart!

Appendix A. Comparative Statics for risk

A. An increase in risk in the absence of ambiguity: the standard case

A.1. Project value after exercise, $W_T \geq W^*$ *:*

Parameter σ^2 representing risk has no impact on the value of the project once launched. Indeed, if there is no ambiguity, then $m = 2c - 1 = 0$ and $W(\pi_t) = \dfrac{\pi_t}{\rho - \mu}$. In the absence of ambiguity, a change in risk does not modify the project value in the stopping region (the agent is *risk neutral* by hypothesis).

A.2. Project value in the continuation region, $W_T < W^*$ *:*

Regarding the option value in the continuation region, let's recall that $V(W_t)$ is given by (20): $V(W_t) = (\dfrac{I}{\alpha - 1})^{1-\alpha} \alpha^{-\alpha} W_t^{\alpha}$. This time, as parameter σ^2 plays a key role in computation of α in (23), the value of the project will change in the continuation region. We need to look at the sign of a few derivatives to identify the impact of an increase in risk, which implies some calculations (Nishimura and Ozaki, 2007): sign of $\dfrac{\partial \alpha}{\partial \sigma^2}$ < 0, sign of $\dfrac{\partial V(W_t)}{\partial \alpha}$ < 0 ; hence, by combining, $\dfrac{\partial V(W_t)}{\partial \sigma^2}$ > 0. An increase in risk increases the value of the project in the continuation region.

A.3. Reservation value $W^* = \dfrac{\alpha}{\alpha - 1} I$ *:*

Again, parameter σ^2 plays a key role in computation of α , so that we again need to establish the signs of: $\dfrac{\partial W^*}{\partial \alpha}$ < 0, and $\dfrac{\partial \alpha}{\partial \sigma^2}$ < 0, (Nishimura and Ozaki, 2007) ; hence, $\dfrac{\partial W^*}{\partial \sigma^2}$ > 0

B.An increase in risk in the presence of ambiguity: a striking impact on project value!

If the decision maker is not neutral towards ambiguity, a change in risk in the presence of ambiguity will impact the project value in the stopping region (*ceteris paribus*). Indeed, if σ^2 increases, then $(\mu + m\sigma)$ now increases if and only if m>0, that is if c>1/2, which in turn implies that $W(\pi_t) = \dfrac{\pi_t}{\rho - (\mu + m\sigma)}$ increases. Consequently, project value in the stopping region increases for an ambiguity seeker when risk increases. The opposite holds true if the decision maker is ambiguity averse[43].

[43] Let's note that in the multiple-priors, as $\kappa > 0$, an increase in risk also leads to a decrease in project value, as $W_t = \dfrac{\pi_t}{\rho - (\mu - \kappa\sigma)}$. This is just a special case in our model, that of ambiguity aversion under *maxmin*

Appendix B. Comparative Statics for ambiguity (1/2)

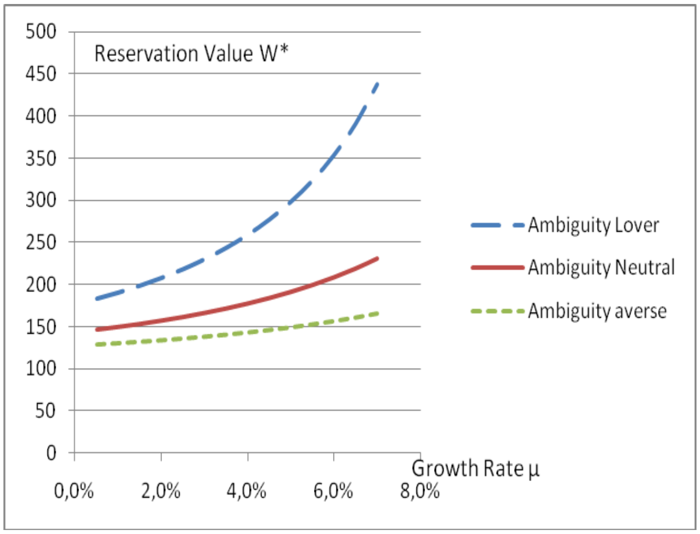

Fig. 1. Reservation Value W^* as a function of μ for decision makers expressing various attitudes towards ambiguity (with $\sigma = 20\%$; $\rho = 15\%$; $\psi = \{-0.1; 0; 0.1\}$ and $I = 100$).

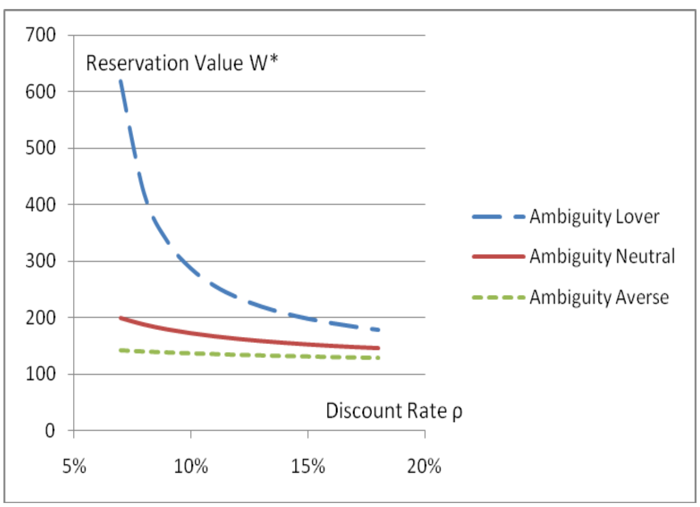

Fig.2. Reservation Value W^* as a function of ρ for decision makers expressing various attitudes towards ambiguity (with $\sigma = 20\%$; $\mu = 1.5\%$; $\psi = \{-0.1; 0; 0.1\}$ and $I = 100$).

Appendix B. Comparative Statics for ambiguity (2/2)

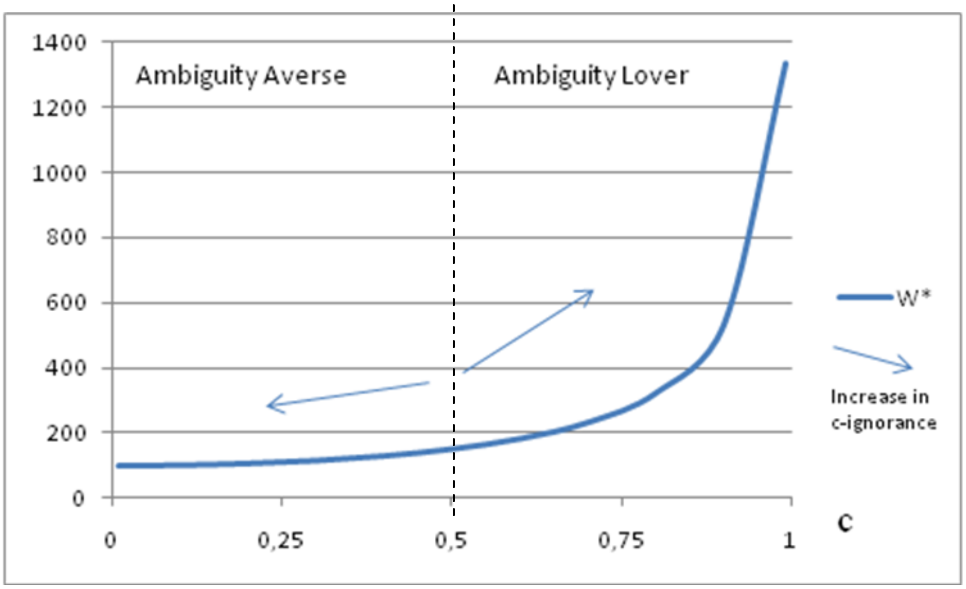

Fig.3. Reservation Value W* as a function of the degree of c-ignorance (with $\sigma = 5\%$, $\mu = 2.5\%$, $\rho = 8\%$ and $I = 100$).

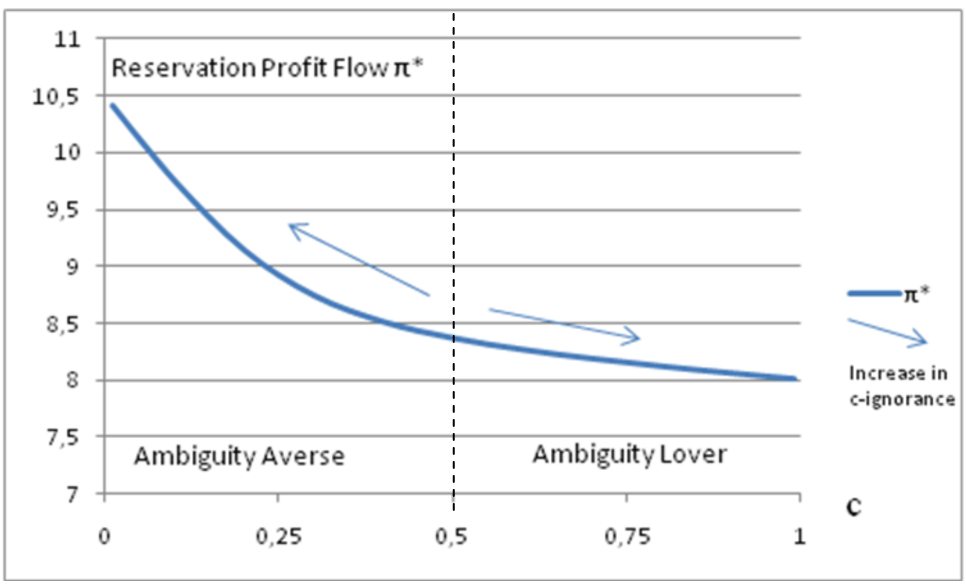

Fig.4. Reservation Profit Flow π* as a function of the degree of c-ignorance (with $\sigma = 5\%$, $\mu = 2.5\%$, $\rho = 8\%$ and $I = 100$).

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