Investment hysteresis under stochastic interest rates

José Carlos Dias and Mark B. Shackleton*

4th February 2005

Abstract

Most decision making research in real options focuses on revenue uncertainty assuming discount rates remain constant. However for many decisions, revenue or cost streams are relatively static and investment is driven by interest rate uncertainty, for example the decision to invest in durable machinery and equipment. Using interest rate models from Cox et al. (1985b), we generalize the work of Ingersoll and Ross (1992) in two ways. Firstly we include real options on perpetuities (in addition to "zero coupon" cash flows). Secondly we incorporate abandonment or disinvestment as well as investment options and thus model interest rate hysteresis (parallel to revenue uncertainty, Dixit (1989a)). Under stochastic interest rates, economic hysteresis is found to be significant, even for small sunk costs.

Key Words: real options, interest rate uncertainty, perpetuities, investment hysteresis.
JEL: G31; D92; D81; C61.

*Dias is with ISCTE and ISCAC, Portugal. Shackleton is with Department of Accounting and Finance at Lancaster University, LA1 4YX, United Kingdom. Tel: +44 (0) 1524 594131, e-mail: m.shackleton@lancaster.ac.uk. Dias gratefully acknowledges the financial support of the PRODEP III Programme. Corresponding author: José Carlos Dias, Department of Finance, Instituto Superior de Contabilidade e Administração de Coimbra, Quinta Agrícola, Bencanta, 3040-316 Coimbra, Portugal. Tel: +351 919337791, e-mail: jdias@iscac.pt.

1 Introduction

In the seminal book of Dixit and Pindyck (1994) it is argued that most major investments share three important characteristics in varying degrees: (1) the investment is partially or completely irreversible. By definition, an investment entails some sunk cost since it cannot be totally recouped if the action is reversed later; (2) there is uncertainty about the future rewards from the investment. Investment decisions are made in an economic environment with ongoing uncertainty, where information arrives gradually; and (3) firms have some leeway about the timing of investment. Usually, an investment opportunity does not disappear if it is not taken immediately. As a result, firms have to decide whether to invest as well as when to invest. It is the interaction between these characteristics that will determine the firm's optimal investment decision. Moreover, when these conditions are met, there is some positive value of waiting for better information. Under these premises, an investment expenditure involves exercising the option to invest, i.e., an option to optimally invest at any time in the future. As a result, the full cost of investment must be the sum of two terms: the cost of investment itself and the opportunity cost value of the lost option.

Investment rules that ignore these characteristics can lead to very wrong answers[1]. For example, McDonald and Siegel (1986) show that the value of this opportunity cost can be large, even for moderate levels of uncertainty. The importance of such characteristics was also made clear by Ross (1995) in his description on the good, the bad, and the ugly of the traditional net present value rule (NPV)[2]. For example, one of the implicit assumptions in the traditional NPV rule is that either the investment decision is reversible, and thus firms can undo some of their investment decisions and recover at least a fraction of the investment costs, or the investment decision is irreversible, in the sense that it is a now or never decision. This

[1]Ignoring these characteristics means that the specific stream of cash flows are completely described in a certain context, that is, the investment projects are treated simply as a stream of cash flows that are exogenously determined without any implicit or explicit contingency admitted. Therefore, managers do not have the ability to affect the cash flows of the project over time. In fact, they do not have anything to manage because flexibility is not taken into account.

[2]The good, the bad, and the ugly of the NPV implies, respectively, rejecting an investment when it should be rejected, rejecting an investment when it should be accepted and accepting an investment when it should be rejected.

means that if the firm does not invest now is not able to do it in the future. Although some investments can be reversible, the majority of them are at least partly irreversible because firms cannot recover all the investment costs. On the contrary, in some cases additional costs of detaching and moving machinery may exist. Since most of the capital expenditures are firm or industry specific they cannot be used in a different firm or different industry. Therefore, they should be considered as largely a sunk cost. But, even if the capital expenditures would not be firm or industry specific, they could not be totally recovered due to the "lemons" problem of Akerlof (1970). Hence, major investment costs are in a large part irreversible.

The irreversible assumption was first introduced by Arrow (1968) and Nickell (1974a,b), among others, but under a certainty framework. Several irreversible investment decision problems with various types of uncertainty were considered afterwards.

Section 2 of this paper describes the uncertain interest rate processes used while Section 3 details the behaviour near the natural zero interest rate boundary. Section 4 develops analytical solutions for the perpetuity as far as possible while Section 5 presents numerical solutions. Section 6 solves the hysteresis problem numerically for the same process as Ingersoll and Ross (1992). Section 7 concludes.

2 Term structure dynamics for spot rates with mean reversion

The first models attempting to describe the term structure dynamics for spot rates did not restrict spot rates to non-negative values, that is, they assumed a positive probability of negative interest rates. For example, Merton (1973a) models the spot rate as an arithmetic Brownian motion and Vasicek (1977) models it as an Ornstein-Uhlenbeck process. Since these processes generate normally distributed interest rates, these can become negative with strictly positive probability. This may make it applicable to real interest rates, but less appropriate for nominal interest rates. Therefore, the applicability of these models to describe spot rate dynamics is limited. Rendleman and Bartter (1980) assume that the instantaneous riskless interest rate follows a geometric Brownian motion[3]. Therefore, they assume that the short-term behavior of

[3]Dothan (1978) also assume that the rate of interest follows a geometric Brownian motion, but with no drift.

interest rates is like the behavior of stock prices under the Black and Scholes (1973) framework. Although this diffusion process precludes negative interest rates (similar to the limited liability effect of stock prices), there is one important difference between interest rates and stock prices. Thus, empirical evidence on interest rate behaviour seems to indicate that interest rates are pulled back to some long-run mean value over time, a phenomenon that is usually called mean reversion[4]. Apart from empirical evidence, this behavior also has economic intuition. For example, Hull (2002, chap. 23) argues that when interest rates are high, the economy tends to slow down and borrowers will require less capital (e.g., mean reversion tends to imply a negative drift) and, as a result, interest rates decline. When interest rates are low, borrowers tend to require more funds (e.g., mean reversion tends to imply a positive drift) and, therefore, the interest rates tend to rise as a result of a high capital demand. Therefore, other models were developed to describe the term structure dynamics for spot rates with mean reversion[5].

The first models to assume that the instantaneous riskless interest rate follows a diffusion process with a reverting mean were Vasicek (1977) and Cox et al. (1985b)[6]. The classical setting in which the term structure dynamics for spot rates follows a mean-reverting diffusion process can be stated as follows:

$$dr_t = \kappa(\theta - r_t)dt + \sigma {r_t}^{\gamma} dW_t \ , \qquad r(0) = r_0 \tag{1}$$

where κ, θ, σ and γ are constants, dW_t is a standard Gauss-Wiener process and it is usually assumed that $\gamma = 0$ or $1/2 \leq \gamma \leq 1$ (see, for example, Geman and Yor (1993)). From this general setting, several models can be analyzed. Chan et al. (1992) presents one of the first attempts to do an empirical comparison of alternative models of the short-term interest rate.

[4]For a more detailed description of the empirical evidence on this issue, see, for example, Campbell et al. (1997, chap. 11) and the references contained therein.

[5]It should be noted that our focus is on the so-called single-factor time-homogeneous models class. This name comes from the fact that in a single-factor equilibrium model, the process for the interest rate involves only one source of uncertainty. Thus, the instantaneous drift and the instantaneous volatility are assumed to be functions of the interest rate, but are independent of time. Different assumptions taken for the instantaneous drift and volatility will, of course, lead to different models. For a more detailed description of this type of models as well as other interest rate models see, for example, Björk (2004) and Musiela and Rutkowski (1998).

[6]It should be noted that although this later paper was published only in 1985, it was already circulating in the academia since 1976 as a working paper.

One of the key results found in their study is that the value of γ is the most important feature differentiating dynamic models of the short-term riskless rate. Moreover, they showed that models which allow $\gamma \geq 1$ can capture the dynamics of the short-term interest rate better than those with $\gamma < 1$. This is because the volatility of the process in the short-run is highly sensitive to the level of the interest rate. Under these premises, they argue that the model used by Cox et al. (1980) to study variable-rate securities, where $\gamma = 3/2$ and the process has no drift, or the model of Dothan (1978) perform better relative to other more well-known models. Of course the lack of mean reversion simplifies term structure models and there is an apparent weak evidence of mean reversion in the short-term rate as it is shown in Chan et al. (1992). In the long-run, however, the empirical evidence on interest rate behaviour seems to indicate that there is a mean reversion effect, (as stated before). Therefore, for optimal investment decisions or capital budgeting problems in a competitive environment under interest rate uncertainty the assumption of no mean reversion of interest rates may not be adequate. Moreover, the conclusion of Chan et al. (1992) that the models of Dothan (1978) and Cox et al. (1980) perform better than the Vasicek or the standard Cox-Ingersoll-Ross square-Gaussian models must be interpreted with caution, since the diffusions of the Dothan and the Cox-Ingersoll-Ross variable-rate models are not ergodic (see Rogers (1995)).

Taking these issues into account, the cases where $\gamma = 0$ and $\gamma = 1/2$ are of primary interest because they lead to analytic solutions, at least for finite maturities[7]. Both cases will result in single-factor models and may be criticized on these grounds. By passing to multi-factor models one should get an improved fit to observed prices, but there is a heavy price to pay since the resulting partial differential equation would have a higher dimension. If our objective were to calculate prices of some interest rate derivatives then other factors could be included in the analysis in order to match observed prices. However, since our focus is on the effects of interest

[7]As it was stated before, one drawback of assuming $\gamma = 0$ is that the short-term interest rate can, in some circumstances, become negative which is not a very appealing feature. The case where $\gamma = 1$ is used by Brennan and Schwartz (1980) to analyze convertible bonds and Courtadon (1982) to price options on default-free bonds. The main advantage of this mean-reverting process (with $\gamma = 1$) is that it presents a boundary at zero. Therefore, it avoids the negative interest rates problem. However, analytical solutions can be found only in very special cases and, thus, numerical methods have to be used since all the analytical tractability is lost. Moreover, as it is suggested by Rogers (1995) none of these two models, that also possess the mean reversion feature, appears to be conclusively superior to the Vasicek (1977) and the Cox et al. (1985b) models.

rate uncertainty on investment decisions a single-factor model of interest rates seems adequate.

2.1 The Vasicek term structure model

If we set $\gamma = 0$ we obtain the one-dimensional Ornstein-Uhlenbeck process that become familiar in finance due to Vasicek (1977). Thus, its short-term structure dynamics for the interest rates is a particular form of equation (1) that we will state as follows:

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t \ , \qquad r(0) = r_0 \tag{2}$$

where κ is the parameter that determines the speed of adjustment (reversion rate), i.e., it measures the intensity with which the interest rate is drawn back towards its long-run mean, θ is the long-run mean of the instantaneous interest rate (asymptotic interest rate), σ is the volatility of the process, r_t is the instantaneous interest rate and dW_t is a standard Gauss-Wiener process. Moreover, it is usually assumed that κ, θ and σ are strictly positive constants. The drift term of the process, $\kappa(\theta - r_t)$, is a restoring force which always pull the stochastic interest rate toward a long-term value (or mean value) of θ. The diffusion term of the process, σ^2, represents the variance of instantaneous changes in the interest rates.

Under this framework, the fundamental partial differential equation to price a default-free discount bond, P, promising to pay one unit of capital at time T, is equal to:

$$\frac{1}{2}\sigma^2\frac{\partial^2 P(r)}{\partial r^2} + \kappa(\theta - r)\frac{\partial P(r)}{\partial r} + \frac{\partial P(r)}{\partial T} - \lambda\sigma\frac{\partial P(r)}{\partial r} - rP(r) = 0 \tag{3}$$

where λ represents the market price of risk in the Vasicek's model. This partial differential equation has to be solved subject to boundary condition $P(r, T, T) = 1$. Under this terminal condition, the solution for the bond prices is given as[8]:

$$P(r, t, T) = e^{A(t,T) - B(t,T)\, r(t)} \tag{4}$$

where

[8]Using Vasicek's interest rate model and bond pricing formulas, Jamshidian (1989) provide analytical solutions for both European options on pure discount bonds and European options on coupon-bearing bonds. The respective European put options can be obtained using the put-call parity relationship. Since these issues are beyond the scope of this paper we will not reproduce the results here.

$$B(t,T) = \frac{1 - e^{-\kappa(T-t)}}{\kappa} \tag{5a}$$

$$A(t,T) = \frac{(B(t,T) - T + t)(\kappa^2\theta - \sigma^2/2)}{\kappa^2} - \frac{\sigma^2 B(t,T)^2}{4\kappa} \tag{5b}$$

This model is one of the very popular models for the term structure dynamics of interest rates, because it allows many of the bond and derivative prices to be computed easily in closed form. The Gaussian feature of the Vasicek's model has, however, the drawback that the interest rate process occasionally take negative values. This undesirable feature may imply that bond prices can grow exponentially which is absurd (see, for example, Rogers (1995)). As a result, the prices of long-term pure discount bonds can exceed their face value, which is obviously a spurious result (see Rogers (1996)). Therefore, we will use the Cox-Ingersoll-Ross framework to price perpetuities, although the use of the Vasicek's framework would lead to a mathematically simple valuation of perpetuities.

2.2 The Cox-Ingersoll-Ross (CIR) term structure model

The well-known valuation framework of asset pricing in a continuous-time competitive economy developed by Cox et al. (1985a) has been the basis for many equilibrium models of contingent claims valuation. For example, the general equilibrium approach to term structure modelling developed by Cox et al. (1985b) is an application of their more general equilibrium framework. In their single factor model of the term structure of interest rates they assume that the interest rate dynamics can be expressed as a diffusion process known as the mean-reverting square root process:

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t\ , \qquad r(0) = r_0 \tag{6}$$

where the parameters of the model have the same meaning as before. The drift term of the process, $\kappa(\theta - r_t)$, is again a restoring force which always pulls the stochastic interest rate toward a long-term value of θ. The diffusion term of the process, $\sigma^2 r_t$, represents the variance of instantaneous changes in the interest rates.

Due to the presence of the square-root in the diffusion coefficient, this process takes only positive values. It can reach zero if $\sigma^2 > 2\kappa\theta$, but it never becomes negative. If $2\kappa\theta \geq \sigma^2$, the upward drift will dominate preventing r_t of reaching the origin[9]. Thus, we can sum up the properties of this stochastic process as follows: (i) if the interest rate starts positive it can never subsequently become negative; (2) if the interest rate hits zero, it can subsequently become positive; (3) the absolute variance of the interest rate increases as the interest rate itself increases; (4) there is a steady state distribution for the interest rate.

The distribution of the future interest rates has some properties, with respect to expected value and variance, that are worthwhile to review[10]. If κ approaches infinity, the long-run mean goes to θ and the variance to zero. But, as κ approaches zero the mean value goes to the current interest rate and the variance to $\sigma^2 r(t) \times (s-t)$, where t and s denotes current time and future time respectively. Thus, taking the limiting case of κ approaching zero, we are setting a new process without drift, i.e., with a zero-expected change in the interest rates. However, as it was pointed out by Ingersoll and Ross (1992), the qualitative properties of the zero-drift process would also be true with their more general drift term.

Under this framework, the fundamental partial differential equation to price a default-free discount bond, P, promising to pay one unit of capital at time T, is equal to:

$$\frac{1}{2}\sigma^2 r \frac{\partial^2 P(r)}{\partial r^2} + \kappa(\theta - r)\frac{\partial P(r)}{\partial r} + \frac{\partial P(r)}{\partial T} - \lambda r \frac{\partial P(r)}{\partial r} - rP(r) = 0 \tag{7}$$

with the boundary condition $P(r,T,T) = 1$. Since the first three terms of equation (7), which come from Ito's formula, represent the expected price change for the bond, the expected return on the bond is $r + (\lambda r \frac{\partial P(r)}{\partial r} \times \frac{1}{P})$. The factor λr represents the covariance of changes in the interest rate with percentage changes in optimally invested wealth and λ is the market risk parameter or price of interest rate risk. Due to the fact that $\frac{\partial P(r)}{\partial r} < 0$, positive premiums will exist if $\lambda < 0$, i.e., if this covariance is negative. The discount bond price is then equal to[11]:

$$P(r,t,T) = A(t,T)\, e^{-B(t,T)\, r(t)} \tag{8}$$

[9]Additional details concerning this relationship will be discussed in the next section.

[10]See Cox et al. (1985b) for additional details.

[11]Cox et al. (1985b) further extended their term structure model to price European call and put options written on discount bonds. Their formulae is also suitable to price American call options on discount bonds because they will never be exercised before maturity, given the absence of interim coupons on these bonds (see Merton (1973a)).

where

$$A(t,T) = \left[\frac{2\omega e^{[(\kappa+\lambda+\omega)(T-t)]/2}}{(\omega+\kappa+\lambda)(e^{\omega(T-t)}-1)+2\omega}\right]^{2\kappa\theta/\sigma^2} \tag{9a}$$

$$B(t,T) = \frac{2(e^{\omega(T-t)}-1)}{(\omega+\kappa+\lambda)(e^{\omega(T-t)}-1)+2\omega} \tag{9b}$$

$$\omega = \left[(\kappa+\lambda)^2 + 2\sigma^2\right]^{1/2} \tag{9c}$$

This model is also one of the very popular models for the term structure dynamics of interest rates. However, the criticism that was applied to Vasicek's arbitrage model does not apply to the Cox et al. (1985b) intertemporal general equilibrium term structure model, because the latter does not allow negative interest rates which is a desirable and more realistic feature for the term structure dynamics of interest rates (see Rogers (1995)). Also, it appears that a square-root process is more suited that an Ornstein-Uhlenbeck process, because there is empirical evidence (e.g., Chan et al. (1992)) showing that the proportionality factor (the γ parameter of equation (1)) assumed in the CIR model might be even to weak, since it is of order 1.5 on the US T-Bill market. Therefore, the valuation of perpetuities under the CIR framework seems more suited than under the Vasicek's model.

3 Boundary conditions on CIR's interest rate dynamics

The so-called Bessel processes play a key role in financial mathematics because of their strong relation with several diffusion processes that are usually used in finance. Examples of these diffusions are the geometric Brownian motion, the Ornstein-Uhlenbeck process and the square-root process. Although this later diffusion was firstly introduced in finance by Cox et al. (1985b), several of its properties were already studied before as squared Bessel processes. These processes are, by definition, Markov processes[12]. One of the key issues of the square-root diffusion is the role played by the term $\kappa\theta$, which is closely related with the dimension δ of a squared Bessel process ($\delta = 4\kappa\theta/\sigma^2$), and have important implications for the boundary conditions of the problem (see, for example, Feller (1951); for a complete description of the

[12]This means that the probability distribution for all future values of the process depends only on its current value and is not affected by past values of the process or by any other current information.

boundary classification for one-dimensional diffusions see Karlin and Taylor (1981, chap. 15)). The values of the function both at $r = 0$ and $r = +\infty$ are of particular interest when we are dealing with interest rate problems. From these two points, only the first one deserves particular attention since no key phenomenon occurs at infinity, because the infinite point is a natural boundary for all specifications of $\kappa\theta$. But, at $r = 0$ the specification of the $\kappa\theta$ term completely changes the behaviour of the problem.

Three important properties are of particular interest[13]: (i) if $2\kappa\theta \geq \sigma^2$, $r = 0$ is an entrance, but not exit, boundary point for the process. This means that 0 acts both as absorbing and reflecting barrier such that no homogeneous boundary conditions can be imposed there. Thus, the origin is inaccessible and the CIR process stays strictly positive[14]; (ii) if $0 < 2\kappa\theta < \sigma^2$, $r = 0$ is a reflecting boundary (exit and entrance), i.e., 0 is chosen to be an instantaneously reflecting regular boundary; (iii) if $\kappa\theta = 0$, $r = 0$ is a trap or an absorbing point and no boundary condition can be imposed there. Thus, when the CIR diffusion process hits 0 it is extinct, i.e., it remains at 0 forever (absorbing or exit boundary). This last property is implausible for a spot rate process because it predicts that the interest rate will remain forever at 0 once this level has been reached. Thus, a reflecting boundary model seems quite reasonable for an interest rate process, but an absorbing boundary model does not[15].

[13] We are assuming that $\kappa\theta \geq 0$, $\sigma > 0$ and the CIR diffusion process is valued in $[0, +\infty)$. Square-root diffusions with these characteristics have been well studied (see, for example, Revuz and Yor (1999, chap. 11)). But it seems also natural to consider the case where the dimension of the process is negative, resulting in $\kappa\theta < 0$, or even extend it to negative starting points (see, for example, Göing-Jaeschke and Yor (2003)). This last extention may be useful for some applications, but for the interest rate dynamics is of no interest.

[14] This means that an entrance boundary cannot be reached from the interior of the state space that is considered, $[0, +\infty)$ in this case. Thus, if the process starts with a positive interest rate, it cannot subsequently become negative or reach the origin. However, it is possible for some other applications to assume that the state variable begins at the left boundary, but then the stochastic process will move to the interior of the state space and will not return to the entrance boundary again.

[15] For example, a slowly reflecting boundary, in which the interest rate can remain at 0 for a finite period of time before returning to positive values again, is more plausible than an absorbing boundary at 0.

4 Valuation of perpetuities under CIR stochastic interest rates

Following Cox et al. (1985a,b), the price of any interest-rate contingent claims satisfies the following partial differential equation:

$$\frac{1}{2}\sigma^2 r\frac{\partial^2 F(r)}{\partial r^2}+\kappa(\theta-r)\frac{\partial F(r)}{\partial r}+\frac{\partial F(r)}{\partial T}-\lambda r\frac{\partial F(r)}{\partial r}-rF(r)+C(r,t)=0 \tag{10}$$

This equation is similar to equation (7). The only difference is the new term $C(r,t)$ which represents the cash rate paid out to the claim[16]. For the valuation of a default-free discount bond $C(r,t)=0$, but for a perpetuity its value is 1 since a perpetuity is a default-free financial instrument that pays a constant stream of one unit of capital p.a.[17]. In addition, for a perpetuity the term $\frac{\partial F(r)}{\partial T}$ will vanish as T goes to infinity. Thus, equation (10) can be restated as:

$$\frac{1}{2}\sigma^2 r\frac{\partial^2 F(r)}{\partial r^2}+\kappa(\theta-r)\frac{\partial F(r)}{\partial r}-\lambda r\frac{\partial F(r)}{\partial r}-rF(r)+1=0 \tag{11}$$

As it was already stated, equation (8) is the solution to the partial differential equation (7), which allow us pricing a default-free discount bond, i.e., pricing a bond for finite maturity. Although this solution was first introduced in finance by Cox et al. (1985b), the formula was already obtained by Pitman and Yor (1982) but in a different context (see, for example, Delbaen (1993) and Geman and Yor (1993)). Thus, the price at time $t=0$ of a zero coupon bond maturing at time T is also equal to:

$$P(r,0,T)=\mathbb{E}_0^{\mathcal{Q}}\left[e^{-\int_0^T r(s)\,ds}\right]=A(0,T)\,e^{-B(0,T)\,r(0)} \tag{12}$$

where $\mathbb{E}_0^{\mathcal{Q}}$ denotes the expectation under the risk-neutral probability $\mathcal{Q}$ (or martingale measure $\mathcal{Q}$), at time $t=0$, with respect to the risk-adjusted process for the instantaneous interest rate that can be written as the following stochastic differential equation:

$$dr_t=\left[\kappa\theta-(\lambda+\kappa)\,r_t\right]dt+\sigma\sqrt{r_t}\,dW_t \tag{13}$$

[16] We also changed the function notation to distinguish the value of a default-free discount bond, $P(r)$, from the value of a perpetuity, $F(r)$.

[17] Another common name for a perpetuity is consol.

and where dW_t is a standard Brownian motion under $\mathcal{Q}$. It should be noted that option pricing analysis usually resort in the so-called risk-neutral valuation which is essentially based in replication and continuous trading arguments[18]. However, the interest rate r is not the price of a traded asset, since there is no asset on the market whose price process is given by r. This means that the present framework is somewhat more complicated than a Black-Scholes setting due to the appearance of the market price of risk λ, which is not determined within the model. We see that the value at time $t = 0$ of a zero coupon bond with maturity date T is given as the expected value of the final payoff of one dollar discounted to present value. This expected value is stated by equation (12), but in this case the expectation is not to be taken using the objective probability measure $\mathcal{P}$. Instead, a martingale measure $\mathcal{Q}$ must be used to denote that the expectation is taken with respect to a risk-adjusted process, where the risk adjustment is determined by reducing the drift of the underlying variable by a factor risk premium λr. Therefore, the risk-adjusted drift of the interest rate square-root process is denoted by the term $[\kappa\theta - (\lambda + \kappa)\, r_t]$. It should also be emphasized that although risk premiums for interest rates may be introduced, they cannot be observed or measured. Thus, in order to be able to solve such problem, an exogenously given λ must be specified. Moreover, the risk factor term to be introduced is determined by things such as the forms of risk aversion possessed by the various agents on the market. This means that if one makes an *ad hoc* choice of $\lambda = 0$, then he is implicitly making an assumption concerning the aggregate risk aversion on the market[19].

Under this framework, we can set the value of a perpetuity, that we will denote as $F(r)$, as follows[20]:

---

[18]For example, it is possible to compute the Black-Scholes arbitrage free prices using such arguments and a risk-neutral valuation approach, because there is a risk-neutral probability measure $\mathcal{Q}$ equivalent to the real world probability measure $\mathcal{P}$ [see, for example, Cox and Ross (1976), Harrison and Kreps (1979) and Harrison and Pliska (1981)]. Since in the case of most real option problems the underlying asset is often not traded, for contingent-claims valuation it is not needed to invoke replication and continuous trading arguments. Instead, it is usually assumed that an intertemporal capital asset pricing model holds, like the one of Merton (1973b), and then a general equilibrium model, such as the one of Cox et al. (1985a), is developed.

[19]For a detailed technical exposition regarding these issues see, for example, Björk (2004).

[20]The valuation of perpetuities using the methodology of Bessel processes under stochastic interest rates within the CIR's framework can be found in Delbaen (1993), Geman and Yor (1993) and Yor (1993). Several mathematical properties are also investigated.

$$F(r) = \mathbb{E}_0^{\mathbb{Q}}\left[\int_0^{\infty} e^{-\int_0^t r(s)\,ds}\,dt\right] = \int_0^{\infty} P(r,0,t)\,dt \tag{14}$$

As we will see later, we need to use the first derivative of the perpetuity function. Differentiation under the integral sign is allowed, even when a limit is infinite, and this gives us:

$$F'(r) = \frac{d}{dr}\int_0^{\infty} P(r,0,t)\,dt = \int_0^{\infty} \frac{\partial P(r,0,t)}{\partial r}\,dt = -\int_0^{\infty} A(0,t)B(0,t)\,e^{-B(0,t)\,r(0)}\,dt \tag{15}$$

In order to compute the value of a perpetuity and its derivative we will use the parameter values taken from the empirical work of Chan et al. (1992). Their values will be considered as our base case parameter values. Additionally, we are also interested in the special case where the term $\kappa\theta = 0$. As it was pointed out by Ingersoll and Ross (1992) there is no consensus about the appropriate λ value to use. As a result, throughout this work we will generally consider no term premia ($\lambda = 0$), yet in some specific cases we will use some positive term premia for comparative reasons only. The parameter values for both cases are presented in Table 1[21].

Table 1: Parameter values for the base and special cases.

Parameter	Base Case Value	Special Case Value
κ	0.2339	0
θ	0.0808	0
σ	0.0854	0.0854
λ	0	0

[21]For the special case we are considering that both κ and θ are zero, but it is a sufficient condition that only κ be zero to generate such case, since it is this parameter that plays a key role on the distribution of the future interest rates. As Cox et al. (1985b) have shown as $\kappa \to 0$ the conditional mean goes to the current interest rate and the conditional variance of $r(s)$ given $r(t)$ (where $s > t$) goes to $\sigma^2\, r(t) \times (s-t)$. Therefore, this lead to the single-factor pure diffusion process of Ingersoll and Ross (1992) that we will use as our special case. But θ also plays a significant role even if the κ parameter is not zero. In this case, if we consider that $\theta = 0$ it still would not be possible to impose a boundary condition at $r = 0$. However, we would continue to have a mean-reverting process but now with an asymptotic interest rate equal to zero.

5 Numerical integration of stochastic interest rates

Now it is necessary to present some numerical computations in order to understand the behaviour of the functions and the impact of considering, or not, a $\kappa\theta$ term equal to zero and different volatility and risk premium levels. Table 2 shows the values of both functions for the base case parameter values, considering different maturities[22] and volatilities and with $r(0) = 0$[23].

Table 2: Values of the perpetuity function and the first derivative of the perpetuity function using the base case parameter values for different levels of volatility.

		$r(0) = 0.00$		
Function	T	$\sigma = 0.03$	$\sigma = 0.0854$	$\sigma = 0.3$
$F(r)$	100	16.136	16.595	21.275
$F'(r)$	100	-52.888	-52.877	-52.913
$F(r)$	500	16.141	16.604	21.275
$F'(r)$	500	-52.913	-52.913	-52.913
$F(r)$	1000	16.141	16.604	21.275
$F'(r)$	1000	-52.913	-52.913	-52.913
$F(r)$	∞	16.141	16.604	21.275
$F'(r)$	∞	-52.913	-52.913	-52.913

The results from Table 2 seems to indicate that the use of a fixed number T in the upper limit

[22]It should be noted that with this approach the resulting formulae does still hold even if the upper bound of the integral in $F(r)$ is a fixed number T. For example, since the stochastic nature of interest rates is particularly relevant for actuarial purposes, considering a fixed number T in the upper limit of the integral may be a better description of the finite nature of human life for such applications.

[23]We choose this interest rate since the values of the functions at this point are of particular interest for the one-factor model that we are using, but similar computations can be done for any positive interest rate wanted. For the special case, however, the value of the perpetuity at $r = 0$ diverges for infinity since the term $\kappa\theta = 0$. For this reason it is not necessary to reproduce the results here.

of the integral, instead of using infinity, will not generate any problem for the base case. We also have tried other interest rate values and we reach the same conclusions. Thus, considering $T = 500$ or $T = 1000$ seems quite reasonable for the analysis and it will simplify the numerical computations if we use this approach. Even the use of $T = 100$ will not produce to much differences. But instead of using equations (14) and (15) to compute the value of a perpetuity and its derivative, we can use, as an alternative, the analytic functions proposed by Delbaen (1993) and Geman and Yor (1993). Using such formulae we also achieve the same values that we present in the table when $T = \infty$. Therefore, the choice between one of the two ways to compute perpetuities under the CIR framework is at the decision of the user. Using a finite maturity may be useful in some insurance applications. However, for other applications such as real options, where it is usually considered that the problems under study are time-independent, any of the alternative approaches are quite suitable. Yet, the use of $T = 500$ or $T = 1000$ in the first approach would generate similar results.

Figure 1 presents the value of a perpetuity as a function of the interest rate using the base case parameter values. It has finite value and slope at zero because rates being stochastic do not remain there for long. The value of the function at $r = 0$ is 16.604 and $F'(0) = -52.913$. It should be noted that for the particular case where $r = 0$, $F'(0) = -1/\kappa\theta$ (see Delbaen (1993)). This shows why the value of $F'(0)$ is the same for different volatility levels (see Table 2) and even for different risk premiums (see Table 3 that we will present below), because at $r = 0$ it only depends on κ and θ. This also clearly demonstrates why it is not possible to impose a boundary condition at $r = 0$ when the term $\kappa\theta = 0$.

Figure 2 presents the value of a perpetuity as a function of the interest rate using the base case parameter values for different levels of volatility. Thus, a perpetuity is an increasing function of the volatility level. However, the way the volatility level affects the perpetuity value is not exactly the same when we consider, or not, that the term $\kappa\theta = 0$.

In Figure 3 we present the value of a perpetuity as a function of the interest rate volatility using the base case parameter values with $r(0) = 0.06$ and two levels of risk premium λ. In this case, the perpetuity value is increasing and strictly convex. But for the special case we have a different behaviour as it is shown in Figure 4. Thus, for lower levels of volatility an increase in the interest rate volatility level does not produce a significant impact on the perpetuity value.

This issue is even more significant for positive risk premiums. But for higher levels of volatility an increase in the volatility level generates a sharply increase on the perpetuity value. This behaviour is justified based on the fact that the volatility level plays a key role in the special case, when compared with the base case, since the term $\kappa\theta$ is zero[24]. We also used other values for $r(0)$ and we reached similar conclusions for both cases.

Until now we are considering only positive volatility levels. It is also interesting to compare both cases with the most basic perpetuity that we can use in finance, the case where we have a zero volatility level (also no mean reversion or risk premium). In this case the perpetuity function is just $F(r) = 1/r$. Figures 5 and 6 present the value of a perpetuity as a function of the interest rate using the base case values and special case values, respectively. The particular case with zero volatility is just the case where $F(r) = 1/r$. Thus, it is easy to see that volatility plays a key role for the special case. In addition, we can conclude that for very low levels of interest rate volatility and moderate interest rate levels the use of a perpetuity of the form $F(r) = 1/r$ is quite reasonable if we are considering that there is no mean reversion effect. But if the true process is mean-reverting this is not true. For lower levels of interest rates the perpetuity value with zero volatility will be overvalued and for higher interest rates it will be undervalued. The conclusion that we can take from this is that even when we are considering low levels of interest rate volatility, the use of a perpetuity function of the type $F(r) = 1/r$ can produce gross errors for the analysis if the true generating rate process is mean-reverting.

Table 3 presents the values of the perpetuities and its derivatives for the base case parameter values, considering different maturities and risk premiums and with $r(0) = 0$. Thus, pricing perpetuities in a general equilibrium framework under different degrees of risk aversion is possible, but an additional unobserved parameter value is required, the market price of risk which we assume to be constant. The values of perpetuities under the special case will, obviously, diverge and it is not necessary to reproduce the results here. Once again, the use of $T = 500$ or $T = 1000$ is quite acceptable, but the use of $T = 100$ is also perfectly reasonable especially for positive risk premiums. We also have tried other interest rate levels and we achieve the same conclusions about this issue.

[24] It should be noted that in this case, equation (8) will be simplified to $P(r, t, T) = e^{-B(t,T)\, r(t)}$ since $A(t, T) = 1$, resulting in a higher role of the interest rate volatility for the stochastic process.

Table 3: Values of the perpetuity function and the first derivative of the perpetuity function using the base case parameter values for different levels of the risk premium.

		$r(0) = 0.00$		
Function	T	$\lambda = 0.0$	$\lambda = -0.1$	$\lambda = -0.2$
$F(r)$	100	16.595	12.954	10.255
$F'(r)$	100	-52.877	-52.912	-52.913
$F(r)$	500	16.604	12.954	10.255
$F'(r)$	500	-52.913	-52.913	-52.913
$F(r)$	1000	16.604	12.954	10.255
$F'(r)$	1000	-52.913	-52.913	-52.913
$F(r)$	∞	16.604	12.954	10.255
$F'(r)$	∞	-52.913	-52.913	-52.913

Figure 7 presents the value of a perpetuity as a function of the interest rate using the base case parameters for different values of risk premium, the higher the risk premium the lower the price. A perpetuity is a decreasing function of the risk premium level. In this case, the pattern the risk premium level affects the perpetuity value is similar for both the base and special cases (strictly decreasing and strictly convex), but it is obviously more pronounced for the later as Figure 8 shows.

It is also interesting to analyse the behaviour of the perpetuity function under different values of κ and θ. The perpetuity is a decreasing convex function of the speed of adjustment for lower levels of interest rates (i.e., interest rates $< \theta$), but it is an increasing concave function for higher levels of interest rates (i.e., interest rates $> \theta$) as it is shown in Figures 9 and 10. The interest rate level of $r = 0.1241$ corresponds to the intersection point between the cases with κ_0 and κ_2 resulting in a perpetuity value of 11.287. Thus, the value of the κ parameter is highly relevant since different levels of speed of adjustment will lead to an intersection between any two of the κ levels. This is particularly relevant for the choice of using an interest rate process with or without mean reversion.

Figure 11 shows that the value of a perpetuity is a decreasing function of the asymptotic interest rate θ.

As Figure 12 highlights there is an interest rate level for which the consideration of mean reversion or no mean reversion makes the perpetuity value the same. A simple numerical computation shows that under the parameters that we are using the intersection point is where $r = 0.3046$ resulting in a perpetuity value of 6.684. Thus, ignoring the mean reversion effect in order to gain simplicity, when the true generating rate process is mean reverting and its existence is strongly documented in the vast empirical evidence that we have previously cited, may cause gross errors for applications in which computing perpetuity values are crucial. For instance, for the case where the mean reversion effect best describe the reality of the economy, the use of a $\kappa\theta$ term equal to zero will lead to a perpetuity value overvalued for interest rates lower than 0.3046 and a perpetuity value undervalued for interest rates higher than 0.3046. This issue is highly relevant for investment and disinvestment decisions of firms where the interest rate uncertainty is a key factor for the decision, since the upper interest rate threshold (trigger point that will induce the firm to disinvest) and the lower interest rate threshold (trigger point that will induce the firm to invest) will be surely influenced by this fact. These ideas will be analysed in the following sections.

6 Investment hysteresis without mean reversion

6.1 Perpetual investment and disinvestment opportunities

To concentrate on the effects of interest rates on investment decisions we use a particular model of real interest rates. To do so, we follow the single-factor pure diffusion process of Ingersoll and Ross (1992) assuming that changes in the instantaneous interest rate, r, satisfy the following process (with a zero mean reversion coefficient):

$$dr_t = \sigma\sqrt{r_t}\, dW_t \tag{16}$$

where σ is constant. This is equivalent to the interest rate dynamics of the risk-adjusted stochastic process $dr_t = -\lambda r_t\, dt + \sigma\sqrt{r_t}\, dW_t$ for risk-neutral pricing in the case of a nonzero term premium λ where it is assumed that λ is constant and $\lambda < 0$ corresponds to positive risk premiums. The process followed here restricts the more general mean-reverting drift process of Cox et al. (1985b), a general equilibrium model of the term structure of default-free securities, and that is an application of their general equilibrium framework for asset pricing in continuous-time (see Cox et al. (1985a)). Since we want to focus on the effects of interest rate uncertainty on the investment and disinvestment decisions, the Ingersoll and Ross (1992) process with a zero expected interest rate change will allow the simplification of our analysis. But in the next section we will also consider the general mean-reverting square root process to analyze the impact of stochastic interest rates under mean reversion on those decisions. Additionally, we will be able to understand the implications of considering, or not, the mean reversion effect for capital budgeting purposes under stochastic interest rates.

According to Cox et al. (1985a,b), the price of any interest-rate contingent claims satisfies the following partial differential equation (for the case where mean reversion is not considered):

$$\frac{1}{2}\sigma^2 r \frac{\partial^2 F(r)}{\partial r^2} - \lambda r \frac{\partial F(r)}{\partial r} + \frac{\partial F(r)}{\partial T} - rF(r) + C(r,t) = 0 \tag{17}$$

where C is the net cash paid out to the claim and λ measures the price of interest-rate risk. If $\lambda < 0$ it corresponds to a case of positive risk premium. Throughout the analysis we will assume that λ is a constant.

Following the ideas that underlies most of the real options' framework, we assume a very long time to maturity options. This technique was firstly raised by Merton (1973a) to obtain closed-form solutions for the perpetual calls and puts options. Using this technique the problem stated in equation (17) becomes time independent since the term $\frac{\partial F(r)}{\partial T}$ will vanish as T becomes very long $\left(\frac{\partial F(r)}{\partial T} \rightarrow 0\right)$. In addition, cash is paid out at a rate of a dollar per annum for ever. For this perpetual case, equation (17) reduces to an ordinary differential equation of the form:

$$\frac{1}{2}\sigma^2 r \frac{\partial^2 F(r)}{\partial r^2} - \lambda r \frac{\partial F(r)}{\partial r} - rF(r) + 1 = 0 \tag{18}$$

Looking at equation (18) it is easy to see that it does not have constant coefficients since they are dependent on r. But with a single change we can turn the problem easier. Thus dividing both sides of the equation by r and rearranging we get:

$$\frac{1}{2}\sigma^2 \frac{\partial^2 F(r)}{\partial r^2} - \lambda \frac{\partial F(r)}{\partial r} - F(r) = -\frac{1}{r} \tag{19}$$

Now, equation (19) is a linear nonhomogeneous constant coefficient equation. The general solution to this equation, $F(r)$, is the sum of the complementary solution, $y(r)$, and the particular solution, $Y(r)$. The general solution can be interpreted using the following economic intuition. $Y(r)$ is interpreted as the expected present value payoff if the state variable r is allowed to fluctuate without any regulation or barrier control, while $F(r)$ is interpreted in the same way, but now with the stochastic process being regulated by some form of control (we will discuss this issue below). Therefore, $y(r)$ must represent the additional value of control. A possible and natural lower barrier for an interest rate process would be $r = 0$, but for this single-factor pure diffusion process such control is not possible because the term $\kappa\theta$ is equal to zero and the slope infinite. As a result, we have to determine the barriers, as well as the constants of the complementary solution, numerically since no closed-form solution is available. In our case, the barrier controls will be determined by the decision to invest or disinvest, that will lead to a lower trigger point or lower barrier point ($\underline{r}$) and an upper threshold or upper barrier point ($\overline{r}$). Thus, the solution to $y(r)$ is the sum of two terms, where one of them will be interpreted as the perpetual investment opportunity and the other interpreted as the perpetual disinvestment opportunity. Since we want to consider models of investment and disinvestment we will add a new state variable to the decision problem, a discrete variable that will indicate if the firm is active (1) or idle (0). When we consider combined entry and exit decisions simultaneously the firm will have, in each state, a call option on the other. For example, if an idle firm exercise its option to invest, it will get an operating profit plus a call option to abandon. Similarly, if an active firm exercises its option to abandon it will return to the idle state and get a new option to invest. In such case, the values of an idle firm and an active firm are interlinked and must be determined simultaneously. Although the combined entry and exit strategy is the most interesting one to analyze, we will also consider the isolated strategies of invest and disinvest for comparisons purposes.

The value of an idle or not active firm, $F_0(r)$, is obtained by the solution of the complementary function of equation (19):

$$\frac{1}{2}\sigma^2\frac{\partial^2 F_0(r)}{\partial r^2} - \lambda\frac{\partial F_0(r)}{\partial r} - F_0(r) = 0 \tag{20}$$

and the value of an active firm, $F_1(r)$, is the solution of the entire equation (19):

$$\frac{1}{2}\sigma^2\frac{\partial^2 F_1(r)}{\partial r^2} - \lambda\frac{\partial F_1(r)}{\partial r} - F_1(r) = -\frac{1}{r} \tag{21}$$

Let us now proceed with the solution of the complementary functions together, since they are similar linear homogeneous equations with constant coefficients. Trying a solution of the form $F(r) = e^{mr}$, we find that $F'(r) = me^{mr}$ and $F''(r) = m^2e^{mr}$. Substitution yields:

$$\left(\frac{1}{2}\sigma^2m^2 - \lambda m - 1\right)e^{mr} = 0 \tag{22}$$

Hence $F(r) = e^{mr}$ is a solution of Equation (19) when m is a root of

$$\frac{1}{2}\sigma^2m^2 - \lambda m - 1 = 0 \tag{23}$$

or

$$\phi(m) = m^2 - vm - w = 0 \tag{24}$$

where we define $v = 2\lambda/\sigma^2$ and $w = 2/\sigma^2$. The convergence condition of equation (24) is $w > 1 - v$. Then, it turns out that $\phi(0) = -w < 0$ and $\phi(1) = 1 - v - w < 0$. Since $\phi''(m) = 2 > 0$ it means that the auxiliary equation has two roots, where one of them must be greater than one (we will call it a) and the other one must be less than zero (we will call it b). The discriminant of the characteristic equation is positive, $\Delta = v^2 + 4w > 0$, which means that the respective solutions are real. Therefore, the two roots can be written out as:

$$a = \frac{+v + \sqrt{v^2 + 4w}}{2} > 1 \tag{25a}$$

$$b = \frac{+v - \sqrt{v^2 + 4w}}{2} < 0 \tag{25b}$$

Thus, we can write the general solution of equation (20) as:

$$F_0(r) = C_1e^{ar} + C_2e^{br} \tag{26}$$

and the general solution of equation (21) as:

$$F_1(r) = C_3e^{ar} + C_4e^{br} + Y(r) \tag{27}$$

where C_1, C_2, C_3 and C_4 are constants to be determined from boundary conditions. A simple economic intuition tells us that for very high interest rate levels idle firms are not induced to invest. Therefore, the option of activating the firm should be nearly worthless for this level rates. As a result, we need that the constant $C_1 = 0$ (associated with the positive root a). This means that the expected net present value of making an investment in the idle state is:

$$F_0(r) = C_2 e^{br} \tag{28}$$

Since an idle firm is not operating does not have any return from the project yet. Therefore, equation (28) is just the option value of a perpetual investment opportunity, $IO(r)$. Over the range interval of interest rates $(\underline{r}, \infty)$, an idle firm will not exercise its option to invest. To simplify our analysis, we will consider that once the investment commitment has been made, the investment project return is identical to a perpetuity making a continuous payment of one unit over time. Thus, no additional resources or expenditures apart from the initial investment are required to maintain the rights over the project or to sustain the project after it has been accepted. A similar assumption is also used by Ingersoll and Ross (1992), but in their case the project returns are identical to a T-period zero-coupon bond with a real face value of one dollar since they are considering finite maturities, whereas we are considering infinite maturities. This assumption implies that operating profits never become negative in our project. Such assumption is also used by, among others, McDonald and Siegel (1986), Pindyck (1988) and Bertola (1998).

The value of an active firm is the sum of two components, the expected present value of the profits and an option value of terminating the project. We know that for very low interest rates an active firm will be induced to continue its operations and not disinvest. Since the value of the abandonment option should go to zero as r becomes very low, we must set $C_4 = 0$ (associated with the negative root b). Therefore, the value of a firm for the active state is:

$$F_1(r) = C_3 e^{ar} + F(r) \tag{29}$$

where $F(r)$ is the particular solution $Y(r)$ of the ordinary differential equation (21). It follows that a particular solution to this equation is, as it was already stated before, the value of a perpetuity making a continuous payment of one unit over time, i.e, $\int_0^\infty P(r, 0, t)\ dt$[25]. Since the perpetuity value represents the expected present value that can be obtained from the project if it is maintained active forever, the remaining part of equation (29) must be the value of a perpetual option to disinvest optimally, i.e., $DO(r) = C_3 e^{ar}$. Over the interest rate range $(0, \overline{r})$ an active firm will continue its operations, holding its option to abandon alive. It should be emphasized that as a simplifying setting we are considering that the investment option, as well as the option to disinvest, are both perpetual since it is possible to delay both decisions in an infinite time frame, which gives them some resemblance with the perpetual calls and puts of Merton (1973a). Finally, it should be noted that for an option to invest, as well as an option to disinvest, have an economically meaningful interpretation they must be non-negative. Therefore, we must have, respectively, $C_2 \geq 0$ and $C_3 \geq 0$. All the numerical simulations that we will present later have shown that this is the case.

6.2 Option to invest

Let us suppose that an idle firm has an option to invest in a particular investment project where interest rate uncertainty is a key factor for the decision to invest, but where, for now, the disinvestment opportunity is not considered. Thus, the firm has to decide whether to continue being idle or to enter in the market. If interest rates drop to a low level, the firm may be induced to change an option to invest paying an investment cost $\overline{I}$ by a perpetuity making a continuous payment of one unit over time. This investment cost is considered a sunk capital cost since it cannot be totally recouped if the firm should decide to quit at a later date. This issue can be explained using the "lemons" problem of Akerlof (1970). The investment strategy can be stated as follows:

$$\overline{I} + IO(r) \rightarrow F(r)$$

The optimal investment policy is determined using one value matching condition and one smooth pasting condition (also called high contact condition)[26]. This yields a system of two

[25]Therefore, we use equation (14) as $F(r)$ for the special case but we have to impose a fixed number T in the upper limit of the integral because the $\kappa\theta$ parameter is zero.

[26]Our real options problems are of American-type nature since they are time-independent and, therefore, can

non-linear equations in two variables (C_2 and $\underline{r}$):

$$\bar{I} + C_2 e^{b\underline{r}} = \int_0^\infty A(t) e^{-B(t)\underline{r}}\, dt \tag{30a}$$

$$bC_2 e^{b\underline{r}} = \int_0^\infty -B(t)A(t) e^{-B(t)\underline{r}}\, dt \tag{30b}$$

The economic intuition of the value matching condition (30a) is that the gain in value from exercising the option to invest (i.e. exercising the control) is exactly equal to the cost of doing so. If this condition was not respected, then arbitrage profits would be possible. The optimality condition of the problem does not arise from the value matching condition, but from the smooth pasting condition. This condition (stated by equation (30b)) says that the first derivative of the option value function must have the same value before and after the option has been exercised, i.e., the marginal utility must take the same value before and after the action has been taken[27].

be exercised at any time before maturity. Thus, they are optimal stopping problems. The optimality conditions for such problems were introduced in the financial economics literature by Samuelson (1965), McKean (1965) and Merton (1973a). For a general treatment of such conditions in a simpler setting see, for example, Dixit (1991b, 1993) and Dumas (1991).

[27]It should be noted that in this case the smooth pasting condition is obtained using the first derivative of the value function, and it holds only for the optimum policy. This happens since the investment cost (as well as the "disinvestment cost") is a lump-sum cost, i.e., it originates a discrete adjustment and the stochastic control problem is regulated using the first derivative of the state variable function. This is what is usually called by stochastic impulse control (see, for example, Constantinides and Richard (1978) and Harrison et al. (1983)). Optimal impulse control has been applied for example by Brennan and Schwartz (1985) to model the opening and closing of a mine and by Grossman and Laroque (1990) for an optimal consumption and portfolio choice problem under the presence of fixed transaction costs. There are situations, however, where linear costs or proportional costs of adjustment exist and for such cases the first-order smooth pasting condition holds for any given barrier control, which requires a higher form of tangency. As a result, a second-order smooth pasting condition (using the second derivative of the value function) replaces the first-order smooth pasting condition at the optimal barrier leading to what Dumas (1991) called "super contact" condition. This is what is usually called by stochastic instantaneous control, where the order of differentiation moves up, i.e., the value matching condition involves the first derivative of the value function and the smooth pasting condition involves the second derivative (see, for example, Harrison and Taylor (1978), Harrison and Taksar (1983) and Harrison (1985)). Examples of such application include the dynamic equilibrium model of Dumas (1992) for two separated countries where it is possible to trade goods and/or capital abroad under proportional transfer costs and the investor's portfolio choice problem under proportional transaction costs of Dumas and Luciano (1991).

If this condition failed, then moving the critical interest rate would raise the value of the option.

The state variable of stochastic control problems referred in the previous footnote follows a Brownian motion or a geometric Brownian motion. In our problem, the state variable r follows a mean reverting square-root process in the base case and a zero-drift pure diffusion process in the special case. However, as Dixit (1991b) points out the general framework of optimal stochastic control can be adapted for other stochastic processes. Therefore, we will have a stochastic impulse control problem for our both cases since the investment cost and the disinvestment proceeds are discrete adjustments that are made at a discrete point in time, i.e., the moment where the decision is taken.

In order to obtain the values of C_2 and $\underline{r}$ we can solve the system of the two non-linear equations numerically, or, as an alternative, we can transform the system of two equations in just one equation inverting the matrix system. Then, the lower threshold can be computed solving the following equation:

$$\overline{I} = F(\underline{r}) - \frac{F'(\underline{r})}{b} \tag{31}$$

Although we can reduce the system to just one equation we still have a non-linear equation and, as a result, we have to use numerical simulations to obtain the lower threshold. Once we have the value of the lower trigger point it is extremely easy to compute the value of the corresponding constant. For an idle firm, no action is taken until r reaches a lower barrier $\underline{r}$, when it is optimal to switch the value of the option to invest plus an investment cost by a perpetuity value. The corresponding numerical simulations will be presented below.

6.3 Option to disinvest

Let us now suppose that an active firm is operating and its payoff is a perpetuity making a continuous payment of one unit over time. But if interest rates start rising to very high rates the firm may be induced to temporarily shut down or even abandon the project. If a project is closed temporarily it turns out that the firm will incur some fixed maintenance costs, but may be opened up again without having to pay again entry costs, i.e., $\overline{I}$[28]. If the project is to

[28]It should be noted that if the maintenance costs are zero then it is never optimal to abandon a closed project as long as there is a real possibility that it will be optimal to reopen the project again. But this assumption of

be permanently abandoned it will incur no maintenance costs, but if the firm wants to enter again in the market has to pay a new lump-sum cost $\overline{I}$. This possibility (i.e., a re–entry option) will be ignored for now. In our case we will assume that once the state variable reaches the upper trigger point it is optimal to abandon the project, and such abandonment policy will not involve any costs. The disinvestment strategy can be stated as follows:

$$\underline{I} \leftarrow F(r) + DO(r)$$

where $\underline{I}$ takes a positive value since when the firm close its operations will not incur any cost to disinvest. Obviously, there may be situations where firms have to incur an extra cost when they want to close, such as the cases of a copper mine or a nuclear power station where some environmental clean up costs have to be supported. In our case, we want to focus our analysis on the possibility that some fraction of the lump-sum cost $\overline{I}$ can be recouped if firms decide to abandon its operations. Therefore, we will define a new variable α that will measure the degree of reversibility, i.e., $\alpha = \underline{I}/\overline{I}$. $\alpha = 0$ corresponds to an option in which the decision taken is irreversible and can be exercised only once. The case $0 < \alpha < 1$ corresponds to partial reversibility. We will consider three cases: $\alpha = 0.25$, $\alpha = 0.50$ and $\alpha = 0.75$. The case where $\alpha = 1$ represents perfect reversibility, a situation that gives rise to a flow option in which two flows can be switched continuously and costlessly (see Shackleton and Wojakowski (2001))[29].

The optimal policy to disinvest is determined using one value matching and one smooth pasting conditions. This yields a system of two non-linear equations in two variables (C_3 and $\overline{r}$):

$$\int_0^\infty A(t)e^{-B(t)\overline{r}}\, dt + C_3 e^{a\overline{r}} = \underline{I} \tag{32a}$$

temporarily suspend operations incurring no costs and restarting at a later point is quite unrealistic, or even almost impossible, for many projects. For such cases, suspension is equivalent to an outright abandonment. The situation where temporary suspension is still possible is analyzed by McDonald and Siegel (1985). In this case, the owner of a physical capital has a Merton (1973a) European call option to produce a given commodity at time t, but he can avoid a loss by shutting down permanently the plant if the variable cost of production exceeds the operating revenues at the option's maturity.

[29]This limiting situation corresponds to the case where the two threshold will collapse to one common switching level that will determine the optimal exercise strategy. The strategy change will occur when the so-called Jorgenson (1963) user costs of capital are equals.

$$\int_0^\infty -B(t)A(t)e^{-B(t)\overline{r}}\,dt + aC_3e^{a\overline{r}} = 0 \tag{32b}$$

Once again, we can compute the values of C_3 and $\overline{r}$ solving the system of the two non-linear equations numerically, or, as an alternative, we can compute the upper threshold numerically using the following equation:

$$\underline{I} = F(\overline{r}) - \frac{F'(\overline{r})}{a} \tag{33}$$

For an active firm, no action will be taken until r reaches the upper barrier $\overline{r}$.

6.4 Switching options

The most interesting problem is the one where optimal investment and disinvestment decisions are considered together. Thus, entry and exit decisions are valued simultaneously originating a lower bound ($\underline{r}$) and an upper bound ($\overline{r}$) with $\underline{r} < \overline{r}$, and where an idle firm is induced to invest once the state variable r crosses the action trigger point $\underline{r}$ and an active firm will be induced to disinvest if the state variable crosses the threshold point $\overline{r}$. The middle band of interest rates without entry or exit actions yields what is usually called by economic hysteresis, since the optimal policy is to maintain the actual status quo, whether the firm is operating or not. Examples of models of investment and disinvestment policies include the optimal policy of a mine producing a single homogeneous commodity whose spot price is assumed to follow an exogenously given geometric Brownian motion (Brennan and Schwartz (1985)), the optimal entry and exit decisions of a firm using a particular production technology (Dixit (1989a)) and the problem in a competitive industry with home and foreign firms with entry and exit costs and where the real exchange rate follows a geometric Brownian motion (Dixit (1989b)).

The corresponding strategy for the entry and exit case can be stated as follows:

$$\overline{I} + IO(r) \rightarrow F(r) + DO(r)$$

$$\underline{I} + IO(r) \leftarrow F(r) + DO(r)$$

In this case, the optimal policy is determined using two value matching and two smooth pasting conditions resulting in a two-sided $(\underline{r}, \overline{r})$ policy. Other examples of two-sided policies include a dynamic menu cost model with state-dependent pricing in which monetary shocks have systematic effects on output (Caplin and Leahy (1991)), a two-asset intertemporal portfolio selection model incorporating proportional transaction costs and where the investor consumes a fixed proportion of his wealth in each period of time (Constantinides (1986)), an investment portfolio choice problem under transaction costs when trading a single risky asset and where the investor accumulates wealth until some terminal point in time in which he consumes all (Dumas and Luciano (1991)), a consumer's decision problem of selling an existing illiquid durable good, such as a house or a car, to purchase a new one when his wealth fluctuates (Grossman and Laroque (1990)), an optimal intertemporal consumption and investment policy of an investor facing both fixed and proportional transaction costs when trading multiple risky assets (Liu (2004)) and a real options holder strategy of continually choosing between two flows (Shackleton and Wojakowski (2001)).

It is important to note that the investment and disinvestment opportunities at the lower threshold are, respectively, $IO(r = \underline{r}) = C_2 e^{b\underline{r}}$ and $DO(r = \underline{r}) = C_3 e^{b\underline{r}}$. Similarly, the investment and disinvestment opportunities at the upper threshold are, respectively, $IO(r = \overline{r}) = C_2 e^{b\overline{r}}$ and $DO(r = \overline{r}) = C_3 e^{b\overline{r}}$. This yields a system of four non-linear equations in four variables (C_2, C_3, $\overline{r}$ and $\underline{r}$)[30]:

$$\overline{I} + C_2 e^{b\underline{r}} = \int_0^{\infty} A(t) e^{-B(t)\underline{r}}\, dt + C_3 e^{a\underline{r}} \tag{36a}$$

[30]It should be noted that this system of four non-linear equations can also be written in the form that is usually presented in some entry and exit problems (see, for example, Dixit and Pindyck (1994, pg. 218)):

$$F_1(\underline{r}) - F_0(\underline{r}) = \overline{I}$$
$$F_1^{'}(\underline{r}) - F_0^{'}(\underline{r}) = 0$$
$$F_1(\overline{r}) - F_0(\overline{r}) = \underline{I}$$
$$F_1^{'}(\overline{r}) - F_0^{'}(\overline{r}) = 0$$

In our case we are considering that a part of the investment cost $\overline{I}$ is not sunk, since it can be recouped if the firm decides to abandon its operations. As a result we have a positive signal in $\underline{I}$. But if it was necessary to pay a lump-sum cost to abandon operations we would have $-\underline{I}$. To rule out a money machine of entry and exit cycles it is necessary that $\overline{I} > \underline{I}$. Thus, the case where $\alpha = 1$ is just the limiting case.

$$bC_2 e^{b\underline{r}} = \int_0^\infty -B(t)A(t)e^{-B(t)\underline{r}}\, dt + aC_3 e^{a\underline{r}} \tag{36b}$$

$$\int_0^\infty A(t)e^{-B(t)\overline{r}}\, dt + C_3 e^{a\overline{r}} = \underline{I} + C_2 e^{b\overline{r}} \tag{36c}$$

$$\int_0^\infty -B(t)A(t)e^{-B(t)\overline{r}}\, dt + aC_3 e^{a\overline{r}} = bC_2 e^{b\overline{r}} \tag{36d}$$

The above equations are highly non-linear and as before a closed form solution is not available. Although we have to rely on numerical methods to get the solution for the two thresholds and the two constants, such numerical solutions are quite easy to obtain using the numerical routines for solving simultaneous non-linear equations that are available in many scientific computing software, such as *Mathematica*. However, some important economic properties of the solution can also be obtained by analytical methods similar to the ones employed by Dixit (1989a). We will use such technique to get a better understanding of the hysteresis effect.

6.5 Economic hysteresis effect

Decisions made under an uncertainty environment where it is costly to reverse economic actions, will lead to an intermediate range of the state variable, called a hysteresis band, where inaction is the optimal policy. It turns out that this range of inertia will produce hysteresis, i.e., permanent effects of temporary shifts. This means that if the state variable starts from a value inside the hysteretic band (in our case, between $\underline{r}$ and $\overline{r}$), and afterwards crosses one of the action trigger points and then returns to its previous level, the action will be taken but not reversed. Thus, hysteresis is an effect that persists after the cause that brought it about has been removed, i.e., a temporary change will originate a permanent effect. The key issue is that the economic actions should be taken only when the underlying state variable becomes extremely favorable and reversed only when its value becomes extremely unfavorable.

Several models of entry and exit decisions of irreversible investments have shown that the range of inaction can be remarkable large. Such examples of economic hysteresis have been provided by, among others, Brennan and Schwartz (1985) who have analyzed a firm's joint decision of entry and exit a mine that will produce a single homogeneous commodity whose spot price is assumed to follow a geometric Brownian motion, Dixit (1989a) for the case of a similar decision of a particular firm using a given production technology and whose output price also follows a geometric Brownian motion and Dixit (1989b) for the case where foreign firms that have entered in the U.S. market when the dollar was high do not have abandoned their investments when the dollar started to fall. In this case the state variable is the exchange rate, namely the price of a dollar in yen, which is assumed to follow once again a geometric Brownian motion. The economic hysteresis effect is also found to be wide in the optimal consumption and portfolio choice literature, as it is shown in the intertemporal portfolio selection model of Constantinides (1986). Therefore, such effect seems to be extremely relevant for many economic applications. Since interest rates are also an important determinant of investment and disinvestment decisions it is important to analyze the economic hysteresis effect provoked by interest rate uncertainty. To our knowledge, this effect has not been previously analyzed under stochastic interest rates.

The economic hysteresis effect produces a range of values for the state variable that is usually defined by highly non-linear equations that need numerical solutions. In some cases, it is possible to use analytic approximations that allow the use of explicit solutions to help understanding the importance of the hysteresis effect (see, for example, Dixit (1991a)). In our case, we have highly non-linear equations that use functions with integrals. As a result, such analytic approximations are very difficult to obtain and we do not attempt to use them. However, we can established a general property of the solution that yields economic hysteresis. A procedure like this was previously used by Dixit (1989a) to examined the nature of hysteresis when the source of uncertainty arises from the output market price, whereas in our case the uncertainty comes from the stochastic nature of the interest rate term structure. To do so, let us define the following function:

$$V(r) = F_1(r) - F_0(r) \tag{37}$$

Using the solutions stated by equations (28) and (29) we have:

$$V(r) = C_3 e^{ar} - C_2 e^{br} + F(r) \tag{38}$$

where $F(r)$ represents the perpetuity value. For small values of r the term with the negative root b dominates. The term is negative, increasing and concave. For very high interest rate levels the dominant term is the one associated with the positive root a. The term is positive, increasing and convex. For the intermediate range, it is the perpetuity value that plays a critical role.

Now, the two value matching and the two smooth pasting conditions can be defined in terms of V as:

$$V(\underline{r}) = \overline{I}\,, \quad V'(\underline{r}) = 0\,, \quad V(\overline{r}) = \underline{I}\,, \quad V'(\overline{r}) = 0 \tag{39}$$

There exits an optimal policy of the type pictured in Figure 13, which is similar in spirit to the diagrams presented by Constantinides and Richard (1978), Harrison et al. (1983) and Dixit (1989a, 1991b). The graph seems to indicate that there always exists an unique optimal policy for the entry and exit decisions problem under stochastic interest rates and this policy is of a simple form. The existence and form of the optimal policy is addressed using the optimal control technique of impulse control. From the figure we can also see that the optimum policy is characterized by the interest rate values $\underline{r}$ and $\overline{r}$. Thus, the firm will invest when it falls to $\underline{r}$, disinvest when it rises to $\overline{r}$ and continue the status quo over the range $(\underline{r}, \overline{r})$, i.e., an idle firm does not invest and an active firm does not abandon. To solve the problem we need to adjust the constants C_2 and C_3 until the value function $V(r)$ becomes tangent to the horizontal lines $\overline{I}$ and $\underline{I}$. Then, it turns out that the tangency points are the thresholds $\underline{r}$ and $\overline{r}$.

To get some analytical results it is important to note that:

$$V''(\underline{r}) < 0\,, \quad V''(\overline{r}) > 0 \tag{40}$$

since $V(r)$ is concave at $\underline{r}$ and convex at $\overline{r}$. Subtracting equation (20) from equation (21) we see that the function $V(r)$ satisfies the following ordinary differential equation:

$$\frac{1}{2}\sigma^2 \frac{\partial^2 V(r)}{\partial r^2} - \lambda \frac{\partial V(r)}{\partial r} - V(r) = -\frac{1}{r} \tag{41}$$

Now evaluating this differential equation at $\underline{r}$ and using the conditions (39) and (40) we get:

$$-\frac{1}{\underline{r}} = \frac{1}{2}\sigma^2 \frac{\partial^2 V(\underline{r})}{\partial \underline{r}^2} - \lambda \frac{\partial V(\underline{r})}{\partial \underline{r}} - V(\underline{r}) < -\overline{I} \tag{42}$$

Using the same approach at the $\overline{r}$ we obtain:

$$-\frac{1}{\overline{r}} = \frac{1}{2}\sigma^2 \frac{\partial^2 V(\overline{r})}{\partial \overline{r}^2} - \lambda \frac{\partial V(\overline{r})}{\partial \overline{r}} - V(\overline{r}) > -\underline{I} \tag{43}$$

Rearranging we get, respectively:

$$\underline{r} < 1/\overline{I} \equiv M_{\underline{r}} \tag{44}$$

and

$$\overline{r} > 1/\underline{I} \equiv M_{\overline{r}} \tag{45}$$

where $M_{\underline{r}}$ and $M_{\overline{r}}$ can be viewed, respectively, as the Marshallian trigger interest rates for investment and disinvestment. Taking decisions using this traditional concept can lead to myopic actions since we are implicitly assuming a static expectation for the interest rate dynamics. Thus, the differences between our thresholds and the Marshallian ones comes from the uncertainty effect. Conditions (44) and (45) highlights that uncertainty widens the Marshallian range of inaction. In the next section, we will provide some numerical results about this issue. For now, we will provide some theoretical insights about it.

The economic theory of investment based on the Marshall's foundations tells us that firms are induced to invest if the output price exceeds long run average cost and are induced to suspend or even abandon its operations if the price falls below long run average costs. Let us suppose that a project can be launched by a firm incurring a sunk cost K which is represented in our case by $\overline{I}$. Once the project is accepted it will return a net operating revenue of one unit over time, i.e., a perpetuity value. Let this perpetuity be discounted at a rate r, that represents the opportunity cost of capital in a risk-neutral world. Therefore, the Marshallian criteria would be to invest when the project has a positive expected net worth, i.e., $1/r > K$. It should be noted that the value of rK, or in our notation $r\overline{I}$, represents the full cost of making and operating the investment, since we are assuming that once the investment is made the return from the project is identical to a perpetuity making a continuous payment of one unit over time. The interest rate level that would make one indifferent between investing or not investing is given by $1/K$, or, in our notation, $1/\overline{I}$. As a result, using the Marshall's economic foundations we would invest when r was lower than the Marshallian investment trigger point $M_{\underline{r}} = 1/\overline{I}$. Similarly, the Marshallian disinvestment trigger point would be $M_{\overline{r}} = 1/\underline{I}$. Now, using these two trigger points it is extremely easy to explain the hysteresis nature through a simple numerical example. Let us suppose that the investment cost is 10 and if a firm wants to abandon operations can recoup 50% of the investment cost (i.e., $\alpha = 0.50$). As a result, the investment and disinvestment Marshallian trigger points are, respectively, 0.10 ($1/\overline{I}$) and 0.20 ($1/\underline{I}$). If the interest rate was initially in the range $(0.10, 0.20)$ and then it falls below the lower threshold the firm would invest, assuming it was in the idle state. But if the interest rate returned again to the middle range, this was insufficient to induce the firm to abandon. For such action, it was necessary that the interest rate would rise above 0.20. Thus, a temporary change in the interest rate levels has originated a permanent effect. However, the Marshallian criteria ignores the option value embedded in investment and disinvestment decisions, since the choice is made between acting right now or do nothing at all, i.e., a now or never decision type. As a result, the gap between lower and upper thresholds will be even higher, which enhances the hysteresis effect.

Using these theoretical insights we can consider some limiting cases. We have mentioned before that we are considering that once the investment commitment has been made, the investment project return is identical to a perpetuity making a continuous payment of one unit over time. Thus, no additional resources or expenditures apart from the initial investment are required to maintain the rights over the project or to sustain the project after it has been accepted. In addition, if the firm wants to abandon the operating project it can recoup a fraction α of the investment cost. This means that in our simplifying setting the operating profits from the project never become negative (the variable costs are zero). As a result, $r\overline{I}$ represents the full cost of investment and $r\alpha\overline{I}$ represents the full disinvestment proceeds. Thus, our theoretical results have to be restricted to these two issues.

The limiting case where both $\overline{I}$ and $\underline{I}$ tend to zero is not very interesting, since both $\underline{r}$ and $\overline{r}$ would diverge. When they tend to extremely large values, both $\underline{r}$ and $\overline{r}$ tend to the common limit 0. But these two issues indicate that sunk costs and disinvestment proceeds are essential for the hysteresis effect. If we fix α at a level of 0.50 for example, and we impose a higher investment cost both the lower and the upper thresholds diminish ($d\underline{r} < 0$ when $d\overline{I} > 0$ and $d\overline{r} < 0$ when $d\underline{I} > 0$, respectively). But the upper threshold falls at a higher rate. Thus, there is a tendency for narrowing the hysteresis band in this case. Let us now fix the investment cost and analyze the impact of different levels of investment recoup. Maintaining the level of investment cost fixed and rising the α parameter, gives a very small rise in the lower threshold

($d\underline{r} > 0$ when $d\underline{I} > 0$) and a sharply decrease in the upper trigger point ($d\overline{r} < 0$ when $d\underline{I} > 0$). Thus, the band of inaction will be narrower.

If we set $\alpha = 0$ the upper threshold diverges to ∞. Thus, an active firm would never abandon the project. But there is still a finite interest rate level that will induce an idle firm to invest in a project that afterwards will not be abandoned. This issue derives from the fact that the profits from the project never become negative. As a result, the firm will continue its operations. It should be noted that in this case ($\alpha = 0$) the constant C_3 goes to zero in equations (36a) - (36d), since the option to disinvest becomes worthless. Thus, equations (36a) and (36b) turn out to be the equations (30a) and (30b). As a result, the lower threshold for the case where $\alpha = 0$ can be computed using equation (31). Once the state variable r crosses this lower trigger point an idle firm will invest immediately and will maintain its operations forever. An active firm will never abandon. But if there were some variable costs these could lead to negative profits and then there would be an upper trigger point that would induce the firm to shut down optimally. Similarly, if the investment costs to reenter again (if a firm decides to shut down its operations previously) goes to ∞, the option to reinvest becomes worthless and, as a result, the constant C_2 goes to zero in equations (36a) - (36d). In this case, equations (36c) and (36d) turn out to be the equations (32a) and (32b). Therefore, the upper threshold point is computed using equation (33). An idle firm never invests due to high entrance costs and an operating firm may be induced to optimally abandon its operations when the interest rate crosses the upper trigger point. This point indicates how bad things must be, i.e., how high interest rates should rise before an active firm abandon its operations, since it knows that due to high entrance costs it can never reinvest later again.

If interest rate uncertainty goes to zero, then $\underline{r} \to M_{\underline{r}}$ and $\overline{r} \to M_{\overline{r}}$. This imply that without uncertainty the range of inaction is determined by the Marshallian trigger points. As the interest rate volatility starts rising, the lower threshold will fall and the upper threshold will rise, which will lead to a wider hysteresis band. Now, maintaining the volatility fixed at a positive level and letting $\overline{I} \to 0$, we have $d\underline{r}/d\overline{I} \to -\infty$ and $d\overline{r}/d\overline{I} \to \infty$. This means that when there is some level of interest rate uncertainty, the hysteresis level emerges very quickly even for very small investment costs. This also means that apart from the output price uncertainty (see, for example, Dixit (1989a)), the interest rate uncertainty also plays a critical role for widening the hysteretic band. All these theoretical insights can be confirmed with the numerical results that we present in the next section.

We know that, on one hand side, in practice firms do not invest until price rises substantially above long run cost. In fact, it is not difficult to find situations where firms require hurdle rates that are much higher than the cost of capital or the internal rate of return. On the other hand side, firms will continue its operations even when they are facing operating losses, and will only disinvest when the output price falls substantially bellow the variable cost. Thus, this upside and downside aspect of the decisions taken by firms may be possibly explained by the economic hysteresis phenomena. In fact, looking at conditions (44) and (45) it is proved that the Marshallian trigger points are not optimal if waiting for better information about the interest rate behaviour is possible. This explains why our lower threshold interest rate is smaller than the Marshallian trigger entry level and why our critical point of abandonment exceeds the Marshallian trigger exit point. For example, at the lower Marshallian trigger point the investment opportunity is equivalent to an option that is only just in the money. It is not optimal for the firm to exercise it unless it goes deeper in the money. Since immediate action has an opportunity cost, namely loss of the option to wait for better information, the economic hysteresis effect is enhanced when waiting is possible. It should be noted that most of the upside potential remains even if an economic action is not taken immediately, but only after a small time delay. Thus, it is the possibility of a downturn and the ability to avoid a wrong economic action that makes waiting valuable. In other words, it is the downside uncertainty that matters most for optimal investment decisions when waiting is possible. This is an implication of the "bad news principle" of Bernanke (1983a). By waiting, the firm can improve its chances of making the correct decision. Thus, under ongoing uncertainty and when waiting is possible the zone of inaction that takes option values into account is wider. We may conclude that when interest rate uncertainty is a key factor for capital budgeting decisions, idle firms are more reluctant to invest and active firms are more reluctant to exit. Thus, we may claim that the reversibility degree and the interest rate uncertainty can help explain investment and disinvestment cyclical fluctuations. Moreover, the analysis of the effect of option values under interest rate uncertainty on investment and disinvestment decisions can also help explain why there is a gap in practice between required hurdle rates and the cost of capital.

6.6 Numerical analysis

After providing some theoretical insights about the economic hysteresis effect under stochastic interest rates, let us now proceed with some numerical results that will confirm our analytical results. We will consider the base case volatility level, $\sigma = 0.0854$, and a smaller and a higher volatility ($\sigma = 0.03$ and $\sigma = 0.3$, respectively) for comparative purposes. In addition, we use two different levels of investment cost, $\overline{I} = 10$ and $\overline{I} = 7.5$. We also establish three degrees of reversibility, $\alpha = 0.25$, $\alpha = 0.50$ and $\alpha = 0.75$. We also present the case of perfect reversibility, $\alpha = 1.00$, to illustrate the limiting case. The case where $\alpha = 0$ is also illustrated since it falls in the single investment strategy situation.

It should be noted that, for now, we are considering the no mean-reverting case. Therefore, we have to use equations (14) and (15) to compute the perpetuity and the derivative of the perpetuity functions, respectively. In addition, we have to set a fixed upper limit for the integrals in both functions, otherwise their values would be $+\infty$ and $-\infty$, respectively. A question now arises. What value T should we use at the upper limits of the integrals? We have tried several time values such as $T = 100$, $T = 500$, $T = 1000$, $T = 3000$, etc. It turns out that from a practical point of view any of these values can be considered as a sufficient large maturity, which gives a time-independent resemblance for the problem. For the single investment strategy the choice of T is not sensible for the lower threshold point. For the disinvestment single strategy, however, this is not the case. Indeed, the upper threshold is rising with T, although not to much strongly marked, especially for low and moderate volatility levels. But for the combined entry and exit strategy, which is the most interesting one, the use of any of those T values does not produce any significative change on both lower and upper trigger points. Yet, we present two different time values, $T = 500$ and $T = 1000$, which confirms what we have described.

Table 4 presents the lower trigger points for the single investment strategy considering different investment cost levels and different interest rate volatilities. Under this strategy, an active firm never shuts its project, since the profits never become negative. Thus, these thresholds indicate which is the interest rate level that will induce an idle firm to enter in a project and continue its operations forever since the option to shut down is worthless (i.e., it corresponds to the case where $\alpha = 0$). For example, considering the base case volatility level and $\overline{I} = 10$ it would be necessary that the interest rate falls to 1.94% to induce an idle firm to invest. At this rate level, the firm will change the full cost of investment (i.e., option to invest plus investment cost) by a project paying a perpetuity. From the table we can conclude that: (i) for a given level of volatility the lower threshold falls as the investment cost rises; and (ii) for a given investment cost level the lower trigger point falls as the volatility level rises. In addition, all investment thresholds when uncertainty is considered are lower than the Marshallian investment trigger point (0.10 and 0.1333 for $\overline{I} = 10$ and $\overline{I} = 7.5$, respectively). It should be noted that for high levels of uncertainty firms would never invest.

Table 4: Lower thresholds for the investment option of the special case under different levels of investment costs and interest rate volatility.

	$\overline{I} = 10$			$\overline{I} = 7.5$		
T	$\sigma = 0.03$	$\sigma = 0.0854$	$\sigma = 0.3$	$\sigma = 0.03$	$\sigma = 0.0854$	$\sigma = 0.3$
500	0.0809	0.0194	-0.4470	0.1145	0.0642	-0.3450
1000	0.0809	0.0194	-0.4470	0.1145	0.0642	-0.3450

Figure 14 presents the project's value rights (i.e., the value of the option to invest) and the NPV from the project for the volatility level $\sigma = 0.0854$. For greater visual appeal we present the functions for $T = 100$. It is easily seen that for high interest rate values firms should not invest since the value of rights are greater than the NPV of the project. The interest rate level that turns the traditional NPV equal to zero is $r = 0.1833$. But at this rate level the rights from the project have a value of 4.3384. Even if interest rates fall bellow $r = 0.1833$, which would turn the NPV positive, it is not optimal that an idle firm invests immediately, because waiting for better information about the term structure dynamics has value. Therefore, it would be necessary that the interest rate level falls to the optimal trigger point $\underline{r} = 0.0194$ to induce an idle firm to invest.

Table 5 presents the upper trigger points for the single disinvestment strategy considering different investment cost levels and interest rate volatilities and different fractions of disinvestment proceeds to investment costs. This case corresponds to the one where the option to reinvest becomes worthless since the investment costs to reenter again are extremely high. Thus, idle firms do not invest and active firms will only disinvest if interest rates goes to sufficiently high values, because they know they cannot reinvest again. Considering the volatility level of $\sigma = 0.0854$, an investment cost of $\overline{I} = 10$ and $\alpha = 0.50$, it would be necessary that interest rates would be around 0.40 to induce an active firm to abandon its operations, knowing that its action could not be reversed later. From the table we can take the following conclusions: (i) for a given level of volatility and investment cost the upper threshold falls when the parameter α rises (i.e., the disinvestment proceeds rises); (ii) for a given volatility level and α parameter the upper trigger point falls as the investment cost rises (at the end it originates a rise on the disinvestment proceeds); and (iii) for a given investment cost and α parameter the optimal disinvestment threshold rises as volatility rises. All disinvestment thresholds when uncertainty is considered are higher than the Marshallian disinvestment trigger points. Thus, higher uncertainty produces higher upper trigger points. It should be noted that for high levels of uncertainty firms would never disinvest almost for sure. Even for moderate and low levels of uncertainty it would be necessary that firms could recoup a very high fraction of the investment costs to induce firms to abandon permanently their operations. This issue arises from the fact that operating profits can never become negative and there is no option to reenter again. As we will see below, when the option to invest again is considered these thresholds will fall significantly due to the reentry option effect.

Table 5: Upper thresholds for the disinvestment option of the special case under different levels of disinvestment proceeds and interest rate volatility and different ratios of the disinvestment proceeds to the investment costs.

		$\overline{I} = 10$			$\overline{I} = 7.5$		
T	α	$\sigma = 0.03$	$\sigma = 0.0854$	$\sigma = 0.3$	$\sigma = 0.03$	$\sigma = 0.0854$	$\sigma = 0.3$
500	0.25	0.4225	0.5091	1.3648	0.5555	0.6138	1.4557
1000		0.4225	0.5286	1.4998	0.5555	0.6222	1.5871
500	0.50	0.2251	0.3818	1.1746	0.2900	0.4229	1.2499
1000		0.2260	0.4145	1.3148	0.2901	0.4520	1.3884

500	0.75	0.1659	0.3365	1.0739	0.2044	0.3675	1.1447
1000		0.1707	0.3723	1.2160	0.2062	0.4013	1.2856
500	1.00	0.1402	0.3095	1.0053	0.1659	0.3365	1.0739
1000		0.1479	0.3467	1.1484	0.1707	0.3723	1.2160

Figure 15 presents the project's exit rights (i.e., the value of the option to abandon) and the NPV of the abandonment decision for the volatility level $\sigma = 0.0854$, $\alpha = 0.50$ and $T = 500$. The convex function in the graph represents the abandonment option (i.e., C_3e^{ar}) and the concave function pictures the NPV of the abandonment decision (i.e., $\underline{I} - F(r)$). We can see that for low interest rate levels firms should not disinvest since it is better to keep the option to abandon alive. For low rate values the disinvestment proceeds are lower than the operating profits from the project which turns the NPV of the decision to abandon negative. The interest rate level that turns the traditional NPV of the decision to abandon equal to zero is $r = 0.3373$. But at this rate level the exit rights from the project have a value of 0.6475. Thus, the firm prefers to keep its abandonment option alive. Even if interest rates rise a little above $r = 0.3373$, which would turn the NPV of the abandonment decision positive, it is not optimal that an active firm disinvests immediately. Therefore, it would be necessary that the interest rate level rises to the optimal trigger point $\overline{r} = 0.3818$ to induce an active firm to shut down permanently its operations knowing that the reentry investment cost makes the decision to reinvest later prohibitive.

Looking to the graph we can note that the convex curve representing the option to abandon lies above the concave function representing the NPV of the abandonment decision to the right of the upper trigger point $\overline{r}$. Thus, someone could wonder if the disinvestment decision is optimal only at the upper threshold, and keeping the option to abandon alive the preferred policy for higher interest rate values. Obviously, this thinking is not correct because the exit option has only a valid economic interpretation for interest rate levels lower that the upper trigger point. If this was not the case it would be created a pure speculative bubble. Thus, the value of keeping the exit option alive would be very high because the possibility of reaching a higher interest rate would offer an even higher value of the abandonment option, with no disinvestment action ever in sight. This would turn the value of waiting for better information infinite, which is not economically intuitive. If this was the case, firms would never invest and disinvest because it was always better to wait for better information. In conclusion, the entry and exit options only have a valid interpretation above $\underline{r}$ and below $\overline{r}$, respectively.

Tables 6 and 7 presents both the lower and upper trigger points for the entry and exit combined strategy considering different investment cost levels, $\overline{I} = 10$ for table 6 and $\overline{I} = 7.5$ for table 7 and different interest rate volatilities and fractions of disinvestment proceeds to investment costs. This case corresponds to the one where idle firms are induced to invest if the interest rate value falls to a sufficient low level, but they own an option to abandon later if interest rates rise to very high values. Once the project is abandoned, firms own an option to reinvest again if interest rates reverse to very low levels again. It turns out that this combined strategy originates a range where inaction is the optimal policy, i.e., idle firms do not invest and active firms do not abandon their operations. Considering a volatility level of $\sigma = 0.0854$, an investment cost of $\overline{I} = 10$ and $\alpha = 0.50$, the lower and upper trigger points are, respectively, $\underline{r} = 0.0199$ and $\overline{r} = 0.2641$, which originates a range of inaction of 0.2442 ($\overline{r} - \underline{r}$). Figure 16 depicts this numerical example. It is possible to see that an unique optimal solution exists. In this case, the Marshallian trigger points would be $M_{\underline{r}} = 1/\overline{I} = 0.10$ and $M_{\overline{r}} = 1/\underline{I} = 0.20$, originating a band of inaction of 0.10 ($M_{\overline{r}} - M_{\underline{r}}$). In this case, our lower trigger point is approximately 80 percent below the Marshallian investment threshold and our upper trigger point is approximately 32 percent above the corresponding Marshallian exit point. Therefore, uncertainty and the embedded option values are responsible for the hysteretic band widening.

Considering these parameters, it is possible to see that the lower trigger point for the single investment strategy (i.e., without the exit option) is $\underline{r} = 0.0194$ (see table 4). For the combined strategy, this lower trigger point is now $\underline{r} = 0.0199$ (see table 6). The small difference in value arises from the firm's possibility to shut down later if interest rates start rising for very high levels. Similarly, the upper trigger point for the single disinvestment strategy (i.e., without the reentry option) is $\overline{r} = 0.3818$ (see table 5). For the combined strategy, the upper trigger point is now $\overline{r} = 0.2641$ (see table 6). This difference comes from the value of the reentry option that is owned by the firm. But now the difference is much more pronounced. There is an economic explanation for this fact. Thus, when the firm invests in a project and has an option to shut down in the future, the disinvestment proceeds present value is very small because the prospect of close its operations is sufficiently far in the future. As a result, the impact on the lower trigger point is very diminutive. However, when a firm is operating and decides to shut down its activities it will receive almost immediately the disinvestment proceeds value, which originates a bigger present value and justifies the greater differences between the two upper trigger points.

We can take the following conclusions from the tables: (i) for a given level of volatility and investment cost the lower threshold rises and the upper threshold falls when the parameter α rises (i.e., the disinvestment proceeds rises). Thus, the range of inaction will be narrower; (ii) for a given volatility level and α parameter both the lower and upper trigger points falls as the investment cost rises (at the end it originates a rise on the disinvestment proceeds); and (iii) for a given investment cost and α parameter the optimal lower trigger point falls and the upper threshold rises as volatility rises, widening the hysteretic range[31].

Table 6: Upper and lower thresholds for the switching option of the special case under an investment cost of 10 for different ratios of the disinvestment proceeds to the investment costs and different interest rate volatilities.

		$\overline{I} = 10$					
		$\sigma = 0.03$		$\sigma = 0.0854$		$\sigma = 0.3$	
T	α	$\underline{r}$	$\overline{r}$	$\underline{r}$	$\overline{r}$	$\underline{r}$	$\overline{r}$
500	0.25	0.0809	0.4225	0.0194	0.4725	−0.4467	0.5922
1000		0.0809	0.4225	0.0194	0.4725	−0.4467	0.5922
500	0.50	0.0810	0.2239	0.0199	0.2641	−0.4427	0.2217
1000		0.0810	0.2239	0.0199	0.2641	−0.4427	0.2217

[31]It should be noted that this is not always true for the upper trigger point. Thus, for very high values of α and investment costs, which produces high disinvestment proceeds, the upper threshold starts rising as the volatility rises, but for very high volatility levels the upper threshold turns its behaviour and starts falling. A possible explanation for this issue may come from the "bird-in-the-hand" argument. Thus, since uncertainty is so high and the value of the disinvestment proceeds is significant, firms may be induced to shut down at lower rates in order to get a safe present value of the disinvestment proceeds.

500	0.75	0.0815	0.1556	0.0233	0.1717	−0.4273	0.0061
1000		0.0815	0.1556	0.0233	0.1717	−0.4273	0.0061
500	1.00	0.0991	0.0991	0.0671	0.0671	−0.2872	−0.2872
1000		0.0991	0.0991	0.0671	0.0671	−0.2872	−0.2872

Table 7: Upper and lower thresholds for the switching option of the special case under an investment cost of 7.5 for different ratios of the disinvestment proceeds to the investment costs and different interest rate volatilities.

		$\overline{I} = 7.5$					
		$\sigma = 0.03$		$\sigma = 0.0854$		$\sigma = 0.3$	
T	α	$\underline{r}$	$\overline{r}$	$\underline{r}$	$\overline{r}$	$\underline{r}$	$\overline{r}$
500	0.25	0.1145	0.5555	0.0642	0.6029	−0.3447	0.7614
1000		0.1145	0.5555	0.0642	0.6029	−0.3447	0.7614
500	0.50	0.1145	0.2899	0.0645	0.3355	−0.3412	0.3497
1000		0.1145	0.2899	0.0645	0.3355	−0.3412	0.3497
500	0.75	0.1148	0.2005	0.0676	0.2271	−0.3262	0.1200
1000		0.1148	0.2005	0.0676	0.2271	−0.3262	0.1200
500	1.00	0.1331	0.1331	0.1127	0.1127	−0.1838	−0.1838
1000		0.1331	0.1331	0.1127	0.1127	−0.1838	−0.1838

We know that it is extremely complicated to analyze analytically the impact of the σ and λ parameters on the optimal trigger points due to the effects they produce on the value function $V(r)$, because both enter in the quadratic equation with roots a and b. However, we can resort some numerical simulations that can highlight their effects. For greater visual appeal we show the corresponding pictures as continuous curves and not as step functions. Figure 17 presents the impact on the entry and exit thresholds as the volatility rises, considering $\overline{I} = 10$, $\alpha = 0.5$

and $\lambda = 0.0$. Clearly, there is a tendency to a wider range of inaction as the volatility rises, as we have already mentioned before. Figure 18 shows the impact on the upper and lower trigger points as the price of interest rate risk rises in absolute value, considering $\overline{I} = 10$, $\alpha = 0.5$ and $\sigma = 0.0854$. It should be noted that positive premiums exist if $\lambda < 0$, but for easy visualization we use the parameter absolute value in this simulation. Thus, as the price of interest rate risk rises there is a tendency for a fall in the upper trigger point and a rise in the lower threshold, which originates a narrower hysteretic band. There is an economic explanation for this. Under the equivalent martingale economy, a higher positive term premia indicates that long-term rates are high relative to the instantaneous rate. This means that interest rates are expected to increase. Thus, with higher positive premiums it is more important to make the investment before the likely rise in the interest rate (see Ingersoll and Ross (1992)). This explains the tendency for a rise in the lower threshold. A similar argument can be used for the upper trigger point. Thus, with the expected increase in the interest rates firms are induced to abandon earlier otherwise the present value of the disinvestment proceeds will be lower due to the prospect of a rise in the interest rate. As a result, the upper threshold has a tendency to fall.

7 Conclusions

When interest rates fall, firms make durable investments, that is to say that they switch from cash (an immediate asset) to longer lived assets with cash flows further ahead in time. When interest rates rise, they will stop undertaking any durable projects. Furthermore if flexibility exists they will also try and reverse the investment process, i.e. divest away from projects with long lived cash flows into projects with more immediate payoffs.

We model this investment hysteresis explicitly using the most tractable form of interest rate uncertainty and describe the sensitivity of the interest rate band to the model parameters. We do this to analyze the beneficial effects of waiting to invest in the presence of uncertainty as well as to shed light on the macro influence of interest rate changes on investment policies.

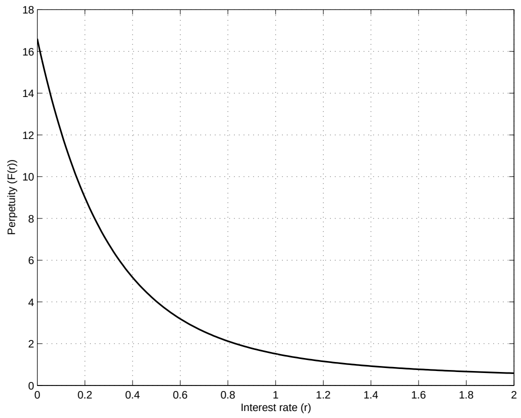

Figure 1: Value of a perpetuity as a function of the interest rate using the base case parameter values.

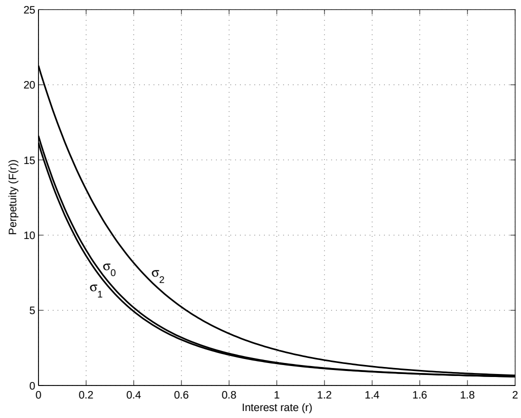

Figure 2: Value of a perpetuity as a function of the interest rate using the base case parameter values for different levels of volatility ($\sigma_0 = 0.0854, \sigma_1 = 0.03$ and $\sigma_2 = 0.3$).

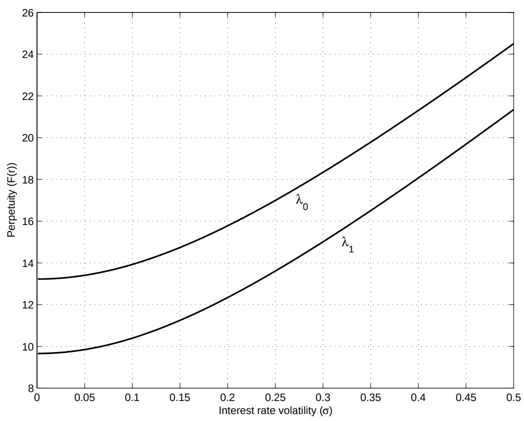

Figure 3: Value of a perpetuity as a function of the interest rate volatility using the base case parameter values for different levels of risk premium ($\lambda_0 = 0$ and $\lambda_1 = -0.1$) and $r(0) = 0.06$.

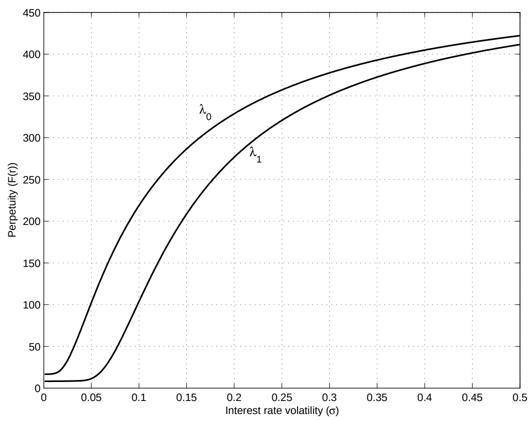

Figure 4: Value of a perpetuity as a function of the interest rate volatility using the special case parameter values for different levels of risk premium ($\lambda_0 = 0$ and $\lambda_1 = -0.1$) and $r(0) = 0.06$.

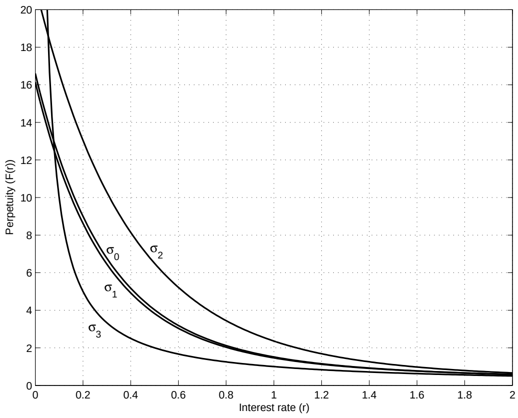

Figure 5: Value of a perpetuity as a function of the interest rate using the base case parameter values for different levels of volatility ($\sigma_0 = 0.0854$, $\sigma_1 = 0.03$, $\sigma_2 = 0.3$ and $\sigma_3 = 0$).

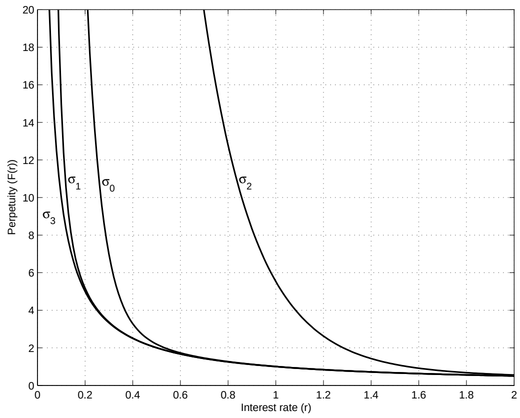

Figure 6: Value of a perpetuity as a function of the interest rate using the special case parameter values for different levels of volatility ($\sigma_0 = 0.0854$, $\sigma_1 = 0.03$, $\sigma_2 = 0.3$ and $\sigma_3 = 0$).

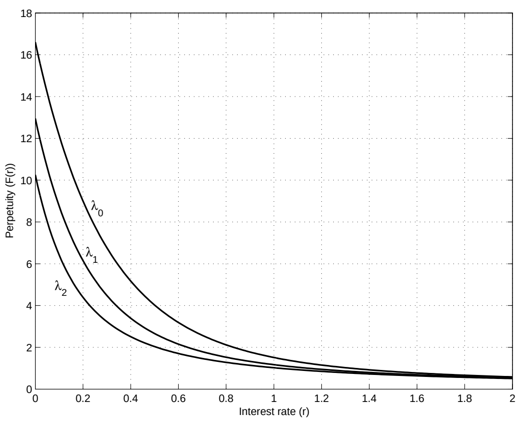

Figure 7: Value of a perpetuity as a function of the interest rate using the base case parameter values for different levels of risk premium ($\lambda_0 = 0.0$, $\lambda_1 = -0.1$ and $\lambda_2 = -0.2$).

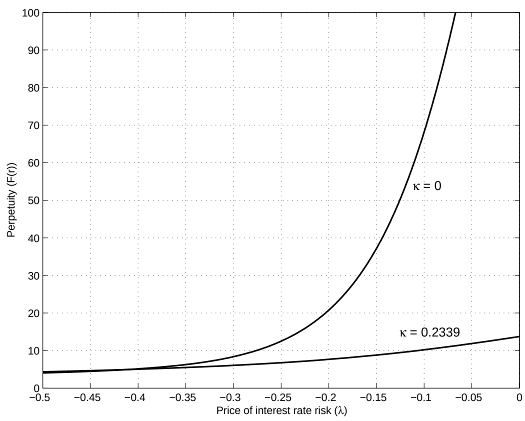

Figure 8: Value of a perpetuity as a function of the risk premium using the base and special cases parameter values and $r(0) = 0.06$.

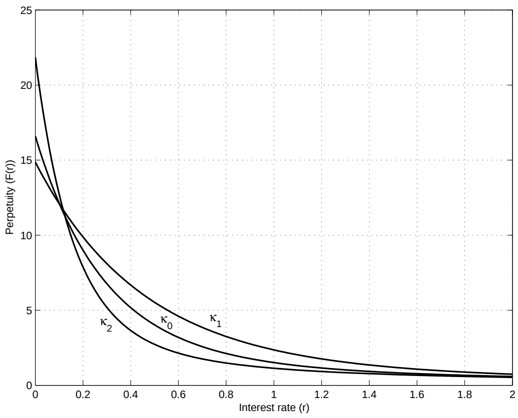

Figure 9: Value of a perpetuity as a function of the interest rate using the base case parameter values for different levels of speed of adjustment ($\kappa_0 = 0.2339$, $\kappa_1 = 0.4$ and $\kappa_2 = 0.1$).

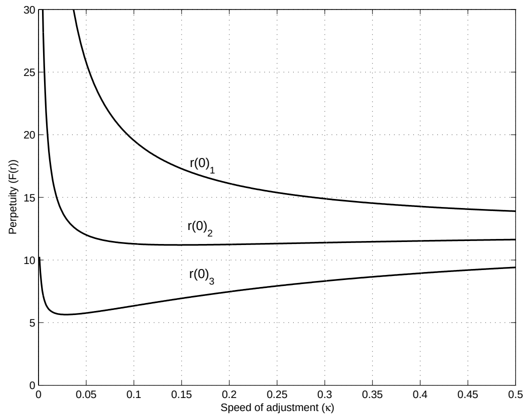

Figure 10: Value of a perpetuity as a function of the speed of adjustment using the base case parameter values for different levels of interest rate ($r(0)_1 = 0.02$, $r(0)_2 = 0.1241$ and $r(0)_3 = 0.25$).

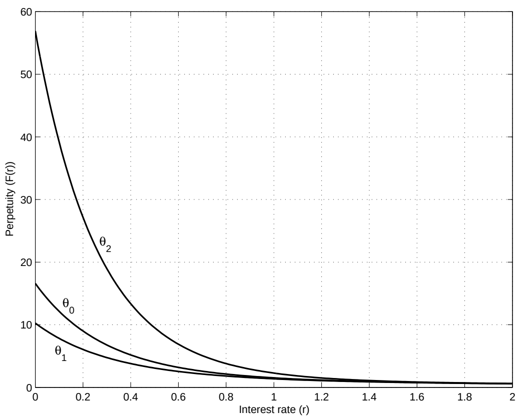

Figure 11: Value of a perpetuity as a function of the interest rate using the base case parameter values for different levels of asymptotic interest rate ($\theta_0 = 0.0808$, $\theta_1 = 0.15$ and $\theta_2 = 0.02$).

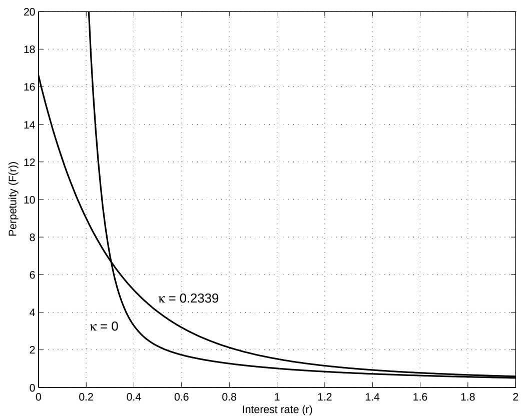

Figure 12: Value of a perpetuity as a function of the interest rate using the base and special cases parameter values.

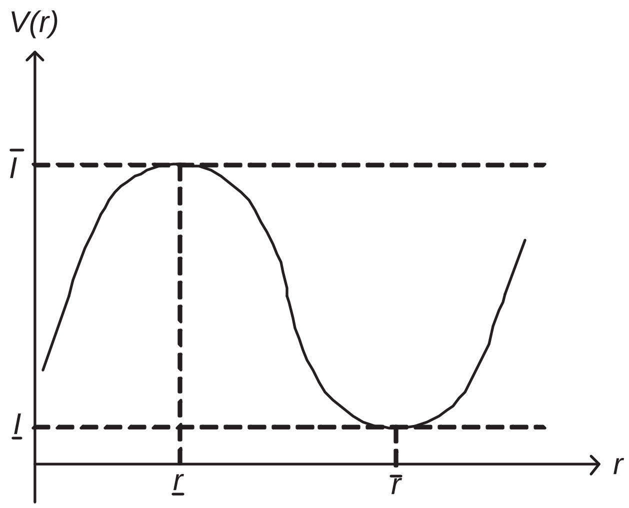

Figure 13: Determination of the theoretical upper and lower thresholds.

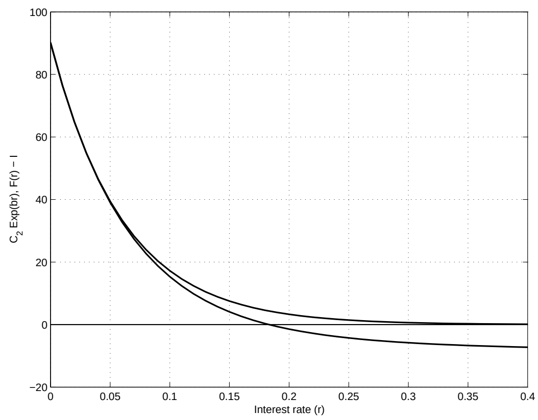

Figure 14: Value of entry rights and NPV of the project.

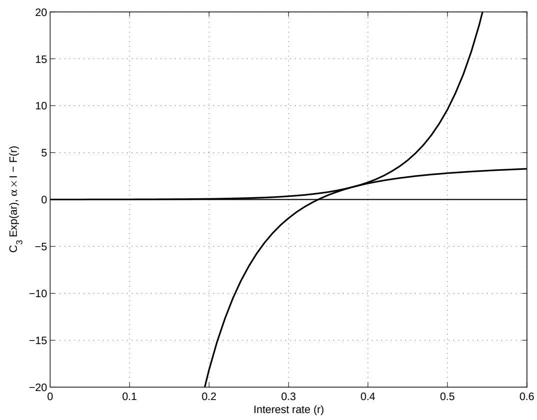

Figure 15: Value of exit rights and NPV of the abandonment decision.

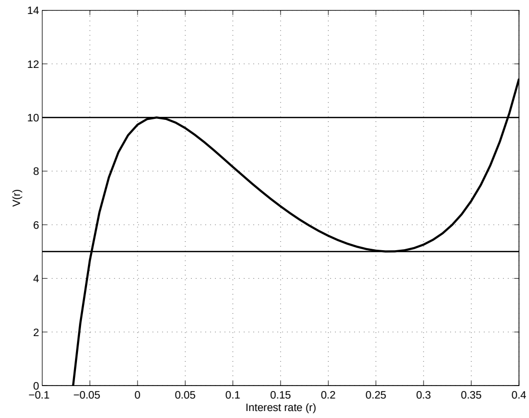

Figure 16: Determination of the numerical upper and lower thresholds.

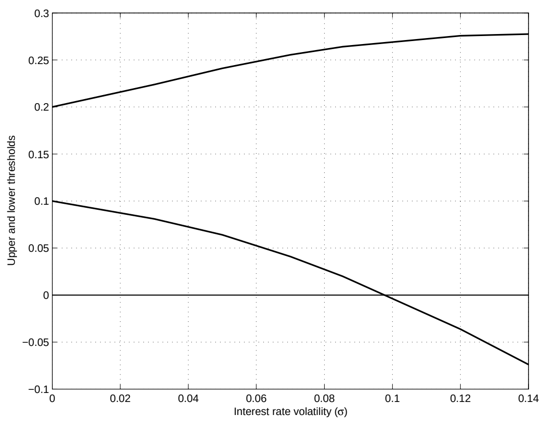

Figure 17: Entry and exit thresholds as functions of interest rate volatility.

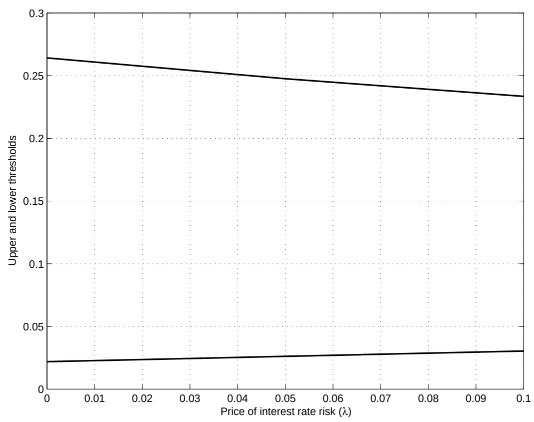

Figure 18: Entry and exit thresholds as functions of interest rate risk price.

References

Akerlof, G. A. (1970). The market for "lemons": Quality uncertainty and the market mechanism. *Quarterly Journal of Economics*, 84(3):488–500.

Arrow, K. J. (1968). Optimal capital policy with irreversible investment. In Wolfe, J. N., editor, *Value, Capital, and Growth: Papers in Honour of Sir John Hicks*, pages 1–19. Edinburgh University Press, Edinburgh.

Bernanke, B. S. (1983a). Irreversibility, uncertainty, and cyclical investment. *Quarterly Journal of Economics*, 98(1):85–106.

Bertola, G. (1998). Irreversible investment. *Research in Economics*, 52(1):3–37.

Björk, T. (2004). *Arbitrage Theory in Continuous Time*. Oxford University Press, Oxford, UK, 2nd edition.

Black, F. and Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3):637–654.

Brennan, M. J. and Schwartz, E. S. (1980). Analyzing convertible bonds. *Journal of Financial and Quantitative Analysis*, 15(4):907–929.

Brennan, M. J. and Schwartz, E. S. (1985). Evaluating natural resource investments. *Journal of Business*, 58(2):135–157.

Campbell, J. Y., Lo, A. W., and MacKinlay, A. C. (1997). *The Econometrics of Financial Markets*. Princeton University Press, Princeton, New Jersey.

Caplin, A. and Leahy, J. (1991). State-dependent pricing and the dynamics of money and output. *Quarterly Journal of Economics*, 106(3):683–708.

Chan, K. C., Karolyi, G. A., Longstaff, F. A., and Sanders, A. B. (1992). An empirical comparison of alternative models of the short-term interest rate. *Journal of Finance*, 47(3):1209–1227.

Constantinides, G. M. (1986). Capital market equilibrium with transaction costs. *Journal of Political Economy*, 94(4):842–862.

Constantinides, G. M. and Richard, S. F. (1978). Existence of optimal simple policies for discounted-cost inventory and cash management in continuous time. *Operations Research*, 26(4):620–636.

Courtadon, G. (1982). The pricing of options on default-free bonds. *Journal of Financial and Quantitative Analysis*, 17(1):75–100.

Cox, J. C., Ingersoll, Jr., J. E., and Ross, S. A. (1980). An analysis of variable rate loan contracts. *Journal of Finance*, 35(2):389–403.

Cox, J. C., Ingersoll, Jr., J. E., and Ross, S. A. (1985a). An intertemporal general equilibrium model of asset prices. *Econometrica*, 53(2):363–384.

Cox, J. C., Ingersoll, Jr., J. E., and Ross, S. A. (1985b). A theory of the term structure of interest rates. *Econometrica*, 53(2):385–408.

Cox, J. C. and Ross, S. A. (1976). The valuation of options for alternative stochastic processes. *Journal of Financial Economics*, 3:145–166.

Delbaen, F. (1993). Consols in the CIR model. *Mathematical Finance*, 3(2):125–134.

Dixit, A. K. (1989a). Entry and exit decisions under uncertainty. *Journal of Political Economy*, 97(3):620–638.

Dixit, A. K. (1989b). Hysteresis, import penetration, and exchange rate pass-through. *Quarterly Journal of Economics*, 104(2):205–228.

Dixit, A. K. (1991a). Analytical approximations in models of hysteresis. *Review of Economic Studies*, 58(1):141–151.

Dixit, A. K. (1991b). A simplified treatment of the theory of optimal regulation of brownian motion. *Journal of Economic Dynamics and Control*, 15(4):657–673.

Dixit, A. K. (1993). *The Art of Smooth Pasting*, volume 55 of *Fundamentals of Pure and Applied Economics*. Harwood Academic Publishers, Chur, Switzerland.

Dixit, A. K. and Pindyck, R. S. (1994). *Investment Under Uncertainty.* Princeton University Press, Princeton, New Jersey.

Dothan, L. U. (1978). On the term structure of interest rates. *Journal of Financial Economics*, 6:59–69.

Dumas, B. (1991). Super contact and related optimality conditions. *Journal of Economic Dynamics and Control*, 15(4):675–685.

Dumas, B. (1992). Dynamic equilibrium and the real exchange rate in a spatially separated world. *Review of Financial Studies*, 5(2):153–180.

Dumas, B. and Luciano, E. (1991). An exact solution to a dynamic portfolio choice problem under transaction costs. *Journal of Finance*, 46(2):577–595.

Feller, W. (1951). Two singular diffusion problems. *Annals of Mathematics*, 54(1):173–182.

Geman, H. and Yor, M. (1993). Bessel processes, asian options, and perpetuities. *Mathematical Finance*, 3(4):349–375.

Göing-Jaeschke, A. and Yor, M. (2003). A survey and some generalizations of Bessel processes. *Bernoulli*, 9(2):313–349.

Grossman, S. J. and Laroque, G. (1990). Asset pricing and optimal portfolio choice in the presence of illiquid durable consumption goods. *Econometrica*, 58(1):25–51.

Harrison, J. M. (1985). *Brownian Motion and Stochastic Flow Systems.* John Wiley and Sons, New York, NY.

Harrison, J. M. and Kreps, D. M. (1979). Martingales and arbitrage in multiperiod securities markets. *Journal of Economic Theory*, 20:381–408.

Harrison, J. M. and Pliska, S. R. (1981). Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Processes and their Applications*, 11:215–260.

Harrison, J. M., Sellke, T. M., and Taylor, A. J. (1983). Impulse control of brownian motion. *Mathematics of Operations Research*, 8(3):454–466.

Harrison, J. M. and Taksar, M. I. (1983). Instantaneous control of brownian motion. *Mathematics of Operations Research*, 8(3):439–453.

Harrison, J. M. and Taylor, A. J. (1978). Optimal control of a brownian storage system. *Stochastic Processes and Their Applications*, 6:179–194.

Hull, J. C. (2002). *Options, Futures, and Other Derivatives.* Prentice Hall, Upper Saddle River, NJ, 5th edition.

Ingersoll, Jr., J. E. and Ross, S. A. (1992). Waiting to invest: Investment and uncertainty. *Journal of Business*, 65(1):1–29.

Jamshidian, F. (1989). An exact bond option formula. *Journal of Finance*, 44(1):205–209.

Jorgenson, D. W. (1963). Capital theory and investment behavior. *American Economic Review*, 53(2):247–259.

Karlin, S. and Taylor, H. M. (1981). *A Second Course in Stochastic Processes.* Academic Press, San Diego.

Liu, H. (2004). Optimal consumption and investment with transaction costs and multiple risky assets. *Journal of Finance*, 59(1):289–338.

McDonald, R. L. and Siegel, D. R. (1985). Investment and the valuation of firms when there is an option to shut down. *International Economic Review*, 26(2):331–349.

McDonald, R. L. and Siegel, D. R. (1986). The value of waiting to invest. *Quarterly Journal of Economics*, 101(4):707–728.

McKean, Jr., H. P. (1965). Appendix: A free boundary problem for the heat equation arising from a problem of mathematical economics. *Industrial Management Review*, 6(2):32–39.

Merton, R. C. (1973a). Theory of rational option pricing. *Bell Journal of Economics and Management Science*, 4(1):141–183.

Merton, R. C. (1973b). An intertemporal capital asset pricing model. *Econometrica*, 41(5):867–887.

Musiela, M. and Rutkowski, M. (1998). *Martingale Methods in Financial Modelling.* Springer-Verlag, Berlin.

Nickell, S. J. (1974a). On the role of expectations in the pure theory of investment. *Review of Economic Studies*, 41(1):1–19.

Nickell, S. J. (1974b). On expectations, government policy and the rate of investment. *Economica*, 41(163):241–255.

Pindyck, R. S. (1988). Irreversible investment, capacity choice, and the value of the firm. *American Economic Review*, 78(5):969–985.

Pitman, J. and Yor, M. (1982). A decomposition of Bessel bridges. *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, 59:425–457.

Rendleman, Jr., R. J. and Bartter, B. J. (1980). The pricing of options on debt securities. *Journal of Financial and Quantitative Analysis*, 15(1):11–24.

Revuz, D. and Yor, M. (1999). *Continuous Martingales and Brownian Motion.* Springer-Verlag, Berlin, 3rd edition.

Rogers, L. C. G. (1995). Which model for term-structure of interest rates should one use? In *Mathematical Finance, IMA Volume 65*, pages 93–116, New York. Springer.

Rogers, L. C. G. (1996). Gaussian errors. *Risk*, 9(1):42–45.

Ross, S. A. (1995). Uses, abuses, and alternatives to the net-present-value rule. *Financial Management*, 24(3):96–102.

Samuelson, P. A. (1965). Rational theory of warrant pricing. *Industrial Management Review*, 6(2):13–32.

Shackleton, M. and Wojakowski, R. (2001). Flow options: Continuous real caps and floors. Working Paper 2001/007, Lancaster University.

Vasicek, O. (1977). An equilibrium characterization of the term structure. *Journal of Financial Economics*, 5:177–188.

Yor, M. (1993). On some exponential-integral functionals of Bessel processes. *Mathematical Finance*, 3(2):231–240.