Debt Financing Irreversible Investment

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Abstract

In this paper, we explore the impact of debt financing on the timing of an irreversible investment and the value of waiting to invest. As a benchmark, we consider the case where the market for loans is perfectly competitive. Alternatively, a small firm has limited access to financial markets and must bargain with its bank to get financing. The debt contract is a Consol and as soon as the firm cannot meet the required coupon payment, liquidation takes place. In the competitive case, when default occurs, the higher the debt level, the higher the coupon, the lower the investment trigger which dampens the option value. Under imperfect competition, the higher the bargaining power of the lender, the higher the coupon charged, the higher the investment trigger but the lower the value of waiting to invest. Earnings volatility has an ambiguous impact on the value of the firm. In particular, more uncertainty negatively affects the option value when investment is close to be undertaken. Overall, the impact of debt on the investment timing depends on the loan market structure, but the possibility of default raises the cost of capital lowering the option value, which may be a reason why firms seem to mainly rely on internal sources to finance investment.

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1. INTRODUCTION

Building on some earlier works on investment by Jorgenson (1963) and Arrow (1968), McDonald and Siegel (1986) were among the first to study the implications of irreversibility on the timing of investment decisions under uncertainty. Since then, an extensive literature in real options has emphasized the benefits from delaying an irreversible investment. When the payoffs of an irreversible investment are stochastic, the investor has an option and when investing she chooses to kill her option. This implies that at the optimal date for investing the present discounted value of future cash-flows exceeds the investment cost by the option value, the marginal benefits of investing being equal to the marginal cost of investing *and* giving up the option. For more details, the reader can refer to Pindyck (1991) as well as the seminal book, Investment under Uncertainty, by Dixit and Pindyck (1994) which represents a comprehensive review on real options.

In the standard real option model, the analysis conducted assumes that the firm can afford the cost of the project. However, many companies must rely on external funds to finance their investment. The objective of this paper is to explore the impact of debt financing on the timing of an irreversible investment and the value of waiting to invest under different loan market structures.

1.1. Related Literature

Bernanke (1983) highlights that only unfavorable outcomes actually matter for the decision to undertake or postpone an investment. In other words, the distribution of payoffs is truncated and actually, only the left tale of the distribution is to be considered. He calls this effect the "bad news principle of irreversible investment". Ingersoll and Ross (1992) study the effects of uncertain interest rates on the investment timing. In particular, they find that uncertainty have an ambiguous impact on the option value of waiting. One of the central issues of this paper, the investment-uncertainty relationship, is related to the work by Caballero (1991) who demonstrates that when relaxing the hypothesis of symmetric adjustment costs, the positive relationship between investment and uncertainty may still hold. In addition, he identifies the nature of competition as the key determinant of the relationship. Actually, under imperfect competition, the investment-uncertainty relationship can become negative when the adjustment costs are highly asymmetric and there is a strong negative relationship between marginal profitability of capital and the level of capital. Along with debt arises the issue of capital structure and its implications on investment. The Modiggliani-Miller theorem (1958) states that companies should be indifferent between using debt or cash flows to finance their investment projects. Merton (1974) and (1977) was the first to use a non-arbitrage approach to evaluate a risky corporate debt. Lehand (1994) focuses on the optimal capital structure by explicitly computing the value of time independent long term risky debts using the contingent claim techniques. Paseka (2004) endogenizes default on debt and looks at the implications on credit spreads. From an empirical point of view, as documented by Ross, Westerfiled and Bradford (1993), 80 percents of firms prefer relying on internal sources of funds for their investments. Jensen and Meckling (1976) argue that when using external funds, managers tend to make the firm's activities riskier at the expense of debt holders. As a consequence, the cost of external funds is higher, which induces firms to mainly self finance their projects. More recently, Gomes (2001) examines investment behavior when firms face a costly access to external funding. He develops a general equilibrium and his findings are quite insightful but he takes as given the cost function. In this paper, coupon or equivalently interest on debt is endogenously determined. Closely related is the paper by Sabarwal (2003) who studies debt financing under limited liability. He assumes perfect competition for the loan market and finds that debt reduces the wedge between the investment trigger and the cost of investing with respect to the standard NPV rule of the irreversible investment theory (McDonald and Siegel (1986)). The option value of waiting shrinks since the project risk is now shared between equity and debt holders, so "bad news" are less costly for the firm.

1.2. Results

The main contribution of the paper is to clarify some effects of debt financing on irreversible investment decisions. As a benchmark, we consider the case where the market for loans is perfectly competitive. Alternatively, a small firm has limited access to financial markets and must bargain with its bank to get financing. The debt contract is a Consol and as soon as the firm cannot meet the required payment, liquidation takes place. In the competitive case, the probability of default induces a higher coupon, a lower investment trigger which dampens the option value. When bargaining takes place, the more power the lender has, the higher the coupon charged, the higher the investment trigger and the lower the value of waiting to invest. Regarding the effects of uncertainty, in the competitive case, the results are the same as when the firm does not use debt. Under bargaining, it is still true more uncertainty raising the investment trigger. However, the impact of the volatility of the project on the option value is now more ambiguous. In particular, more uncertainty dampens the value of waiting to invest when the optimal investment date is close.

The paper is organized as follows. Section 2 describes the economic setting and provides some analytical results. In section 3, we assume that the market for loans is perfectly competitive. Conversely, in section 4, we use Nash bargaining to model the negotiation process between the firm and its bank. Section 5 concludes. Proofs of all results are collected in the appendix.

2. THE ECONOMIC SETTING

We consider a standard irreversible investment problem. Time is continuous; a firm has to choose optimally the timing of its investment under uncertainty while partially relying on debt to cover the cost of the project. We examine two distinct market structures for the loan market. As a benchmark, we consider the case of perfect competition. Alternatively, we assume that the firm is small, does not have access to financial markets and must negotiate with its bank the financing of its project.

2.1. Investment Opportunity and Information Structure

Uncertainty is modeled by a probability space $(\Omega, \mathcal{F}, P)$ on which is defined a two dimensional (standard) Brownian motion w. A state of nature ω is an element of Ω. $\mathcal{F}$ denotes the tribe of subsets of Ω that are events over which the probability measure P is assigned. Let $\mathcal{F}_t$ be the σ-algebra generated by the observations of the value of the project, $\{P(s); 0 \leq s \leq t)\}$ and augmented. At time t, the investor's information set is $\mathcal{F}_t$. The filtration $\mathbb{F} = \{\mathcal{F}_t,\ t \in \mathbb{R}_+\}$ is the information structure and satisfies the usual conditions (increasing, right-continuous, augmented).

A risk neutral firm has to choose when to invest into a project whose gross revenues P fluctuates across time according to a geometric Brownian motion

$$dP(t) = P(t)\left(\alpha dt + \sigma dw(t)\right),$$

where $dw(t)$ is the increment of a standard Wiener process under the probability P, α is the average growth rate of future revenues and σ captures the magnitude of the uncertainty. The investment is irreversible with cost $I > 0$, the risk free rate is $r > 0$. Let μ be the average return of an asset portfolio perfectly correlated with P. As presented in Dixit and Pindyck (1994), we denote $\delta = \mu - \alpha$ and we assume that $\delta > 0$ for the value of the project to be bounded. Assuming that the output of the project is tradable, under complete markets, μ is the market risk-adjusted rate of return and by the CAPM formula, we have

$$\mu = r + \rho_{Pm}\phi\sigma,$$

where ϕ is the market price of risk and ρ_{Pm} is the coefficient of correlation between P and the whole market. It follows that under the risk neutral probability Q, the dynamics of the gross revenues P are given by

$$dP(t) = P(t)\left((r-\delta)dt + \sigma dw_Q(t)\right),$$

with

$$dw_Q(t) = dw(t) + \frac{\alpha - (r-\delta)}{\sigma}dt,$$

where $dw_Q(t)$ is the increment of a standard Wiener process under the probability Q. In the sequel, E_t^Q denotes the conditional probability at time t given the information set $\mathcal{F}_t$

under the risk neutral probability Q.

Contract

The contract between the firm and the lender is specified as follows: The lender agrees to deliver an amount $D \leq I$ when the decision to invest is undertaken and immediately after, the firm agrees to deliver a perpetual fixed coupon $C > 0$ (Consol) provided that its revenues are above C. When its revenues fall below C, the firm must turn over its entire revenues. As soon as earnings hit a minimum level $0 \leq L \leq C$, bankruptcy is declared[1], the firm is liquidated at no cost and the lender receives the minimum value between the value of the project and the perpetuity $\frac{C}{r}$. A particular case is when $L = C$.

We start by briefly recalling the standard irreversible investment decision problem as presented in Dixit and Pindyck (1994).

2.2. Benchmark Case: The standard Irreversible Investment Problem

A firm has to choose when to invest into a project whose cash flows P fluctuate across time according to a geometric Brownian motion

$$dP(t) = P(t)\left((r-\delta)dt + \sigma dw_Q(t)\right).$$

The problem can be seen as an infinite horizon American Call option with strike price I and underlying security P. Between time t and $t+dt$, as long as the investment is not completed, there is no cash outflows or inflows. Thus the option value to invest F evolves according to the following dynamics

$$F(P) = 0 + e^{-rdt}E_t\left[F(P+dP)\right].$$

Using Ito's lemma, it is easy to show that F satisfies the following ODE

$$\frac{\sigma^2}{2}P^2F''(V) + (r-\delta)PF'(P) = rF(P).$$

The interpretation goes as follows: The expected value of waiting is equal to the risk free return on the amount $F(P)$. The general solution is

$$F(P) = AP^{\beta_1} + BP^{\beta_2},$$

where (A, B) is a couple of constants to be determined and (β_1, β_2) are respectively the positive and negative root of the quadratic equation

$$\frac{\sigma^2}{2}x^2 + (r - \delta - \frac{\sigma^2}{2})x - r = 0.$$

[1] In Paseka (2003), as soon as the firm cannot fulfill its payment, a court supervises a mediation between bondholders and the management who can propose a reorganization plan if the asset value goes up to a certain level. The firm is liquidated if the value of the company drops to a floor level.

Since we must have $F(0) = 0$, this implies that $B = 0$. Thus

$$F(P) = AP^{\beta_1}.$$

As long as the value of waiting $F(P)$ is greater that the net benefit of investing

$$E_0^Q \left[\int_0^\infty P(s) e^{-rs} ds \right] = \frac{P}{\delta} - I.$$

The investment trigger value P_F^* is such that

$$\begin{aligned} F(P_F^*) &= \frac{P_N^*}{\delta} - I \\ F'(P_F^*) &= \frac{1}{\delta}. \end{aligned}$$

The last condition is known as the smooth pasting condition. We obtain

$$\begin{aligned} P_F^* &= \frac{\beta_1 \delta}{\beta_1 - 1} I \\ A &= \frac{P_F^{*(1-\beta_1)}}{\beta_1}. \end{aligned}$$

The value of the project is

$$\begin{array}{ll} \left(\frac{P_F^*}{\delta} - I\right)\left(\frac{P}{P_F^*}\right)^{\beta_1} & \text{for } P \leq P_F^* \\ \frac{P}{\delta} - I & \text{for } P \geq P_F^*. \end{array}$$

Here the implicit value of the coupon is

$$C = \frac{I}{r}.$$

The investment decision is: Invest as soon as P hits the trigger value P_F^*. Dixit and Pindyck (1994) conclude that the NPV rule is simply incorrect. There exists a wedge between the value of the project and the cost of undertaking it, the size of the wedge being the factor $\frac{\beta_1}{\beta_1 - 1} > 1$. In the sequel, we investigate how bargaining can affect this wedge. We first look at the firm problem.

2.3. The Firm Problem

We start by determining the reward by investing into the project. Let us assume that it is optimal to invest when $P > C$ (we will check this conjecture later). Let us denote τ the stopping time defined by

$$\tau = \inf \ \{t \geq 0, \ P(t) = L\},$$

with $P_0 > L$. The firm net cash flows are

$$[P-C]^+ \ 1_{\{t\leq\tau\}},$$

where 1_A is the indicator function for the set A and $x^+ = \max\{x, 0\}$ is the positive part of x. At time τ, the firm is liquidated and the shareholders receive R with

$$R(C,L) = \left[\frac{L}{\delta} - \frac{C}{r}\right]^+,$$

since when $P_0 = L$

$$E_0^Q\left[\int_0^\infty P(s)e^{-rs}ds\right] = \frac{L}{\delta}.$$

Hence, the reward value $V^F(P)$ of investing at time 0 (when cash flow is P) is

$$V^F(P) = E_0^Q\left[\int_0^\tau [P(s)-C]^+ e^{-rs}ds + e^{-r\tau}R(C,L)\right].$$

As shown in appendix 1.,

$$E_0^Q\left[e^{-r\tau}R(C,L)\right] = \left(\frac{P}{L}\right)^{\beta_2} R(C,L).$$

Then, as shown in Dixit and Pindyck, chapter 6, p. 187

$$E_0^Q\left[\int_0^\tau [P(s)-C]^+ e^{-rs}ds\right] = \begin{cases} A_1\left(\frac{P}{C}\right)^{\beta_1} + A_2\left(\frac{P}{C}\right)^{\beta_2} + \frac{P}{\delta} - \frac{C}{r},\ P \geq C \\ B_1\left(\frac{P}{C}\right)^{\beta_1} + B_2\left(\frac{P}{C}\right)^{\beta_2},\ L \leq P \leq C \end{cases},$$

where A_1, A_2, B_1, B_2 are constants to be determined. To rule out bubbles, we must have $A_1 = 0$. Then, the boundary condition is

$$B_1\left(\frac{L}{C}\right)^{\beta_1} + B_2\left(\frac{L}{C}\right)^{\beta_2} = 0$$

and the value matching and smooth pasting conditions at $P = C$ respectively are

$$\begin{aligned} A_2 + \frac{C}{\delta} - \frac{C}{r} &= B_1 + B_2 \\ \beta_2 A_2 C^{-1} + \frac{1}{\delta} &= \beta_1 B_1 C^{-1} + \beta_2 B_2 C^{-1}. \end{aligned}$$

We are only interested in the constant A_2. It follows that

$$A_2 = \frac{\beta_1 - \beta_2\left(\frac{L}{C}\right)^{\beta_1-\beta_2}}{\beta_2(\beta_1-\beta_2)}\left((1-\beta_2)\frac{C}{\delta} + \beta_2\frac{C}{r}\right) - \frac{C}{\beta_2\delta}.$$

Finally we obtain

$$V^F(P) = A_2\left(\frac{P}{C}\right)^{\beta_2} + \frac{P}{\delta} - \frac{C}{r} + \left(\frac{P}{L}\right)^{\beta_2} R(C,L).$$

Hence, the option value starting at P_0 is given by

$$F(P_0) = \sup_{\tau \geq 0} E_0^Q \left[e^{-r\tau} \left(V^F(P(\tau)) - I + D \right) \right].$$

Dropping the time index, we have

$$F(P) = \begin{cases} AP^{\beta_1} + BP^{\beta_2}, \; P \leq P^* \\ V^F(P) - I + D, \; P \geq P^*, \end{cases}$$

where P^* is the optimal investment threshold. Since $F(0) = 0$, this implies that $B = 0$ and A is a positive constant to be determined. The value matching and smooth pasting conditions are

$$\begin{aligned} A(P^*)^{\beta_1} &= \frac{P^*}{\delta} - \frac{C}{r} + A_2(\frac{P^*}{C})^{\beta_2} + \left(\frac{P^*}{L}\right)^{\beta_2} R(C,L) - I + D \\ \beta_1 A(P^*)^{\beta_1} &= \frac{P^*}{\delta} + \beta_2 A_2(\frac{P^*}{C})^{\beta_2} + \beta_2 \left(\frac{P^*}{L}\right)^{\beta_2} R(C,L). \end{aligned}$$

Eliminating A leads to

$$(\beta_1 - 1)\frac{P^*}{\delta} + (\beta_1 - \beta_2)\left(A_2(\frac{P^*}{C})^{\beta_2} + \left(\frac{P^*}{L}\right)^{\beta_2} R(C,L) \right) = \beta_1 \left(I + \frac{C}{r} - D \right).$$

Finally the option value F is given by

$$F(P) = \frac{(1-\beta_2)\frac{P^*}{\delta} + \beta_2(I + \frac{C}{r} - D)}{\beta_1 - \beta_2} \left(\frac{P}{P^*}\right)^{\beta_1}. \tag{2.1}$$

2.4. The Lender Problem

At time τ, the firm is liquidated and debt holders receive $T(C,L)$ with

$$T(C,L) = \min\ \{\frac{L}{\delta}, \frac{C}{r}\}.$$

As long as $t \leq \tau$, they receive

$$\min\ \{C, P\}.$$

It follows that the value $V^L(P)$ of the lender is given by

$$V^L(P) = E_0^Q \left[\int_0^\tau \min\ \{C, P(s)\} e^{-rs} ds + e^{-r\tau} T(C,L) \right].$$

As seen before,

$$E_0^Q \left[e^{-r\tau} T(C,L) \right] = \left(\frac{P}{L}\right)^{\beta_2} T(C,L).$$

Given the preliminary result in appendix 1., it is easy to show that

$$E_0^Q\left[\int_0^\tau \min\ \{C,P(s)\}e^{-rs}ds\right]=\begin{cases} M_1\left(\frac{P}{C}\right)^{\beta_1}+M_2\left(\frac{P}{C}\right)^{\beta_2}+\frac{C}{r},\ P\geq C \\ N_1\left(\frac{P}{C}\right)^{\beta_1}+N_2\left(\frac{P}{C}\right)^{\beta_2}+\frac{P}{\delta},\ L\leq P\leq C \end{cases},$$

where M_1, M_2, N_1, N_2 are constants to be determined. To rule out bubbles, we must have $M_1 = 0$. Then, the boundary condition is

$$N_1\left(\frac{L}{C}\right)^{\beta_1}+N_2\left(\frac{L}{C}\right)^{\beta_2}+\frac{L}{\delta}=0,$$

and the value matching and smooth pasting conditions at $P = C$ respectively are

$$\begin{aligned} M_2+\frac{C}{r} &= N_1+N_2+\frac{C}{\delta} \\ \beta_2 M_2 C^{-1} &= \beta_1 N_1 C^{-1}+\beta_2 N_2 C^{-1}+\frac{1}{\delta}. \end{aligned}$$

We are only interested in the constant M_2. It follows that

$$M_2=-\left(\frac{L}{C}\right)^{-\beta_2}\frac{L}{\delta}-\frac{\beta_1-\beta_2\left(\frac{L}{C}\right)^{\beta_1-\beta_2}}{\beta_1-\beta_2}\frac{C}{r}+\frac{\beta_1-1-(\beta_2-1)\left(\frac{L}{C}\right)^{\beta_1-\beta_2}}{\beta_1-\beta_2}\frac{C}{\delta}.$$

Finally we obtain

$$V^L(P)=\frac{C}{r}+M_2\left(\frac{P}{C}\right)^{\beta_2}+\left(\frac{P}{L}\right)^{\beta_2}T(C,L).$$

We now determine the optimal contract (P^*, C^*) under two different market structures for the loan market.

3. Perfect Competition Case

In this paragraph, we assume that the market for loans is perfectly competitive so the no-profit entry condition implies when $P = P_C^*$

$$V^L(P_C^*)-D=0,$$

or equivalently

$$\frac{C}{r}-D+M_2\left(\frac{P_C^*}{C}\right)^{\beta_2}+\left(\frac{P_C^*}{L}\right)^{\beta_2}T(C,L)=0.$$

Definition 1. *An equilibrium is an optimal couple (P_C^*, C^*) that satisfies*

$$\begin{aligned} (\beta_1-1)\frac{P_C^*}{\delta}+(\beta_1-\beta_2)\left(A_2(\frac{P_C^*}{C^*})^{\beta_2}+\left(\frac{P_C^*}{L}\right)^{\beta_2}R(C^*,L)\right) &= \beta_1\left(I+\frac{C^*}{r}-D\right) \\ \frac{C^*}{r}-D+M_2\left(\frac{P_C^*}{C^*}\right)^{\beta_2}+\left(\frac{P_C^*}{L}\right)^{\beta_2}T(C^*,L) &= 0. \end{aligned}$$

A special case: $L = C$. In this case

$$\begin{aligned} A_2 &= -(\frac{C}{\delta} - \frac{C}{r}) \\ R(C,C) &= C\left[\frac{1}{\delta} - \frac{1}{r}\right]^+ \\ M_2 &= -\frac{C}{r} \\ T(C,C) &= C\min\{\frac{1}{\delta}, \frac{1}{r}\}. \end{aligned}$$

Therefore the contract (P_C^*, C^*) is defined by

$$\begin{aligned} (\beta_1 - 1)\frac{P_C^*}{\delta} + (\beta_1 - \beta_2)(\frac{P_C^*}{C^*})^{\beta_2}\left(\left[\frac{1}{\delta} - \frac{1}{r}\right]^+ - \frac{1}{\delta} + \frac{1}{r}\right)C^* &= \beta_1\left(I + \frac{C^*}{r} - D\right) \\ \frac{C^*}{r} - D + \left(\frac{P_C^*}{C^*}\right)^{\beta_2}\left(\min\{\frac{1}{\delta}, \frac{1}{r}\} - \frac{1}{r}\right)C^* &= 0. \end{aligned}$$

If $r > \delta$, then there is no default (in the sense that even if the firm is liquidated, its resale value can cover the perpetuity $\frac{C}{r}$) so

$$\begin{aligned} C^* &= rD \\ P_C^* &= P_F^* = \frac{\beta_1\delta}{\beta_1 - 1}I. \end{aligned}$$

If $r < \delta$, then

$$\frac{C^*}{r} - D + \left(\frac{P_C^*}{C^*}\right)^{\beta_2}\left(\frac{1}{\delta} - \frac{1}{r}\right)C^* = 0,$$

so

$$C > rD,$$

and

$$\frac{P_C^*}{\delta} = \frac{\beta_1}{\beta_1 - 1}I + \frac{\beta_2}{\beta_1 - 1}\left(\frac{C^*}{r} - D\right). \tag{3.1}$$

In appendix 3., the existence and uniqueness of a solution is established. The wedge between NPV and investment trigger threshold is reduced. Debt financing introduces risk sharing between the entrepreneur and the financial institution. The investment up-front is reduced which fosters early investment decision. However, Jorgenson user cost is capital is

$$r(I - D) + C^*.$$

As shown is appendix 3, the investment trigger P_C^* is always strictly greater than $r(I - D) + C^*$ and $\frac{P_C^*}{\delta}$ is always above the cost of the project I. In addition, from relationship (3.1), it is easy to see that $P_C^* < P_F^*$. Hence, we have

$$1 < \frac{P_C^*}{\delta} \leq \frac{P_F^*}{\delta},$$

which means that Tobin's q is still above unity but it is reduced with respect to the self financing case. Also worth noticing is the fact that at the time of the investment, the firm pays $I-D$, and should the firm invest without relying on debt into a project with the same characteristics but with cost $I-D$, the investment trigger value will be

$$\frac{\beta_1}{\beta_1-1}(I-D).$$

It turns out that

$$\frac{P_C^*}{\delta} > \frac{\beta_1}{\beta_1-1}(I-D),$$

as shown in appendix 3. The manager takes into account the possibility of not being able to fulfill her commitment in the future and the associated cost, i.e., liquidation of the company. Consequently she requires a larger wedge between P_C^* and C^* than the myopic value $\frac{\beta_1}{\beta_1-1}(I-D)$. Equivalently, one can realize the investment payoffs are reduced due to the liquidation threat since

$$\begin{aligned} V^F(P^*)-(I-D) &= \frac{P^*}{\delta}-\frac{C}{r}+\left(\frac{1}{r}-\frac{1}{\delta}\right)\left(\frac{P^*}{C}\right)^{\beta_2}-(I-D) \\ &= \frac{P^*}{\delta}-I \\ &< \frac{P^*}{\delta}-(I-D), \end{aligned}$$

which leads the manager to wait longer. In addition, an upper bound for the coupon is δD, so $\delta - r$ is an upper bound for the premium on the riskfree rate r. Finally, using relationship (3.1), the option value F can be written

$$F(P)=\left(\frac{P_C^*}{\delta}-I\right)\left(\frac{P}{P_C^*}\right)^{\beta_1}.$$

Then, given P

$$\begin{aligned} \frac{\partial F(P)}{\partial P_C^*} &= \frac{1}{P_C^*}\left(\frac{P}{P_C^*}\right)^{\beta_1}\left(\frac{P_C^*}{\delta}-\beta_1\left(\frac{P_C^*}{\delta}-I\right)\right) \\ &= \frac{\beta_1-1}{P_C^*}\left(\frac{P}{P_C^*}\right)^{\beta_1}\left(\frac{\beta_1 I}{\beta_1-1}-\frac{P_C^*}{\delta}\right) \geq 0. \end{aligned}$$

The standard McDonald Siegel (1986) investment thresholds corresponds to the case where this is no debt, $D=0$. Since $P_C^* \leq P_F^*$, we can conclude that relying on debt decreases the value of waiting to invest. Finally, in appendix 3, we also prove that an increase of the debt level increases the value of the coupon, i.e. $\frac{\partial C^*}{\partial D} > 0$ and reduces the investment trigger threshold, i.e. $\frac{\partial P_C^*}{\partial D} < 0$. We conclude the analysis by presenting some numerical simulations on the effects of debt level and uncertainty on the equilibrium couple (P_C^*, C^*).

Table I: Impact of debt level on investment threshold and coupon

D/I	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
C^*	0	0.414	0.849	1.302	1.775	2.270	2.787	3.331	3.906	4.518	5.179
P_C^*	9.464	9.455	9.433	9.399	9.353	9.293	9.219	9.128	9.017	8.882	8.716

$$r = 0.04,\ \delta = 0.06,\ \sigma = 0.2,\ I = 100.$$

Table II: Impact of uncertainty on investment threshold and coupon

σ	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
C^*	2.084	2.158	2.270	2.366	2.447	2.515	2.572	2.621	2.663	2.699	2.729
P_C^*	6	7.081	9.293	12.314	16.174	20.942	26.592	33.210	40.793	49.352	58.894

$$r = 0.04,\ \delta = 0.06,\ I = 100,\ D/I = 0.5.$$

Both the investment trigger and the coupon increase with uncertainty. However, we notice that P_C^* is much more sensitive to uncertainty than C^*. Then

$$\frac{\partial F(P)}{\partial \sigma} = \frac{\partial F(P)}{\partial P_C^*}\frac{\partial P_C^*}{\partial \sigma} + \left(\tfrac{P_C^*}{\delta} - I\right) \ln \frac{P}{P_C^*} \left(\frac{P}{P_C^*}\right)^{\beta_1} \frac{\partial \beta_1}{\partial \sigma}.$$

Since $P_C^* \leq P_F^*$, it follows that $\frac{\partial F(P)}{\partial P_C^*} > 0$ and thus we can conclude that when uncertainty rises, so does the option value.

3.1. Extension to Senior and Junior debts

In this paragraph, we assume that the face values of the senior and junior debts are respectively D_1 and D_2 with $D_1 + D_2 \leq I$. The conditions of the contracts are the same as before: the firm agrees to pay a perpetual coupon C_1 for its senior debt and similarly a perpetual coupon C_2 for its junior debt. To keep things simple, we assume that the company is liquidated as soon as earnings P hits $C_1 + C_2$. As before, the condition for the firm is

$$(\beta_1 - 1)\frac{P_C^*}{\delta} + (\beta_1 - \beta_2)\left(\frac{P_C^*}{C_1^* + C_2^*}\right)^{\beta_2} \left(\left[\frac{1}{\delta} - \frac{1}{r}\right]^+ - \frac{1}{\delta} + \frac{1}{r}\right)(C_1^* + C_2^*) = \beta_1 \left(I + \frac{C_1^* + C_2^*}{r} - D_1 - D_2\right) \tag{3.2}$$

For the senior debt, we still have

$$T_1(C_1, C_1 + C_2) = \min \left\{\frac{C_1 + C_2}{\delta}, \frac{C_1}{r}\right\}.$$

However, for the junior debt, we have

$$\begin{aligned} T_2(C_2, C_1+C_2) &= \min\ \{\frac{C_1+C_2}{\delta} - T_1(C_1, C_1+C_2), \frac{C_2}{r}\} \\ &= \min\ \{\left[\frac{C_1+C_2}{\delta} - \frac{C_1}{r}\right]^+, \frac{C_2}{r}\}. \end{aligned}$$

Then the no profit entry conditions for senior and junior debt holders respectively are

$$\frac{C_1^*}{r} - D_1 + \left(\frac{P_C^*}{C_1^*+C_2^*}\right)^{\beta_2}\left(\min\ \{\frac{C_1^*+C_2^*}{\delta}, \frac{C_1^*}{r}\} - \frac{C_1^*}{r}\right) = 0 \tag{3.3}$$

$$\frac{C_2^*}{r} - D_2 + \left(\frac{P_C^*}{C_1^*+C_2^*}\right)^{\beta_2}\left(\min\ \{\left[\frac{C_1^*+C_2^*}{\delta} - \frac{C_1^*}{r}\right]^+, \frac{C_2^*}{r}\} - \frac{C_2^*}{r}\right) = 0. \tag{3.4}$$

There are three possible outcomes. Case 1, there is no default for both the senior and junior debts. In this case, $C_1^* = rD_1$ and $C_2^* = rD_2$. Again, a necessary and sufficient condition for this to happen is $r > \delta$. Case 2, there is default only for the junior debt. In this case, $C_1^* = rD_1$ and $C_2^* > rD_2$. We need $r < \delta$, but δ should not be too high. Case 3 is when there is default for both the junior and the senior debt ($r << \delta$) and therefore $C_1^* > rD_1$ and $C_2^* > rD_2$. Note that in case 2 and 3, the investment trigger is still below P_F^*. Note that summing up relationships (3.3) and (3.4), we have

$$\frac{C_1^*+C_2^*}{r} - (D_1+D_2) + \left(\frac{P_C^*}{C_1^*+C_2^*}\right)^{\beta_2}\left(\min\{\frac{1}{\delta}, \frac{1}{r}\} - \frac{1}{r}\right)(C_1^*+C_2^*) = 0. \tag{3.5}$$

This implies that given a total amount of debt $D = D_1 + D_2$, the investment trigger P_C^* is same as in the case of senior debt with face value D. The composition of the total debt does not affect P_C^*. It is then easy to realize that the result also holds for the option F that is unaffected by the composition of the total debt. Moreover, the existence and uniqueness of (P_C^*, C_1^*, C_2^*) is guaranteed since given we already know that there is unique solution to equations (3.2) and (3.5). Then, given a value for $C_1 + C_2$, it is easy to see that (3.3) and (3.4) has a unique solution (C_1^*, C_2^*). Finally, we present some numerical simulations for the junior and senior debt coupons.

Table III: Investment threshold, Senior and Junior debt coupons

	$\delta = 0.06$	$\delta = 0.25$
C_1^*	1.2	1.312
C_2^*	1.587	6.024
P_C^*	9.21877	26.848

$r = 0.04,\ \sigma = 0.2,\ I = 100,\ D_1/I = D_2/I = 0.3.$

For an equal face value, junior and senior coupons can be quite different, reflecting a much larger probability of default for the junior debt. Next, we move away from perfectly competitive loan markets and analyze the case of a small company that needs to bargain with some credit institution to get financing.

4. Nash Bargaining

In this section, we assume that as soon as the price of the project drops below C liquidation takes place. We use Nash Bargaining to model the interaction between the firm and its banks as for instance presented in Osborne and Rubinstein (1990), chapter 2. As shown in the sequel, a nice feature of this bargaining process is that the outcome is independent of the initial condition (here the value of earnings) at the date of the agreement between the two parties. At time 0, the borrower and the lender agree on a couple (P_N^*, C_N^*) that is solution of the program

$$\max_{(P,C)\in S} F(P_0)V^L(P_0), \tag{N}$$

where

$$S = \left\{ (P,C),\ (\beta_1 - 1)\frac{P}{\delta} + (\beta_1 - \beta_2)(\frac{P}{C})^{\beta_2}\left(\left[\frac{1}{\delta} - \frac{1}{r}\right]^+ - \frac{1}{\delta} + \frac{1}{r}\right)C = \beta_1\left(I + \frac{C}{r} - D\right)\right\}.$$

Since the contract is closed before the investment is realized, $P_0 < P_N^*$ and we have

$$F(P_0) = \frac{(1-\beta_2)\frac{P_N^*}{\delta} + \beta_2(I + \frac{C}{r} - D)}{\beta_1 - \beta_2}\left(\frac{P_0}{P_N^*}\right)^{\beta_1}.$$

$$V^L(P_0) = (V^L(P_N^*) - D)E_0^Q\left[e^{-r\tau}\right],$$

where τ is the first time P hits P_N^*. Thus

$$V^L(P_0) = (V^L(P_N^*) - D)\left(\frac{P_0}{P_N^*}\right)^{\beta_1},$$

Case 1: $r > \delta$ In this case, we have

$$\begin{aligned} P_N^* &= \frac{\beta_1\delta}{\beta_1 - 1}\left(I + \frac{C}{r} - D\right), \\ V^L(P_N^*) &= \frac{C}{r}, \end{aligned}$$

so program N is equivalent to

$$\max_C\ (\frac{C}{r} - D)(I + \frac{C}{r} - D)^{1-2\beta_1}.$$

Notice that the bargaining problem is independent from the initial price P_0 and the optimal solution is

$$C_N^* = rD + \frac{rI}{2(\beta_1 - 1)}.$$

In the competitive case, the coupon C is equal to rD. Here, the lender has more bargaining power and uses it to charge a fixed premium (independent from the amount lent) equal to

$\frac{rI}{2(\beta_1-1)}$. In addition, since $\frac{\partial\beta_1}{\partial\sigma} < 0$, we can conclude that the higher the uncertainty, the higher the risk premium $\frac{rI}{2(\beta_1-1)}$. This leads to an investment threshold P_N^* given by

$$P_N^* = \frac{\beta_1(2\beta_1-1)\delta I}{2(\beta_1-1)^2} > P_C^*.$$

Investment takes place at a later date with respect to the competitive case and note that P_N^* is independent of the debt level D. The wedge between NPV and investment trigger threshold is enhanced. We have seen that, given P

$$\frac{\partial F(P)}{\partial P^*} = \frac{\beta_1-1}{P_C}\left(\frac{P}{P_C}\right)^{\beta_1}\left(\frac{\beta_1 I}{\beta_1-1} - \frac{P^*}{\delta}\right).$$

Since $P_N^* > \frac{\beta_1\delta I}{\beta_1-1}$, we find that the value of waiting to invest F is reduced. In appendix 4, we indeed prove that $P_N^* > C_N^*$.

By maximizing the product of the utility functions, implicitly, we have assumed that the firm and the lender have equal bargaining power. Alternatively, we can consider the family of asymmetric Nash solutions for which that the firm has a bargaining ability with weight $1-\theta$ whereas the lender has a bargaining ability with weight θ for some θ in $[0,1]$. Program N is now

$$\max_{(P,C)} \ (F(P_0))^{1-\theta}\left(V^L(P_0)\right)^{\theta},$$

or equivalently

$$\max_C \ (\frac{C}{r} - D)^{\theta}(I + \frac{C}{r} - D)^{1-\beta_1-\theta}.$$

The solution is

$$\begin{aligned} C_N^* &= rD + \frac{r\theta I}{\beta_1-1} \\ P_N^* &= \frac{\beta_1(\beta_1-1+\theta)\delta I}{(\beta_1-1)^2} > P_C^*. \end{aligned}$$

Indeed, the higher the bargaining power of the lender, the higher the coupon and the investment threshold and therefore the lower the option value F. Note that the maximum coupon is

$$C_N^* = rD + \frac{rI}{\beta_1-1}.$$

Even when the lender has all the bargaining power, optimally she does not charge a too high coupon as she understands that the higher the coupon, the longer the investment is postponed and therefore the more she has to wait before receiving some payment. The proof to show that $P_N^* > C_N^*$ is the same as before. We now investigate the effects of uncertainty on the option value.

4.1. Uncertainty Effects on the Option Value

The volatility of the project now plays an ambiguous role. On the one hand, as in the standard case, the direct effect of uncertainty is to raise the value of waiting to wait due the convexity of the payoffs. On the other hand, the higher the magnitude of uncertainty, the higher the cost of capital and the lower the value of the firm. We now examine in details the two effects. We already know that $\frac{\partial \beta_1}{\partial \sigma} < 0$ so

$$\frac{\partial P_N^*}{\partial \sigma} = -\frac{(\beta_1 - 1)(1+\theta) + 2\theta}{(\beta_1 - 1)^3} \frac{\partial \beta_1}{\partial \sigma} > 0.$$

Then, given $P \leq P_N^*$, we have

$$\frac{\partial F(P)}{\partial \sigma} = \underbrace{\frac{\partial F(P)}{\partial P_N^*} \frac{\partial P_N^*}{\partial \sigma}}_{\text{indirect effect}} + \underbrace{\left(\frac{P_N^*}{\delta} - I\right) \ln \frac{P}{P_N^*} \left(\frac{P}{P_N^*}\right)^{\beta_1} \frac{\partial \beta_1}{\partial \sigma}}_{\text{direct effect}}.$$

The direct effect is positive whereas the indirect is negative since $\frac{\partial P_N^*}{\partial \sigma} > 0$ and $\frac{\partial F(P)}{\partial P_N^*} < 0$. Overall, as proved in appendix 4.

$$\frac{\partial F(P)}{\partial \sigma} = \frac{\beta_1}{(\beta_1 - 1)^4 P_N^*} \left(\frac{P}{P_N^*}\right)^{\beta_1} \left(\theta((\beta_1 - 1)(1+\theta) + 2\theta) + \delta\left((\beta_1 - 1 + \theta)(\beta_1(1+\theta) - 1\right) \ln \frac{P}{P_N^*}\right) \frac{\partial \beta_1}{\partial \sigma}.$$

When $P << P_N^*$, the direct effect overcomes the indirect effect and the option value increases with uncertainty. However, when P is close to P_N^*, the direct effect is very small and the indirect effect dominates. This means that when $\frac{P}{P_N^*}$ is small, more uncertainty increases the option value, when $\frac{P}{P_N^*}$ is large enough, more uncertainty decreases the option value, and in the between, the relationship between option value and uncertainty is U shape. Ingersoll and Ross (1992) also find that changes in uncertainty about interest rates may have an ambiguous impact on the option value of waiting. Moreover, our results corroborate Caballero's (1991) findings on the investment-uncertainty relationship that can become negative when the adjustment costs are highly asymmetric and there is a strong negative relationship between marginal profitability of capital and the level of capital.

Case 2: $r < \delta$ In this case, we have

$$\begin{aligned} \frac{\beta_1 - 1}{\delta} P^* - (\beta_1 - \beta_2)(\frac{P^*}{C})^{\beta_2} \left(\frac{1}{\delta} - \frac{1}{r}\right) C &= \beta_1 \left(I + \frac{C}{r} - D\right) \\ V^L(P^*) &= \frac{C}{r} - D + C(\frac{1}{\delta} - \frac{1}{r}) \left(\frac{P^*}{C}\right)^{\beta_2}, \end{aligned}$$

so

$$\begin{aligned} V^L(P^*) - D &= \frac{1}{\beta_1 - \beta_2} \left(\frac{\beta_1 - 1}{\delta} P^* - \beta_2(\frac{C}{r} - D) - \beta_1 I\right) \\ F(P_0) &= \frac{\frac{1-\beta_2}{\delta} P^* + \beta_2(I + \frac{C}{r} - D)}{\beta_1 - \beta_2} \left(\frac{P_0}{P^*}\right)^{\beta_1}. \end{aligned}$$

Again, program N is independent from the initial cash flow P_0 and can be rewritten

$$\begin{array}{ll} \max\limits_{C,P^*} & \left(\frac{\beta_1-1}{\delta}P^* - \beta_2(\frac{C}{r}-D) - \beta_1 I\right)\left(\frac{1-\beta_2}{\delta}P^* + \beta_2(I+\frac{C}{r}-D)\right)(P^*)^{-2\beta_1} \\ \text{s.t.} & \frac{\beta_1-1}{\delta}P^* - (\beta_1-\beta_2)(\frac{P^*}{C})^{\beta_2}\left(\frac{1}{r}-\frac{1}{\delta}\right)C = \beta_1\left(I+\frac{C}{r}-D\right) \end{array}$$

The first order conditions with respect to P^* and C respectively are

$$\begin{aligned} & \frac{\frac{\beta_1-1}{\delta}}{\frac{\beta_1-1}{\delta}P^* - \beta_2(\frac{C}{r}-D) - \beta_1 I} + \frac{\frac{1-\beta_2}{\delta}}{\frac{1-\beta_2}{\delta}P^* + \beta_2(I+\frac{C}{r}-D)} - \frac{2\beta_1}{P^*} \\ = \ & \psi\left(\frac{\beta_1-1}{\delta} - \beta_2(\beta_1-\beta_2)\left(\frac{1}{r}-\frac{1}{\delta}\right)(\frac{P^*}{C})^{\beta_2-1}\right) \end{aligned}$$

$$\begin{aligned} & \frac{-\frac{\beta_2}{r}}{\frac{\beta_1-1}{\delta}P^* - \beta_2(\frac{C}{r}-D) - \beta_1 I} + \frac{\frac{\beta_2}{r}}{\frac{1-\beta_2}{\delta}P^* + \beta_2(I+\frac{C}{r}-D)} \\ = \ & \psi\left(-\frac{\beta_1}{r} + (\beta_2-1)(\beta_1-\beta_2)\left(\frac{1}{r}-\frac{1}{\delta}\right)(\frac{P^*}{C})^{\beta_2}\right), \end{aligned}$$

where ψ is the Lagrange multiplier. Hence

$$\begin{aligned} & \frac{\frac{\beta_1-1}{\delta}P^*}{\frac{\beta_1-1}{\delta}P^* - \beta_2(\frac{C}{r}-D) - \beta_1 I} + \frac{\frac{1-\beta_2}{\delta}P^*}{\frac{1-\beta_2}{\delta}P^* + \beta_2(I+\frac{C}{r}-D)} - 2\beta_1 \\ = \ & \psi\left(\frac{(\beta_1-1)(1-\beta_2)}{\delta}P^* + \beta_1\beta_2\left(I+\frac{C}{r}-D\right)\right) \end{aligned}$$

$$\begin{aligned} & \frac{-\beta_2\frac{C}{r}}{\frac{\beta_1-1}{\delta}P^* - \beta_2(\frac{C}{r}-D) - \beta_1 I} + \frac{\beta_2\frac{C}{r}}{\frac{1-\beta_2}{\delta}P^* + \beta_2(I+\frac{C}{r}-D)} \\ = \ & \psi\left(-\frac{(\beta_1-1)(1-\beta_2)}{\delta}P^* - \beta_1\beta_2\frac{C}{r} - \beta_1(\beta_2-1)(I-D)\right) \end{aligned}$$

The couple (P^*, C) is solution of the following 2 by 2 non-linear system

$$\frac{\beta_1-1}{\delta}P^* - (\beta_1-\beta_2)(\frac{P^*}{C})^{\beta_2}\left(\frac{1}{r}-\frac{1}{\delta}\right)C = \beta_1\left(I+\frac{C}{r}-D\right)$$

$$\begin{aligned} & \frac{\frac{P^*}{\delta}\left(\frac{(\beta_1-1)(1-\beta_2)}{\delta}P^* + \beta_2(\beta_1+\beta_2-2)(\frac{C}{r}-D) + (2\beta_1\beta_2-\beta_1-\beta_2)I\right) - 2\beta_1(\frac{\beta_1-1}{\delta}P^* - \beta_2(\frac{C}{r}-D) - \beta_1)(\frac{1-\beta_2}{\delta}P^* + \beta_2(I+\frac{C}{r}-D))}{\beta_2\frac{C}{r}(\frac{\beta_1+\beta_2-2}{\delta}P^* - 2\beta_2(\frac{C}{r}-D) - (\beta_1+\beta_2)I)} \\ = & \frac{\frac{(\beta_1-1)(1-\beta_2)}{\delta}P^* + \beta_1\beta_2\left(I+\frac{C}{r}-D\right)}{-\frac{(\beta_1-1)(1-\beta_2)}{\delta}P^* - \beta_1\beta_2\frac{C}{r} - \beta_1(\beta_2-1)(I-D)} \end{aligned}$$

Numerical Simulations.

Table VI: Impact of debt level on investment threshold and coupon

D/I	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
C^*	1.258	1.719	2.194	2.684	3.190	3.713	4.254	4.817	5.404	6.021	6.675
P_N^*	12.202	12.176	12.136	12.084	12.018	11.937	11.840	11.724	11.588	11.426	11.233

$r = 0.04,\ \delta = 0.06,\ \sigma = 0.2,\ I = 100.$

As in the competitive case, P_N^* is decreasing in the level of debt D whereas C^* is increasing. We notice that with respect to Table I, for any given value of D/I, both the coupon and the investment trigger threshold are higher.

Table V: Impact of uncertainty on investment threshold and coupon

σ	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
C^*	2.084	2.599	3.713	5.222	7.118	9.428	12.177	15.386	19.07	23.237	27.894
P_N^*	6	7.744	11.937	19.002	30.280	47.587	73.211	109.925	160.99	230.164	321.701

$r = 0.04,\ \delta = 0.06,\ I = 100,\ D/I = 0.5.$

As before, more uncertainty implies a higher probability of default, so the coupon must be greater and so is the investment threshold.

5. CONCLUSION

We used a very simple model of irreversible investment to explore the implications of debt financing on investment timing decisions and the value of waiting to invest. Two market structures for the loan market have been considered: Perfect competition and Nash bargaining. In the first case, the investment trigger is below the usual NPV value of the irreversible investment theory indicating that the decision to invest is hastened. Conversely, the opposite occurs when markets for external funds are not competitive. In this last case, we also find that the relationship uncertainty - option value is now ambiguous and possibly negative if the investment is close to be undertaken. The possibility of debt default induces a higher cost of capital, and not surprisingly, the more bargaining power the lender has, the higher the coupon charged. Consequently, in both cases, the value of waiting to invest is reduced with respect to the self financing case, which may be a reason why firms seem to mainly rely on internal sources to finance investment.

Our model is very sterilized and in particular, we have ignored the effects of tax benefits to leverage which could be worth considering. Another possible extension to the model would be to assume a finite horizon debt in order to investigate the impact of short term versus long term debt financing on the investment timing and the option value. This is left for future research.

6. APPENDIX

6.1. APPENDIX 1

Preliminary result. It τ is a stopping time, then for all continuous function f

$$F(P_0) = E_0^Q\left[\int_0^\tau f(P(s))e^{-rs}ds\right],$$

satisfies the following ODE

$$rF(P) = f(P) + (r-\delta)PF'(P) + \frac{\sigma^2}{2}P^2F''(P).$$

See Karling and Taylor (1981). Then, let $P_0 > L$ and define

$$\tau = \inf\{t \geq 0,\ P(t) = L\}.$$

We want to compute

$$E_0^Q\left[e^{-r\tau}\right].$$

Let us write

$$1 - r\int_0^\tau e^{-rs}ds = e^{-r\tau}.$$

Given the preliminary result

$$F(P_0) = E_0^Q\left[\int_0^\tau e^{-rs}ds\right]$$

satisfies

$$rF(P) = 1 + (r-\delta)PF'(P) + \frac{\sigma^2}{2}P^2F''(P).$$

The general solution to this equation is given by

$$F(P_0) = \frac{1}{r} + AP_0^{\beta_1} + BP_0^{\beta_2},$$

where β_1 and β_2 are respectively the positive and negative root of the quadratic

$$\frac{\sigma^2}{2}x^2 + (r - \delta - \frac{\sigma^2}{2})x = r.$$

Hence we have

$$E_0^Q\left[e^{-r\tau}\right] = -AP_0^{\beta_1} - BP_0^{\beta_2}.$$

Note that the LHS is bounded which implies that we must have $A = 0$ and for $P_0 = L$, the LHS is equal to 1, so we must have $-BL^{\beta_2} = 1$. It follows that

$$E_0^Q e^{-r\tau} = \left(\frac{P_0}{L}\right)^{\beta_2}.$$

6.2. APPENDIX 2

We want to show existence and uniqueness of the following system

$$\begin{aligned} \frac{C^*}{r} - D + \left(\frac{P_C^*}{C^*}\right)^{\beta_2}\left(\frac{1}{\delta} - \frac{1}{r}\right)C^* &= 0 \\ \frac{\beta_1}{\beta_1 - 1}I + \frac{\beta_2}{\beta_1 - 1}\left(\frac{C^*}{r} - D\right) &= \frac{P_C^*}{\delta}, \end{aligned}$$

Define $x = \frac{P_C^*}{C^*}$ and manipulating the two equations of the system, we obtain that x must be solution of

$$\left(\frac{1}{r} - \frac{1}{\delta}\right)x^{\beta_2} + \frac{(\beta_1 - 1)}{\beta_1 u - \beta_2}\frac{x}{\delta} - \frac{\beta_1 u}{\beta_1 u - \beta_2} = 0,$$

with $u = \frac{I}{D} \geq 1$. It is then enough to that the function

$$\begin{array}{llll} & [1,\infty) & \rightarrow & \mathbb{R} \\ F: & x & \mapsto & \left(\frac{1}{r} - \frac{1}{\delta}\right)x^{\beta_2} + \frac{(\beta_1 - 1)}{\beta_1 u - \beta_2}\frac{x}{\delta} - \frac{\beta_1 u}{(\beta_1 u - \beta_2)r}, \end{array}$$

has a unique root. Note that F is a continuous function with

$$\begin{aligned} F(1) &= \frac{1}{(\beta_1 u - \beta_2)r\delta}\left((r-\delta)\beta_2 + r((1-u)\beta_1 - 1\right) \\ &\leq \frac{1}{(\beta_1 u - \beta_2)r\delta}\left((r-\delta)\beta_2 - r\right). \end{aligned}$$

Since

$$(r-\delta)\beta_2 - r = -\frac{\sigma^2}{2}\beta_2(\beta_2 - 1) < 0,$$

we can conclude that $F(1) < 0$. Then F is differentiable with

$$\begin{aligned} F'(x) &= \beta_2\left(\frac{1}{r} - \frac{1}{\delta}\right)x^{\beta_2 - 1} + \frac{(\beta_1 - 1)}{\beta_1 u - \beta_2}\frac{1}{\delta} \\ F''(x) &= \beta_2(\beta_2 - 1)\left(\frac{1}{r} - \frac{1}{\delta}\right)x^{\beta_2 - 2} > 0. \end{aligned}$$

F' is either always strictly positive or non-positive on some interval $[1, x^*]$ and positive on (x^*, ∞). Since $\lim_{x\rightarrow\infty} F(x) = \infty$, we can conclude that F has a unique root on $[1,\infty)$. It follows that the couple (P_C^*, C^*) exists and is unique with $P_C^* \geq C^*$.

6.3. APPENDIX 3

The Jorgenson user cost of capital is

$$r(I - D) + C.$$

Thus we want to show that

$$\frac{\beta_1 \delta}{\beta_1 - 1} I + \frac{\beta_2 \delta}{\beta_1 - 1} \left(\frac{C^*}{r} - D \right) > r(I - D) + C^*,$$

or equivalently

$$\beta_1 (r - \delta) I + rI > \left(\frac{C^*}{r} - D \right) (r(\beta_1 - 1) - \beta_2 \delta).$$

Notice that

$$\begin{aligned} \frac{C^*}{r} - D &= \left(\frac{P_C^*}{C^*} \right)^{\beta_2} \left(\frac{1}{r} - \frac{1}{\delta} \right) C^* \\ &< \left(\frac{1}{r} - \frac{1}{\delta} \right) C^* \text{ since } \left(\frac{P_C^*}{C^*} \right)^{\beta_2} < 1, \end{aligned}$$

which implies that

$$C^* < \delta D.$$

Hence, it is enough to prove that

$$\beta_1 (r - \delta) I + rI \geq D \left(\frac{\delta}{r} - 1 \right) (r(\beta_1 - 1) - \beta_2 \delta),$$

or

$$(\beta_1 (r - \delta) + r) u \geq \left(\frac{\delta}{r} - 1 \right) (r(\beta_1 - 1) - \beta_2 \delta),$$

with $u = \frac{I}{D} \geq 1$. since $\beta_1 (r - \delta) + r > 0$, it is in fact enough to show that

$$\beta_1 (r - \delta) + r \geq \left(\frac{\delta}{r} - 1 \right) (r(\beta_1 - 1) - \beta_2 \delta),$$

or after some cancellation that

$$0 > -1 - \beta_2 \left(\frac{\delta}{r} - 1 \right),$$

which indeed is true since

$$r + (\delta - r) \beta_2 > 0.$$

In addition, we have

$$\frac{P_C^*}{\delta} > I,$$

since

$$\begin{aligned} \frac{P_C^*}{\delta} - I &= \frac{I}{\beta_1 - 1} + \frac{\beta_2}{\beta_1 - 1} \left(\frac{C^*}{r} - D \right) \\ &> \frac{D}{\beta_1 - 1} \left(u + \beta_2 \left(\frac{\delta}{r} - 1 \right) \right) \\ &> \frac{D}{(\beta_1 - 1) r} \left(r + \beta_2 (\delta - r) \right) > 0. \end{aligned}$$

$$
\begin{aligned}
\frac{P_C^*}{\delta} - \frac{\beta_1}{\beta_1 - 1}(I - D) &= \frac{1}{\beta_1 - 1}\left(\beta_2\left(\frac{C^*}{r} - D\right) + \beta_1 D\right) \\
&> \frac{D}{(\beta_1 - 1)r}\left(\beta_1 r + (\delta - r)\beta_2\right) \\
&> \frac{D}{(\beta_1 - 1)r}\left(r + (\delta - r)\beta_2\right) > 0.
\end{aligned}
$$

On the one hand

$$\frac{\beta_1 - 1}{\delta}\frac{\partial P_C^*}{\partial D} = -\beta_2\left(1 - \frac{\partial C^*}{\partial D}\right) \tag{6.1}$$

and on the other hand since

$$\left(\frac{1}{r} - \frac{1}{\delta}\right)(P_C^*)^{\beta_2} = \left(\frac{C^*}{r} - D\right)(C^*)^{\beta_2 - 1},$$

we have

$$\beta_2\left(\frac{1}{r} - \frac{1}{\delta}\right)(P_C^*)^{\beta_2 - 1}\frac{\partial P_C^*}{\partial D} = \left(\beta_2\frac{C^*}{r} + (1 - \beta_2)D\right)(C^*)^{\beta_2 - 2}\frac{\partial C^*}{\partial D}. \tag{6.2}$$

Notice that since $C < \delta D$, we have

$$\beta_2\frac{C^*}{r} + (1 - \beta_2)D > \frac{D}{r}(r + (\delta - r)\beta_2) > 0.$$

This implies that $\frac{\partial P_C^*}{\partial D}$ and $\frac{\partial C^*}{\partial D}$ must have opposite sign. Then, from relationship (6.2), $\frac{\partial P_C^*}{\partial D} = 0$ exactly when $\frac{\partial C^*}{\partial D} = 0$. But $\frac{\partial P_C^*}{\partial D} = 0$ is then incompatible with relationship (**??**). It follows that $\frac{\partial P_C^*}{\partial D}$ and $\frac{\partial C^*}{\partial D}$ must have a constant sign. Since $P_C^* \leq P_F^*$ and P_F^* corresponds to the case when $D = 0$, we can conclude that $\frac{\partial P_C^*}{\partial D} < 0$ and therefore $\frac{\partial C^*}{\partial D} > 0$.

6.4. APPENDIX 4

We want to show that $P_N^* > C_N^*$ or equivalently that

$$\frac{\beta_1(2\beta_1 - 1)\delta I}{2(\beta_1 - 1)^2} > rD + \frac{rI}{2(\beta_1 - 1)},$$

i.e.

$$\beta_1(2\beta_1 - 1)\delta I > r(\beta_1 - 1)\left(2(\beta_1 - 1)D + I\right).$$

Since $D \leq I$, it is enough to show that

$$\beta_1\delta > r(\beta_1 - 1).$$

Then recall that β_1 is the positive root of the quadratic Q with

$$Q(x) = \frac{\sigma^2}{2}x^2 + (r - \delta - \frac{\sigma^2}{2})x - r.$$

Since $\beta_1 > 1$, from $Q(\beta_1) = 0$, we obtain that

$$\frac{\sigma^2}{2}\beta_1(\beta_1 - 1) + (r - \delta)\beta_1 = r,$$

so

$$\beta_1\delta - r(\beta_1 - 1) = \frac{\sigma^2}{2}\beta_1(\beta_1 - 1) > 0,$$

and the proof is complete.

Uncertainty and Option Value: $\delta < r$. We already know that $\frac{\partial\beta_1}{\partial\sigma} < 0$. Then, for $P \le P_N^*$, we have

$$\begin{aligned}
\frac{\partial F(P)}{\partial\beta_1} &= \frac{\partial F(P)}{\partial P_N^*}\frac{\partial P_N^*}{\partial\beta_1} + \left(\tfrac{P_N^*}{\delta} - I\right)\ln\frac{P}{P_N^*}\left(\frac{P}{P_N^*}\right)^{\beta_1} \\
&= \frac{1}{P_N^*}\left(\frac{P}{P_N^*}\right)^{\beta_1}\left(\left(\beta_1 I - (\beta_1 - 1)\tfrac{P_N^*}{\delta}\right)\frac{\partial P_N^*}{\partial\beta_1} + P_N^*\left(\tfrac{P_N^*}{\delta} - I\right)\ln\frac{P}{P_N^*}\right).
\end{aligned}$$

Since $P_N^* = \frac{\beta_1(\beta_1 - 1 + \theta)\delta I}{(\beta_1 - 1)^2}$, it follows that

$$\frac{\partial P_N^*}{\partial\beta_1} = -\frac{(\beta_1 - 1)(1 + \theta) + 2\theta}{(\beta_1 - 1)^3},$$

so finally

$$\frac{\partial F(P)}{\partial\beta_1} = \frac{\beta_1}{(\beta_1 - 1)^4 P_N^*}\left(\frac{P}{P_N^*}\right)^{\beta_1}\left(\theta((\beta_1 - 1)(1 + \theta) + 2\theta) + \delta\left((\beta_1 - 1 + \theta)(\beta_1(1 + \theta) - 1)\ln\frac{P}{P_N^*}\right).$$

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