# The Value of Flexibility in Sequencing Growth Investment*

Peter M. Kort[1,2], Pauli Murto[3], Grzegorz Pawlina[4],
[1]Department of Econometrics & Operations Research and CentER,
Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands
[2]Department of Economics, University of Antwerp, Prinsstraat 13,
2000 Antwerp 1, Belgium
[3]Department of Economics, Helsinki School of Economics,
P.O. Box 1210, FIN-00101, Finland
[4]Department of Accounting and Finance, Management School,
Lancaster University, LA1 4YX, United Kingdom

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**Abstract**

We analyze the investment decision of a firm that has an option to complete an investment project either in one lump or in two smaller parts at distinct points in time. The firm faces a trade-off between the cost savings that arise when the project is completed at once and the additional flexibility that arises when the firm is able to respond to resolving uncertainty by choosing optimal timing individually for each stage. We derive the optimal investment policy and show that, contrary to our initial presumption, higher uncertainty makes the lump investment more attractive relative to the apparently more flexible alternative of splitting the investment.

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## 1 Introduction

One of the central issues in the recent literature on investment is the relationship between uncertainty and investment. The classic result is that higher uncertainty leads to a higher critical level of the relevant state variable at which the investment optimally occurs (see e.g. Dixit and Pindyck, 1994). However, the impact of uncertainty on investment goes far beyond a mere relationship between the volatility of the state variable and the optimal investment threshold. For example, uncertainty also influences the optimal size of an investment project (cf. Capozza and Li, 1994, Bar-Ilan and Strange, 1998). This is basically due to the fact that when a firm faces a choice between mutually exclusive real options, the degree of uncertainty may determine which of them will be optimally exercised, as explained in Dixit (1993).

There are also other types of choices that firms may face when investing. In this paper we look at the choice between different degrees of flexibility in proceeding with investment. More precisely, we study optimal investment policy under circumstances, where it is possible to accomplish a given project either in one lump or in pieces. To be more concrete, let us think, for example, of: an entry into a new market segment with the possibility to either proceed in steps or through a single launch; construction of production capacity with the possibility to build either many small plants or one big plant; or adoption of a new technology with the possibility to either learn about its properties by switching first only partly or to switch completely at once. In all such cases, one may envision firms considering whether it is best to wait for the right time to carry out the whole project at once, or whether it is better to carry out a part of the project earlier, and to accomplish the rest later when the market has grown enough.

This leads to a trade-off of the following kind. If the whole project is undertaken in one lump, the firm exploits economies of scale and typically saves on costs. By building a large plant instead of two or three smaller ones, a firm might be able to reduce its average cost and increase its profitability. On the other hand, by sequencing the project, the firm has more freedom in responding to the gradually resolving uncertainty as each step may be accomplished at its individually optimal time. In this sense, the first alternative is more cost-efficient while the latter is more flexible. In this paper we ask what the optimal investment policy in such a situation is like and how it is affected by changes in the firms' environment. In particular, we are interested to see how the degree of uncertainty affects the firm's optimal investment policy, that is, the choice between an apparently more flexible alternative of sequencing the investment, and the less flexible alternative of undertaking the whole project in one lump.

To answer this, we set up a stylized model that follows closely the spirit of standard models of irreversible investment under uncertainty. As in the prototype model of McDonald and Siegel (1986), we model a firm that must choose the optimal time to invest in an irreversible project whose payoff depends on an exogenous stochastic process. However, in our model the firm faces two possibilities. One is to undertake the whole project at once, and the other is to undertake the project in two separate stages at different points in time. If the firm decides to undertake the project in two separate stages, it gains flexibility in choosing the optimal timing of investment separately for each stage and it can refrain from committing resources in the second stage if the market conditions become unfavorable. On the other hand, undertaking the project in two stages is assumed to be more costly than a lump investment. Thus, there is a trade-off between flexibility and the level of investment expenditure necessary to complete the entire project. Hence, it is important to value this flexibility. To do so, we apply the theory of real options (see, e.g. Dixit and Pindyck, 1994).[1]

The main question we address is how the degree of uncertainty affects the choice between the lumpy and the sequential investment. Consider for instance demand uncertainty. If the firm irreversibly invests in a large addition to capacity, and demand grows only slowly or even shrinks, it will find itself holding capital it does not need. Hence when the growth of demand is uncertain there is a tradeoff between scale economies and the flexibility that is gained by investing more frequently in small increments to capacity as they are needed. This would lead to the conclusion that higher uncertainty favors sequential investment, and in fact this is confirmed by Dixit and Pindyck (1994, pp. 51-54), who illustrate this by analyzing the choice between investing in a large coal-fired plant or in two smaller oil-fired plants. In this light, our model provides a surprising answer: the higher the uncertainty, the more preferable the lumpy project is relative to sequencing the project. This may seem counterintuitive indeed given that one of the main lessons of standard real options is that the higher the level of uncertainty, the more firms benefit from various forms of flexibility. In our case, quite on the contrary, the higher the level of uncertainty, the less valuable is the more flexible alternative is relative to the more cost-efficient alternative.

To understand the result, it is important to know that the real investment options are convex functions of the project values. Now, based on the argument analogous to Jensen's inequality, it follows that adding up the option values of each stage of the sequential project gives a greater value than the option on the sum of all stages of the sequential project. These two values can be made equal by making the sequential project more expensive, thus having as investment expenditure $\hat{\kappa}I$ with $\hat{\kappa} > 1$. Now, we use the property that an increase in uncertainty reduces convexity of the options as functions of project values. As a consequence, the effect of Jensen's inequality is smaller. This leads to a lower level of $\hat{\kappa}$.

To conclude this section, it is worthwhile to emphasize that our model allows a broader interpretation than a single investment project accomplished in stages. Optimal adoption of

[1] A related model is presented in Décamps, Mariotti and Villeneuve (2003). They study the choice between a small and a large project, where choosing the small project allows one later to re-invest in the large project. Technically the difference is that we allow an arbitrary division of the costs between the two stages of the sequential project, whereas Décamps et al. (2003) assume that undertaking the large project costs the same irrespective of whether or not the small project has already been undertaken.

new technologies provides an important example that also fits to our model framework. By adopting an intermediate technology, the firm will be able to implement the next-generation technology at a lower cost (due to learning, for example) and will produce more efficiently in the meantime. This corresponds to sequential investment. However, the total cost of technology adoption will be higher for such a firm than for the one that decides to "leapfrog" directly to the next-generation technology. This latter alternative corresponds to lumpy investment. In a similar context, Grenadier and Weiss (1997) provide a result that resembles ours. In a model with sequential technological innovations, they show that increased uncertainty favors waiting until the final technology is invented. However, in their model uncertainty concerns the arrival time of an improved technology, whereas in our model it concerns the market environment. Consequently, in their model the improved technology is adopted at an exogenously determined moment, but in our model the timing is endogenously determined. It should also be noted that Grenadier and Weiss derive their result by numerical simulations, while our results are derived analytically.

Further, our model can also be viewed as a problem, where a firm faces two investment projects, whose rewards are dependent on a single state variable. If the projects are considered separately, it is optimal to undertake them at different dates. However, it is possible to bundle the projects (i.e. undertake them simultaneously), which gives a discount on the total cost. The problem of the firm is to decide under which conditions bundling is optimal. Another example of similar flavor would be the purchase decisions of a consumer, who may buy different goods separately each at its individually optimal time, or to purchase them together at a discounted price. Finally, our framework can be applied to analyze the takeover decision of a firm that faces a choice between acquiring a block of shares or the entire target company.

The remainder of the paper is organized as follows. Section 2 contains the description of the model, whereas in Section 3 the optimal investment policy is presented. The role of uncertainty is analyzed in Section 4 and Section 5 concludes. All proofs are relegated to the appendix.

## 2 Model

The model is a variant of the prototype model of irreversible investment under uncertainty presented in McDonald and Siegel (1986), and further elaborated in a large number of papers. An extensive summary of many extensions is given in Dixit and Pindyck (1994).

There is a risk neutral firm, which operates in continuous time with an infinite horizon and discounts its cash flows with a constant rate $r$. The firm faces a single investment opportunity, which it can accomplish either in one lump or in two separate stages. The timing of investment and the type of investment (lumpy vs. sequential) is to be chosen optimally in order to maximize the value of the firm. Initially, the firm earns no revenues.

Once it has invested, it earns an instantaneous profit given by

$$\pi_t = Y_t R_i, \tag{1}$$

where $R_i$ is a constant that stands for the deterministic part of the profit when $i \in \{1, 2\}$ stages have been accomplished. By accomplishing the project in one lump, the firm moves directly from profit flow 0 to $Y_t R_2$ (lumpy investment), while by splitting the project, the firm moves first from 0 to $Y_t R_1$, and later from $Y_t R_1$ to $Y_t R_2$ (sequential investment). To have a sensible problem, we assume $0 < R_1 < R_2$.

The variable $Y_t$ represents a multiplicative shock that follows a geometric Brownian motion:

$$dY_t = \mu Y_t dt + \sigma Y_t d\omega_t, \tag{2}$$

where $Y_0 > 0$, $0 < \mu < r$, $\sigma > 0$, and the $d\omega$'s are independently and identically distributed according to a normal distribution with mean zero and variance $dt$.

The cost of investment depends on whether the project is accomplished in one or two steps. We define two parameters, $\alpha$ and $\kappa$, to specify the interrelation of different cost terms. In case of lumpy investment the investment cost is simply $I$, while in case of sequential investment, the associated investment costs for the first and second stages are $I_1 \equiv \alpha\kappa I$ and $I_2 \equiv (1 - \alpha)\kappa I$, respectively. Here, $\alpha$ represents the cost share of stage 1 of the total cost and $\kappa \geq 1$ reflects the premium for flexibility that must be paid in order to be able to split the project.

We next assume decreasing returns to scale, i.e. $R_1 > \alpha R_2$. If this did not hold, the firm could never benefit from the possibility to split the project, and the lumpy project with no cost premium would trivially dominate.

The firm has to decide about both the timing and the type of investment. In the next section, we will derive the optimal investment policy in three steps: first, we establish the optimal investment policy when only the lumpy investment alternative is available, then we do the same when only the sequential policy alternative is available, and finally, we consider the whole problem where both alternatives are available.

# 3 Optimal Investment Policy

## 3.1 Only single-stage investment available

The situation in which the firm can only invest in the whole project at once corresponds exactly to the basic model of investment under uncertainty as described in McDonald and Siegel (1986), and analyzed further in Dixit and Pindyck (1994). It is well known that the optimal investment policy is a trigger strategy such that it is optimal to invest whenever the current value of $Y$ is above a certain threshold level, which we denote by $Y_L$. The standard procedure to solve the problem is to set up the dynamic programming equation for the value function $F_L(Y)$, where the application of Itô's lemma and appropriate boundary conditions are used to determine the exact form of $F_L(Y)$ and the value of $Y_L$. We merely state the result here, see Dixit and Pindyck (1994) for details. The investment threshold is:

$$Y_L = \frac{\beta_1}{\beta_1 - 1} \frac{(r-\mu) I}{R_2}, \tag{3}$$

where

$$\beta_1 = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 1. \tag{4}$$

In the continuation region, that is when $Y < Y_L$, the value of the option to invest is:

$$F_L(Y) = \left(\frac{Y_L R_2}{r-\mu} - I\right)\left(\frac{Y}{Y_L}\right)^{\beta_1}. \tag{5}$$

### 3.2 Only sequential investment available

Now, consider the case in which the firm splits the project into two stages. The option to invest in the first stage may be seen as a compound option, since accomplishing the first stage gives an option to proceed to the second stage.[2] However, it will turn out that due to our assumption $R_1 > \alpha R_2$, the problem collapses into two single-project investment problems that are identical to that considered in section 3.1. Nevertheless, we derive the solution by first analyzing the optimal investment rule of stage 2, and use that to determine the optimal investment threshold of stage 1.

The second stage problem is clearly identical to the one considered in section 3.1, because one may see the second stage investment as a single project that gives an additional profit flow $Y(R_2 - R_1)$ at cost $I_2$. Thus, the only modification to section 3.1 is to replace $R_2$ by $R_2 - R_1$ and $I$ by $I_2$. The optimal investment threshold is:

$$Y_2 = \frac{\beta_1}{\beta_1 - 1} \frac{(r-\mu) I_2}{(R_2 - R_1)}, \tag{6}$$

and the value of the option to invest in stage 2 given that $Y < Y_2$ is:

$$F_2(Y) = \left(\frac{Y_2 (R_2 - R_1)}{r-\mu} - I_2\right)\left(\frac{Y}{Y_2}\right)^{\beta_1}. \tag{7}$$

Now, consider the investment decision of stage 1. By setting up the Bellman equation, applying Itô's lemma, and employing a boundary condition that sets the value of the investment option to zero as $Y$ approaches zero, one can easily show that the value of the opportunity

[2]See Bar-Ilan and Strange (1998) for a somewhat more complicated model of sequential investment that incorporates investment lags.

to undertake the first stage investment must be of the form $F_S(Y) \equiv A_1 Y^{\beta_1}$. Further, the following value-matching and smooth-pasting conditions must be satisfied at the optimal investment threshold $Y_1$ (again, see Dixit and Pindyck, 1994, for details):

$$A_1 Y_1^{\beta_1} = \frac{Y_1 R_1}{r-\mu} - I_1 + \left(\frac{Y_2 (R_2 - R_1)}{r-\mu} - I_2\right)\left(\frac{Y_1}{Y_2}\right)^{\beta_1} \tag{8}$$

$$\beta_1 A_1 Y_1^{\beta_1 - 1} = \frac{R_1}{r-\mu} + \beta_1 \left(\frac{Y_2 (R_2 - R_1)}{r-\mu} - I_2\right)\left(\frac{Y_1}{Y_2}\right)^{\beta_1 - 1}. \tag{9}$$

From these, one may solve $A_1$ and $Y_1$, the latter being:

$$Y_1 = \frac{\beta_1}{\beta_1 - 1}\frac{(r-\mu) I_1}{R_1}. \tag{10}$$

Clearly, $Y_1 < Y_2$ under our assumption $R_1 > \alpha R_2$, so the first stage is accomplished strictly earlier than the second stage. Solving for $A_1$ from (8) and (9) yields the value of the (compound) option to invest sequentially in the project:

$$F_S(Y) = \left(\frac{Y_1 R_1}{r-\mu} - I_1\right)\left(\frac{Y}{Y_1}\right)^{\beta_1} + \left(\frac{Y_2 (R_2 - R_1)}{r-\mu} - I_2\right)\left(\frac{Y}{Y_2}\right)^{\beta_1} \tag{11}$$

The value function (11) applies in the continuation region $Y < Y_1$. Comparing the expression (10) with (3), one can see that the optimal investment threshold of the first stage, $Y_1$, is equal to the investment threshold of an investor, who only has a standard investment opportunity leading to profit flow $YR_1$ with no further options. In other words, the existence of the second stage has no effect on the optimal exercise time of the first stage. This result is due to the special structure of optimal stopping problems that also underlies the main conclusions of Leahy (1993) and Baldursson and Karatzas (1997), according to which an investor, who must take into account subsequent investments of the competitors, employs the same investment policy as a monopolist who is not threatened by such future events.

In summary, the optimal investment thresholds $Y_1$ and $Y_2$ are obtained by solving separately two investment problems: one providing profit flow $YR_1$ at cost $I_1$, and one providing profit flow $Y(R_2 - R_1)$ at cost $I_2$, respectively. Moreover, note from (11) that the value of the opportunity to invest in the first stage is the sum of the two options considered separately, that is, $F_S(Y) = F_1(Y) + F_2(Y)$, where $F_1(Y) = \left(\frac{Y_1 R_1}{r-\mu} - I_1\right)\left(\frac{Y}{Y_1}\right)^{\beta_1}$ is the value of a firm which can invest in the first stage only.

### 3.3 General problem

So far, we have determined the option values and the optimal investment thresholds for the lump and the sequential investment separately. Now we derive the investment policy, which includes the optimal choice between those two alternatives. This choice will critically depend on the relation of the separate option values $F_L(Y)$ and $F_S(Y)$, because for low values of

$Y$ one can directly conclude that one should choose the option that is more valuable. Since we are interested in the trade-off between cost efficiency and flexibility, we want to state the relation of the option values in terms of parameter $\kappa$ that represents the cost premium that the firm must pay for the flexibility of splitting the investment. We first claim that there is a single threshold value such that if $\kappa$ is below that level, the option value of the sequential investment dominates that of the lumpy investment, while the converse is true for $\kappa$ above that level (Proposition 1).[3] Then, we point out that a simple comparison of the option values is not sufficient for high $Y$, and derive the optimal investment policy applicable in all regions of the state space (Proposition 2).

**Proposition 1** *Consider the $Y$-interval $(0, Y_1)$. The critical level of premium $\widehat{\kappa}$ at which the option value associated with the lump investment is equal to the option value of the sequential investment is given by*

$$\widehat{\kappa} = \left( \frac{\frac{R_1^{\beta_1}}{\alpha^{\beta_1 - 1}} + \frac{(R_2 - R_1)^{\beta_1}}{(1-\alpha)^{\beta_1 - 1}}}{R_2^{\beta_1}} \right)^{\frac{1}{\beta_1 - 1}} . \tag{12}$$

*For $\kappa < \widehat{\kappa}$, we have $F_S(Y) > F_L(Y)$, whereas for $\kappa > \widehat{\kappa}$, we have $F_L(Y) > F_S(Y)$.*

**Proof.** See the Appendix. ■

Proposition 1 gives us a useful result concerning the option values of the two investment strategies. Once the current realization of $Y$ is such that the firm is in the continuation region of both investment problems analyzed separately, the choice between the lump and the sequential investment strategy is obvious: the firm chooses the strategy that is associated with a higher option value.

Nevertheless, this is not sufficient for determining the investment policy in a situation when the value of the state variable $Y$ is so high that at least one of the options, when analyzed in isolation, should be exercised. This is due to the fact that Proposition 1 compares the option values only and does not say anything about the relative payoffs of strategies in their corresponding stopping regions.

When $\kappa > \widehat{\kappa}$, the lumpy project dominates the sequential project for low values of $Y$, and as $Y$ increases, the lumpy project becomes all the more attractive relative to the sequential project. Then the maximum of the continuation and the stopping value for the lump investment is greater than the corresponding maximum for the sequential investment for every $Y$, and thus it can then never be optimal to invest in the sequential project. In such a situation

[3] More generally, we could present the threshold where the two options are equally valuable as the surface in the parameter space where function $f(\alpha, \kappa, \mu, \sigma, R_1, R_2; Y) \equiv F_S(Y) - F_L(Y)$ gets the value zero for low values of $Y$. Thus, the threshold level $\widehat{\kappa}$ giving the value of $\kappa$ where $F_S(Y) = F_L(Y)$ is implicitly defined by the condition $f(\alpha, \kappa, \mu, \sigma, R_1, R_2; Y) = 0$, and thus of course a function of all other parameters of the model except $\kappa$.

the option to invest in the sequential project is redundant, and the optimal investment policy is the simple trigger strategy that was derived in section 3.1.

When $\kappa < \widehat{\kappa}$, the solution is more complicated. From Proposition 1 we know that for low values of $Y$, the option to invest sequentially is worth more than the option to make the lump investment. On the other hand, once we analyze the values in the stopping regions, we can easily see that the net present value (NPV) of the lump investment project is higher than the NPV of the sequential investment as long as $Y$ is sufficiently high and $\kappa$ is strictly greater than 1. Consequently, analogous to Décamps, Mariotti and Villeneuve (2003), who obtain a similar result for a problem where the firm could choose between undertaking a small and a large project, for intermediate values of $Y$ an inaction region exists, which is part of the stopping region of the first sequential investment. However, the first sequential investment is not carried out here, because it would kill the lump investment option. Speaking in terms of $Y$-valuation, just below this region the firm undertakes the sequential investment, while above the lump investment is undertaken. The boundaries $Y_{P-}$ and $Y_{P+}$ of the inaction region can be determined by the following system of value-matching and smooth-pasting conditions:

$$BY_{P-}^{\beta_1} + CY_{P-}^{\beta_2} = \frac{Y_{P-}R_1}{r-\mu} - I_1 + F_2(Y_{P-}), \tag{13}$$

$$\beta_1 BY_{P-}^{\beta_1} + \beta_2 CY_{P-}^{\beta_2} = \frac{Y_{P-}R_1}{r-\mu} + \beta_1 F_2(Y_{P-}), \tag{14}$$

$$BY_{P+}^{\beta_1} + CY_{P+}^{\beta_2} = \frac{Y_{P+}R_2}{r-\mu} - I, \tag{15}$$

$$\beta_1 BY_{P+}^{\beta_1} + \beta_2 CY_{P+}^{\beta_2} = \frac{Y_{P+}R_2}{r-\mu}, \tag{16}$$

where $B$ and $C$ are constants and $\beta_2$ is given by

$$\beta_2 = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} < 0. \tag{17}$$

As long as the level of uncertainty is strictly positive, interval $[Y_{P-}, Y_{P+}]$ is non-degenerate. The following proposition characterizes the optimal investment policy by splitting the state space into various action and inaction regions:

**Proposition 2** *i) when $\kappa > \widehat{\kappa}$, the optimal investment policy is defined by the single threshold $Y_L$ defined in (3): wait if $Y \in (0, Y_L)$, and invest immediately in the lump project if $Y \in [Y_L, \infty)$.*

*ii) when $\kappa \leq \widehat{\kappa}$, the optimal investment policy is defined by four thresholds $Y_1$, $Y_2$, $Y_{P-}$, and $Y_{P+}$: wait if $Y \in (0, Y_1) \cup (Y_{P-}, Y_{P+})$; invest immediately in the first stage if $Y \in [Y_1, Y_{P-}]$ and subsequently in the second stage as soon as $Y$ hits $Y_2$; invest immediately in the lump project if $Y \in [Y_{P+}, \infty)$.*

*Thresholds $Y_1$ and $Y_2$ are defined in (10) and (6), respectively, while $Y_{P-}$ and $Y_{P+}$ are obtained together with constants $B$ and $C$ by solving the system of equations (13)-(16).*

**Proof.** See the Appendix. ■

In order to fully characterize the solution to our problem, we have defined three threshold levels, $Y_1$, $Y_2$, and $Y_L$, which are defined for all $\kappa > 1$, plus two threshold levels, $Y_{P-}$, and $Y_{P+}$, which are only defined when $\kappa \leq \widehat{\kappa}$ . It can be shown that when $\kappa < (=)\,\widehat{\kappa}$, the threshold levels satisfy $0 < Y_1 < (=)\,Y_{P-} < Y_L < (=)\,Y_{P+} < Y_2$. When $\kappa > \widehat{\kappa}$, we have simply $0 < Y_1 < Y_L < Y_2$ .

It is interesting to observe that $Y_{P+} > Y_L$ for $\kappa < \widehat{\kappa}$. This results from the fact that by investing in the lump project the firm not only commits its full cost but also loses the option to invest in the first stage only. Therefore the opportunity cost of investing in the lump project is higher leading to a higher investment threshold. When $\kappa = \widehat{\kappa}$, it holds that $Y_{P+} = Y_L$, because the option to invest in the sequential project, which is lost upon investing at $Y_{P+}$, is worthless. Furthermore, it holds that $Y_{P-} > Y_1$ for $\kappa < \widehat{\kappa}$, but as $\kappa$ increases towards $\widehat{\kappa}$, $Y_{P-}$ and $Y_1$ approach one another, and in the limit the first stage investment region $[Y_1, Y_{P-}]$ degenerates to a set of measure zero.

# 4 Role of Uncertainty

We are particularly interested in how the choice between the lump and the sequential investment depends on the degree of uncertainty. To answer this question, we examine the effect of the degree of uncertainty on threshold level $\widehat{\kappa}$. According to Proposition 2, the lumpy investment is clearly superior to the sequential investment when $\kappa > \widehat{\kappa}$. On the other hand, when $\kappa < \widehat{\kappa}$, the situation is not so clear since any investment policy may be optimally chosen depending on the initial value of $Y$. However, our interpretation in this case is that the sequential investment dominates the lumpy investment. This can be justified by a natural assumption that the starting value of $Y$ is so low that initially it is not optimal to invest in any project. Note that similar interpretation on the domination relation of mutually exclusive options is adopted, for example, in Dixit (1993).

Consequently, we interpret our model so that $\widehat{\kappa}$ represents the cost advantage for the lumpy investment required to compensate for the loss of flexibility associated with splitting the investment. Thus, an increase (decrease) in $\widehat{\kappa}$ enhances (reduces) the value of flexibility in sequencing the investment, because it makes a larger cost premium necessary to compensate for it. The next proposition states our main result:

**Proposition 3** *The value of flexibility in sequencing investment is negatively related to uncertainty, i.e. the following relationship holds:*

$$\frac{\partial \widehat{\kappa}}{\partial \sigma} < 0. \tag{18}$$

**Proof.** See the Appendix. ■

This means that higher uncertainty *reduces* the value of flexibility in sequencing the project.

The intuition is as follows. To understand the result, it is important to know that the real investment options are convex functions of the project values. To see this, notice that for the lump project inserting (3) into (5) gives

$$F_L(Y) = \frac{(\beta_1 - 1)^{\beta_1 - 1}}{\beta_1^{\beta_1} I^{\beta_1 - 1}} \left(\frac{R_2 Y}{r - \mu}\right)^{\beta_1}. \tag{19}$$

(For stage 1 and 2 of the sequential project analogous formulae hold.) Now, based on the argument analogous to Jensen's inequality it holds that adding up the option values of each stage of the sequential project gives a greater value than the option on the sum of each stage of the sequential project. These two values can be made equal by making the sequential project more expensive, thus having as investment expenditure $\hat{\kappa} I$ with $\hat{\kappa} > 1$. Now, we use the fact that the increase of uncertainty ($\sigma$) causes a decrease of $\beta$ and thus leads to a reduction of the convexity of the options as function of the project values. As a consequence, the effect of Jensen's inequality is smaller. This leads to a lower level of $\hat{\kappa}$.

As we have seen in the previous section, the optimal investment strategy is not a simple trigger strategy when $\kappa < \hat{\kappa}$. Instead, threshold levels $Y_{P-}$ and $Y_{P+}$ exist such that the sequential investment is preferred for $Y \leq Y_{P-}$, the firm will carry out the lump investment for $Y \geq Y_{P+}$, while the firm waits for $Y_{P-} < Y < Y_{P+}$. Within this latter region it may hold that a higher level of uncertainty can lead to a higher probability of investing in the sequential project. The intuition here is as follows: higher uncertainty increases the value of the option to wait for the lump project by more than it increases the corresponding option related to the sequential project. This effect is amplified by an increase of the NPV of the sequential project (since it includes the option to invest in the second stage) and no change of the NPV of the lump project (which is linear in $Y$). In the most extreme case, i.e. when for a given $Y$ an increase in uncertainty will relocate $Y_{P-}$ to the right from $Y$, an increase in uncertainty leads to optimal immediate investment in the first stage. Since an uncertainty level exists at which investment in the large project is optimal, the relationship between uncertainty and the probability of investment in the staged project is non-monotonic in this case.

## 5 Conclusions

In the paper, we analyze the choice between investing in the whole project at once and completing it in two stages. We determine the optimal investment rule of the firm as a function of the premium the firm has to pay for the flexibility to split the project (and not having to commit the cost of the entire project up-front).

We show that higher uncertainty favors the lump investment relative to completing the project in two stages. Depending on the interpretation of our model, this means that when uncertainty is higher, the value-maximizing strategy will be *a*) investing in the whole project at once rather than sequencing the investment, *b*) leapfrogging rather than implementing a progressive technology adoption, *c*) bundling two projects together rather than undertaking them separately, or *d*) taking over an entire firm rather than purchasing a partial stake as a first step possibly followed by a complete takeover.

# A Appendix

**Proof of Proposition 1.** We begin by comparing the two option values $F_L(Y)$ and $F_S(Y)$. It holds (cf. (5) and (11) and the definitions of $I_1$ and $I_2$) that

$$\begin{aligned}\frac{F_L(Y)}{F_S(Y)} &= \frac{\left(\frac{Y_L R_2}{r-\mu} - I\right)\left(\frac{Y}{Y_L}\right)^{\beta_1}}{\left(\frac{Y_1 R_1}{r-\mu} - I_1\right)\left(\frac{Y}{Y_1}\right)^{\beta_1} + \left(\frac{Y_2(R_2-R_1)}{r-\mu} - I_2\right)\left(\frac{Y}{Y_2}\right)^{\beta_1}} \\ &= \frac{\kappa^{\beta_1-1}R_2^{\beta_1}}{\frac{R_1^{\beta_1}}{\alpha^{\beta_1-1}} + \frac{(R_2-R_1)^{\beta_1}}{(1-\alpha)^{\beta-1}}}. \end{aligned} \tag{A.1}$$

Equation (12) follows directly from (A.1).

Now, we prove that a threshold value $\widehat{\kappa}$ exists for which the ratio $F_L(Y)/F_S(Y)$ equals 1. This means that, irrespective from the set of parameter values, a non-empty set $\mathcal{K}$ exists for which the option value to invest in the whole project at once is higher. As a consequence, the value of investment opportunity to invest in two stages is higher for $\kappa \in [1,\infty)\setminus\mathcal{K}$.[4]

It can easily be seen that for $\kappa = 1$ and $\beta_1 \to 1$ the values of both investment opportunities (lumpy and staged) are equal. Define

$$D(\beta_1,\kappa) \equiv \kappa^{\beta_1-1}R_2^{\beta_1} - \frac{R_1^{\beta_1}}{\alpha^{\beta_1-1}} - \frac{(R_2-R_1)^{\beta_1}}{(1-\alpha)^{\beta_1-1}}. \tag{A.2}$$

Let us calculate the following derivative

$$\begin{aligned}\frac{\partial D(\beta_1,\kappa)}{\partial \beta_1} &= \frac{\partial}{\partial \beta_1}\left[\kappa^{\beta_1-1}R_2^{\beta_1} - \frac{R_1^{\beta_1}}{\alpha^{\beta_1-1}} - \frac{(R_2-R_1)^{\beta_1}}{(1-\alpha)^{\beta_1-1}}\right] \\ &= \beta_1(\kappa R_2)^{\beta_1-1} - \left(\frac{R_1}{\alpha}\right)^{\beta_1-1}(\beta_1 - R_1\ln\alpha) \\ &\quad - \left(\frac{R_2-R_1}{1-\alpha}\right)^{\beta_1-1}(\beta_1 - (R_2-R_1)\ln(1-\alpha)). \end{aligned} \tag{A.3}$$

[4] In order to compare the payoffs of two investment strategies (lump vs. sequential), it is sufficient to compare the present values of the corresponding investment costs. In other words, it holds that

$$\frac{F_L(Y)}{F_S(Y)} = \frac{PV(I)}{PV(I_1)+PV(I_2)},$$

where $PV(.)$ denotes the present value operator.

For $\kappa = 1$ (A.2) is always negative since

$$\begin{aligned} & \beta_1 (\kappa R_2)^{\beta_1 - 1} - \left(\frac{R_1}{\alpha}\right)^{\beta_1 - 1} (\beta_1 - R_1 \ln \alpha) \\ < \;\; & \beta_1 R_2^{\beta_1 - 1} - \beta_1 \left(\frac{R_1}{\alpha}\right)^{\beta_1 - 1} < 0. \end{aligned}$$

Therefore, for $\kappa = 1$ the value of a staged investment opportunity is higher than the value of a lumpy project. On the basis of (A.1) it can be concluded that the ratio $F_L(Y)/F_S(Y)$ increases monotonically (and in a unbounded way) in $\kappa$. Therefore, $\widehat{\kappa}$ exists and is unique. ■

**Proof of Proposition 2.** The proposition is proven by examining the value functions corresponding to the lump and sequential investments in the corresponding continuation and stopping regions. For $\kappa > \widehat{\kappa}$, the value of the single-stage project always exceeds the value of the sequential project (as it is shown for continuation values in Proposition 1 and can directly be shown for stopping values by comparing net present values). For $\kappa < \widehat{\kappa}$, The value of the single-stage project still exceeds the value of the sequential project in the stopping region, but is smaller in the continuation region (again, see Proposition 1). This implies that in the latter case, an inaction region exists for intermediate values of $Y$. (Formal proof will follow). ■

**Proof of Proposition 3.** Calculating the total derivative of $D(\beta_1, \kappa)$ yields (see (A.2))

$$\frac{\partial D}{\partial \beta_1} + \frac{\partial D}{\partial \kappa}\frac{d\widehat{\kappa}}{d\beta_1} = 0, \tag{A.4}$$

which, in turn, implies

$$\frac{d\widehat{\kappa}}{d\beta_1} = -\frac{\left.\frac{\partial D}{\partial \beta_1}\right|_{\kappa=\widehat{\kappa}}}{\frac{\partial D}{\partial \widehat{\kappa}}}. \tag{A.5}$$

Since, we already know that $D$ increases in $\widehat{\kappa}$, all we have to prove is that $\left.\frac{\partial D}{\partial \beta_1}\right|_{\kappa=\widehat{\kappa}}$ is negative. The trigger premium $\widehat{\kappa}$ is a solution to $D(\beta_1, \kappa) = 0$ and is given by

$$\widehat{\kappa} = \left(\frac{\frac{R_1^{\beta_1}}{\alpha^{\beta_1 - 1}} + \frac{(R_2 - R_1)^{\beta_1}}{(1-\alpha)^{\beta_1 - 1}}}{R_2^{\beta_1}}\right)^{\frac{1}{\beta_1 - 1}}. \tag{A.6}$$

Consequently,

$$\begin{aligned} \left.\frac{\partial D}{\partial \beta_1}\right|_{\kappa=\widehat{\kappa}} = \; & \left(\frac{R_1}{\alpha}\right)^{\beta_1 - 1} \left(\beta_1 \left(\frac{R_1}{R_2} - 1\right) + R_1 \ln \alpha\right) \\ & + \left(\frac{R_2 - R_1}{1 - \alpha}\right)^{\beta_1 - 1} \\ & \times \left(\beta_1 \left(\frac{R_2 - R_1}{R_2} - 1\right) - (R_2 - R_1) \ln (1 - \alpha)\right) \\ < \; & 0. \end{aligned} \tag{A.7}$$

This completes the proof. ■

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