Valuing Options to Learn: Optimal Timing of Information Acquisition

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Abstract

This article considers the value of information and optimal timing to acquire it in a model of irreversible investment. There are two types of uncertainties: first, the value of the investment project depends on an observable stochastic process, and second, it depends on an ex-ante uncertain parameter, whose true value may be learnt at a cost. The former type of uncertainty implies that the opportunity to acquire information has an option-like character: besides owning a real option on the actual project, the firm also owns an option to learn. We derive and illustrate the value of such learning options and the optimal timing to learn. Choosing the timing involves a trade-off: by postponing it the firm benefits from the delayed cost, but on the other hand suffers from increased likelihood that the revealed information would have been more beneficial earlier.

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1 Introduction

The standard theory of real options focuses on investment projects, whose values are subject to a given exogenous stochastic process. By waiting and observing the development of the exogenous process, the firm under consideration continuously updates the present value of the project. This continuously resolving uncertainty induces a value to waiting, which makes the firm more reluctant to carry out the investment than would be the case under certainty. For an extensive treatment on the topic, see Dixit and Pindyck (1994).

This paper adds into this consideration another type of uncertainty, which is not resolved by passive waiting. It is clear that such uncertainties are present in most real investment projects. Accordingly, the net present values on which investment decisions are based, are estimates of eventual uncertain cash flows. The reason why this is not emphasized in the standard real options theory is obvious: it would not really change anything in the basic model as long as the only way to reveal such uncertainty is to actually undertake the project. However, the situation changes substantially if it is also possible to activate costly information acquisition prior to investing. Then it is reasonable to ask what the optimal information acquisition strategy is like.

As an example, consider a firm, which is planning an oil field development. There are two types of uncertainties associated with the project. First, the price of oil develops unpredictably, which is a typical example of the type of uncertainty considered in standard real options models. Second, the amount of oil in the field is uncertain. Assume that having a prior probability distribution on the amount of oil, the firm faces the following decision problem: is it better to go directly ahead with the investment on the basis of the prior estimate, or is it better to first carry out a costly exploratory drilling and then make the actual investment decision on the basis of this additional information? The firm observes continuously the development of oil price and faces two choices: 1) which of these two alternatives to take (exploratory drilling or going straight ahead with the project) and 2) when to act. In this situation, the firm owns not only a standard real option on the oil investment, but also an option to acquire more information.

In this paper I consider the value of such learning options and the optimal timing of exercising them.

The value of learning actions in real options settings has earlier been studied by Martzoukos (2000). He models the learning actions as random jumps in the underlying stochastic process, which are activated by the firm. While providing insight on how learning may potentially increase the value of the investment opportunities, the model does not provide a fully satisfactory characterization of information updating, because the sense in which new information changes the accuracy of the firm's beliefs is not defined. A more consistent framework would set a prior distribution on the unknown parameter, and let new information change beliefs in Bayesian fashion (see Moscarini and Smith, 2002, for the value of information under Bayesian updating). Along this line, Decamps et al. (2001) and Roche (2003) consider models of investment, where the drift of the stochastic process describing the development of the project value is unknown, and the firm updates its belief by Bayesian updating while observing the process. However, such learning is passive in the sense that the firm updates its belief continuously as long as it is waiting, whereas my focus is on active and discrete information acquisition, where the firm must decide *when* to learn. Roberts and Weitzman (1981) and Weitzman (1981) consider sequential development projects, where by completing subsequent stages of the project the firm updates its beliefs over the final benefits of the project. In these models there is only uncertainty that is resolved by the firm's actions, whereas my focus is on the interaction of stochastically evolving environment (exogenous uncertainty that resolves by waiting) and uncertainty that is resolved by the firm's actions. This allows one to focus on the conditions that determine *when* information actually becomes worthy.

The setting considered in this paper involves a single investment opportunity, the value of which is subject to 1) an exogenous stochastic process and 2) a parameter on which the investor has incomplete information. The firm may undertake a learning activity that reveals the true value of the parameter at any moment by paying a given cost. On the other hand, the firm may also choose to neglect the possibility to learn, and undertake the project on the basis of prior distribution of the parameter. The main point is that since the value of the project develops in time according to the exogenous stochastic process, also the value of the option to learn depends on this process. The value of information depends on the extent to which the firm's optimal reaction in the face of information increases its payoffs under different possible realizations of the learning outcome. When considering the optimal timing to learn, the firm faces a basic trade-off: by postponing the learning activity the firm postpones the cost of learning, but on the other hand increases the probability that learning will reveal information that would have been more beneficial if received earlier. The firm must balance these two counteracting effects in determining the optimal timing of learning.

The paper considers an extreme form of learning, where through a single information acquisition action the firm is able to learn perfectly the true value of the unknown parameter. As such, perfect learning is of course consistent with Bayesian updating: the posterior distribution is completely concentrated on the true value of the parameter. More importantly, the ex-post project value as a function of the revealed parameter value may be interpreted as the expected project value after purchasing a noisy signal and adopting the posterior distribution by Bayesian updating. Thus, the assumption of perfect learning is actually just a notational simplification. The benefit of this simplification is the fact that it allows one to introduce the concept of *option to learn* in the most illuminating form with straight forward formulas for the value of information and optimal timing of learning. The drawback is that the model is now restricted to the purchase of only one signal with a predetermined precision. However, since to my knowledge this type of information acquisition options have not been analyzed in a similar setting before, it is natural to start with the simplest and most intuitive version. Naturally, it may be a possible future direction to extend this kind of analysis to incorporate the purchase of multiple noisy signals or the choice of signal precision.

The paper is organized as follows. Section 2 sets up the model. Section 3 derives the value of the firm's investment opportunity and the value of information. Section 4 derives the conditions for optimal timing of learning. Section 5 presents a more specified version of the model in order to illustrate the main points. Section 6 concludes.

2 Model

A risk-neutral investor faces a single investment opportunity. The investment is irreversible, but its timing can be freely chosen. Time is continuous, horizon is infinite, and the investor discounts cash flows at rate r. Let the net present value of the investment project be defined by a function $V : R^+ \times R \to R$. The first argument to this function is a stochastic state variable X that shifts according to a geometric Brownian motion:[1]

$$\frac{dX}{X} = \alpha dt + \sigma dz, \tag{1}$$

where α and σ are constants and dz is the standard Brownian motion increment. More precisely, X refers generally to a solution process of (1), X_t to the value of the process at time t, x to an arbitrary value of the process at an undefined time, and $\{X_t^x\}$ to a solution process of (1) that starts at $X_0 = x$.

The second argument is a random variable θ that is distributed according to a piecewise continuous density function f with a connected support $\Theta \subseteq R$.[2]

Notice that the assumption of risk neutralily can be interpreted so that the investor acts so as to maximize the market value of the investment opportunity given that there is a market for traded assets that prices the risk associated with the fluctuations of X, and on the other hand uncertainty in θ is fully diversifiable. In this case the firm's objective is interpreted using equivalent risk-neutral valuation principle, which in essence means that the drift rate in (1) is adjusted appropriately (e.g. Cox and Ross, 1976). Since θ must be understood to represent project specific uncertainty, the assumption of diversifiable risk seems reasonable.

[1] It would perhaps seem more natural to use a standard Brownian motion as the state variable, but geometric Brownian motion was chosen in order to make the notation easily comparable with standard real options literature. It does not really matter which specification we use, since the state variable is simply an argument in the project value V, which is the focus of the model. Standard Brownian motion and geometric Brownian motion can be made equivalent by a simple transformation of the function V.

[2] In other words, Θ is the set of possible values that the variable θ may take.

Given the values of the two variables representing the state of the world, x and θ, the net present value of the project is $V(x,\theta)$. We adopt the following assumptions on the function V (and parameters α and σ):

Assumption 1:

$$\begin{aligned}\frac{\partial V(x,\theta)}{\partial x} &> 0 \text{ for all } x>0,\ \theta\in\Theta,\\ \frac{\partial V(x,\theta)}{\partial \theta} &> 0 \text{ for all } x>0,\ \theta\in\Theta.\end{aligned}$$

Assumption 2:

$$\exists M\in R \text{ s.t. } V(x,\theta)>0 \text{ for all } \theta\in\Theta \text{ whenever } x>M.$$

Assumption 3:

$$\exists N\in R \text{ s.t. } \frac{\alpha x V'(x,\theta)+\frac{1}{2}\sigma^2x^2V''(x,\theta)}{V(x,\theta)}<r \text{ for all } x>N \text{ and } \theta\in\Theta,$$

where $V'(\cdot,\cdot)$ and $V''(\cdot,\cdot)$ stand for the first and second derivatives of V with respect to the first argument.

These assumptions ensure that the optimal stopping problems that will be considered have a certain simple structure. Assumption 1 simply says that V is increasing in both of its arguments, meaning that both variables X and θ represent something for which the investor prefers as high value as possible. Assumption 2 means that no matter how low the value of θ, the net present value will be positive as soon as X reaches a high enough level. This ensures that sooner or later the project will be profitable. Assumption 3 is made in order to ensure that the project value does not increase too fast in X. The main implication of this assumption is that the value of the investment opportunity can not appreciate so fast that it would be optimal to wait forever.

The main issue of the model is the information that the investor possesses. We assume that while observing perfectly the development of X, the investor does not initially know the realization of θ. There are two possibilities how the value of θ may be revealed. First, it is possible at any time to learn its value by paying a fixed fee C. Alternatively, if no such information acquisition activity is undertaken, the value of θ is ultimately revealed when the project is undertaken. The problem is to decide whether to learn θ before going ahead with the project, or whether to invest in the project without learning θ in advance. The timing of information acquisition and/or investment should be chosen optimally in order to maximize the present value of the investment opportunity. Mathematically, given an initial value $X_0 = x$, the problem is to find the maximum:

$$\max\left\{\sup_{\tau} E_x\left(e^{-r\tau}\left[E_\theta\left(\sup_{\tau'\geq\tau} E_x\left[e^{-r\tau'}V\left(X_{\tau'}^x,\theta\right)\right]\right)-C\right]\right);\sup_{\tau''} E_x\left[e^{-r\tau''}E_\theta V\left(X_{\tau''}^x,\theta\right)\right]\right\} \quad (2)$$

where τ, τ', and τ'' are stopping times adapted to X. Here, τ stands for the moment of information acquisition, τ' stands for the moment when the actual investment is undertaken once the value of θ has been learnt, and τ'' stands for the moment when the investment is undertaken without learning θ in advance. Note that the left hand side term of (2) is the value of the investment opportunity in case learning takes place, and the right hand side is the value of the investment opportunity in case investment is undertaken without learning. Thus, whether or not learning is optimal depends on which term is greater. The value of the investment opportunity is the maximum of these two terms as (2) indicates. It should now be quite clear what the nature of the optimal behavior must be: either let X rise to a certain level to trigger the investment without learning θ in advance (the right hand term is greater) or let X rise to a certain level to trigger information acquisition, and subsequently depending on the learning outcome invest in the actual project straight ahead or let X rise to an even higher level to trigger the investment (the left hand term is greater).

It is useful to emphasize here that even if the model has been formulated so that the learning reveals fully the true value of θ, the model may be interpreted so that θ is the value of a noisy signal that allows one to update the posterior distribution of another underlying random variable. Thus, we have actually a Bayesian learning model so long as $V(x, \theta)$ is interpreted as the *expected* value of the project after optimally updating beliefs using the signal realization θ.

We will next consider in more detail the value of the investment opportunity proceeding through two special cases (sections 3.1 and 3.2) to finally consider the maximum value of (2) in section 3.3.

3 Value of the investment opportunity

3.1 Full information

Proceeding backwards, consider first the case of full information, where the value of θ is already known. Denote by $F(x, \theta)$ the value of such an investment opportunity with full information given the current value $X_t = x$:

$$F(x, \theta) = \sup_{\tau} E_x \left[e^{-r\tau} V(X_\tau^x, \theta) \right]. \tag{3}$$

Finding the optimal investment timing τ^*, that is, the stopping time that maximizes (3) is a standard real-option problem analyzed in detail for example in Dixit and Pindyck (1994). It is well known that the solution is characterized by an investment threshold such that it is optimal to invest at the first moment when X hits this threshold from below. Note that this threshold depends now on the value of θ, and we may thus denote the optimal investment threshold under full information by $X^*(\theta)$. For a given value of θ, $X^*(\theta)$ can be solved by the principle of dynamic programming with an application of Ito's lemma and two special conditions (value-matching and smooth-pasting conditions, see Øksendal, 2000, or Dixit and Pindyck, 1994, for more details). It can be shown that $X^*(\theta)$ must be continuous in θ under the assumptions already made. However, to facilitate the analysis, we impose here one final assumption to our model:

Assumption 4:

$$\frac{\partial X^* (\theta)}{\partial \theta} < 0 \text{ for all } \theta \in \Theta. \tag{4}$$

Of course, this assumption concerns mainly the form of function V. To formulate it so that it explicitly derives conditions for V would be desirable, but it seems challenging to find reasonable restrictions on V that imply it. However, it is easy to check that (4) holds under all typical situations, while on the other hand it is also possible to construct examples of V for which it does not hold even if Assumptions 1-3 hold.

Since by Assumption 4 we restrict to cases where $X^* (\cdot)$ is monotonic, it has an inverse that we denote by $\theta^* (\cdot)$. In other words, $\theta^* (x)$ is the value of θ for which $x = X^* (\theta^* (x))$. To simplicity notation in the following sections, we now extend the definition of θ^* to all $X \in R$:

$$\theta^* (x) \equiv -\infty \text{ for all } x > X^* (\inf \Theta), \tag{5}$$

$$\theta^* (x) \equiv \infty \text{ for all } x < X^* (\sup \Theta). \tag{6}$$

It is clear that

$$\frac{\partial \theta^* (x)}{\partial x} < 0$$

for all such x that $\theta^* (x) \in \Theta$.

The application of Bellman's principle of optimality and Ito's lemma results in a second order differential equation for F with respect to x, which together with appropriate boundary conditions implies that the full information value of the investment opportunity must be of the form (again, see Dixit and Pindyck, 1994, for details) :

$$F(x,\theta) = \begin{cases} A(\theta)\, x^{\beta}, x < X^*(\theta) \\ \\ V(x,\theta), x \geq X^*(\theta) \end{cases} \tag{7}$$

where

$$\beta \equiv \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{\left[\frac{\alpha}{\sigma^2} - \frac{1}{2}\right]^2 + \frac{2r}{\sigma^2}} > 1 \tag{8}$$

and $A(\theta)$ is some continuous function, whose exact form depends on the form of $V(x,\theta)$.

3.2 Incomplete information, no learning opportunity

Consider next a case, where θ is not known, and it is not even possible to learn it before the actual investment is undertaken. This is also a standard real options problem, where the net present value of the project is the expectation with respect to θ. We denote this expectation by $\overline{V}(x) \equiv E_\theta V(x,\theta)$. Given the current value $X_t = x$, denote the value of the opportunity to undertake the project by $\overline{F}(x)$:

$$\overline{F}(x) \equiv \sup_\tau E_x\left[e^{-r\tau} E_\theta V(X_\tau^x, \theta)\right] = \sup_\tau E_x\left[e^{-r\tau}\overline{V}(X_\tau^x)\right]. \tag{9}$$

Again, the optimal investment timing is the moment, when X crosses a certain threshold from below. This optimal investment threshold, which we denote $\overline{X}$, can be solved following the same principles as in solving $X^*(\theta)$. Consequently, $\overline{F}$ must take the form:

$$\overline{F}(x) = \begin{cases} \overline{A}x^{\beta}, x < \overline{X} \\ \\ \overline{V}(x), x \geq \overline{X} \end{cases}, \tag{10}$$

where β is as given in (8), and $\overline{A}$ is a constant whose value depends on the form of V.

3.3 Incomplete information with possibility to learn

Finally, consider the case of our special interest where θ is not known, yet it is possible to learn it before undertaking the project. We denote the value of the investment opportunity prior to learning by $F^0(x)$.

Its value is given by (2), which using the definitions given in sections 3.1 and 3.2 can be written as:

$$\begin{aligned} F^0(x) &= \max\left\{\sup_{\tau} E_x\left(e^{-r\tau}\left[E_\theta F\left(X_\tau^x,\theta\right)-C\right]\right);\sup_{\tau''} E_x\left[e^{-r\tau''}\overline{V}\left(X_{\tau''}^x\right)\right]\right\} \\ &= \max\left\{\sup_{\tau} E_x\left(e^{-r\tau}\left[E_\theta F\left(X_\tau^x,\theta\right)-C\right]\right);\overline{F}(x)\right\}. \end{aligned} \tag{11}$$

Note that this ex-ante value $F^0(x)$ is a function of x only, since the value of θ is not known. We may now define the value of the *option to learn* as the difference between the value of the investment project when learning is possible and when learning is not possible:

$$F_I(x) \equiv F^0(x) - \overline{F}(x). \tag{12}$$

It is now easy to see that if the right hand side component of (11) is greater than the left hand side component, it is not optimal to learn, and consequently $F^0(x) = \overline{F}(x)$ implying that the option to learn is worthless: $F_I(x) = 0$. On the other hand, if the left hand side component is greater, we have $F^0(x) > \overline{F}(x)$ implying that the option to learn is valuable: $F_I(x) > 0$.

Rearranging (12), we may write the total value of the firm's investment opportunity as a sum of two components:

$$F^0(x) = F_I(x) + \overline{F}(x), \tag{13}$$

where $F_I(x)$ is the value of the option to learn the true value of θ before undertaking the project and $\overline{F}(x)$ is the value of the option to undertake the project without the possibility to learn θ. It is of course possible that information acquisition is too costly, and it is not optimal to learn at all before undertaking the project, in which case we have $F_I(x) = 0$ and $\overline{F}(x) = F^0(x)$ for all x.

Having now defined what is meant by the value of the option to learn, consider next the actual value of information. Learning θ gives the investor at the moment of information acquisition a standard opportunity to invest in a project with full information. As stated above, this *ex-post* value of the investment opportunity (the value under full information) is $F(x,\theta)$. *Ex-ante*, the value of this investment opportunity is $E_\theta[F(x,\theta)]$. Therefore, given the state value x, the instantaneous ex-ante present value of information is (hereafter referred as *value of information*):

$$V_I(x) \equiv E_\theta[F(x,\theta)] - \overline{F}(x), \tag{14}$$

where, using (7):

$$E_\theta[F(x,\theta)] = \int_{-\infty}^{\infty} F(x,\theta) f(\theta)\, d\theta = \int_{-\infty}^{\theta^*(x)} A(\theta) x^\beta f(\theta)\, d\theta + \int_{\theta^*(x)}^{\infty} V(x,\theta) f(\theta)\, d\theta. \tag{15}$$

As seen in (15), the ex-ante value of the full information investment opportunity, which is the expected value of the asset obtained through learning, is composed of two terms. The first integral in (15) reflects the possibility that the revealed value of θ is so low that it is not yet optimal to undertake the project, while the second term reflects the possibility that θ is found so high that it is optimal to go ahead with the project straight away. This latter alternative is the potential payoff from learning now instead of doing so later, since it represents the case where the increased information immediately alters the optimal behavior. It is clear that the probability of this latter alternative must be sufficiently high in order to make information acquisition optimal, otherwise it would be better to delay costly learning as the loss from doing so would be small.

Using (10) and (15), V_I can be written in three separate parts:

$$V_I(x) = \begin{cases} \int_{-\infty}^{\infty} A(\theta) x^\beta f(\theta)\, d\theta - \overline{A} x^\beta, & 0 \le x < X^*(\sup\Theta) \\ \int_{-\infty}^{\theta^*(x)} A(\theta) x^\beta f(\theta)\, d\theta + \int_{\theta^*(x)}^{\infty} V(x,\theta) f(\theta)\, d\theta - \overline{A} x^\beta, & X^*(\sup\Theta) \le x < \overline{X} \\ \int_{-\infty}^{\theta^*(x)} A(\theta) x^\beta f(\theta)\, d\theta + \int_{\theta^*(x)}^{\infty} V(x,\theta) f(\theta)\, d\theta - \overline{V}(x), & x \ge \overline{X} \end{cases} \tag{16}$$

We now state the following proposition:

Proposition 1 *Value of information $V_I(x)$ as defined in (14) is a positive and continuous function defined for all $x > 0$. It has the properties*

$$\lim_{x\to 0^+} V_I(x) = \lim_{x\to\infty} V_I(x) = 0,$$

implying that it attains a unique maximum value, which we denote

$$V_I^* \equiv \max_{x>0} V_I(x).$$

Proof. The properties of V_I in question are easy to establish from (16). First, each of the three distinct parts are clearly composed of continuous functions of x. Further, using (6),

$$\lim_{x\to X^*(\sup\Theta)^-}\left[\int_{-\infty}^{\infty} A(\theta)\, x^\beta f(\theta)\, d\theta\right] = \lim_{x\to X^*(\sup\Theta)^+}\left[\int_{-\infty}^{\theta^*(x)} A(\theta)\, x^\beta f(\theta)\, d\theta + \int_{\theta^*(x)}^{\infty} V(x,\theta)\, f(\theta)\, d\theta\right]$$

and from (10),

$$\lim_{x\to \overline{X}^-} \overline{A}x^\beta = \lim_{x\to \overline{X}^-} \overline{V}(x),$$

which means that $V_I(x)$ is continuous. That V_I is positive is clear from the way how it is defined in (14), and follows also quite easily by checking each of the three parts in (16) separately (but we do not work this out in detail here).

Finally, note that $\int_{-\infty}^{\infty} A(\theta)\, x^\beta f(\theta)\, d\theta - \overline{A}x^\beta = x^\beta\left(\int_{-\infty}^{\infty} A(\theta)\, f(\theta)\, d\theta - \overline{A}\right)$, which obviously goes to zero as $x \to 0^+$, meaning that $\lim_{x\to 0^+} V_I(x) = 0$, and on the other hand $\int_{-\infty}^{\theta^*(x)} A(\theta)\, x^\beta f(\theta)\, d\theta + \int_{\theta^*(x)}^{\infty} V(x,\theta)\, f(\theta)\, d\theta \to \int_{-\infty}^{\infty} V(x,\theta)\, f(\theta)\, d\theta = \overline{V}(x)$ as $x\to\infty$, meaning that $\lim_{x\to\infty} V_I(x) = 0$. ■

Building on Proposition 1, it is now straight-forward to pose the following:

Proposition 2 *It is optimal to undertake the information acquisition if and only if $V_I^* > C$. Expressed in another way:*

$$V_I^* > C \Longleftrightarrow F^0(x) > 0 \text{ for small enough } x.$$

Proof. The value of information as defined in (14) is derived from the assumption of maximizing (2). Therefore, if V_I would be greater than C without this implying that the information is ever used would contradict the very definition of V_I. In other words, there can not be a positive net value for something that does not affect the firm's payoff. Conversely, if $V_I^* < C$, acquiring information at any moment would incur a net loss to the firm, which would contradict value maximizing behavior. ■

Proposition 2 simply says that if the maximum value of information is greater than the cost of acquiring it, it must be optimal at some point to acquire it, which is equivalent to the fact that the option to learn is valuable for small values of x. The reason for this *small values* -condition lies in the fact that if for some reason the investor fails to act optimally and lets X rise too high without doing anything, the value of the option to learn disappears as the possibility to invest straight ahead in the project begins to dominate the possibility to first check its value by learning.

Note that even if we have now shown that the question whether it is optimal to learn in the first place can be answered by checking whether the *maximum value* of information is greater than the cost of acquiring it, it would be wrong to conclude that the optimal moment to learn is the particular moment when this maximum value is reached. We now turn to the issue of optimal timing of information acquisition.

4 Optimal timing of learning

In this section the conditions for the optimal timing of information acquisition are derived. We proceed as follows. First, the conditions that must hold at the optimal learning point are derived mathematically. Then, a proposition is stated that clarifies the principle for which this optimal timing rule is based on. Finally, the intuition behind this principle is discussed. Throughout this section it is assumed that the cost of information acquisition is so small that $V_I^* > C$ holds, and thus by Proposition 2 it is optimal to learn.

Since the value of information develops in time according to X, it seems clear that the optimal time to learn is when X hits some threshold level for the first time. We denote this optimal threshold to acquire information, hereafter referred as *learning threshold*, by X^I. Setting up the Bellman equation for F_I, applying Ito's lemma, and applying a boundary condition that requires $\lim_{x \to 0} F_I(x) = 0$, determine the value of the option to learn to be of the form:

$$F_I(x) = A_I x^{\beta}, \text{ when } x < X^I,$$

where A_I is a positive parameter to be solved and β is given by (8). Following the standard procedure used in this type of optimal stopping problems (again, see Dixit and Pindyck, 1994), the optimal learning threshold X^I is obtained by setting A_I and X^I so that the following value matching and smooth pasting conditions hold:

$$\begin{aligned} F_I\left(X^I\right) &= V_I\left(X^I\right) - C = E_\theta\left[F\left(X^I,\theta\right)\right] - \overline{F}\left(X^I\right) - C \qquad (17) \\ &= \int_{-\infty}^{\theta^*\left(X^I\right)} F\left(X^I,\theta\right) f(\theta)\, d\theta + \int_{\theta^*\left(X^I\right)}^{\infty} V\left(X^I,\theta\right) f(\theta)\, d\theta - \overline{F}\left(X^I\right) - C, \\ F_I'\left(X^I\right) &= V_I'\left(X^I\right) = \frac{\partial}{\partial x}\left[E_\theta\left[F(x,\theta)\right]\right]_{x=X^I} - \overline{F}'\left(X^I\right) \qquad (18) \\ &= \frac{\partial}{\partial x}\left[\int_{-\infty}^{\theta^*(x)} F(x,\theta) f(\theta)\, d\theta + \int_{\theta^*(X^I)}^{\infty} V(x,\theta) f(\theta)\, d\theta\right]_{x=X^I} - \overline{F}'\left(X^I\right) \\ &= \frac{\partial\theta^*\left(X^I\right)}{\partial x} F\left(X^I,\theta^*\left(X^I\right)\right) f\left(\theta^*\left(X^I\right)\right) + \int_{-\infty}^{\theta^*\left(X^I\right)} \frac{\partial F\left(X^I,\theta\right)}{\partial x} f(\theta)\, d\theta \\ &\quad - \frac{\partial\theta^*\left(X^I\right)}{\partial x} V\left(X^I,\theta^*\left(X^I\right)\right) f\left(\theta^*\left(X^I\right)\right) + \int_{\theta^*\left(X^I\right)}^{\infty} \frac{\partial V\left(X^I,\theta\right)}{\partial x} f(\theta)\, d\theta - \overline{F}'\left(X^I\right) \\ &= \int_{-\infty}^{\theta^*\left(X^I\right)} \frac{\partial F\left(X^I,\theta\right)}{\partial x} f(\theta)\, d\theta + \int_{\theta^*\left(X^I\right)}^{\infty} \frac{\partial V\left(X^I,\theta\right)}{\partial x} f(\theta)\, d\theta - \overline{F}'\left(X^I\right), \end{aligned}$$

where primes denote derivatives with respect to x. It can be shown that $X^I < \overline{X}$.[3] From (16), the value of information is:

[3]This can be seen by noting that when $x \geq X^I$, $V_I(x) = E_\theta(F(x,\theta)) - E_\theta(V(x,\theta))$. Since $\frac{\partial}{\partial x}F(x,\theta) \leq \frac{\partial}{\partial x}V(x,\theta)$, this implies that $\frac{\partial}{\partial x}V_I(x) \leq 0$ when $x \geq X^I$. Therefore, there is no point in delaying learning beyond X^I, because the value of information is already decreasing there.

$$V_I(x) = \int_{-\infty}^{\theta^*(x)} A(\theta)\, x^{\beta} f(\theta)\, d\theta + \int_{\theta^*(x)}^{\infty} V(x,\theta)\, f(\theta)\, d\theta - \overline{A}x^{\beta}, x < X^I$$

and the conditions (17) and (18) for X^I can be written as:

$$A_I\left(X^I\right)^{\beta} = \left(X^I\right)^{\beta} \int_{-\infty}^{\theta^*\left(X^I\right)} A(\theta)\, f(\theta)\, d\theta + \int_{\theta^*\left(X^I\right)}^{\infty} V\left(X^I,\theta\right) f(\theta)\, d\theta - \overline{A}\left(X^I\right)^{\beta} - C, \quad (19)$$

$$\beta A_I\left(X^I\right)^{\beta-1} = \beta\left(X^I\right)^{\beta-1} \int_{-\infty}^{\theta^*\left(X^I\right)} A(\theta)\, f(\theta)\, d\theta + \int_{\theta^*\left(X^I\right)}^{\infty} \frac{\partial V\left(X^I,\theta\right)}{\partial x} f(\theta)\, d\theta - \beta\overline{A}\left(X^I\right)^{\beta-1}. \quad (20)$$

This means that the optimal learning threshold can be found by solving X^I together with parameter A_I from theses equations. To guide towards a more intuitive principle of determining X^I, note that an equivalent way to formulate the problem is to consider the value of the option to not only learn, but also subsequently invest in the actual project: $F^0(x) = F_I(x) + \overline{F}(x)$. By exactly the same procedure as with F_I, it can be shown that this must be of the form:

$$F^0(x) = A_0 x^{\beta}, \text{ when } x < X^I,$$

where A_0 is a parameter to be solved. If it is going to be optimal to learn before investing, then at the optimal learning point X^I, the following conditions must hold:

$$\begin{aligned} F^0\left(X^I\right) &= E_{\theta}\left[F\left(X^I,\theta\right)\right] - C && (21)\\ &= \int_{-\infty}^{\theta^*\left(X^I\right)} F\left(X^I,\theta\right) f(\theta)\, d\theta + \int_{\theta^*\left(X^I\right)}^{\infty} V\left(X^I,\theta\right) f(\theta)\, d\theta - C \\ \left(F^0\right)'\left(X^I\right) &= \frac{\partial}{\partial x}\left[E_{\theta}\left[F(x,\theta)\right]\right]_{x=X^I} && (22)\\ &= \int_{-\infty}^{\theta^*\left(X^I\right)} \frac{\partial F\left(X^I,\theta\right)}{\partial x} f(\theta)\, d\theta + \int_{\theta^*\left(X^I\right)}^{\infty} \frac{\partial V\left(X^I,\theta\right)}{\partial x} f(\theta)\, d\theta \end{aligned}$$

or

$$A_0\left(X^I\right)^\beta = \left(X^I\right)^\beta \int_{-\infty}^{\theta^*\left(X^I\right)} A(\theta) f(\theta)\, d\theta + \int_{\theta^*\left(X^I\right)}^{\infty} V\left(X^I,\theta\right) f(\theta)\, d\theta - C, \tag{23}$$

$$\beta A_0\left(X^I\right)^{\beta-1} = \beta\left(X^I\right)^{\beta-1} \int_{-\infty}^{\theta^*\left(X^I\right)} A(\theta) f(\theta)\, d\theta + \int_{\theta^*\left(X^I\right)}^{\infty} \frac{\partial V\left(X^I,\theta\right)}{\partial x} f(\theta)\, d\theta. \tag{24}$$

These are the same as (19) and (20) with $A_0 = A_I + \overline{A}$. These conditions may be even further simplified by defining a function

$$B\left(\theta'\right) \equiv A_0 - \int_{-\infty}^{\theta'} A(\theta) f(\theta)\, d\theta. \tag{25}$$

Substituting this in (23) and (24) yields:

$$B\left(\theta^*\left(X^I\right)\right) \cdot \left(X^I\right)^\beta = \int_{\theta^*\left(X^I\right)}^{\infty} V\left(X^I,\theta\right) f(\theta)\, d\theta - C, \tag{26}$$

$$\beta \cdot B\left(\theta^*\left(X^I\right)\right) \cdot \left(X^I\right)^{\beta-1} = \int_{\theta^*\left(X^I\right)}^{\infty} \frac{\partial V\left(X^I,\theta\right)}{\partial x} f(\theta)\, d\theta. \tag{27}$$

where $B\left(\theta^*\left(X^I\right)\right)$ is the function (25) evaluated at $\theta^*\left(X^I\right)$. Therefore, to determine the optimal learning rule, the problem is to find two positive real numbers X^I and $B\left(\theta^*\left(X^I\right)\right)$such that (26) and (27) are satisfied. Then A_0 and A_I are easily found by evaluating $A_0 = B\left(\theta^*\left(X^I\right)\right) + \int_{-\infty}^{\theta^*\left(X^I\right)} A(\theta) f(\theta)\, d\theta$ and $A_I = A_0 - \overline{A}$.

Conditions (26) and (27) mean that the optimal investment threshold X^I is equal to the optimal threshold of a firm, which has an opportunity to invest in a project with present value $\int_{\theta^*\left(X^I\right)}^{\infty} V\left(X^I,\theta\right) f(\theta)\, d\theta$ at an investment cost C. We elaborate this finding in a proposition, for which we need some new notation. Define

$$\widetilde{V}\left(x,\theta'\right) \equiv \int_{\theta'}^{\infty} V(x,\theta) f(\theta)\, d\theta,$$

that is, $\widetilde{V}\left(x,\theta'\right)$ is the expected value of a special kind of a project, which pays exactly the same as $V\left(x,\theta\right)$ if $\theta>\theta'$, but zero otherwise. Consider an auxiliary investment problem, where the investor must choose the optimal time to invest in project $\widetilde{V}\left(x,\theta'\right)$ at cost C:

$$\sup_{\tau} E_x\left[e^{-r\tau}\left(\widetilde{V}\left(X_\tau^x,\theta'\right)-C\right)\right]. \tag{28}$$

Define $\widetilde{X}\left(\theta'\right)$ to be the optimal investment threshold that solves this problem. The optimal learning threshold of the original problem is now determined by the following proposition:

Proposition 3 *The mapping $\widetilde{X}\circ\theta^*$ has a unique fixed point, which is the optimal learning threshold X^I. This means that the learning threshold X^I satisfies:*

$$X^I=\widetilde{X}\left(\theta^*\left(X^I\right)\right).$$

Proof. Function θ^* is decreasing as discussed before. On the other hand, it is clear that $\widetilde{X}$ is increasing, since the greater its argument, the smaller the value of the project associated with (28), and thus less anxious the investor is to invest in it. This implies that $\widetilde{X}\circ\theta^*$ is a decreasing mapping defined for all $x>0$. It is also straight forward to show that θ^* and $\widetilde{X}$ are continuous, implying that also $\widetilde{X}\circ\theta^*$ is continuous. Consequently, $\widetilde{X}\circ\theta^*$ must have exactly one fixed point. Conditions (26) and (27) ensure that X^I is the fixed point of $\widetilde{X}\circ\theta^*$. ■

We now give some intuition for the optimal learning rule. Proposition 3 means that the optimal learning point is the same point where it would be optimal to invest in a project with value $\widetilde{V}\left(x,\theta^*\left(X^I\right)\right)$, which can be written as $\widetilde{V}\left(x,\theta^*\left(X^I\right)\right)=\int_{\theta^*(X^I)}^{\infty}V\left(x,\theta\right)f\left(\theta\right)d\theta=E_\theta\left[V\left(x,\theta\right)\left|\theta>\theta^*\left(X^I\right)\right.\right]P\left(\theta>\theta^*\left(X^I\right)\right)$, thus representing the expected value of the project on condition that θ is so high that it is optimal to invest as soon as $X>X^I$, weighted by the probability that this is really the case. Thus, it can be seen that the optimal timing to learn balances the cost of delaying the learning, which is due to the possibility that the project is so valuable that the delay in learning causes also a delay in the actual investment, and the payoff of delaying the learning, which is the standard riskless rate of return on the cost C.

To strengthen the intuition, it is helpful to think of the project V as a portfolio of an infinite number of indivisible small projects indexed by θ, each weighted by the probability density f. Then, the lack of information on θ may be interpreted as a constraint that forces the investor to use a single decision rule for all of these projects. The payoff from learning θ is due to the fact that it allows the investor to use a separate decision rule for each project, which is obviously desirable. Thus, when considering the optimal timing of learning, the investor must balance the fact that postponing learning decreases the present value cost of learning, but on the other hand, it increases the "mass" of projects with high θ, which will then be undertaken too late, because the constraint that hinders using separate decision rules for each value of θ is removed too late. This explains why the optimal learning point can be characterized by a solution to an auxiliary problem in which the investor only cares about the high values of θ.

In summary, the basic trade-off that the investor faces consists of two counteracting effects caused by an additional delay in learning: 1) delayed cost of learning and 2) increased probability that the project investment will be undertaken too late. In the next section, I illustrate the optimal learning policy with a more specified version of the model.

5 Illustration

This section illustrates the main properties of the model. For this purpose, the model is specified further by choosing a particular form for the function V and probability distribution of θ. In particular, it is assumed that the present value of the investment is the product of θ and x, and there is a constant cost of investment, I. Further, θ is assumed to be uniformly distributed along $\left[0, \overline{\theta}\right]$, where without loss of generality I pick $\overline{\theta} = 2$:

$$
\begin{aligned}
V(x,\theta) &= \theta x - I, && (29)\\
f(\theta) &= \begin{cases} 0, \text{ when } \theta < 0 \text{ or } \theta > 2 \\ \frac{1}{2}, \text{ when } 0 \leq \theta \leq 2 \end{cases}. && (30)
\end{aligned}
$$

With the functional form of V in (29), it is standard to show that:

$$
F(x,\theta) = \begin{cases} A(\theta)\, x^{\beta}, x < X^{*}(\theta) \\ \theta x - I, x \geq X^{*}(\theta) \end{cases},
$$

where

$$
\begin{aligned}
X^{*}(\theta) &= \frac{\beta I}{(\beta-1)\,\theta}, && (31)\\
A(\theta) &= \frac{(\beta-1)^{\beta-1} I^{1-\beta}}{\beta^{\beta}} \theta^{\beta}.
\end{aligned}
$$

From (30), $E(\theta) = 1$, which implies that $\overline{V}(x) = x - I$. Then:

$$
\overline{F}(x) = \begin{cases} \overline{A} x^{\beta}, x < \overline{X} \\ \overline{V}(x) = x - I, x \geq \overline{X} \end{cases}, \tag{32}
$$

where

$$
\begin{aligned}
\overline{X} &= \frac{\beta I}{(\beta-1)},\\
\overline{A} &= \frac{(\beta-1)^{\beta-1} I^{1-\beta}}{\beta^{\beta}}.
\end{aligned}
$$

From (31),

$$
\theta^{*}(x) = \frac{\beta I}{(\beta-1)\,x} \text{ when } x \geq \frac{\beta I}{(\beta-1)\,2}. \tag{33}
$$

Figure 1 illustrates $X^{*}(\theta)$, the optimal investment threshold under full information as a function of θ. Note that the restriction $x \geq \frac{\beta I}{(\beta-1)2}$ in (33) is due to the fact that for lower values of x it is not yet optimal to invest even when the project is as good as it can be, that is, when $\theta = 2$.

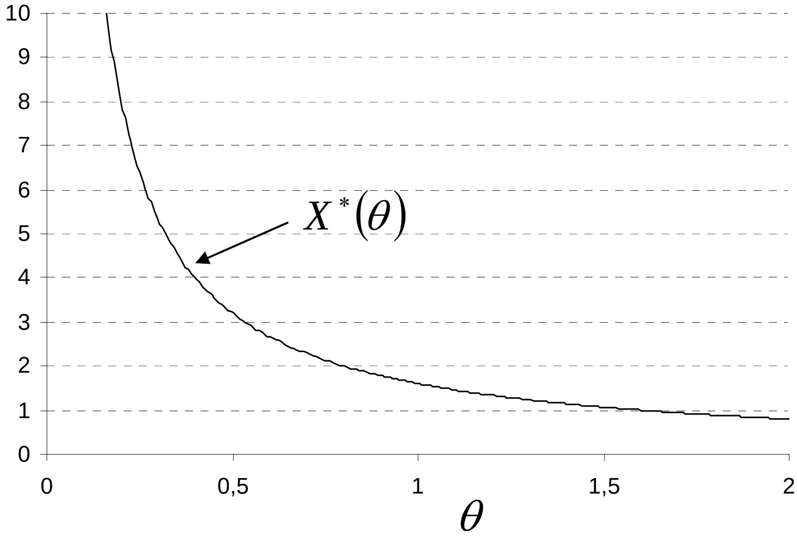

Figure 1: Optimal investment threshold under full information. The parameter values are $\alpha = 0$, $\sigma = 0.15$, $r = 0.05$, $I = 1$.

From (14) and (15), the value of information is:

$$
\begin{aligned}
V_I(x) &= \int_{-\infty}^{\theta^*(x)} A(\theta)\, x^{\beta} f(\theta)\, d\theta + \int_{\theta^*(x)}^{\infty} V(x,\theta)\, f(\theta)\, d\theta - \overline{F}(x) \\
&= \frac{1}{2} x^{\beta} \frac{(\beta-1)^{(\beta-1)}\, I^{(1-\beta)}}{\beta^{\beta}} \int_{0}^{\min\left(\frac{\beta I}{(\beta-1)x},2\right)} \theta^{\beta} d\theta + \frac{1}{2} \int_{\min\left(\frac{\beta I}{(\beta-1)x},2\right)}^{2} (\theta x - I)\, d\theta - \overline{F}(x) \\
&= \begin{cases} 2^{\beta} x^{\beta} (\beta-1)^{\beta-1} I^{1-\beta} \frac{\beta^{-\beta}}{\beta+1} - \overline{F}(x), & 0 \le x < \frac{\beta I}{(\beta-1)2} \\ x - I + \frac{\beta^2 I^2}{4(\beta^2-1)x} - \overline{F}(x), & x \ge \frac{\beta I}{(\beta-1)2} \end{cases}
\end{aligned}
$$

Inserting (32) we have:

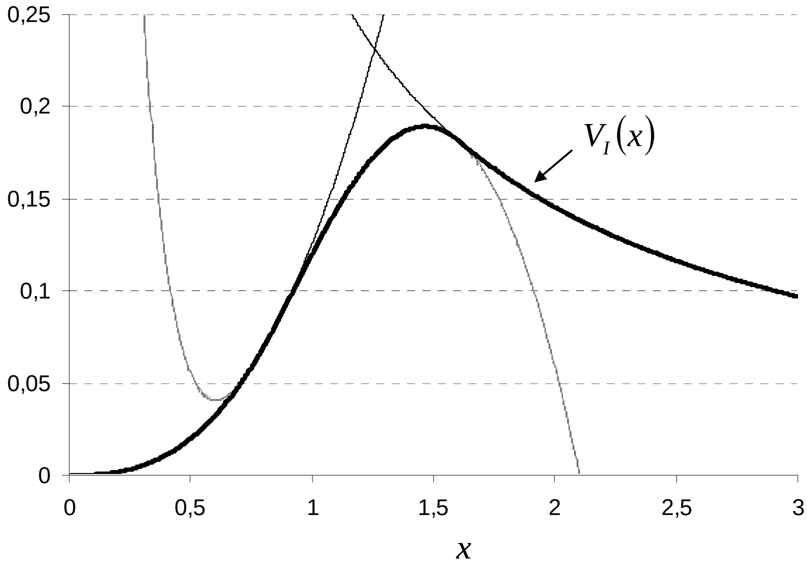

Figure 2: Value of information.

$$
\begin{aligned}
V_I(x) &= \begin{cases} 2^{\beta} x^{\beta} (\beta-1)^{\beta-1} I^{1-\beta} \frac{\beta^{-\beta}}{\beta+1} - \frac{(\beta-1)^{\beta-1} I^{1-\beta}}{\beta^{\beta}} x^{\beta}, & 0 \le x < \frac{\beta I}{(\beta-1)2} \\ x - I + \frac{\beta^2 I^2}{4(\beta^2-1)x} - \frac{(\beta-1)^{\beta-1} I^{1-\beta}}{\beta^{\beta}} x^{\beta}, & \frac{\beta I}{(\beta-1)2} \le x < \frac{\beta I}{(\beta-1)} \\ x - I + \frac{\beta^2 I^2}{4(\beta^2-1)x} - x + I & x \ge \frac{\beta I}{(\beta-1)} \end{cases} \\
&= \begin{cases} x^{\beta} (\beta-1)^{\beta-1} I^{1-\beta} \frac{2^{\beta}\beta^{-\beta} - \beta^{1-\beta} - \beta^{-\beta}}{\beta+1}, & 0 \le x < \frac{\beta I}{(\beta-1)2} \\ x - I + \frac{\beta^2 I^2}{4(\beta^2-1)x} - \frac{(\beta-1)^{\beta-1} I^{1-\beta}}{\beta^{\beta}} x^{\beta}, & \frac{\beta I}{(\beta-1)2} \le x < \frac{\beta I}{(\beta-1)} \\ \frac{\beta^2 I^2}{4(\beta^2-1)x}, & x \ge \frac{\beta I}{(\beta-1)} \end{cases}
\end{aligned} \tag{34}
$$

Figure 2 shows V_I with a given set of parameter values and illustrates that it is composed of three different curves, as (34) indicates.

Applying the value matching and smooth pasting conditions (17) and (18), we find:

$$X^I = \frac{1}{2}\frac{\beta}{\beta-1}\left(I + C + \sqrt{(2IC + C^2)}\right)$$

$$A_I = \frac{X^I - I + \frac{\beta^2 I^2}{4(\beta^2-1)X^I} - \frac{(\beta-1)^{\beta-1} I^{1-\beta}}{\beta^\beta}\left(X^I\right)^\beta - C}{\left(X^I\right)^\beta}$$

Figure 3 illustrates the solution with a given set of parameters. It shows as functions of x the net present value of information, $V_I(x) - C$, the ex-ante present value of full information investment option minus the cost of learning, $E_\theta[F(x,\theta)] - C$, and the net present value of the project with no possibility to learn, $\overline{V}(x)$. The values of the corresponding options to take these assets, $F_I(x)$, $F^0(x)$, and $\overline{F}(x)$ are also shown. Notice that the optimal learning threshold X^I is lower than $\overline{X}$, the optimal investment threshold under the assumption that learning is not possible. It can also be seen that the smooth pasting conditions hold for $F_I(x)$ and $F^0(x)$ at X^I, as required by (17) and (18) for the former and (21) and (22) for the latter. Note also that (13) holds, that is, $F^0(x) = F_I(x) + \overline{F}(x)$. Finally, note that the value of the option to learn, $F_I(x)$, is quite a considerable fraction of the total option value $F^0(x)$. This emphasizes the importance for the firm to manage optimally its learning options.

It is interesting to look at how the value of information is affected by changes in key parameters. Figure 4 illustrates $V_I(x) - C$ and the corresponding option value $F_I(x)$ at different values of σ. It is interesting to note that the increased uncertainty in the exogenous state variable x decreases the value of the option to learn. This is in contrast with the usual effect of uncertainty on option values. For example, it is well known that in the standard investment model with complete information on the payoff, increased uncertainty increases the value of the option to invest. In our model, increased uncertainty indeed increases the value of the option to undertake the project (as in standard models), but on the other hand decreases the value of the learning option. The explanation is that the greater the exogenous uncertainty, the less it seems to matter if the investment is undertaken at the wrong time, thus less valuable the information that helps to make correct timing decisions. Note that if C is suitably chosen, it may very well be the degree of exogenous uncertainty that determines whether it is optimal

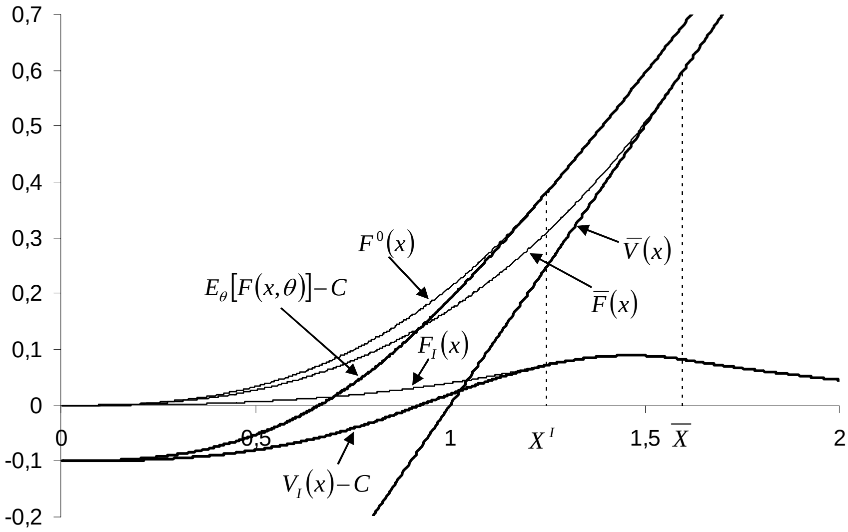

Figure 3: Optimal learning rule and the associated option values and payoff functions. The cost of learning is $C = 0.1$, other parameters are as in Figure 1.

to learn in the first place.

Figure 5 illustrates $V_I(x) - C$ and the corresponding option value $F_I(x)$ at different values of α. The effect is similar to that of σ: increased α decreases the value of the learning option. The intuition can be explained as follows. When α is low, the prospects for high values of x in the future are low. Thus, with low α, an underestimation of the true value of θ has a high cost, because it postpones the investment further into the future than with high α. Therefore, the potential loss due to incomplete information is high at low α, and thus the value of learning is high.

6 Conclusions

We have considered the optimal decision rule of a firm that faces an uncertain environment, and has an opportunity to undertake an irreversible investment project on which it has incomplete information. The firm is allowed to learn the true value of the unknown parameter at any time at a given cost. It has

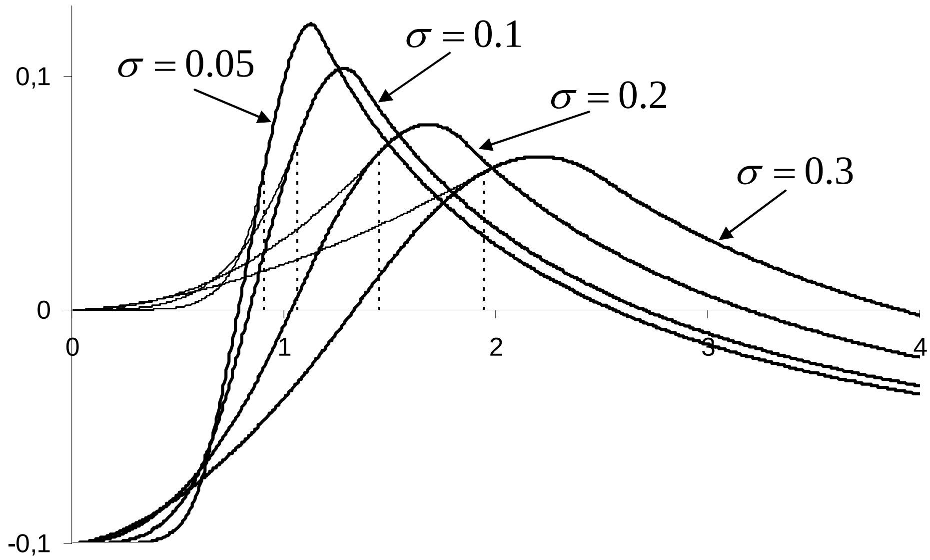

Figure 4: Value of information at different values of σ. The other parameters are as in Figure 3.

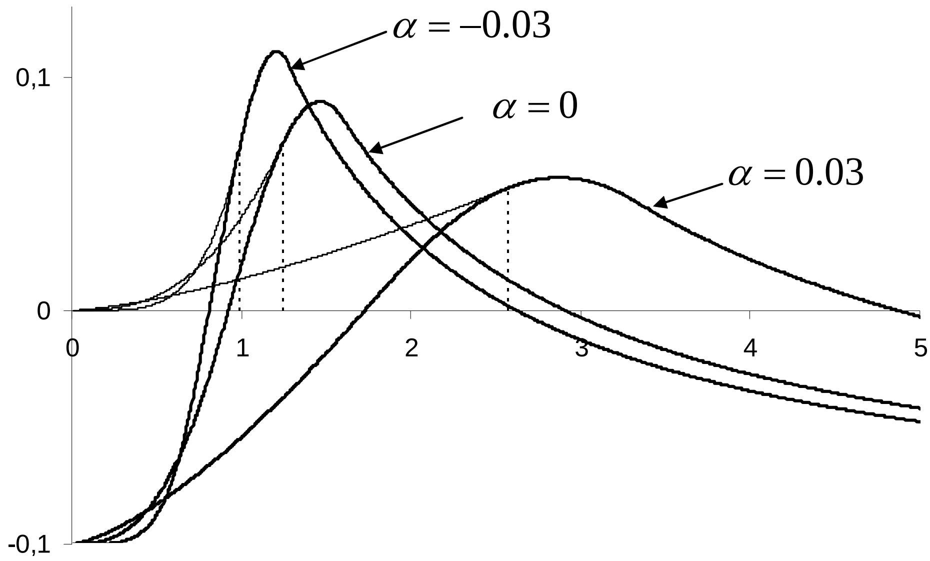

Figure 5: Value of information at different values of α. The other parameters are as in Figure 3.

been shown that the instantaneous value of information is driven by the exogenous stochastic variable. The value of the option to learn has been characterized, and the conditions for the optimal timing of information acquisition have been identified. Through a more specified version of the model, it has been demonstrated that the increased exogenous uncertainty and increased rate of growth of the project value decrease the value of information.

There are some potential directions for further work. The possibility to extend the framework to the purchase of multiple informative signals or the determination of the signal precision was already mentioned in the introduction section. It is also possible to consider interaction of the learning options with other decision variables besides timing of investment, for example the scale of investment or market entry with a choice of quality level. An interesting, but more distant direction for future research would be to incorporate market interactions and information externalities. One could envision the production and sale of information to be conducted by other parties than the investor (see Admati and Pfleiderer, 1990, for the sale of information in a different context). There is also a recent literature that combines real options analysis and game theory. Papers that incorporate incomplete information include Grenadier (1999), which considers a setting where firms with private information update their beliefs by observing each other's investment decisions, and Lambrecht and Perraudin (2002), which considers competition over a single investment project, where competing firms have incomplete information on each other's investment cost. On the other hand, there are numerous deterministic models that study strategic effects of information externalities. A possible direction for future work would be to incorporate concepts from such literature in our framework in order to consider the *strategic* value of learning options.

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