REAL OPTIONS WITH UNKNOWN-DATE EVENTS

Óscar Gutiérrez Arnaiz[†]

nector@terra.es

Assistant professor

CPS- Universidad de Zaragoza (*Spain*)

Mª de Luna, 3. 50018-Zaragoza (Spain)

Tel: +34 976 761000

Francisco Ruiz-Aliseda

Fran.RUIZ@insead.edu

Assistant profesor

INSEAD Fointenebleau (*France*)

Boulevard de Constance. 77305-Fontainebleau (France)

Tel: +33 (0) 160724434

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Abstract

The real options literature has provided new insights on how to manage irreversible capital investments whose payoffs are always uncertain. Two of the most important predictions from such theory are: (i) greater risk delays a firm's investment timing, and (ii) greater risk increases the option value of waiting. This paper challenges such conclusions in a setting in which the relevant random variable is the arrival time of an unfavorable event. In addition, we avoid using stochastic calculus by introducing a novel framework in which a firm updates its beliefs about the profitability of an investment opportunity by waiting to invest.

Key words: Investment under Uncertainty, Option Value, Entry Timing, Bad News Principle, Hazard Rate and Bayesian Updating.

JEL Classification: D81, G31, L12.

INTRODUCTION

The real world is full of situations in which the main source of uncertainty regarding the value of an investment opportunity stems from the arrival date of a crucial event. For instance, let us consider in the first place a firm that has to choose the timing of investment in an R&D project whose outcome can be patented. In winner-takes-all industries, the time until discovery of the product or technology is clearly one of the most critical factors to take into account because of competitive forces. Yet, the time-to-discovery is usually unknown ex ante, so it is reasonable to assume that the relevant randomness of such R&D project largely stems from the uncertain discovery date, as well as those of competitors.[1]

In the second place, suppose that a firm has to make a decision about when to build a factory subject to a probable change in environmental or tax policy. As discussed by Dixit and Pindyck (1994, p. 304), the date of policy change can be considered to be unknown to the firm and "it is commonly believed that expectations of shifts of policy can have powerful effects on decisions to invest."

In the last place, consider product launching decisions. In particular, let us suppose that a firm has to decide when to introduce a recently developed product under conditions of uncertainty about the future date at which a substitute product may be launched. Competition of this product may entail the gradual decline in the demand of the product sold by the firm, and, consequently, randomness about the maturity date of the market would be an aspect that would critically affect the firm's launching decision.[2]

The real options literature has certainly been aware of the importance of these situations in which the date of occurrence of a key event is uncertain. Indeed, this explains

[1] See Weeds (2002) for a formalization of such situation in a game-theoretic real options setting.

[2] This example is drawn from Bollen (1999).

the use of Poisson processes in real option valuation. Yet, while Poisson arrivals seem the correct way to model such phenomena, this modeling approach is not particularly suited to perform many analyses beyond real option pricing. For example, the effects of mean-preserving spreads on investment timing and option values cannot be determined. For this reason, the purpose of this paper is to reexamine some of the conclusions of the theory of real options in a fairly general setting in which all uncertainty refers to the arrival date of an unfavorable event that critically affects a firm's payoff.[3] To this end, we introduce a novel framework that is based on Bayesian updating: by simply waiting to invest, the firm can update its beliefs about the profitability of an investment opportunity. Moreover, our setup requires no stochastic calculus, which significantly reduces the difficulty of the model. Furthermore, the firm's problem is amenable to an intuitive analysis in terms of marginality conditions linked to Bernanke's (1983) "bad news principle of irreversible investment."

The consideration of uncertain-date events yields that some of the fundamental results of the real options approach are not clear-cut. Firstly, the canonical real options model predicts that the value of an investment opportunity is non-decreasing in the variance parameter of the Geometric Brownian Motion that governs the return of the underlying asset.[4] Greater uncertainty cannot be harmful because the company always has the option to wait for better times or even not to invest should conditions turn out to be adverse. But, at the same time, the firm can capitalize on favorable market evolution and invest right on. Therefore, there exists an asymmetry in that waiting to invest shields the firm against adverse realizations of uncertainty, but does not prevent it from taking advantage of

[3] The date of occurrence of such event can be viewed to originate from a non-homogeneous Poisson process with a time-dependent arrival rate in which only the first event that occurs matters.

[4] See, e.g., the classic paper by McDonald and Siegel (1986).

favorable ones. This asymmetry, always present when there exist options whose payoffs are convex in the realization of the random variable, accounts for the non-negative effects of greater uncertainty on the value of investment opportunities. Notwithstanding, one of the contributions of our paper is to show that such asymmetry is not present when there exists uncertainty about the arrival date of an unfavorable event. In principle, in our setup, the event may occur before or after investing, but we show that the firm finds it optimal to undertake the investment only if uncertainty has not been resolved. Given this result and that the space of outcomes coincides with that of the control variable (namely, time), it follows that the firm would be insured against bad realizations because of waiting, and would take advantage of (very) good ones. However, it would be damaged by realizations shortly after the optimal time of investment. That is, if the event occurred right after investing, then the project may turn out to be unprofitable ex post, even though it might have seemed an excellent investment in expectation. As a result, more uncertainty may destroy option values, depending on whether the probability of occurrence of the event after the date of investment is increased by a sufficiently large amount.

Secondly, real options theory usually predicts that increased risk delays investment timing, a relevant aspect for both public policy and business purposes. Such conclusion basically follows from Bernanke's (1983) bad news principle. Intuitively, the benefit of waiting arises from the avoidance of making a poor investment when news is bad (i.e., when events are unfavorable). Given that only adverse events matter and a mean-preserving spread increases their probability of occurrence, the marginal benefit of delaying investment increases with uncertainty. Since the marginal opportunity cost of waiting (namely, current profits forgone) is unaffected by the spread, the net marginal benefit of deferring investment increases with uncertainty, which in turn induces a delay in entry. We show that this need not be true in our setting. More specifically, in our model only adverse events matter too. Yet, the firm endogenously chooses whether to position itself in a situation in which a mean-preserving spread increases or decreases the conditional probability of immediate occurrence of the (bad) event, which explains why the conclusions may differ.

It is widely believed that real options theory predicts that greater risk depresses investment, and at the same time increases the value of an investment opportunity. These results can already be found in McDonald and Siegel (1986, p. 714). Indeed, the prediction that more uncertainty leads to less investment appears to be supported by empirical evidence, as shown by Ferderer (1993) using aggregate data, or Leahy and Whited (1996) and Guiso and Parigi (1999) using micro data. These two last studies suggest that real options theory is the most solid theory of investment under uncertainty, which reinforces the need for a more comprehensive framework that helps to determine the factors and conditions that drive theoretical conclusions. Our work tries to be a step in this direction.

The paper is organized as follows. Section 2 introduces the model and Section 3 solves it. Section 4 identifies a necessary and sufficient condition for an increase in risk to speed up investment, while Section 5 provides a necessary and sufficient condition for the value of an investment opportunity to be a non-decreasing function of risk. Section 6 concludes. A mathematical appendix with all proofs is included at the end of the paper.

2. FOUNDATIONS OF THE THEORETICAL MODEL

Let time, denoted by t, be a continuous variable, i.e. $t \in [0,\infty)$. Suppose that a *risk-neutral* firm –probably a monopolist– has to decide the instant at which it wants to enter a market by launching a product for which there already exist some potential buyers. Such decision is complicated by the existence of uncertainty about the temporal evolution of the market, which in turn affects the pattern of profit evolution. Uncertainty unravels partially over time, implying that the market (and, as a result, profit) evolves in the following manner. In a first stage, the profit flow, which is positive at date $t = 0,$ grows over time. However, the market reaches its ephemeral maturity at an instant of time τ, where τ is a continuous random variable with density $f(\tau)$ defined on $[0, \infty)$. (We will slightly abuse the notation and τ will also denote its realization.) Hence, in a second phase whose beginning is unknown at date 0, instantaneous profit decreases over time and converges to 0 as $t \to \infty$, perhaps because consumers perceive that there is another product that can better serve their needs and choose to switch gradually. Formally, we assume the following:

Assumption 1: Given a realization τ of the random maturity date, the flow of profits made by the firm if active in the market evolves continuously over time as follows:

$$\Pi(t, \tau) = \begin{cases} \pi \exp(\alpha t) & \text{if } 0 \leq t \leq \tau \\ \pi \exp[\alpha(2\tau - t)] & \text{if } t > \tau \end{cases}$$

$\pi > 0$ denotes the profit the firm would make at date $t = 0,$ while $\alpha > 0$ denotes the growth rate of the profit flow (if the market is in expansion; otherwise, it is its decay rate).[5]

Assumption 2: The maturity date of the market τ is a random variable with continuous density function $f(\tau)$ with support $[0, \infty)$.

In order to set up a real options framework, we require investment to be irreversible:

[5] It is straightforward to introduce different rates of growth and decay, or even more general functions for the growth and decline stages, but we choose not to do so to keep the model simple. Note also that we are implicitly assuming that operating costs are small enough, in order to avoid making exit an issue and focus only on entry timing. However, zero cost is not necessary for positive profits. For example, a sufficiently low marginal cost in a standard Hotelling model with a single firm located at one extreme would suffice, since costs would always be transferred to consumers, who in turn would be willing to pay a high price.

Assumption 3: The firm bears an entirely sunk cost of entry $K > 0$ and discounts future payoffs at a constant risk-free interest rate $r \geq 0$.

Finally, we bound the value of the firm's investment opportunity by assuming that the expected discounted value of one dollar that is capitalized at an instantaneous rate of α is finite no matter what the length of the ascending phase of the life cycle is:[6]

Assumption 4: $\int_0^\infty e^{(\alpha - r)\tau} f(\tau) d\tau < \infty$.

3. RESOLUTION OF THE MODEL

The firm's objective at time $t = 0$ is to choose an entry rule that maximizes the expected discounted stream of cash flows conditional upon information available at the time of entry. We proceed now to characterize such optimal entry rule for the two possible states of the system, depending on whether the maturity date of the market has been revealed or not.

In the first place, it is clear that it is not optimal for the firm to exit at some date after having invested, given our assumptions of no scrap value and positive profit flow at any possible situation. This holds no matter if τ is known or not. In turn, Lemma 1 below describes the firm's behavior once the maturity of the market has been reached. According to this result, the firm prefers to invest immediately once τ is revealed, but only if such date is sufficiently large; otherwise, it prefers not to invest:

[6] In the traditional real options framework, a parallel convergence condition requires the rate of expected growth of the investment to be smaller than the risk-free rate.

Lemma 1: *Immediate investment at the revealed maturity date* τ *is optimal* $\forall \tau > t^{\max} = \max\left\{0, \frac{1}{\alpha}\log\left(\frac{K(\alpha+r)}{\pi}\right)\right\}$, *whereas investment during the declining phase of the market is not profitable* $\forall \tau \le t^{\max}$.

Hence, to characterize the optimal entry rule fully, it only remains to focus on the time t_1 at which the firm would enter the market if τ had not been revealed yet, and thus the profit cycle were in its ascending phase. Using Lemma 1, the value of the firm's investment opportunity at date $t = 0$ as a function of its entry time $t_1 \ge 0$ is:

$$V(t_1) = \int_0^{t_1} f(\tau)\max(0, \int_\tau^\infty \Pi(s,\tau)e^{-rs}ds - Ke^{-r\tau})d\tau + \int_{t_1}^\infty f(\tau)(\int_{t_1}^\infty \Pi(s,\tau)e^{-rs}ds - Ke^{-rt_1})d\tau$$

If the firm chooses to wait until t_1, then, for realizations of τ smaller than t_1, it seizes the payoff to immediate investment at τ if and only if it is positive. In contrast, if the firm ends up entering at t_1 while the profit cycle is growing, then its expects to gain a certain payoff that is contingent on the information gathered by the firm until time t_1 (namely, that τ must be greater than t_1). By another application of the lemma, $V(t_1)$ can be rewritten as follows:

$$V(t_1) = \begin{cases} \int_{t_1}^\infty f(\tau)(\int_{t_1}^\infty \Pi(s,\tau)e^{-rs}ds - Ke^{-rt_1})d\tau & \text{if } t_1 \in [0, t^{\max}) \\ \int_{t^{\max}}^{t_1} f(\tau)(\int_\tau^\infty \Pi(s,\tau)e^{-rs}ds - Ke^{-r\tau})d\tau + \int_{t_1}^\infty f(\tau)(\int_{t_1}^\infty \Pi(s,\tau)e^{-rs}ds - Ke^{-rt_1})d\tau & \text{else} \end{cases}$$

$V(t_1)$ can be easily shown to be continuously differentiable. Therefore, unlike conventional real options analysis, in which expected payoff functions depend on Ito processes and thereby are not differentiable in the classical sense, the firm's optimization program can be solved using standard differentiation techniques:[7]

$$\max_{t_1 \geq 0} V(t_1)$$
$$\text{s.t.}\, V(t_1) > 0$$

Before examining the firm's optimal decision rule when the market is growing, let us introduce some notation. In particular, let $\lambda(t) = \frac{f(t)}{\int_t^\infty f(\tau)d\tau}$ denote the hazard rate, that is, the probability of the market immediately reaching its maturity at date t given that this event has not occurred previously. To ensure that $V(t_1)$ is single-peaked, we also assume that the environment is such that $\alpha + 2r \geq \frac{f'(t)}{f(t)}$ for all $t \geq 0,$ which automatically holds if the density is decreasing.[8] We can now characterize the firm's optimal investment rule:

[7] It is worth remarking that the optimal entry time to be derived will not change as time goes by and no event occurs. In particular, let $V(t_1|t)$ be the value of the firm if it chooses to invest at t_1 (conditional on the market still expanding), given its current information at $t \geq 0.$ Note that, conditional upon the cycle still growing at $t,$ we have that $V(t_1|t) = V(t_1|0)e^{rt} / \Pr(t \leq \tau),$ so the set of maximizers of $V(t_1|t)$ coincides with that of $V(t_1|0),$ and there is no loss of generality in letting $t = 0.$ Intuitively, at date 0 the firm already takes into account the underlying Bayesian updating process of which it can benefit just by delaying investment. As a result, it anticipates having better information at the time of entry if the market keeps on growing.

[8] A decreasing density is implied by a non-increasing hazard rate in the prior distribution over market size, which appears to be empirically supported in the light of the work by Barbarino and Jovanovic (2004). At a theoretical level, a decreasing hazard rate follows if the true hazard rate is constant but unknown to the firm, which can update its beliefs in a Bayesian fashion as time goes by (see, e.g, Choi 1991, footnote 9). Familiar probability distributions with non-increasing hazard rate and support $[0, \infty)$ (other than the exponential)

Proposition 1: *The firm's optimal entry rule is "enter at* $t^e = \min\{t_1 \geq 0 : W(t_1) \leq 0\}$ *if* $t^e < \tau$; *else do not enter," where* $W(t_1) = \dfrac{K(\alpha + r)}{\pi} - e^{\alpha t_1}\left(1 + \dfrac{\alpha}{r + \lambda(t_1)}\right)$.

The following result allows for an economic interpretation of the firm's optimal solution:

Corollary 1: $t^e > 0$ *satisfies* $\pi e^{\alpha t^e} = rK + \lambda(t^e)(K - \int_{t^e}^{\infty} \pi e^{\alpha(2t^e - s)} e^{-r(s - t^e)} ds)$.

In words: at an interior solution, the firm decides to enter at the instant of time such that the marginal value of waiting equals the marginal cost of delaying entry. The marginal cost is the profit flow forgone by waiting dt, $\pi e^{\alpha t^e} dt$, while the marginal value is the part of sunk cost saved by delaying entry plus the marginal option value of waiting and avoiding an irreversible action. The latter value stems from the "bad news principle of irreversible investments," which can be found in Bernanke (1983). According to this principle, the firm must only care about the bad news that may arrive in the next instant of time when deciding whether or not to undertake an irreversible project.[9] Thus, the firm believes there is some positive probability that the market may suddenly start declining right after time t^e (this is

include the gamma, log-logistic, Weibull and F distributions for certain values of the parameters that define them. However, neither the density nor the hazard rate need to be non-increasing for the assumption to hold. For example, the density is decreasing even though the hazard rate is monotone increasing in Bass (1969) if $q \leq p$. Additional examples of common random variables with non-monotonic densities that may satisfy the requirement include the lognormal and the Gompertz (for certain parameter values), among several others.

[9] Irreversibility yields no advantages but implies some costs that arise because the firm cannot recoup its investment if conditions turn out to be adverse, which creates the asymmetry that the firm cares about adverse events (which would not be regrettable were investment reversible) but not favorable ones.

the bad news). Hence, waiting allows it to avoid making a negative payoff with probability $\lambda(t^e)dt$.[10] Overall, total marginal value is equal to:

$$rKdt + \lambda(t^e)dt(K - \int_{t^e}^{\infty} \pi e^{\alpha(2t^e - s)} e^{-r(s-t^e)} ds).$$

Note from the second term that waiting allows the firm to update its beliefs about the maturity date in a Bayesian fashion. By delaying entry, the firm benefits from learning what some events cannot be (via the denominator of the hazard rate; see its definition), thus allowing for a better assessment of the probabilities of still-not-occurred-events.[11] At the same time, waiting implies that the firm faces a different ex ante probability of the market immediately reaching its maturity (via the numerator of the hazard rate).

4. IMPACT OF GREATER UNCERTAINTY ON ENTRY TIMING

One of the standard predictions of real options models is that higher uncertainty delays the optimal time of investment. There are some exceptions in certain contexts, though, as for example in Dixit and Pindyck (1994, pp. 370-372).

We now show that the effect on investment timing of a greater spread is ambiguous when the payoff to the firm crucially depends on the unknown arrival time of an

[10] By Lemma 1, the payoff if the firm invests at date t is negative if demand suddenly decays $\forall\, t < t^{\max}$. In particular, for $t^e < t^{\max}$ (which always holds, as shown in the proof of Proposition 1).

[11] The reason being that conditioning reduces the outcome space, which enhances the firm's information set. This contrasts with Roberts' and Weitzman's (1981) model of staged investment, in which the value of a project is also unknown to the company but the firm can reduce uncertainty by going ahead in a sequential fashion. In our stylized model, unlike theirs, information gathering does not require an earlier investment. Rather, it requires waiting for information to arrive.

unfavorable event. In particular, we give a necessary and sufficient condition for a mean-preserving increase in the spread (MPIS) to shorten the optimal time of entry whenever $t^e > 0$. Since we are to compare members of a family of distribution functions based on (small) differences in the spread, with the restriction that their means be the same, we assume for the remainder of the paper that the density can be parameterized by σ^2, i.e., $f(\tau|\sigma^2)$. σ^2 is a parameter of increasing risk, so we will usually call it the variance of the random variable. As a result, the hazard rate is also a function of σ^2, $\lambda(t|\sigma^2)$, and, for convenience, we assume that it is differentiable in both arguments, denoting partial derivatives by subscripts. Lastly, notice that $W(t^e,\sigma^2) = \dfrac{K(\alpha+r)}{\pi} - e^{\alpha t^e}\left(1+\dfrac{\alpha}{r+\lambda(t^e|\sigma^2)}\right)$ is a continuously differentiable function on the neighborhood of any pair (t_0^e,σ_0^2) such that $W(t_0^e,\sigma_0^2)=0$. Then we can establish the following:

Proposition 2: *Consider a family of distributions that can be parameterized by* σ^2, *a parameter such that a rise in it represents an MPIS. Then increasing* σ^2 *hastens investment if and only if* $\lambda_{\sigma^2}(t_0^e|\sigma_0^2) < 0$.

Let us assume for expositional purposes that $f(\tau|\cdot)$ is a continuously differentiable function $\forall\,\tau$, so that $\lambda_{\sigma^2}(t_0^e|\sigma_0^2) < 0$ if and only $f_{\sigma^2}(t_0^e|\sigma_0^2) + \lambda(t_0^e|\sigma_0^2)\int_0^{t_0^e} f_{\sigma^2}(\tau|\sigma_0^2)d\tau < 0$. Hence, an MPIS has two distinct effects on optimal investment timing. On the one hand, it has an impact on the unconditional probability of immediate decay after investing, $f(t_0^e|\sigma_0^2)$. At an intuitive level, the firm tends to speed up entry if the MPIS decreases the (ex ante) probability of making losses immediately after investing. On the other hand, there is an additional impact on the probability that the firm does not end up investing because of a low realization of τ, $\int_0^{t_0^e} f(\tau|\sigma_0^2)d\tau$. Intuitively, the firm is more inclined to hasten investment if the probability of a high realization that would allow it to enter increases.

If $\lambda_{\sigma^2}(t_0^e|\sigma_0^2) < 0$,[12] the marginal option value of waiting –that is, the cost of irreversibility– would be reduced due to the decrease of the weight put on the losses avoided by delaying entry. Such result is in stark contrast with the standard conclusion from the real options literature, according to which an MPIS uniformly increases risk at every point in time, which in turn increases the marginal value of delaying entry and thus delays investment. In our setup with a single random variable, an MPIS does not have this property,[13] as exemplified by the following numerical example. In particular, let us assume

[12] It is worth remarking that the restrictions that an MPIS imposes on the cumulative distribution function are consistent with $\lambda_{\sigma^2}(t_0^e|\sigma_0^2) < 0$. A sketch of the proof is as follows. Let $H(t) = \int_0^t \int_0^s f_{\sigma^2}(\tau|\sigma_0^2)d\tau ds$. Then an MPIS requires $H(t) \geq 0 \,\forall\, t$, with $H(0) = H(\infty) = 0$. Note that $H(t)$ is twice continuously differentiable if $f_{\sigma^2}(\cdot|\sigma_0^2)$ is continuous, with $H^{'}(t) = \int_0^t f_{\sigma^2}(\tau|\sigma_0^2)d\tau$ and $H^{"}(t) = f_{\sigma^2}(t|\sigma_0^2)$. Since $\lambda_{\sigma^2}(t|\sigma_0^2) < 0$ if and only if $f_{\sigma^2}(t|\sigma_0^2) + \int_0^t f_{\sigma^2}(\tau|\sigma_0^2)d\tau < 0$, it suffices to find a non-empty region in which $H(t)$ is both decreasing and convex. The properties that $H(0) = H(\infty) = 0$ and $H(t) \geq 0$, together with the continuous differentiability of the function, yield such result for sufficiently large t, as can be readily seen graphically.

[13] A uniform increase in the hazard rate when augmenting σ^2 would delay the optimal entry time. Yet, this would call for something other than an MPIS. Namely, "a distributional upgrade," as defined by Arozamena and Cantillon (2004), which is derived from a notion of first-order conditional stochastic dominance, rather than one of second order stochastic dominance. Therefore, not only risk would be affected when varying σ^2.

that τ follows a gamma distribution with parameters $\gamma > 0$ and $\rho > 0$: $\tau \sim Gamma(\gamma, \rho)$. Recall that $E(\tau) = \frac{\rho}{\gamma}$ and $Var(\tau) = \frac{\rho}{\gamma^2}$, which means that we can perform an MPIS by simply multiplying both γ and ρ by any positive scalar smaller than 1. In addition, let the firm be a monopolist facing a linear demand with intercept $a = 50$ and slope $b = 1$. If costs are assumed to be zero, then it is well-known that $\pi = \frac{50^2}{4}$. Finally, let $K = 7000$ and $r = \alpha = 5\%$. Considering that the hazard rate of the gamma is non-increasing –and, hence, the density is decreasing– if $\rho \leq 1$, we have the following results, summarized in Table 1:

ρ	γ	$\boldsymbol{E(\tau)}$	$\boldsymbol{Var(\tau)}$	$\boldsymbol{t^e}$
0.8	0.4	2	5	0.58
0.5	0.25	2	8	0.56
0.1	0.05	2	40	0.2

Table 1: Greater uncertainty speeds up entry

5. IMPACT OF GREATER UNCERTAINTY ON OPTION VALUES

Traditional real options theory predicts that an increase in uncertainty does not harm the value of a firm's investment opportunity (see Dixit and Pindyck 1994, Chapters 5, 6, or Trigeorgis 1996, Chapters 4, 7, 11). We next study conditions under which this may not happen in our setting. More precisely, we perform a comparative static analysis of the impact of an MPIS on the value of the investment opportunity.

First note that $t^e = 0$ if and only if $\pi \geq \pi^e = \frac{K(\alpha + r)(r + f(0))}{\alpha + r + f(0)}$, since $\lim_{t \downarrow 0} \lambda(t) = \frac{\lim_{t \downarrow 0} f(t)}{\lim_{t \downarrow 0} \int_t^\infty f(\tau) d\tau} = \lim_{t \downarrow 0} f(t) \equiv f(0)$. Given that the density function can be parameterized by σ^2, the threshold that optimally triggers immediate investment at date 0 is also a function of σ^2: $\pi^e(\sigma^2) = \frac{K(\alpha+r)(r+f(0|\sigma^2))}{\alpha+r+f(0|\sigma^2)}$. As a result, the maximal value of the investment opportunity as a function of both π and σ^2 is as follows:

$$V(\pi,\sigma^2) = \begin{cases} \pi\left(\frac{1}{\alpha+r}+\frac{1}{\alpha-r}\right)\int_{t^e(\sigma^2)}^{\infty} e^{(\alpha-r)\tau} f(\tau|\sigma^2)d\tau - & \\ \left(\frac{\pi e^{(\alpha-r)t^e(\sigma^2)}}{\alpha-r}+Ke^{-rt^e(\sigma^2)}\right)\int_{t^e(\sigma^2)}^{\infty} f(\tau|\sigma^2)d\tau & \text{if } \pi \geq \pi^e(\sigma^2) \\ \pi\left(\frac{1}{\alpha+r}+\frac{1}{\alpha-r}\right)\int_0^{\infty} e^{(\alpha-r)\tau} f(\tau|\sigma^2)d\tau - \left(\frac{\pi}{\alpha-r}+K\right) & \text{else} \end{cases}$$

As readily seen from this expression, comparative statics are slightly complicated, for the value of the investment opportunity is a piecewise differentiable function of π, and the non-differentiability point π^e depends on σ^2. For this reason, we make a mild assumption that can be relaxed for some specific probability distributions that do not satisfy it. In particular, we assume from now on that $f(0|\sigma^2)$ does not increase when performing an MPIS,[14] which allows us to characterize some relevant properties of $\pi^e(\sigma^2)$:

Lemma 2: $\pi^e(\sigma^2)$ *is a continuous and non-decreasing function with range bounded by the interval* $[rK,(\alpha+r)K]$.

Lemma 2 states that $\pi^e(\sigma^2)$ is non-decreasing in σ^2, so an MPIS usually makes investment at $t=0$ more difficult as happens in traditional real options models. Yet, note that, as Proposition 2 shows, investment timing need not be delayed, because the firm may not wish to invest at $t=0$ even for a low variance. Let $F(\tau|\sigma^2)$ denote the cumulative

[14] For example, the lognormal distribution satisfies such assumption, since $f(0|\sigma^2)=0\ \forall\, \sigma^2$.

distribution function of τ, and let us impose the regularity condition that $\lim_{\tau\to\infty} e^{(\alpha-r)\tau} F_{\sigma^2}(\tau|\sigma^2) = 0.$[15] Then we can establish a necessary and sufficient condition for an MPIS not to reduce the value of the firm's investment opportunity:

Proposition 3: *An MPIS does not reduce the value of the investment opportunity if and only if* $2(r+\lambda(t_0^e|\sigma_0^2))\int_{t_0^e}^{\infty} e^{(\alpha-r)(\tau-t_0^e)} F_{\sigma^2}(\tau|\sigma_0^2)d\tau \leq \int_0^{t_0^e} f_{\sigma^2}(\tau|\sigma_0^2)d\tau.$

Intuitively, the usual asymmetry due to the fact that the firm can benefit from upside risk without being affected by downside risk is no longer present in this situation. The ex post value of the investment opportunity is not convex in the realization of the random variable unless $t^e = 0.$ Indeed, it is not a continuous function of τ, as illustrated by Figure 1. It is this fact that makes the results differ from conventional setups, noting that the envelope theorem implies that a change in t^e due to the MPIS has a negligible effect on the firm's maximal payoff. In particular, the outcome space coincides with that of t, which allows the discontinuity to appear. The firm does not invest when its worst-scenario payoff is 0 (the only way to prevent the discontinuity from arising), but rather when its expected payoff is maximal, that is, at t^e. Yet, the firm faces risk of losses if the maturity of the market arrives shortly after investing (since $t^e < t^{\max}$ if $t^e > 0$), as readily seen in Figure 1(a). It is this risk of losses that explains why increasing the spread may partially destroy option values.

[15] For instance, this property is automatically satisfied if $\alpha \leq r$.

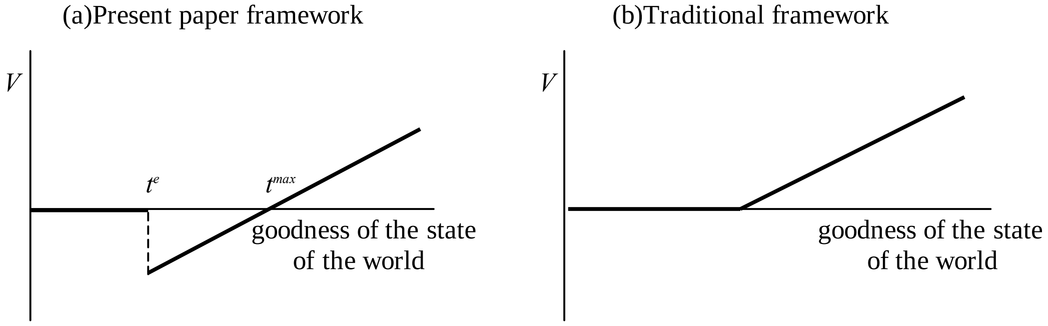

Figure 1: Ex post value of the investment opportunity

As a result, the firm is insured against bad realizations because of waiting, and definitely takes advantage of good realizations, but is damaged by realizations sufficiently close to $t^e > 0$. The overall effect of greater uncertainty is thus ambiguous. In particular, Proposition 3 shows that if an MPIS increases the probability that the firm does not invest (i.e., $\int_0^{t_0^e} f_{\sigma^2}(\tau|\sigma_0^2)d\tau \geq 0$), then the value of the investment opportunity does not decrease (since the left hand side of the expression in Proposition 3 is always non-positive,[16] while the right hand side would be positive). The point is that the firm would lose nothing if $\tau < t_0^e$ and would take advantage of a smaller probability of decay on dates following its entry. Matters may be different, though, if the probability that the firm invests increases when the spread is augmented, since $\int_0^{t_0^e} f_{\sigma^2}(\tau|\sigma_0^2)d\tau$ could be negative and very small indeed. In such a case, the probability of the market maturity date occurring at some date between t_0^e and $t^{\max}$ (and thus the probability of making losses) may increase so much so as to destroy part of the value of the option to invest.

[16] See Tirole (1988, p. 397, exercise 10.6) or Mas-Colell, Whinston and Green (1995, p. 198, expressions 6.D.1 and 6.D.2).

Continuing with the example of Section 4, we next provide a numerical illustration that greater uncertainty harms option values. Since a gamma distribution has $f(0|\sigma^2)=\infty$ $\forall\,\sigma^2$ if and only if its hazard rate is decreasing, we have the results depicted in Table 2.

ρ	γ	$\boldsymbol{E(\tau)}$	$\boldsymbol{Var(\tau)}$	$\boldsymbol{V}$
0.8	0.4	2	5	560
0.5	0.25	2	8	627
0.1	0.05	2	40	598

Table 2: Greater uncertainty harms option values

6. CONCLUSION

This paper has focused on investment contexts in which the only source of uncertainty affecting the value of a project stems from the unknown date of arrival of an unfavorable event. We have not modeled uncertainty over time by the means of a full-fledged stochastic process, and we have assumed under fairly general conditions that the date of occurrence of the event is a single random variable. In particular, it need not be exponentially distributed, which would rule out Poisson processes, since they are not well suited for certain analyses.

The consideration of unknown-date events as a source of uncertainty dramatically affects some of the core conclusions that characterize conventional real options theory. More specifically, we have identified a necessary and sufficient condition for the value of an investment opportunity to decrease with uncertainty. This would occur because the time space –which is the control space– and the outcome space –which is the state space– coincide. Consequently, waiting allows the firm to be insured against adverse states of the world while taking advantage of favorable states. The firm cannot avoid being damaged (and thus making losses) if the adverse event occurs shortly after investing, though. The reason is that the firm chooses to invest when expected net present value is maximal, not when downside risk vanishes, and thus greater uncertainty may increase the probability of occurrence of the adverse event right after investing by a sufficiently large amount.

In addition, we have shown that greater uncertainty may speed up entry timing in certain situations. Pinpointing the conditions under which this occurs is certainly relevant for empirical work on the investment-uncertainty relationship. In our model, the firm anticipates that it will have better information at the time of investment, so at the margin, it only cares about the conditional probability of immediate arrival of the unfavorable event. Greater uncertainty may reduce this probability of receiving bad news, thus decreasing the cost of making an irreversible investment, which would hasten entry. Intuitively, the firm would face a smaller (ex ante) probability of making losses immediately after investing and/or a larger probability of being able to invest because of favorable market conditions.

Lastly, we have set up an alternative theoretical framework for continuous-time real options models of investment whenever the demand of a product follows a life cycle that is unknown to the firm.[17] The setting is simpler than that proposed by Bollen (1999), and is a potentially useful building block for issues that we do not deal with, such as the option to add or contract capacity,[18] R&D investment opportunities that open up the option to enter new markets that are expected to evolve as the one described, or game-theoretic setups.

[17] Such a model may be relevant based on empirical evidence. Thus, Bowman and Moskowitz (2001, p. 775) suggest that one of the mistakes made by Merck when valuing its Project Gamma was the use of the Black-Scholes formula, instead of taking into account that some biotechnology products follow a life cycle.

[18] In particular, Bollen (1999) examines these situations numerically. He shows that traditional real options models tend to overvalue (undervalue) the option to expand (respectively, contract) a project.

APPENDIX

Proof of Lemma 1: Since $NPV(t) = \int_t^{\infty} \pi e^{2\alpha\tau} e^{-(\alpha+r)s} ds - Ke^{-rt}$ is strictly quasi-convex on $(-\infty,+\infty)$,[19] it follows that, for all $\tau < t' < t''$, $NPV(t') < \max(NPV(\tau), NPV(t''))$. Taking the limit as $t'' \to \infty$, $NPV(t') < \max(NPV(\tau),0)$. Finally, note that $NPV(\tau) > 0 \Leftrightarrow \tau > t^{\max} = \max\left\{0, \frac{1}{\alpha}\log\left(\frac{K(\alpha+r)}{\pi}\right)\right\}$, which completes the proof.■

Proof of Proposition 1: We first show that $V(t_1)$ is monotone decreasing on $[t^{\max}, \infty)$. On this set, the function becomes:

$$V(t_1) = \int_{t^{\max}}^{t_1} f(\tau)(\int_{\tau}^{\infty} \Pi(s,\tau)e^{-rs} ds - Ke^{-r\tau})d\tau + \int_{t_1}^{\infty} f(\tau)(\int_{t_1}^{\infty} \Pi(s,\tau)e^{-rs} ds - Ke^{-rt_1})d\tau =$$
$$\int_{t^{\max}}^{t_1} f(\tau)(\int_{\tau}^{\infty} \pi e^{2\alpha\tau} e^{-(\alpha+r)s} ds - Ke^{-r\tau})d\tau + \int_{t_1}^{\infty} f(\tau)(\int_{t_1}^{\tau} \pi e^{(\alpha-r)s} ds + \int_{\tau}^{\infty} \pi e^{2\alpha\tau} e^{-(\alpha+r)s} ds - Ke^{-rt_1})d\tau.$$

Differentiating with respect to t_1 and performing some algebraic manipulations yields:

$$\frac{dV(t_1)}{dt_1} \equiv V'(t_1) = (rK - \pi e^{\alpha t_1})e^{-rt_1} \int_{t_1}^{\infty} f(\tau)d\tau.$$

We claim that $V'(t_1) < 0$ for $t_1 \geq t^{\max}$. Otherwise, we would reach a contradiction:

$$0 \leq (rK - \pi e^{\alpha t_1})e^{-rt_1} \int_{t_1}^{\infty} f(\tau)d\tau \leq -\alpha K e^{-rt_1} \int_{t_1}^{\infty} f(\tau)d\tau < 0,$$

since $\frac{\pi e^{\alpha t_1}}{\alpha + r} \geq \frac{\pi e^{\alpha t^{\max}}}{\alpha + r} \geq K$ for $t_1 \geq t^{\max}$. Assumption 4 implies that $V(t_1)$ is bounded above, which shows that $V(t_1)$ attains a unique global maximum when $t^{\max} = 0$, so let $t^{\max} > 0$

[19] Formally, because $\frac{dNPV(t)}{dt} = 0 \Rightarrow \frac{d^2 NPV(t)}{dt^2} > 0.$

and note that, to conclude the proof, we can restrict our attention to the set $[0,t^{\max}]$. In this case, $V(t_1)$ is as follows:

$$V(t_1)=\int_{t_1}^{\infty} f(\tau)(\int_{t_1}^{\tau}\pi e^{(\alpha-r)s}ds+\int_{\tau}^{\infty}\pi e^{2\alpha\tau}e^{-(\alpha+r)s}ds-Ke^{-rt_1})d\tau.$$

By Assumption 4, this function is bounded above. Differentiating it with respect to t_1, solving the integrals, taking into account that $\lambda(t_1)=\frac{f(t_1)}{\int_{t_1}^{\infty} f(\tau)d\tau}$ and rearranging yields:

$$\begin{aligned}\frac{dV(t_1)}{dt_1}\equiv V'(t_1)&=-\frac{\pi}{\alpha+r}e^{(\alpha-r)t_1}f(t_1)-\pi e^{(\alpha-r)t_1}\int_{t_1}^{\infty} f(\tau)d\tau+Ke^{-rt_1}(f(t_1)+r\int_{t_1}^{\infty} f(\tau)d\tau)\\&=\left(-\frac{\pi}{\alpha+r}e^{\alpha t_1}\lambda(t_1)-\pi e^{\alpha t_1}+K(r+\lambda(t_1))\right)\int_{t_1}^{\infty} e^{-rt_1}f(\tau)d\tau\end{aligned}$$

As $\frac{\alpha+r}{\pi(r+\lambda(t_1))\int_{t_1}^{\infty} e^{-rt_1}f(\tau)d\tau}>0,$ $signV'(t_1)=signW(t_1),$ so it suffices to show that $W(\cdot)$ is a decreasing function in order to prove that a unique global maximum is attained at $t^e=\min\{t_1\geq 0: W(t_1)\leq 0\}$. The assumption that $\alpha+2r\geq\frac{f'(t_1)}{f(t_1)}$ for all t_1 implies that

$$\left[\frac{f'(t_1)}{f(t_1)}-(\alpha+2r)\right]\lambda(t_1)\leq 0<r(\alpha+r)\Rightarrow$$

(A.1) $$(r+\lambda(t_1))(\alpha+r+\lambda(t_1))>\frac{\lambda(t_1)f'(t_1)}{f(t_1)}+[\lambda(t_1)]^2=\lambda'(t_1).$$

As a result, it follows that $W'(t_1)=-\frac{\alpha e^{\alpha t_1}[(r+\lambda(t_1))(\alpha+r+\lambda(t_1))-\lambda'(t_1)]}{(r+\lambda(t_1))^2}<0.$[20]

[20] If $\alpha=r$, we should derive the functional form of $V(t_1)$ from scratch, as directly plugging in $\alpha=r$ implies that $V(t_1)$ is not well defined. Yet, some calculations show that the same expression obtains for computing t^e.

It is also clear that $V(t^e) > 0,$ since $\lim_{t_1 \to \infty} V(t_1) = 0,$ and $V(t_1)$ is single-peaked. Finally, simple manipulations show that $t^e < t^{\max}$ if $t^e > 0,$ so the firm does not enter if the maturity date of the market turns out to be smaller than t^e (by Lemma 1).■

Proof of Corollary 1: For $t^e > 0,$ the following holds: $W(t^e) = 0.$ So straightforward manipulations after multiplying such expression through by $\dfrac{\pi(r + \lambda(t^e))}{\alpha + r},$ and noticing that $\dfrac{\pi e^{\alpha t^e}}{\alpha + r} = \int_{t^e}^{\infty} \pi e^{\alpha(2t^e - s)} e^{-r(s - t^e)} ds$ imply that $\pi e^{\alpha t^e} = rK + \lambda(t^e)(K - \int_{t^e}^{\infty} \pi e^{\alpha(2t^e - s)} e^{-r(s-t^e)} ds).$■

Proof of Proposition 2: Differentiate $W(t^e, \sigma^2)$ with respect to t^e and σ^2 and rearrange so that:

$$\frac{dt^e}{d\sigma^2}(\sigma_0^2) = -\frac{e^{\alpha t_0^e}\left(\dfrac{-\alpha\lambda_{\sigma^2}(t_0^e|\sigma_0^2)}{(r+\lambda(t_0^e|\sigma_0^2))^2}\right)}{\alpha e^{t_0^e}\left(\dfrac{(r+\lambda(t_0^e|\sigma_0^2))^2 + \alpha(r+\lambda(t_0^e|\sigma_0^2)) - \lambda_t(t_0^e|\sigma_0^2)}{(r+\lambda(t_0^e|\sigma_0^2))^2}\right)} = \frac{\lambda_{\sigma^2}(t_0^e|\sigma_0^2)}{((r+\lambda(t_0^e|\sigma_0^2))^2 + \alpha(r+\lambda(t_0^e|\sigma_0^2)) - \lambda_t(t_0^e|\sigma_0^2)}.$$

Since $(\alpha + r + \lambda(t_0^e|\sigma_0^2))(r + \lambda(t_0^e|\sigma_0^2)) > \lambda_t(t_0^e|\sigma_0^2)$ by expression (A.1), we have that $sign\left(\dfrac{dt^e}{d\sigma^2}(\sigma_0^2)\right) = sign\left(\lambda_{\sigma^2}(t_0^e|\sigma_0^2)\right),$ which completes the proof.■

Proof of Lemma 2: First note that $\pi^e(\sigma^2)$ is a well-defined function (and not a correspondence) by the uniqueness of t^e for a given σ^2. Indeed, it is continuous by the continuity of $f(\cdot|\sigma^2)\,\forall\,\sigma^2$. Furthermore, if an MPIS is performed so that the variance infinitesimally rises from σ_0^2 to σ_1^2, then $f(0|\sigma_1^2) \geq f(0|\sigma_0^2)$, so $\pi^e(\sigma_1^2) \geq \pi^e(\sigma_0^2),$ and thus $\pi^e(\sigma^2)$ is non-decreasing. Finally, note that $f(0|\sigma^2)$ can be neither smaller than 0 nor larger than ∞, which, together with non-decreasingness of $\pi^e(\sigma^2)$, implies that the range of the function must be bounded by $[rK,(\alpha+r)K]$. ■

Proof of Proposition 3: We proceed to prove the statement of the proposition for three different cases:

(i) If $\pi \geq \lim_{\sigma^2\to\infty} \pi^e(\sigma^2)$, then the firm's optimal time of entry is $t^e(\sigma^2)=0\ \forall\ \sigma^2$,[21] so:

$$V(\sigma_0^2)=\pi\left(\frac{1}{\alpha+r}+\frac{1}{\alpha-r}\right)\int_0^\infty e^{(\alpha-r)\tau} f(\tau|\sigma^2)d\tau-\left(\frac{\pi}{\alpha-r}+K\right).$$

Hence, differentiating yields:

$$V^{'}(\sigma_0^2)\equiv\frac{dV(\sigma_0^2)}{d\sigma^2}=\pi\left(\frac{1}{\alpha+r}+\frac{1}{\alpha-r}\right)\int_0^\infty e^{(\alpha-r)\tau} f_{\sigma^2}(\tau|\sigma^2)d\tau$$

Integrating by parts, considering that $\int_0^\infty f_{\sigma^2}(\tau|\sigma_o^2)d\tau=0$ and $F_{\sigma^2}(\tau|\sigma_0^2)=\int_0^\tau f_{\sigma^2}(s|\sigma_0^2)ds$ when an MPIS is performed, and noticing that $\int_0^\infty e^{(\alpha-r)\tau}F_{\sigma^2}(\tau|\sigma^2)$ is well-defined we have:

$$\begin{aligned}V^{'}(\sigma_0^2)&=\pi\left(\frac{1}{\alpha+r}+\frac{1}{\alpha-r}\right)\left\{\left[e^{(\alpha-r)\tau}\int_0^\tau f_{\sigma^2}(s|\sigma^2)ds\right]_0^\infty-\int_0^\infty(\alpha-r)e^{(\alpha-r)\tau}\int_0^\tau f_{\sigma^2}(s|\sigma^2)dsd\tau\right\}\\&=-\left(\frac{2\alpha\pi}{\alpha+r}\right)\int_0^\infty e^{(\alpha-r)\tau}F_{\sigma^2}(\tau|\sigma^2)d\tau\end{aligned}$$

Hence, taking into account that $t^e(\sigma_0^2)=t_0^e=0$ and thus $\int_0^{t_0^e} f_{\sigma^2}(\tau|\sigma_0^2)d\tau=0$:

[21] This follows from the facts that $\pi>\pi^e(\sigma^2)\ \forall\ \sigma^2$ if and only if the firm invests immediately for all σ^2.

$$V'(\sigma_0^2) \geq 0 \Leftrightarrow \int_0^\infty e^{(\alpha-r)\tau} F_{\sigma^2}(\tau|\sigma_0^2)d\tau \leq 0$$

$$\Leftrightarrow 2(\alpha + \lambda(t_0^e|\sigma_0^2))\int_{t_0^e}^\infty e^{(\alpha-r)(\tau-t_0^e)} F_{\sigma^2}(\tau|\sigma_0^2)d\tau \leq \int_0^{t_0^e} f_{\sigma^2}(\tau|\sigma_0^2)d\tau.$$

This completes the proof when $\pi \geq \lim_{\sigma^2\to\infty} \pi^e(\sigma^2)$.

(ii) If $\pi \leq \lim_{\sigma^2\to 0} \pi^e(\sigma^2)$, then we have that $t^e(\sigma_0^2) > 0 \, \forall \, \sigma_0^2$ by Lemma 2, so:

$$V(\sigma^2) = \pi\left(\frac{1}{\alpha+r} + \frac{1}{\alpha-r}\right)\int_{t^e(\sigma^2)}^\infty e^{(\alpha-r)\tau} f(\tau|\sigma^2)d\tau -$$
$$\left(\frac{\pi e^{(\alpha-r)t^e(\sigma^2)}}{\alpha-r} + Ke^{-rt^e(\sigma^2)}\right)\int_{t^e(\sigma^2)}^\infty f(\tau|\sigma^2)d\tau$$

Given that $\frac{1}{\alpha}\log\left(\frac{rK}{\pi}\right) \leq t^e(\sigma^2) \leq \frac{1}{\alpha}\log\left(\frac{(\alpha+r)K}{\pi}\right) \forall \, \sigma^2$ (this trivially follows from Proposition 1), $t^e(\sigma^2)$ is a continuous function by the theorem of the maximum. Indeed, we have that $t^e(\sigma^2)$ is differentiable on a local neighborhood of σ_0^2 by the implicit function theorem, and hence the envelope theorem implies that effects of σ^2 on $V(\sigma^2)$ via $t^e(\sigma^2)$ are of second-order, so letting $t^e(\sigma_0^2) = t_0^e$, we have:

$$V'(\sigma_0^2) = \pi\left(\frac{1}{\alpha+r} + \frac{1}{\alpha-r}\right)\int_{t_0^e}^\infty e^{(\alpha-r)\tau} f_{\sigma^2}(\tau|\sigma_0^2)d\tau - \left(\frac{\pi e^{(\alpha-r)t_0^e}}{\alpha-r} + Ke^{-rt_0^e}\right)\int_{t_0^e}^\infty f_{\sigma^2}(\tau|\sigma_0^2)d\tau$$

Integration by parts noting that $\int_0^\infty e^{(\alpha-r)\tau} F_{\sigma^2}(\tau|\sigma^2)$ is finite and recalling that $\int_0^\infty f_{\sigma^2}(\tau|\sigma_0^2)d\tau = 0$ and $F_{\sigma^2}(\tau|\sigma_0^2) = \int_0^\tau f_{\sigma^2}(s|\sigma_0^2)ds$ when an MPIS is performed, yields:

$$V'(\sigma_0^2) = -\left(\frac{\pi e^{(\alpha-r)t_0^e}}{\alpha-r} + Ke^{-rt_0^e}\right)\int_{t_0^e}^{\infty} f_{\sigma^2}(\tau|\sigma_0^2)d\tau +$$

$$\pi\left(\frac{1}{\alpha+r}+\frac{1}{\alpha-r}\right)\left\{\left[e^{(\alpha-r)\tau}\int_0^{\tau} f_{\sigma^2}(s|\sigma_0^2)ds\right]_{t_0^e}^{\infty} - \int_{t_0^e}^{\infty}(\alpha-r)e^{(\alpha-r)\tau}\int_0^{\tau} f_{\sigma^2}(s|\sigma_0^2)dsd\tau\right\} =$$

$$\left(\frac{\pi e^{(\alpha-r)t_0^e}}{\alpha+r}+\frac{\pi e^{(\alpha-r)t_0^e}}{\alpha-r}-\frac{\pi e^{(\alpha-r)t_0^e}}{\alpha+r} - Ke^{-rt_0^e}\right)\int_{t_0^e}^{\infty} f_{\sigma^2}(\tau|\sigma_0^2)d\tau - \left(\frac{2\alpha\pi}{\alpha+r}\right)\int_{t_0^e}^{\infty} e^{(\alpha-r)\tau}F_{\sigma^2}(\tau|\sigma^2)d\tau =$$

$$\left(\frac{\pi e^{(\alpha-r)t_0^e}}{\alpha+r} - Ke^{-rt_0^e}\right)\int_{t_0^e}^{\infty} f_{\sigma^2}(\tau|\sigma_0^2)d\tau - \left(\frac{2\alpha\pi}{\alpha+r}\right)\int_{t_0^e}^{\infty} e^{(\alpha-r)\tau}F_{\sigma^2}(\tau|\sigma^2)d\tau.$$

Hence, we have that

$$V'(\sigma_0^2) \geq 0 \Leftrightarrow \left(\frac{2\alpha\pi}{\alpha+r}\right)\int_{t_0^e}^{\infty} e^{(\alpha-r)\tau}F_{\sigma^2}(\tau|\sigma_0^2)d\tau \leq \left(-\frac{\alpha\pi e^{(\alpha-r)t_0^e}}{(\alpha+r)(r+\lambda(t_0^e|\sigma_0^2))}\right)\int_{t_0^e}^{\infty} f_{\sigma^2}(\tau|\sigma_0^2)d\tau$$

$$\Leftrightarrow 2(r+\lambda(t_0^e|\sigma_0^2))\int_{t_0^e}^{\infty} e^{(\alpha-r)(\tau-t_0^e)}F_{\sigma^2}(\tau|\sigma_0^2)d\tau \leq -\int_{t_0^e}^{\infty} f_{\sigma^2}(\tau|\sigma_0^2)d\tau$$

$$\Leftrightarrow 2(r+\lambda(t_0^e|\sigma_0^2))\int_{t_0^e}^{\infty} e^{(\alpha-r)(\tau-t_0^e)}F_{\sigma^2}(\tau|\sigma_0^2)d\tau \leq \int_0^{t_0^e} f_{\sigma^2}(\tau|\sigma_0^2)d\tau$$

where first we have used the first-order condition of the optimization program and finally we have used the fact that $\int_0^{\infty} f_{\sigma^2}(\tau|\sigma_0^2)d\tau = 0$.

(iii) If $\lim_{\sigma^2\to\infty}\pi^e(\sigma^2) > \pi \geq \lim_{\sigma^2\to 0}\pi^e(\sigma^2)$, define $\Sigma(\pi) \equiv \{\sigma^2 : \pi^e(\sigma^2) = \pi\}$. Since $\pi^e(\sigma^2)$ is continuous and non-decreasing by Lemma 2, then $\Sigma(\pi)$ must be convex (perhaps a singleton), so the following facts clearly hold: (1) case (i) applies $\forall\,\sigma^2 < \inf\Sigma(\pi)$; (2) case (ii) applies $\forall\,\sigma^2 \geq \inf\Sigma(\pi)$; and (3) $t^e(\sigma_0^2)\downarrow 0$ as $\pi \uparrow \pi^e(\sigma^2)$, so $t^e(\sigma^2)$ is a continuous function (since $\pi^e(\sigma^2)$ is continuous). This shows that, although $V(\sigma^2)$ need not be differentiable at $\inf\Sigma(\pi)$, it is clearly continuous and non-decreasing under the condition stated in the proposition.■

REFERENCES

Arozamena, L. and E. Cantillon, "Investment Incentives in Procurement Auctions," *Review of Economic Studies*, 71 (2004), 1-18.

Barbarino, A. T. and B. Jovanovic, "Shakeouts and Market Crashes," *mimeo*, 2004, Dept. of Economics, University of Chicago.

Bass, F.M., "A New Product Growth for Model Consumer Durables," *Management Science*, 15 (1969), 215-227.

Bernanke, B. S., "Irreversibility, Uncertainty and Cyclical Investment," *Quarterly Journal of Economics*, 98 (1983), 85-106.

Bollen, N.P.B., "Real Options and Product Life Cycles," *Management Science*, 45 (1999), 670-684.

Bowman, E.H. and G. T. Moskowitz, "Real Options Analysis and Strategic Decision Making," *Organization Science*, 12 (2001), 772-777.

Choi, J. P., "Dynamic R&D Competition under "Hazard Rate" Uncertainty," *Rand Journal of Economics*, 22 (1991), 596-610.

Dixit, A. K. and R. S. Pindyck, *Investment under Uncertainty* (Princeton: Princeton University Press, 1994).

Ferderer, J.P., "The Impact of Uncertainty on Aggregate Investment Spending: An Empirical Analysis," *Journal of Money, Credit and Banking*, 25 (1993), 30-48.

Guiso, L. and G. Parigi, "Investment and Demand Uncertainty," *Quarterly Journal of Economics*, 114 (1999), 185-227.

Leahy, J.V. and T. M. Whited, "The Effect of Uncertainty on Investment: Some Stylized Facts," *Journal of Money, Credit and Banking*, 28 (1996), 64-83.

Mas-Colell, A., M. D. Whinston and J. R. Green, *Microeconomic Theory* (Oxford: Oxford University Press, 1995).

McDonald, R.L. and D. Siegel, "The Value of Waiting to Invest," *Quarterly Journal of Economics*, 101 (1986), 707-727.

Roberts, K. and M. L. Weitzman, "Funding Criteria for Research, Development and Exploration Projects," *Econometrica*, 49 (1981), 1261-1288.

Tirole, J., *The Theory of Industrial Organization* (Cambridge: MIT Press, 1988).

Trigeorgis, L., *Real Options: Managerial Flexibility and Strategy in Resource Allocation* (Cambridge: MIT Press, 1996).

Weeds, H., "Strategic Delay in a Real Options Model of R&D Competition," *Review of Economic Studies*, 69 (2002), 729-747.