Investment, Capacity Choice and Outsourcing under Uncertainty

Makoto Goto[a,*], Ryuta Takashima[b],

[a] *Graduate School of Finance, Accounting and Law, Waseda University*
[b] *Department of Nuclear Engineering and Management, The University of Tokyo*

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Abstract

In this paper, we formulate a firm's investment in a production facility with maximum capacity and outsourcing strategy. Existing literature of capacity choice results in an explosive capacity and long delay in investment. Actually, there are slight possibility that a firm installs a facility with an explosive capacity, instead, a firm can outsource production when the demand exceeds its capacity. Outsourcing strategy is expected to inhibit the capacity, accelerate the investment and increase the value. In order to show these topics, we investigate optimal investment timing, facility's size and outsourcing strategy simultaneously.

Keywords: Capacity choice; outsourcing
JEL classification: D81

1 Introduction

In the industry where demand has wide fluctuation, e.g., semiconductor industry, firms have to choose the capacity of their production facilities.

The most famous real options literature which investigates such a capacity choice problem is Pindyck (1988). He uses incremental investment approach, i.e., a firm chooses the capacity sequentially. However, capacity expansion needs high cost and much time in practice. Dangl (1999) investigates the problem that a firm chooses the capacity only

*Corresponding author. Address: 1-4-1 Nihombashi, Chuo-ku, Tokyo 103-0027, Japan; E-mail: mako.50@aoni.waseda.jp

when the facility is installed. His main result is that uncertainty leads to an explosive capacity and long delay in investment.

Actually, there are slight possibility that a firm installs a facility with an explosive capacity. Instead, a firm has the option to outsource production when the demand exceeds its capacity. Real options literature which investigate outsourcing are increasing recently. Li and Kouvelis (2008) formulate risk sharing contract. Alvarez and Stenbacka (2007) investigate the relation between market uncertainty and production mode. Jiang et al. (2008) focus venders' loss of option to wait.

The objective of this paper is to investigate optimal investment timing, facility's size and outsourcing strategy simultaneously.

2 The model

We consider a firm that has the option to invest in a production facility with maximum capacity m. The firm must choice the output to maximize the profit at every time, however the output is constrained by fixed m during the conception period.

It is assumed that the demand at time t for the firm follows a geometric Brownian motion:

$$\mathrm{d}X_t = \alpha X_t \mathrm{d}t + \sigma X_t \mathrm{d}W_t, \quad X_0 = x, \tag{1}$$

where α is the instantaneous expected growth rate of X_t, σ (> 0) is the instantaneous volatility of X_t, and W_t is a standard Brownian motion. Then, we assume the output price at time t is

$$P_t = X_t - \delta Q_t, \tag{2}$$

where Q_t is the output and δ describes the dependence of the price on the output. The profit flow the firm receives just after the project is installed is

$$\pi(X_t, Q_t, m) = (P_t - c)Q_t, \quad 0 \leq Q_t \leq m, \tag{3}$$

where c is the marginal production cost. Since the firm must choice the output to maximize the profit flow, we have

$$Q(X_t, m) = \begin{cases} 0, & \text{for } X_t < c, \\ \dfrac{X_t - c}{2\delta}, & \text{for } c \leq X_t < \tilde{X}, \\ m, & \text{for } X_t \geq \tilde{X}, \end{cases} \tag{4}$$

$$\pi(X_t, m) = \begin{cases} 0, & \text{for } X_t < c, \\ \dfrac{(X_t - c)^2}{4\delta}, & \text{for } c \leq X_t < \tilde{X}, \\ (X_t - c)m - \delta m^2, & \text{for } X_t \geq \tilde{X}, \end{cases} \tag{5}$$

where $\tilde{X} = c + 2\delta m$.

Now, we assume that for $X_t > \tilde{X}$, the firm can outsource production at the amount $\overline{Q}$, if the cost of outsourcing exceeds opportunity cost of missing the demand. In case of outsourcing, the firm is assumed to incur the variable cost $\overline{c}$ ($> c$) and the fixed cost k to start outsourcing. On the other hand, the fixed cost ℓ is needed at the end of outsourcing. Therefore, the profit flow after the addition of outsourcing is

$$\overline{\pi}(X_t, m) = (X_t - c)m + (X_t - \overline{c})\overline{Q} - \delta(m + \overline{Q})^2. \tag{6}$$

The investment costs to install a production facility are a function of maximum capacity m:

$$I(m) = bm^\gamma. \tag{7}$$

The firm's decision is to determine simultaneously when to install a production facility, how large the capacity is, and when to start outsourcing with observing uncertain demand.

3 Value of the project

First, we define the value function without outsourcing:

$$V_0(x, m) = \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-\rho t}\pi(X_t, m)\mathrm{d}t\right], \tag{8}$$

where ρ ($> \alpha$) is the individual discount rate of the firm. Equation (8) satisfies the following ordinary differential equation:

$$\frac{1}{2}\sigma^2 x^2 V_0''(x, m) + \alpha x V_0'(x, m) - \rho V_0(x, m) + \pi(x, m) = 0, \tag{9}$$

where the prime denotes differentiation with respect to x. Because equation (5) separates depending on x, equation (9) is devided into three parts:

$$\begin{cases} \frac{1}{2}\sigma^2 x^2 V_1''(x,m) + \alpha x V_1'(x,m) - \rho V_1(x,m) = 0, & \text{for } x < c, \\ \frac{1}{2}\sigma^2 x^2 V_2''(x,m) + \alpha x V_2'(x,m) - \rho V_2(x,m) + \frac{(x-c)^2}{4\delta} = 0, & \text{for } c \le x < \tilde{X}, \\ \frac{1}{2}\sigma^2 x^2 V_3''(x,m) + \alpha x V_3'(x,m) - \rho V_3(x,m) + (x-c)m - \delta m^2 = 0, & \text{for } x \ge \tilde{X}. \end{cases} \tag{10}$$

Then, we have

$$V_0(x,m) = \begin{cases} V_1(x,m), & \text{for } x < c, \\ V_2(x,m), & \text{for } c \le x < \tilde{X}, \\ V_3(x,m), & \text{for } x \ge \tilde{X}, \end{cases} \tag{11}$$

$$V_1(x,m) = A_1 x^{\beta_1} + A_2 x^{\beta_2}, \tag{12}$$

$$V_2(x,m) = B_1 x^{\beta_1} + B_2 x^{\beta_2} + \frac{1}{4\delta}\left(\frac{x^2}{\rho - 2\alpha - \sigma^2} - \frac{2cx}{\rho - \alpha} + \frac{c^2}{\rho}\right), \tag{13}$$

$$V_3(x,m) = C_1 x^{\beta_1} + C_2 x^{\beta_2} + \frac{mx}{\rho - \alpha} - \frac{\delta m^2 + cm}{\rho}, \tag{14}$$

where β_1 (> 1) and β_2 (< 0) are the positive and negative root of the following characteristic equation:

$$\frac{1}{2}\sigma^2 \beta(\beta - 1) + \alpha\beta - \rho = 0, \tag{15}$$

respectively.

Next, we consider the value after the addition of outsourcing:

$$\overline{V}(x,m) = D_1 x^{\beta_1} + D_2 x^{\beta_2} + \frac{(m + \overline{Q})x}{\rho - \alpha} - \frac{cm + \bar{c}\overline{Q} + \delta(m + \overline{Q})^2}{\rho}. \tag{16}$$

Since outsourcing is implemented iff the cost of outsourcing exceeds oppotunity cost of missing the demand, the firm's value is of the following form:

$$V(x,m) = \begin{cases} V_0(x,m) & \text{for } x < \overline{X}, \\ \overline{V}(x,m) & \text{for } x \ge \underline{X}, \end{cases} \tag{17}$$

where $\underline{X} < \overline{X}$ is the optimal threshold to terminate and start outsourcing, respectively.

Then, we need the following boundary conditions in order to find the optimal thresholds

$\overline{X}$, $\underline{X}$ and unknown coefficients A_i, B_i, C_i, D_i:

$$V_1(0, m) = 0, \tag{18}$$

$$V_1(c, m) = V_2(c, m), \tag{19}$$

$$V_1'(c, m) = V_2'(c, m), \tag{20}$$

$$V_2(\tilde{X}, m) = V_3(\tilde{X}, m), \tag{21}$$

$$V_2'(\tilde{X}, m) = V_3'(\tilde{X}, m), \tag{22}$$

$$V_0(\overline{X}, m) = \overline{V}(\overline{X}, m) - k, \tag{23}$$

$$V_0'(\overline{X}, m) = \overline{V}'(\overline{X}, m), \tag{24}$$

$$\overline{V}(\underline{X}, m) = V_0(\underline{X}, m) - \ell, \tag{25}$$

$$\overline{V}'(\underline{X}, m) = V_0'(\underline{X}, m), \tag{26}$$

$$\lim_{x \to \infty} \overline{V}(x, m) = \frac{(p - \overline{c})x}{\rho - \alpha} + \frac{(\overline{c} - c)m}{\rho}. \tag{27}$$

Equations (18) and (27) are the initial conditions or no-bubble conditions. Equations (19)–(22) ensure the continuity and smoothness of the value function. Equations (23) and (24) are the value-matching and smooth-pasting conditions to start outsourcing, respectively. Equations (25) and (26) are the value-matching and smooth-pasting conditions to terminate outsourcing, respectively. Since these result in the system of six non-linear equations, we must find the thresholds and coefficients numerically.

Note that $V_0(x, m)$ in equations (23)–(24) mean that it is not obvious whether outsourcing has to start at the demand level greater than maximum capacity or not beforehand. Furthermore, $V_0(x, m)$ in equations (25)–(26) mean the same thing about the end of outsourcing.

Finally, the firm chooses the maximum capacity to maximize the value of the project:

$$V^*(x) = \max_m [V(x, m) - I(m)] = V(x, m^*) - I(m^*), \tag{28}$$

where m^* is the optimal maximum capacity. Of course, m^* depends on x and is found numerically.

4 Value of the option to invest

To complete the three optimal decisions, the firm has to maximize the value of option to invest in the facility maximized by outsourcing strategy and maximum capacity:

$$F(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}[\mathrm{e}^{-\rho\tau} V^*(X_\tau)] = \mathbb{E}[\mathrm{e}^{-\rho\tau^*} V^*(X_{\tau^*})], \tag{29}$$

$$\tau^* = \inf\{t > 0 : X_t \geq X^*\}, \tag{30}$$

where $\mathcal{T}$ denotes the collection of admissible stopping times and X^* is the optimal threshold to install the facility. Using the standard real options approach again, we have

$$F(x) = \begin{cases} D_1 x^{\beta_1} + D_2 x^{\beta_2}, & \text{for } x < X^*, \\ V^*(x), & \text{for } x \geq X^*. \end{cases} \tag{31}$$

To find D_i and X^*, we need the following boundary conditions:

$$F(0) = 0, \tag{32}$$

$$F(X^*) = V^*(X^*), \tag{33}$$

$$F'(X^*) = V^{*\prime}(X^*). \tag{34}$$

Since $V^*(x)$ is found numerically, we must also find D_i and X^* numerically.

5 Numerical examples

In this section, we use the basic parameter values in Table 1. Then, we have the optimal thresholds for three capacities, $m = 1, 2, 3$, in Table 2.

Table 2 shows all the thresholds $\tilde{X}$, $\underline{X}$, $\overline{X}$ and X^* are increasing with the capacity m. When $m = 1$, the firm should invest at $X^* = 5.38$ and the output should be the maximum ($Q_{\tau^*} = m$), because $X^* > \tilde{X}$. The output is kept the maximum until X_t reaches $\overline{X} = 10.12$, and then the firm should start outsourcing. Afterward, outsourcing is terminated at $\underline{X} = 4.42$ and the output is the maximum because $\underline{X} > \tilde{X}$. When $m = 2$, firm's optimal behavior is the same as when $m = 1$, except that the output should be less than the maximum when outsourcing is terminated, because $\underline{X} < \tilde{X}$. When $m = 3$, the behavior has an important difference that the output should be less than the maximum. It is because $X^* < \tilde{X}$, that is, $\overline{V}(X^*, 3)$ is tangent to $V_2(X^*, 3)$. Therefore, the firm cannot utilize all the capacity which is too large.

To see detail of above facts, we have three figures. Figures 1 and 2 show the value functions when $m = 1$ and $m = 3$, respectively. Figure 3 shows the comparative statics

Table 1: The parameter values for the base case.

parameter		value
expected growth rate	α	0.02
volatility	σ	0.2
discount rate	ρ	0.1
dependence of the price on the output	δ	1
in-house variable cost	c	2
outsourcing variable cost	$\bar{c}$	4
outsourcing quantity	$\overline{Q}$	0.5
outsourcing fixed cost (start)	k	5
(end)	ℓ	1
investment cost: $I(m) = bm^\gamma$	b	10
	γ	0.7

Table 2: The values of thresholds for three capacities.

m	$\tilde{X} = c + 2\delta m$	$\underline{X}$	$\overline{X}$	X^*	value-matching
1.0	4.0	4.42	10.12	5.38	V_3
2.0	6.0	5.97	12.61	6.04	V_3
3.0	8.0	7.55	15.08	6.58	V_2

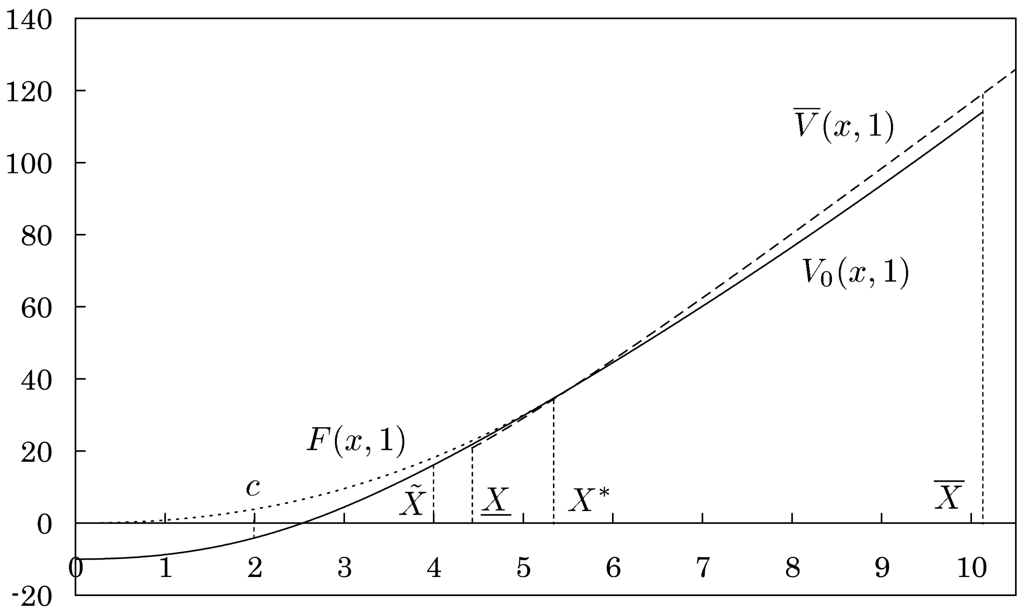

Figure 1: The value functions when $m = 1$.

of the thresholds with respect to m. In figure 3, we can see $c < \tilde{X}, \underline{X}, X^* < \overline{X}$ for all m. It shows an optimality of investment, because $X^* > c$ implies the firm does not invest in the idle state and $X^* < \overline{X}$ implies the firm does not start outsourcing as soon as investment. We need more discussion for details of $\tilde{X}$, $\underline{X}$ and X^*. Capacity threshold $\tilde{X}$ and investment threshold X^* have the highest and lowest sensitivity of the three, respectively. Furthermore, $\tilde{X}$ and X^* have the lowest and highest value when $m = 1$, respectively. Therefore, $\tilde{X}$ crosses $\underline{X}$ and X^*, let m_1 and m_2 denote the corresponding capacities, respectively. There are three ranges for the relation between capacity and output:

1. For $m < m_1$, $Q(X^*) = m$ and $Q(\underline{X}) = m$,

2. for $m_1 \leq m < m_2$, $Q(X^*) = m$ and $Q(\underline{X}) < m$,

3. for $m \geq m_2$, $Q(X^*) < m$ and $Q(\underline{X}) < m$.

This property implies the corresponding value-matching and smooth-pasting conditions vary with m. Due to these properties, we have much difficulty in finding the optimal capacity.

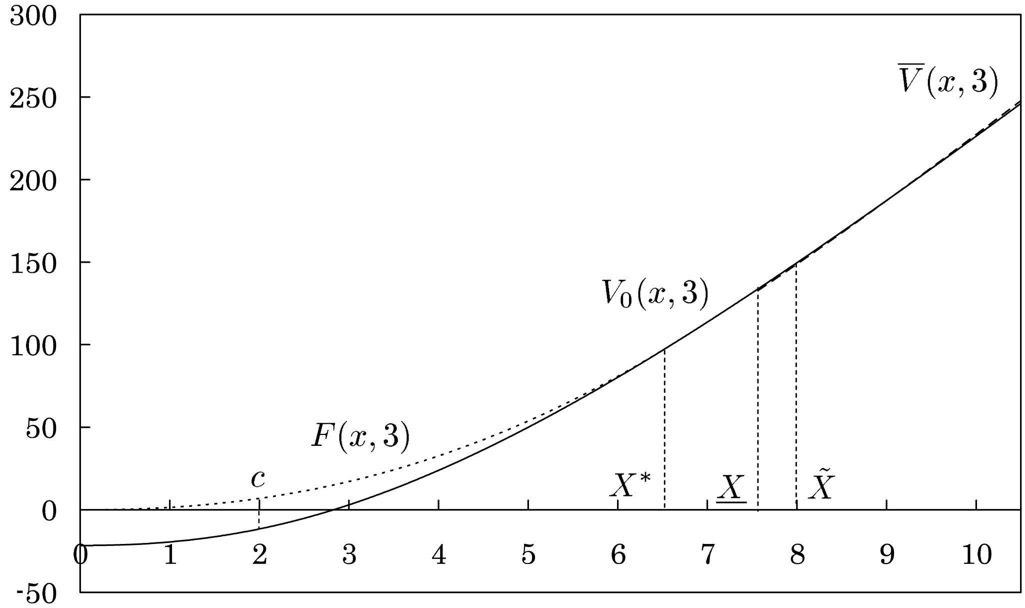

Figure 2: The value functions when $m = 3$.

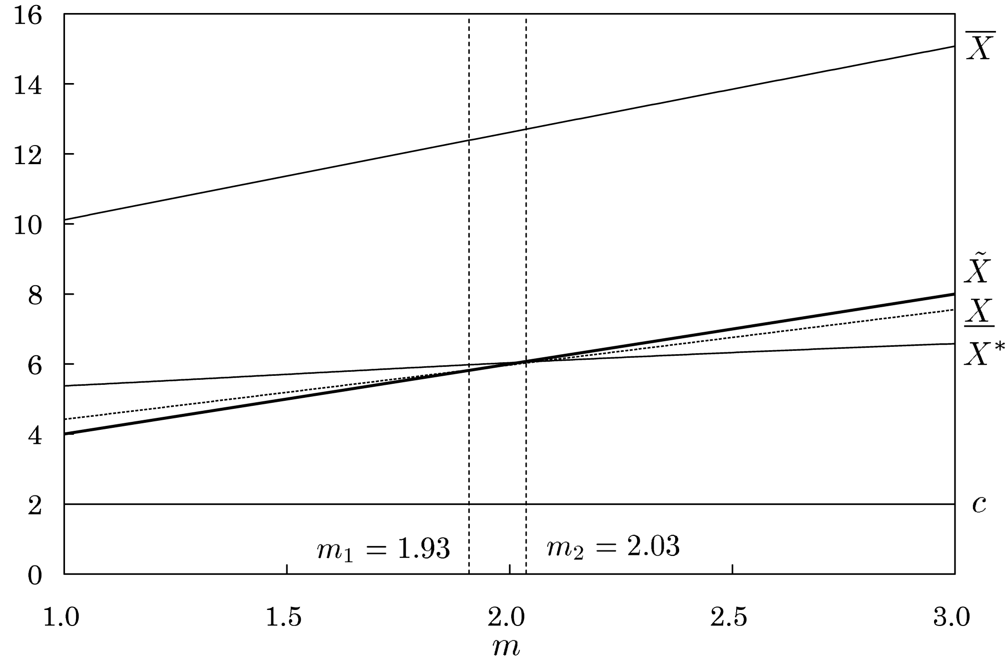

Figure 3: The comparative statics of the thresholds with respect to m.

6 Conclusion

In this paper, we have formulated firm's decision to determine simultaneously when to install a production facility, how large the capacity is, and when to start outsourcing. We have shown all the thresholds are increasing with the capacity, and the necessary value-matching and smooth-pasting conditions vary with the capacity. And a certain optimality of investment has been shown given a capacity.

Several issues are challenges for the future. First, we must find the optimal capacity and compare it with that of Dangl (1999). Second, we will analyze comparative statics with respect to some important parameters.

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