Evaluating Natural Resource Investments Using the Least–Squares Monte Carlo Simulation Approach*

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Abstract

The idea that Monte Carlo simulation can not be applied to the pricing of options (real or financial) with early exercise features has been overridden in the light of new research results in the last decade. This paper attempts to contribute to this revived interest on Monte Carlo simulation valuation, by applying the proposed least–squares simulation method to the valuation of a hypothetical natural resource investment. Results seem to suggest that the Monte Carlo framework might be a natural way forward in the valuation of investments under multiple uncertainties and project–specific complexities.

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1 Introduction

Even though it is now widely accepted amongst academic and corporate practitioners that investment alternatives can in principle be evaluated via contingent claim techniques, it is also often acknowledged that there are several aspects of corporate decision–making that do not conform to the celebrated Black–Scholes–Merton pricing framework (Black and Scholes (1973), Merton (1973)).

One such aspect, which partly motivates this paper, questions the suitability of the Black–Scholes–Merton geometric Brownian motion assumptions in evaluating real investment options. Contrary to the Black and Scholes (1973) setting, most investment opportunities can be undertaken at any point in time and not just at a pre–specified expiration date. Thus their award–winning, European–style option pricing formula can hardly be considered a realistic valuation tool for the options embedded in most investment decisions of interest. Furthermore, unlike the American–style option formula of Merton (1973) which forms the basis of much theoretical research in real options, investment decisions are almost never perpetual in nature. Regardless how proprietary or unique a project is, it is rather simplistic to assume that the option to exploit the project will last forever. Moreover, the option to invest in a project is in many cases exposed to more risk factors than the basic financial option pricing framework will by construction admit.

Incorporating American–style exercise features, finite option maturities and multiple stochastic variables might be desirable features of a real options valuation framework, however the mathematical complexity that such features introduce often makes analytical solutions impossible. Thus, for contingent claim valuation of an investment opportunity to be realistic, decision–makers will most likely have to resort to numerical methods.

Fortunately, there is a plethora of numerical methods (initially proposed in the traded derivatives literature) which could in principle be applied to the valuation of real investment options. In this paper, we initially attempt a brief (and by no means comprehensive) overview of the numerical methods proposed, before concentrating on the applicability of a newly–proposed method based on Least–Squares Monte Carlo simulation.

This new approach, proposed by Longstaff and Schwartz (2001), adds to a recently revived interest in evaluating American–style and/or path–dependent, finite–horizon, multidimensional options by simulation. Prior contributions to this stream include Tilley (1993), Carriere (1996), Barraquand and Martineau (1995), Broadie and Glasserman (1997), Broadie, Glasserman and Jain (1997), Reymar and Zwecher (1997), Ibáñez and Zapatero (1998), Carr (1998) and Tsitsiklis and Van Roy (2001).[1]

In this paper we attempt to assess the applicability of the Longstaff and Schwartz (2001) algorithm for valuing real investment opportunities. In Section 2, we start with an overview of the most important numerical methods applied in option pricing, which could in principle be used in a real options valuation framework. In Section 3 a detailed step–by–step account of the Longstaff and Schwartz (2001) algorithm is presented for completeness. In Section 4 the algorithm is applied to the pricing of American options under constant (Section 4.1) and mean–reverting (Section 4.2) interest rates, and the results are benchmarked against alternative pricing methods and published research. In Section 5 we turn our attention to the valuation of a natural resources investment with several option–like characteristics. The valuation is initially benchmarked against an existing model (Section 5.1), before extended to account for multiple state variables (Section 5.2). Finally, Section 6 concludes.

2 Numerical methods

2.1 Lattice or tree approaches

Discrete trees representations of stochastic processes and their use in option valuation have first been proposed by Cox, Ross and Rubinstein (1979) and Rendleman and Bartter (1979). In its simplest form, the multiplicative binomial lattice has proved an excellent pedagogical and pricing tool. The method's power lies in that it is easy to set up, it can accommodate any pattern of cash flow payments of an underlying and it can handle early exercise features that are essential for American–style derivatives. An example of binomial tree valuation of real asset investments is provided by Smit and Ankum (1993).

[1] Some of these papers are more extensively reviewed in Section 2.3.

However the problem of the method is that handling more than a couple of stochastic factors using a lattice seems computationally infeasible. In particular, the required number of tree nodes grows geometrically when we allow for multiple stochastic factors such as interest rates, dividend yields, volatilities, or multiple underlying assets. Among others Boyle (1988), Boyle, Evnine and Gibbs (1989), Trigeorgis (1991) and Gamba and Trigeorgis (2001) have proposed lattice methods for (real) contingent claims that can handle more than one stochastic variable. However in these contributions, all the underlying stochastic variables have to be jointly lognormally distributed, an assumption difficult to justify for many factors such as the interest rate or the convenience yield of commodities.[2]

2.2 Finite difference method

Brought to finance from engineering, Brennan and Schwartz (1977) used finite difference methods to value a derivative by solving the security differential equation numerically. The differential equation is first approximated by a set of difference equations, which are then solved iteratively from known boundary conditions.

The appealing feature of the method is its speed: in an accompanying paper, Brennan and Schwartz (1978) establish the relation of finite differences with numerical integration, which is available as a standard procedure in most mathematical software packages. In another important paper, these authors demonstrated the applicability of the method in the evaluation of natural resource investments (Brennan and Schwartz (1985)).

However, much like lattice methods, the finite difference approach suffers from the curse of dimensionality, i.e. it is computationally impossible to extend it to several stochastic underlying factors.[3]

[2]For empirical evidence on the stochastic behaviour of interest rates and convenience yields readers are referred to Chan *et al.* (1992) and Schwartz (1997) respectively.

[3]It has been shown that the Kamrad and Ritchken (1991) extension of the basic lattice framework, if properly parameterised, is a special case of the explicit finite difference method (see Rubinstein (2000) for the correspondence). This relation between the two approaches seems to rationalise their common inability to handle multiple state variables.

2.3 Monte Carlo simulation

Boyle (1977) was the first to apply Monte Carlo (MC from now on) simulation to option pricing. Simulation–based techniques are straightforward, easy to set up and can handle both discrete and proportional dividend payments. Their most appealing feature however is that the computational complexity involved only grows linearly with the number of the underlying stochastic processes introduced. Furthermore, MC simulation is perhaps unique vis–á–vis other numerical methods in that the distribution(s) used to generate returns on the underlying asset(s) need not have closed form analytic expressions, thus opening the possibility of deriving prices using empirically observed distributions.

Despite its desirable features, until very recently it has been perceived that it would be impossible to use MC simulation to price American–style derivatives (see Campbell, Lo and MacKinley (1997, p. 390) and Hull (1997, p. 364)). The reason is the early exercise possibility inherent in American options, which can not be captured by a forward–induction technique like simulation.

The first attempt to determine the early exercise strategy of American options via simulation was by Tilley (1993). He proposed a MC algorithm that mimicked the standard lattice method by determining the value of keeping the option alive as the present value of the expected one–period–ahead values. Carriere (1996) used the theory of stopping times to establish that the value of the early exercise privilege of American options is equivalent to calculating a number of conditional expectations. These are usually difficult to evaluate explicitly but can be approximated using nonparametric regression involving simulated stock price trajectories. He also suggested that the estimators proposed by Tilley (1993) were biased. Barraquand and Martineau (1995) developed a method similar in spirit to Tilley (1993), but easier to extend. The key idea (so called "stratified state aggregation along the payoff") was to partition the time–state space into a number of bins, in such a way that the option payoff was approximately the same for all paths in the particular bin. From the simulated paths, the transition probabilities of moving to a different bin next period conditional on the current bin could be calculated and used to determine the expected option continuation value. Broadie and Glasserman (1997) and Broadie, Glasserman and Jain (1997) proposed a method that simulated the evolution of underlying assets via random trees that branch at each of the possible early exercise dates and two consistent price estimates (which constituted a confidence interval for the option price) were obtained from their method. Broadie, Glasserman and Jain (1997) have criticised the methods suggested by Tilley (1993) and Barraquand and Martineau (1995). They suggested that the price estimates by Tilley were upward biased since the same simulated values were used to construct the lattice–like state space and estimate the optimal early exercise policy. They argued that the same should apply to the method by Barraquand and Martineau and provided a simple example that showed that convergence to the correct prices was not guaranteed. They concluded that among the larger class of price estimators that used path stratification and lattice mimicking, there is no unbiased estimator of the American option price.

The method we adopt in this paper, which takes a different approach to the computation of the early exercise value, has been proposed by Longstaff and Schwartz (2001) (henceforth $L\&S$). Their idea is to approximate the conditional expectation of continuation value at each possible exercise date from a cross–sectional regression of simulated paths. In their paper, $L\&S$ demonstrated the applicability of the method by pricing a wide range of option types (American–style, path dependent, multidimensional, options on jump–diffusion processes, etc.). In the next section a description of the $L\&S$ method is presented for the sake of completeness.

3 The Longstaff and Schwartz approach

The $L\&S$ method consists of three sequential steps: First, path trajectories for all the relevant stochastic state variables that determine the price of an option are simulated. Then, working backwards from the option maturity, a matrix detailing the time (if any) at which the option is optimally exercised along each path is determined (the option payoff matrix). This is accomplished by using least–squares regression on the cross–section of in–the–money paths to approximate the early exercise boundary at each point in time. Finally, the option price is determined by discounting and averaging the relevant option payoffs across all matrix entries.

We proceed by providing a detailed description of the $L\&S$, three–step method. The specification is kept as general as possible; however in some instances, reference is made to specific option contracts for clarity.

1. Simulation of underlying asset(s) paths

   (a) Simulate a large number (M) of paths of asset(s) prices or returns. If the option has Bermudan features (i.e. it can only be exercised at a finite number of times up to and including the option maturity), the number of steps is set equal to the number of possible exercise dates. Otherwise the number of steps (N) must be sufficiently large to limit the discretisation bias and approximate the continuous possible exercise. Let $\mathbb{S}_j^q(t_i)$ denote a $1 \times q$ vector of underlying state variable(s) along a given path j at a given time t_i corresponding to step i, where $j = 1, \ldots, M$, $i = 1, \ldots, N$ and q is the number of stochastic factors that the value of the option depends on. In the case of an American put option on a stock, $q = 1$ and $\mathbb{S}_j^1(t_i) = S_j(t_i)$ is the stock price along path j for times t_i.

2. Calculation of the payoff matrix

   (a) Let $F\left(\mathbb{S}_j^q(t_i)\right)$ denote the immediate exercise value of the option along path j at time t_i, $j = 1, \ldots, M$, $i = 1, \ldots, N$. For example, in the case of a put option on a stock, $F\left(\mathbb{S}_j^q(t_i)\right)$ would correspond to $F(S_j(t_i)) = E - S_j(t_i)$ where E is the exercise price. For a put option on the minimum of L assets, we would have $F\left(S_j^1(t_i), \ldots, S_j^L(t_i)\right) = E - \min\left[S_j^1(t_i), \ldots, S_j^L(t_i)\right]$, while for an average strike Asian put option $F(S_j(t_i)) = \frac{1}{i}\sum_{l=0}^{i} S_j(t_l) - S_j(t_i)$. Let $\mathbb{P}$ be the option payoff matrix, with dimensions $M \times N$ and typical elements $f_{j,i}$. At time $t_N \equiv T$ (the expiration date of the option) the payoff along each path would be the maximum of zero and of the intrinsic value of the option, i.e.

$$f_{j,N} = \max\left[0, F\left(\mathbb{S}_j^q(T)\right)\right] \qquad 1 \leq j \leq M \tag{1}$$

If an estimate, $\widehat{p}$, of the "true" European option price p is needed, this can be done immediately by discounting and averaging the last column of the $\mathbb{P}$ matrix,

$$\widehat{p} = \frac{1}{M} \sum_{j=1}^{M} f_{j,N} B_{0,T} \tag{2}$$

where $B_{\tau,s}$ denotes the price at time τ of a zero–coupon bond with a face value of \$1, maturing at time s $(s \geq \tau)$.

(b) For non-European option prices P, working backwards from the option maturity T, at any time t_i, $0 < i < N$ and for the subset of in–the–money paths, $\widetilde{M} \subseteq M$, $\widetilde{M} = \left\{ j : F\left(\mathbb{S}_j^q(t_i)\right) > 0 \right\}$, a decision between immediate exercise or continuation must be made. The optimal decision depends on a comparison between the immediate exercise value, $F\left(\mathbb{S}_j^q(t_i)\right)$, and the conditional expected payoff from keeping the option alive (conditional on the current state, $\mathbb{S}_j^q(t_i)$) which is denoted $\mathbb{E}\left[P|\, \mathbb{S}_j^q(t_i)\right]$. Using next period values along each path to determine the latter would lead to biased price estimates, as this would be equivalent to assuming that the option holder has perfect foresight.[4] On the other hand, again simulating several paths from each possible exercise point would result in the curse of dimensionality that lattice methods suffer. *L&S* suggest approximating the conditional expectation of continuation using the theory of Hilbert spaces and the information in the cross–section of the $\widetilde{M}$ paths.

(c) The theory of Hilbert spaces tells us that any function $g(x)$ belonging to the space can be represented as a countable linear combination of bases for this vector space, i.e.

$$g(x) = \sum_{k=0}^{\infty} \beta_k \phi_k(x) \tag{3}$$

where $\{\phi_k(x)\}_{k=1}^{\infty}$ form a basis. To use this in practice, we represent $g(x)$ using a finite linear combination of only K terms which

[4]This is the point raised by Broadie, Glasserman and Jain (1997) in their critique on previous work by Tilley (1993) and Barraquand and Martineau (1995).

we denote

$$g_K(x) = \sum_{k=0}^{K} \beta_k \phi_k(x) \tag{4}$$

One possible way to approximate $g_K(x)$ is by least–squares regression: use $M \geq K+1$ observation points $\left(g\left(x^i\right), x^i\right)$ to get estimates of the coefficients $\{\beta_k\}_{k=0}^{K}$ by solving

$$\min_{\{\beta_k\}_{k=0}^{K}} \sum_{i=1}^{M} \left(\beta_0\phi_0\left(x^i\right) + \beta_1\phi_1\left(x^i\right) + \ldots + \beta_K\phi_K\left(x^i\right) - g\left(x^i\right)\right)^2 \tag{5}$$

Using the resulting estimates $\left\{\widehat{\beta}_k\right\}_{k=0}^{K}$ from (5), we can calculate the approximation as

$$\widehat{g}_K(x) = \sum_{k=0}^{K} \widehat{\beta}_k \phi_k(x). \tag{6}$$

According to the theory, under very general conditions, $\widehat{g}_K(x) \to g_K(x)$ as $M \to \infty$.

(d) For the problem at hand, the theory translates to the following: for any time t_i, $0 < i < N$, $\mathbb{E}\left[P \middle| \mathbb{S}_j^q(t_i)\right]$, the conditional expectation of keeping the option alive when the current state is $\mathbb{S}_j^q(t_i)$ corresponds to the unknown function $g(x)$ in (3). This is the function we wish to approximate through least–squares regression in order to determine the times at which early exercise is optimal. To employ least–squares for the approximation, we will use the cross-section of the observations provided by the simulated in–the–money paths. That is for any $j \in \widetilde{M}$, the payoff from keeping the option unexercised is the payoff at some point along the path until the expiration date, discounted back to current time

$$y_j(t_i) = \sum_{n=i+1}^{N} f_{j,n} B_{t_i,t_n}. \tag{7}$$

This is the dependent variable, i.e. $g\left(x^i\right)$ in equation (5). The independent variables (the first K terms of the squared parenthesis in equation (5)) are

$$X_j(t_i) = H\left(\mathbb{S}_j^q(t_i)\right) \tag{8}$$

where $H\left(\mathbb{S}_j^q(t_i)\right)$ is a transformation of the state variable(s) based on the choice of the basis functions. Using OLS regression, a set of coefficient estimates can be obtained

$$\widehat{\beta}(t_i) = \left(X(t_i)' X(t_i)\right)^{-1} X(t_i)' y(t_i),$$

which can then be used as in equation (6) to get an approximate estimate of the "true" $g(x) = E\left[y_j(t_i) \,|\, \mathbb{S}_j^q(t_i)\right]$, the expected continuation value, conditional on the current state.

(e) The fitted values from the regression, $\widehat{y}(t_i) = X(t_i)\widehat{\beta}(t_i)$ are now used to determine whether at time t_i it is indeed optimal to exercise the option given the state. They are compared to the value of immediate exercise $F\left(\mathbb{S}_j^q(t_i)\right)$. If the latter is greater, $f_{j,i}$ is set equal to $F\left(\mathbb{S}_j^q(t_i)\right)$ and all other values in $\mathbb{P}$, $f_{j,n}$ $i < n \le N$, are set equal to zero, since options can only be exercised at most once along each path. In all other cases, $f_{j,i}$ is set equal to zero.

$$f_{j,i} = \begin{cases} F\left(\mathbb{S}_j^q(t_i)\right) \text{ and } f_{j,n} = 0,\ i < n \le N & \text{if } F\left(\mathbb{S}_j^q(t_i)\right) > \widehat{y}_j(t_i) \\ 0 & \text{if } F\left(\mathbb{S}_j^q(t_i)\right) \le \widehat{y}_j(t_i) \end{cases} \quad j \in \widetilde{M}$$

3. Calculating the option price

(a) At time $t_i = 0$ an estimate $\widehat{P}$ of the price of the option with early exercise features (i.e. American, Bermudan, etc.) can be found by discounting payoffs to time zero and averaging across paths

$$\widehat{P} = \frac{1}{M}\sum_{j=1}^{M}\sum_{i=1}^{N} f_{j,i} B_{0,t_i}.$$

4 Financial options: Results and benchmarks

In this section we describe how the method is implemented in order to price financial American–style options. In order to benchmark our implementation of the method, results are compared against alternative option pricing numerical methods and other published research.

4.1 One factor American puts

Initially we price American–style put options on a dividend–paying share of stock, where the risk–neutral stock price process follows the stochastic differential equation (SDE)

$$dS(t) = (r - \delta) S(t) dt + \sigma S(t) dZ(t) \tag{9}$$

with r, $\delta \geq 0$, $\sigma > 0$ constants and $Z(t)$ a standard Brownian motion.

Since the solution to the SDE in (9) is known in closed–form, a sequence of stock prices at dates $0 < t_1 \leq t_2 \leq \ldots \leq t_N = T$ can be obtained by

$$S(t_{i+1}) = S(t_i) \exp\left\{\left(r - \delta - \frac{1}{2}\sigma^2\right)(t_{i+1} - t_i) + \sigma\sqrt{t_{i+1} - t_i} Z(t_{i+1})\right\} \tag{10}$$

where $Z(t_{i+1}) \sim$ i.i.d. $\mathcal{N}(0, 1)$.

In each simulation, a total of $M = 100,000$ paths and $N = 50$ time–steps per year are used. The default random number generator in GAUSS is used to simulate the stock price paths[5] (experimentation with other random number generation procedures such as those in Box and Muller (1958) and Marsaglia and Bray (1964) made little difference in the price estimates produced).

Antithetic simulation is used as an easy variance reduction technique. The basic idea of the technique is to introduce negative correlation between the simulated paths. This is done by sampling $\frac{M}{2}$, i.i.d. $\mathcal{N}(0, 1)$ random numbers $Z(t)$, and use these together with $-Z(t)$ to generate the simulated paths. It can be shown that this method reduces the sampling variance of the option price estimator produced.[6]

To approximate the conditional expectation, we follow $L\&S$ and use the family of Laguerre polynomials given by

$$L_k(x) = \frac{e^x}{k!}\frac{d^k\left(x^k e^{-x}\right)}{dx^k} \tag{11}$$

($k = 0, 1, \ldots, K$) as the basis functions in the $H(.)$ transformation in (8). As in Longstaff and Schwartz (2001), we use the first three terms of (11) as

[5] The seed is set equal to $100,000$.

[6] Interested readers can refer to Hammersley and Handscomb (1964) for an introduction to this technique.

a base case, i.e. $k = 0, 1$ and 2 and each of the three terms is multiplied by the weighting function $e^{-\frac{x}{2}}$. Thus, at all times t_i, $0 < i < N$, the following least–squares regression is performed

$$y(t_i) = \beta_0 + \beta_1 e^{-\frac{S(t_i)}{2}} + \beta_2 e^{-\frac{S(t_i)}{2}} (1 - S(t_i)) \\ + \beta_3 e^{-\frac{S(t_i)}{2}} \left(1 - 2S(t_i) + \frac{S^2(t_i)}{2}\right) + \epsilon(t_i) \quad (12)$$

where $y(t_i)$ is given by equation (7) and $\epsilon(t_i)$ is an error term.[7]

We price options ranging from deep out–of–the–money to deep in–the–money and for different volatilities and expiration dates. Simulated American–option prices (denoted $\widehat{P}$) are reported in column 5 of Tables 1–3. In all calculations $E = 40$, $r = 6\%$ and $\delta = 0$. In Table 1, put prices are benchmarked against an implicit finite difference scheme with 40,000 time steps per year and 1,000 steps for the stock price (P_{FD} in column 4).

In all but one case, pricing errors are less than 1 cent (or 0.13% in relative terms). The maximum absolute error is 1.1 cents (0.65% in percentage terms). In just one of the twenty option specifications priced in Table 1 is the hypothesis $H_0 : \widehat{P} = P_{FD}$ rejected at a 5% significance level. We also report simulated European put prices in column 10 (denoted $\widehat{p}$) which are benchmarked against the closed–form Black and Scholes (1973) formula (p_{BS} in column 9). The simulation algorithm also prices the European counterparts very accurately.

In Table 2, the same simulated prices are benchmarked against the binomial tree approach of Cox, Ross and Rubinstein (1979) with 4,000 time steps (P_{CRR}). The comparison yields similar results, with all errors less than 1%. In all but one cases the hypothesis that the simulated put estimates are equal to the binomial model prices is accepted.

Table 3 compares our results with those reported by Longstaff and Schwartz (2001) in their original paper. Results are relatively close, with the highest percentage error being 0.5% (2.5 cents in absolute terms). Unfortunately, Longstaff and Schwartz (2001) do not report any details concerning the random number generation of their implementation, thus a closer calibration can not be achieved.

[7] It is recommended that all payoffs and stock prices are normalised by dividing with the exercise price E to avoid numerical overflow.

S	T	σ	P_{FD}	$\widehat{P}$	$(s.e.)$	Δ_{US}[a]	$\Delta_{US}^{\%}$	p_{BS}	$\widehat{p}$	$(s.e.)$	Δ_{EU}[b]	$\Delta_{EU}^{\%}$
36	1	0.2	4.478	4.481	0.005	0.003	0.073	3.844	3.846	0.006	0.002	0.052
36	2	0.2	4.840	4.845	0.008	0.005	0.112	3.763	3.755	0.007	-0.008	-0.213
36	1	0.4	7.101	7.105	0.023	0.004	0.055	6.711	6.715	0.016	0.004	0.060
36	2	0.4	8.507	8.504	0.031	-0.003	-0.036	7.700	7.690	0.020	-0.010	-0.130
38	1	0.2	3.250	3.249	0.005	-0.001	-0.017	2.852	2.855	0.005	0.003	0.105
38	2	0.2	3.745	3.748	0.008	0.003	0.089	2.991	2.982	0.006	-0.009	-0.301
38	1	0.4	6.147	6.149	0.022	0.002	0.040	5.834	5.840	0.014	0.006	0.103
38	2	0.4	7.670	7.673	0.030	0.003	0.044	6.979	6.972	0.018	-0.007	-0.100
40	1	0.2	2.314	2.315	0.005	0.001	0.049	2.066	2.070	0.003	0.004	0.194
40	2	0.2	2.884	2.891	0.007	0.007	0.246	2.356	2.348	0.004	-0.008†	-0.340
40	1	0.4	5.312	5.316	0.020	0.004	0.071	5.060	5.066	0.012	0.006	0.119
40	2	0.4	6.920	6.917	0.029	-0.003	-0.047	6.326	6.323	0.017	-0.003	-0.047
42	1	0.2	1.617	1.618	0.004	0.001	0.032	1.465	1.466	0.002	0.001	0.068
42	2	0.2	2.212	2.213	0.006	0.001	0.058	1.841	1.837	0.003	-0.004	-0.217
42	1	0.4	4.582	4.585	0.018	0.003	0.066	4.379	4.385	0.011	0.006	0.137
42	2	0.4	6.248	6.242	0.028	-0.006	-0.102	5.736	5.737	0.015	0.001	0.017
44	1	0.2	1.110	1.115	0.003	0.005	0.446	1.017	1.017	0.002	0.000	0.000
44	2	0.2	1.690	1.679	0.005	-0.011†	-0.647	1.429	1.427	0.003	-0.002	-0.140
44	1	0.4	3.948	3.955	0.017	0.007	0.185	3.783	3.789	0.009	0.006	0.159
44	2	0.4	5.647	5.647	0.026	0.000	0.004	5.202	5.209	0.014	0.007	0.135

Table 1: This Table compares American put prices from a finite difference method (P_{FD}) to the $L\&S$ simulation algorithm ($\widehat{P}$). The exercise price is $E = 40$, the interest rate is $r = 6\%$ and the dividend yield $\delta = 0$. S is the initial stock price, T the time to maturity, and σ the volatility of the underlying stock price returns. In each simulation $100,000$ ($50,000$ plus $50,000$ antithetic) paths and 50 possible exercise dates per year are used.

[a] $\Delta_{US} = \widehat{P} - P_{FD}$. A $\dagger$ means that the $H_0 : \widehat{P} = P_{FD}$ is rejected at the 5% level.

[b] $\Delta_{EU} = \widehat{p} - p_{BS}$. A $\dagger$ means that the $H_0 : \widehat{p} = p_{BS}$ is rejected at the 5% level.

S	T	σ	P_{CRR}	$\widehat{P}$	$(s.e.)$	Δ_{US}[a]	$\Delta_{US}^{\%}$
36	1	0.2	4.487	4.481	0.005	-0.005	-0.121
36	2	0.2	4.848	4.845	0.008	-0.003	-0.059
36	1	0.4	7.109	7.105	0.023	-0.004	-0.061
36	2	0.4	8.514	8.504	0.031	-0.010	-0.121
38	1	0.2	3.257	3.249	0.005	-0.008	-0.238
38	2	0.2	3.751	3.748	0.008	-0.003	-0.081
38	1	0.4	6.155	6.149	0.022	-0.005	-0.083
38	2	0.4	7.675	7.673	0.030	-0.002	-0.020
40	1	0.2	2.320	2.315	0.005	-0.004	-0.189
40	2	0.2	2.890	2.891	0.007	0.001	0.043
40	1	0.4	5.318	5.316	0.020	-0.002	-0.044
40	2	0.4	6.923	6.917	0.029	-0.006	-0.094
42	1	0.2	1.621	1.618	0.004	-0.004	-0.230
42	2	0.2	2.217	2.213	0.006	-0.004	-0.160
42	1	0.4	4.588	4.585	0.018	-0.003	-0.070
42	2	0.4	6.251	6.242	0.028	-0.009	-0.142
44	1	0.2	1.113	1.115	0.003	0.002	0.182
44	2	0.2	1.693	1.679	0.005	-0.014†	-0.848
44	1	0.4	3.953	3.955	0.017	0.002	0.055
44	2	0.4	5.647	5.647	0.026	0.000	0.003

Table 2: This Table compares American put prices from a binomial pricing model (P_{CRR}) to the *L&S* simulation algorithm ($\widehat{P}$). The exercise price is $E = 40$, the interest rate is $r = 6\%$ and the dividend yield $\delta = 0$. S is the initial stock price, T the time to maturity, and σ the volatility of the underlying stock price returns. In each simulation $100,000$ ($50,000$ plus $50,000$ antithetic) paths and 50 possible exercise dates per year are used.

[a] $\Delta_{US} = \widehat{P} - P_{CRR}$. A † means that the $H_0 : \widehat{P} = P_{CRR}$ is rejected at the 5% level.

S	T	σ	$P_{L\&S}$	$(s.e.)$	$\widehat{P}$	$(s.e.)$	Δ_{US}[a]	$\Delta_{US}^{\%}$
36	1	0.2	4.472	0.010	4.481	0.005	0.009	0.207
36	2	0.2	4.821	0.012	4.845	0.008	0.024†	0.506
36	1	0.4	7.091	0.020	7.105	0.023	0.014	0.196
36	2	0.4	8.488	0.024	8.504	0.031	0.016	0.187
38	1	0.2	3.244	0.009	3.249	0.005	0.005	0.168
38	2	0.2	3.735	0.011	3.748	0.008	0.013	0.357
38	1	0.4	6.139	0.019	6.149	0.022	0.010	0.170
38	2	0.4	7.669	0.022	7.673	0.030	0.004	0.057
40	1	0.2	2.313	0.009	2.315	0.005	0.002	0.092
40	2	0.2	2.879	0.010	2.891	0.007	0.012	0.420
40	1	0.4	5.308	0.018	5.316	0.020	0.008	0.147
40	2	0.4	6.921	0.022	6.917	0.029	-0.004	-0.061
42	1	0.2	1.617	0.007	1.618	0.004	0.001	0.032
42	2	0.2	2.206	0.010	2.213	0.006	0.007	0.330
42	1	0.4	4.588	0.017	4.585	0.018	-0.003	-0.070
42	2	0.4	6.243	0.021	6.242	0.028	-0.001	-0.022
44	1	0.2	1.118	0.007	1.115	0.003	-0.003	-0.272
44	2	0.2	1.675	0.009	1.679	0.005	0.004	0.243
44	1	0.4	3.957	0.017	3.955	0.017	-0.002	-0.043
44	2	0.4	5.622	0.021	5.647	0.026	0.025	0.449

Table 3: This Table compares American put prices from Table 1 in Longstaff and Schwartz (2001) ($P_{L\&S}$) to our implementation of the simulation algorithm ($\widehat{P}$). The exercise price is $E = 40$, the interest rate is $r = 6\%$ and the dividend yield $\delta = 0$. S is the initial stock price, T the time to maturity, and σ the volatility of the underlying stock price returns. In each simulation $100,000$ ($50,000$ plus $50,000$ antithetic) paths and 50 possible exercise dates per year are used.

[a] $\Delta_{US} = \widehat{P} - P_{L\&S}$. A † means that the $H_0 : \widehat{P} = P_{L\&S}$ is rejected at the 5% level.

However, the results in Tables 1–3 suggest that our implementation of the method is also very accurate.

We also assess the sensitivity of the method on the number of terms K used in the approximation of the expected continuation value (Equation (4)). In Table 4, panel (A) we price two in–the–money (row specifications 8 and 12 of Table 1), two at–the–money (specifications 1 and 20) and two out–of–the–money (specifications 9 and 13) options by increasing the Laguerre family members k in (11) from 1 to $K = 5$. As evident from the results, increasing the number of regressors from one to two increases the price estimate in all six specifications. The average increase is 2.5 cents, and the increase in each specification is significant at the 5% level using a standard population means test. In most cases, increasing the regressors to two makes the prices very close to the benchmark (the finite difference method). Only two prices (specifications 12 and 20) are now significantly different from the benchmark prices at a 5% level. Increasing K to three does not have the same dramatic effect, even though all price estimates increase. None of the price estimates are now different from the finite difference benchmark. Thus using two or three members of the Laguerre polynomials family should produce very accurate option prices.

In the remainder of the Table, we assess the sensitivity of the method to family of the basis functions used for the approximation. Instead of the Laguerre polynomials in (11) we use ordinary polynomials $\left\{S\left(t_i\right)^k\right\}_{k=0}^{\infty}$ in panel (B). Even though not orthogonal, ordinary polynomials are very easy to use from a computational point of view and it has been suggested by *L&S* that they might be sufficient for pricing purposes. As the results of panel (B) suggest, for the case of American put options, monomials can produce very accurate prices and these results seem to converge faster since none of the estimates is significantly different from the benchmark for as few regressors as $K = 2$. Following Stentoft (2001), in panel (C) we use the family of General Chebyshev polynomials $T_k(x) = \cos(k \arccos(2x - 1))$. When weighted with the function $\left(1 - (2x - 1)^2\right)^{-0.5}$ these polynomials are orthogonal on the interval $[0, 1]$ where our regression independent variables reside. The results in panel (C) seem to suggest that nothing is gained by using a more complex polynomials family than simple polynomials.

In all remaining applications, simple polynomials are used in the cross-

Panel A:	Laguerre				
$Nr.$	$K=1$	$K=2$	$K=3$	$K=4$	$K=5$
1	$4.450^{\dagger,\ddagger}$ (0.006)	$4.473^{\ddagger}$ (0.005)	4.481 (0.005)	4.482 (0.005)	4.477 (0.004)
8	$7.623^{\ddagger}$ (0.031)	$7.657^{\ddagger}$ (0.015)	7.673 (0.030)	7.671 (0.029)	7.670 (0.027)
9	$2.287^{\dagger,\ddagger}$ (0.005)	$2.312^{\ddagger}$ (0.005)	2.315 (0.005)	2.315 (0.004)	2.316 (0.004)
12	$6.857^{\dagger,\ddagger}$ (0.030)	$6.887^{\dagger,\ddagger}$ (0.015)	6.917 (0.029)	6.919 (0.024)	6.918 (0.022)
13	$1.601^{\dagger,\ddagger}$ (0.004)	$1.612^{\ddagger}$ (0.004)	1.618 (0.004)	1.619 (0.004)	1.620 (0.004)
20	$5.583^{\dagger,\ddagger}$ (0.027)	$5.610^{\dagger,\ddagger}$ (0.013)	5.647 (0.026)	5.650 (0.024)	5.652 (0.019)

Panel B:	Polynomials				
$Nr.$	$K=1$	$K=2$	$K=3$	$K=4$	$K=5$
1	$4.433^{\dagger,\ddagger}$ (0.006)	$4.474^{\ddagger}$ (0.005)	4.482 (0.005)	4.482 (0.005)	4.481 (0.002)
8	$7.562^{\dagger,\ddagger}$ (0.033)	$7.657^{\ddagger}$ (0.030)	7.674 (0.031)	7.675 (0.030)	7.676 (0.030)
9	$2.277^{\dagger,\ddagger}$ (0.005)	$2.312^{\ddagger}$ (0.005)	2.316 (0.005)	2.319 (0.005)	2.319 (0.004)
12	$6.797^{\dagger,\ddagger}$ (0.031)	$6.892^{\ddagger}$ (0.029)	6.916 (0.029)	6.914 (0.029)	6.913 (0.027)
13	$1.591^{\dagger,\ddagger}$ (0.004)	$1.613^{\ddagger}$ (0.004)	1.617 (0.004)	1.626 (0.004)	1.622 (0.003)
20	$5.531^{\dagger,\ddagger}$ (0.028)	$5.612^{\ddagger}$ (0.026)	5.619 (0.026)	5.622 (0.026)	5.622 (0.026)

Panel C:	Chebyshev				
$Nr.$	$K=1$	$K=2$	$K=3$	$K=4$	$K=5$
1	$4.265^{\dagger,\ddagger}$ (0.003)	$3.953^{\dagger,\ddagger}$ (0.001)	$4.476^{\ddagger}$ (0.006)	$4.484^{\ddagger}$ (0.003)	4.483 (0.005)
8	$6.460^{\dagger,\ddagger}$ (0.016)	$7.010^{\dagger,\ddagger}$ (0.021)	$7.247^{\dagger,\ddagger}$ (0.024)	$7.685^{\ddagger}$ (0.024)	7.677 (0.028)
9	$2.065^{\dagger,\ddagger}$ (0.003)	$1.382^{\dagger,\ddagger}$ (0.002)	$2.316^{\ddagger}$ (0.005)	2.317 (0.005)	2.316 (0.005)
12	$5.843^{\dagger,\ddagger}$ (0.016)	$6.333^{\dagger,\ddagger}$ (0.021)	$6.518^{\dagger,\ddagger}$ (0.023)	$6.943^{\ddagger}$ (0.024)	6.922 (0.027)
13	$1.473^{\dagger,\ddagger}$ (0.002)	$0.931^{\dagger,\ddagger}$ (0.002)	$1.616^{\ddagger}$ (0.004)	1.616 (0.004)	1.617 (0.004)
20	$4.827^{\dagger,\ddagger}$ (0.016)	$5.180^{\dagger,\ddagger}$ (0.020)	$5.297^{\dagger,\ddagger}$ (0.021)	$5.628^{\ddagger}$ (0.022)	5.661 (0.025)

Table 4: Numbers in parentheses are standard errors of the price estimates. A † indicates that the estimate is significantly different from the benchmark values in Table 1. A ‡ indicates that the change is significant at a 5% level, when the number of regressors is increased from $K-1$ to K.

sectional regressions unless otherwise stated. To conclude, the Longstaff and Schwartz (2001) method can provide accurate and reliable price estimates for American–style one factor stock options.

4.2 Two factor American puts

Next we test the method in the context of a two state–variable pricing problem. Namely, we price American style put options on a dividend–paying share of stock, where the stock price follows the dynamics specified in equation (9), and the short rate of interest is stochastic. Specifically, we assume that r in (9) is no longer constant but follows an extended Vasicek diffusion of the form

$$dr(t) = \eta \left(\frac{\theta(t)}{\eta} - r(t) \right) dt + \nu dW(t) \tag{13}$$

This model, initially proposed by Hull and White (1990), has the unique feature that it allows interest rates to revert to a time–dependent level $\frac{\theta(t)}{\eta}$ in the long run. Parameter η is the speed of mean reversion, while the time–dependent function $\theta(t)$ can be calculated from the initially observed term structure of interest rates prevailing in the market. Stock prices and interest rates are assumed to be exposed to correlated shocks, $dZ(t)\, dW(t) = \rho dt$, with ρ a constant correlation coefficient.

Of course, the flexibility of simulation methods permits more general processes for the short rate (for example those studied in Amin and Bodurtha (1995)) to be accommodated easily. The specification in (13) is chosen so that our implementation is benchmarked against the results of Menkveld and Vorst (1991, Table 5). Details of the risk–neutral valuation framework in a stochastic interest rate economy are omitted; interested readers can refer to Ho, Stapleton and Subrahmanyam (1997).

To generate discrete simulation paths for the interest rate process in (13), we use an $AR(1)$ approximation of the form

$$r(t_{i+1}) = \left(1 - e^{-\eta(t_{i+1}-t_i)}\right) \frac{\theta(t_i)}{\eta} + e^{-\eta(t_{i+1}-t_i)} r(t_i) + W(t_{i+1})$$

where $W(t_{i+1}) \sim$ i.i.d. $\mathcal{N}(0, \upsilon(t_{i+1}))$, $\upsilon(t_{i+1}) = \frac{\nu^2}{2\eta}\left(1 - e^{-2\eta(t_{i+1}-t_i)}\right)$.

In each simulation $M = 100,000$ (including antithetic) paths and $N = 50$ time–steps per year are used. Since it was shown in the previous section that ordinary polynomials achieve a good regression fit, the following regression specification is employed for every t_i, $0 < i < N$

$$y(t_i) = \beta_0 + \beta_1 S(t_i) + \beta_2 r(t_i) + \beta_3 S^2(t_i) + \beta_4 r^2(t_i) + \beta_5 S(t_i) r(t_i) + \epsilon(t_i) \quad (14)$$

In Table 5, simulated American–style put prices ($\widehat{P}$) are compared to those reported by Menkveld and Vorst (1991, Table 5) and their numerical integration method (P_{MV} in column 3). Errors range from 2 cents to slightly over half a dollar. The highest discrepancies appear in the case of an upward–sloping initial term structure of interest rates. In half of the specifications priced, estimates are statistically indistinguishable from the benchmark.[8] The correlation coefficient between interest rates and stock price changes seems to play a more important role in the range of put prices than that reported by Menkveld and Vorst (1991).

We conclude that as Longstaff and Schwartz (2001) suggest, the simulation method can price multidimensional American style options easily and accurately.

5 Real Options applications

In the previous sections we have demonstrated that the least–squares MC simulation method of Longstaff and Schwartz (2001) provides a promising and reliable framework for pricing financial options. In this section we turn to assess the appropriateness of this method for valuing projects that involve several decision modes for real operating assets.

5.1 Valuation of a copper mine: The benchmark

We first apply the method in the valuation of the classic copper mine numerical example in Brennan and Schwartz (1985). In this highly influential paper, Brennan and Schwartz laid the foundations for applying option pricing arbitrage arguments to the valuation of flexibility inherent in natural

[8] The other half of the specifications however are statistically different from the benchmark. It should be stressed though that the method of Menkveld and Vorst (1991) against which our prices are benchmarked, is based on numerical integration, which although fast is an approximation.

	σ	ρ	P_{MV}	$\widehat{P}$	$(s.e.)$	Δ_{US}[a]	
Flat	0.2	-0.5	6.52	6.30	0.006	-0.22†	$c_1 = 0.0382$
		0	6.53	6.43	0.007	-0.10	$c_2 = 0$
		0.5	6.54	6.52	0.007	-0.02	$c_3 = 0$
	0.3	-0.5	10.25	10.13	0.015	-0.12	
		0	10.27	10.24	0.015	-0.03	
		0.5	10.28	10.33	0.015	0.05	
	0.5	-0.5	17.85	17.75	0.038	-0.10	
		0	17.87	17.84	0.038	-0.03	
		0.5	17.89	17.92	0.039	0.03	
Upward	0.2	-0.5	6.52	5.96	0.006	-0.56†	$c_1 = 0.08$
		0	6.53	6.07	0.006	-0.46†	$c_2 = -0.05$
		0.5	6.54	6.16	0.006	-0.38†	$c_3 = -0.18$
	0.3	-0.5	10.32	9.75	0.014	-0.57†	
		0	10.33	9.86	0.014	-0.47†	
		0.5	10.34	9.95	0.015	-0.39†	
	0.5	-0.5	17.92	17.39	0.037	-0.53	
		0	17.93	17.48	0.037	-0.45	
		0.5	17.95	17.57	0.038	-0.38	
Downward	0.2	-0.5	6.37	6.70	0.007	0.33†	$c_1 = -0.00353$
		0	6.39	6.82	0.007	0.43†	$c_2 = 0.05$
		0.5	6.41	6.91	0.007	0.50†	$c_3 = -0.18$
	0.3	-0.5	10.18	10.54	0.016	0.36†	
		0	10.2	10.65	0.016	0.45†	
		0.5	10.23	10.74	0.016	0.51†	
	0.5	-0.5	17.79	18.11	0.039	0.32	
		0	17.82	18.23	0.039	0.41	
		0.5	17.84	18.31	0.038	0.47	

Table 5: This Table compares put prices under stochastic interest rates from Menkveld and Vorst (1991, Table 5) (P_{MV}) to the $L\&S$ simulation algorithm ($\widehat{P}$). The term structure is of the form $r(t) = c_1 + c_2 e^{c_3 t}$. The stock and exercise prices are $S = E = 100$, the dividend yield $\delta = 0$, the short rate volatility $\nu = 0.01$, the mean reversion rate $\eta = 0.1$ and the option maturity $T = 1$ year. σ is the volatility of stock returns and ρ the correlation between

resource investments. They demonstrate that the value–maximising policy in the face of stochastic output prices involves the optimal exercise timing of path–dependent, American–style options to initiate, temporarily cease or completely abandon production.

In brief, if the price $S(t)$ of the homogenous commodity produced by the mine is assumed to evolve exogenously according to equation (9) in *nominal* terms (where δ is now interpreted as the constant and proportional convenience yield that accrues to the owner of the physical commodity but not to the holder of a contract for future delivery of the commodity), the value (in *real* terms) of the mine when open $u(s, Q, t)$ and when closed $w(s, Q, t)$ must satisfy

$$\frac{1}{2}\sigma^2 s^2 u_{ss} + (r-\delta) s u_s - q^* u_Q + q^* (s-a) - \tau - (r+\lambda_1) u = 0 \tag{15}$$

and

$$\frac{1}{2}\sigma^2 s^2 w_{ss} + (r-\delta) s w_s - f - (r+\lambda_0) w = 0 \tag{16}$$

respectively, subject to

$$w(s_0^*, Q) = 0 \tag{17}$$

$$u(s_1^*, Q) = \max\left[w(s_1^*, Q) - k_1(Q), 0\right] \tag{18}$$

$$w(s_2^*, Q) = u(s_2^*, Q) - k_2(Q) \tag{19}$$

$$w(s, 0) = u(s, 0) = 0 \tag{20}$$

$$w_s(s_0^*, Q) = 0 \tag{21}$$

$$u_s(s_1^*, Q) = \begin{cases} w_s(s_1^*, Q) & \text{if } w(s_1^*, Q) - k_1(Q) \geq 0 \\ 0 & \text{if } w(s_1^*, Q) - k_1(Q) < 0 \end{cases} \tag{22}$$

$$w_s(s_2^*, Q) = u_s(s_2^*, Q) \tag{23}$$

where r is the real interest rate; Q is the physical inventory of the mine; q^* is the feasible extraction rate assumed constant; s is the commodity price in *real* terms (i.e. deflated, $s = Se^{-\pi t}$ with inflation rate π); a is the average deflated cash cost rate of producing at rate q^*; f is the after–tax fixed–cost rate of maintaining the mine when closed; $\tau = t_1 q^* s + \max\left[t_2 q^* (s(1-t_1) - a), 0\right]$ is the total income tax and royalties in real terms, t_1 is the royalty rate and t_2 is the income tax rate; λ_i, $i = 0, 1$ are property tax rates when the mine

Mine		
	Output rate (q^*):	10 million pounds/year
	Inventory (Q):	150 million pounds
	Initial average production cost ($a(q^*, Q)$):	\$0.50/pound
	Initial cost of opening ($k_2(Q)$):	\$200,000
	Initial cost of closing ($k_1(Q)$):	\$200,000
	Initial maintenance cost (f):	\$500,000/year
Copper		
	Convenience yield (δ):	1%/year
	Price variance (σ^2):	8%/year
Taxes		
	Real estate (λ_1, λ_2):	2%/year
	Income (t_2):	50%
	Royalty (t_1):	0%
	Interest rate (r):	2%/year

Table 6: Parameter inputs for the hypothetical copper mine from Brennan and Schwartz (1985).

is open (λ_1) or closed (λ_0); k_1 and k_2 are the deflated costs of closing and opening the mine respectively; s_0^* is the deflated commodity price at which the mine is abandoned if already closed; s_1^* is the deflated commodity price at which the mine is closed down or abandoned if it was previously open and s_2^* is the commodity price (in real terms) at which the mine is opened up if previously closed.

Equations (17)–(19) and (21)–(23) are standard value–matching and smooth–pasting conditions at the optimal decision thresholds, while equation (20) is the boundary condition that applies when inventory is depleted.

Brennan and Schwartz (1985, pp. 147-150) provide a numerical example of the model by using a finite difference approximation scheme to solve the partial differential equations in (15) and (16). The parameters used in the numerical example are summarised in Table 6.

To evaluate the copper mine using MC simulation we again use equation (10) to produce copper price trajectories. At each time t_i the optimal decision depends on the current state. If the mine is open with n units of reserves remaining, the decision maker has two "options": either suspend operations and close the mine temporarily or permanently abandon it. These alternatives have different payoffs ($-f - k_1$ and zero respectively) while the continuation value assumes the mine remains open and reserves are reduced to $n - 1$. On the other hand if the mine is currently closed with n units remaining, it can either be opened (with payoff $q^* (s - a) - \tau - k_2$ and $n - 1$ units now remaining) or abandoned (zero payoff). The continuation value comes from keeping the mine closed and incurring the maintenance cost $-f$ for another time period. From the discussion above it is clear that in contrast to the financial options treated in Section 4, valuation of the copper mine is highly path–dependent since not only the current copper price $S(t_i)$, but also the current operating mode, as well as the remaining reserves level, need to be monitored for determining the optimal policy. Despite the extra computational burden that path–dependence introduces however, the $L\&S$ method can still be applied.

Table 7 compares the results of the simulation valuation with those reported in Brennan and Schwartz (1985, Table 2) for a range of initial copper prices. In all cases, valuation errors are less than 1%, suggesting that the simulation approach can adequately evaluate investments with multiple option characteristics.

What should be stressed from this valuation exercise however is the wide range of extensions that simulation methods offer compared to other numerical methods in the real options valuation framework. For example, the simple random walk in (9)—used in this section for comparability with the Brennan and Schwartz (1985) model—is a very strong assumption given the extensive research in commodity prices. Given that simulation is much more efficient in solving problems with multi–factor processes, our simple numerical application could easily be extended to include more underlying processes like a stochastic convenience yield (Gibson and Schwartz (1990), Schwartz (1997)) or more complex process specifications.

5.2 Valuation of a copper mine: An extension

To demonstrate this wide range of possible extensions, we re–evaluate the copper mine, this time under a two factor price model. Specifically, the

Copper Price (US$/lb) $S(t_0)$	Brennan–Schwartz Finite Difference		Longstaff–Schwartz Monte Carlo		Error $\Delta(\%)$	
	Open	Closed	Open	Closed	Open	Closed
0.4	4.15	4.35	4.17	4.37	0.006	0.004
0.5	7.95	8.11	7.93	8.06	-0.002	-0.007
0.6	12.52	12.49	12.48	12.49	-0.004	0.000
0.7	17.56	17.38	17.59	17.42	0.002	0.002
0.8	22.88	22.68	22.81	22.63	-0.003	-0.002
0.9	28.38	28.18	28.31	28.11	-0.003	-0.002
1.0	34.01	33.81	33.92	33.73	-0.003	-0.002

Table 7: This Table compares evaluations of the hypothetical copper mine treated in Brennan and Schwartz (1985) with those produced by the Monte Carlo Least–Squares simulation method of Longstaff and Schwartz (2001). All values are in $ millions.

commodity price model of Schwartz (1997) is used, according to which the commodity price $S(t)$ still follows the process in (9), but the proportional convenience yield δ is now stochastic

$$d\delta(t) = \left[\kappa\left(\overline{\delta} - \delta(t)\right) - \lambda\right] dt + \gamma dB(t) \tag{24}$$

and changes in $\delta(t)$ are correlated with changes in the commodity price $S(t)$, i.e. $dZ(t)\, dB(t) = \varrho dt$ (with ϱ constant). Equation (24) implies that the convenience yield of the commodity reverts to a long–run mean $\overline{\delta}$ at a speed of κ, and λ is the market price of convenient yield risk, assumed constant.

To produce trajectories for the convenience yield δ in (24), the following approximation is used

$$\delta(t_{i+1}) = \left(1 - e^{-\kappa(t_{i+1}-t_i)}\right)\left(\overline{\delta} - \frac{\lambda\gamma}{\kappa}\right) + e^{-\kappa(t_{i+1}-t_i)}\delta(t_i) + B(t_{i+1})$$

where $B(t_{i+1}) \sim$ i.i.d. $\mathcal{N}(0, v(t_{i+1}))$, $v(t_{i+1}) = \frac{\gamma^2}{2\kappa}\left(1 - e^{-2\gamma(t_{i+1}-t_i)}\right)$.

Tables 8 and 9 report values (in \$ millions) for the hypothetical copper mine (when initially open and closed respectively) for the parameter inputs of Table 6 and a wide range of input values for the convenience yield process (ϱ, $\overline{\delta}$, γ and κ). In each valuation, $S(t_0) = 0.7$, $\lambda = 0$ and $M = 100,000$ ($50,000$ plus $50,000$ antithetic) simulation paths are used.

As evident from Tables 8–9, the correlation between commodity prices and convenience yields ϱ has a decreasing effect on the value of the investment. In accordance with intuition, the more correlated the two risk factors, the less valuable the flexibility to alter the operating mode of the project. For the same reason, the more volatile the convenience yield of the commodity (the higher γ is), the more valuable the options embedded in the operation of the mine ceteris paribus.

However, mean reversion in the convenience yield process appears to have a less straightforward effect on valuation. First of all, the higher the level towards which the convenience yield reverts ($\overline{\delta}$), the less valuable the project is. This makes intuitive sense, since a relatively high long–run mean increases the probability of lower spot prices in the future, thus making production less attractive.

Secondly, the speed with which the convenience yield mean–reverts seems to have a secondary effect on the value of the investment. The resulting valuations appear relatively insensitive to the speed of mean reversion: project

Open mine, Panel (a): $\varrho = 0.20$

	$\bar{\delta} = \delta_0 = 0.01$			$\bar{\delta} = \delta_0 = 0.10$			$\bar{\delta} = \delta_0 = 0.15$		
	$\gamma = 0.05$	$\gamma = 0.10$	$\gamma = 0.15$	$\gamma = 0.05$	$\gamma = 0.10$	$\gamma = 0.15$	$\gamma = 0.05$	$\gamma = 0.10$	$\gamma = 0.15$
$\kappa = 0.65$	20.11	36.34	104.5	5.083	9.870	31.05	2.379	4.554	15.32
$\kappa = 0.85$	19.12	28.42	57.07	4.820	7.423	16.31	2.279	3.396	7.730
$\kappa = 1.05$	18.65	25.02	42.19	4.707	6.413	11.61	2.237	2.943	5.369

Open mine, Panel (b): $\varrho = 0.50$

	$\bar{\delta} = \delta_0 = 0.01$			$\bar{\delta} = \delta_0 = 0.10$			$\bar{\delta} = \delta_0 = 0.15$		
	$\gamma = 0.05$	$\gamma = 0.10$	$\gamma = 0.15$	$\gamma = 0.05$	$\gamma = 0.10$	$\gamma = 0.15$	$\gamma = 0.05$	$\gamma = 0.10$	$\gamma = 0.15$
$\kappa = 0.65$	17.38	28.24	73.79	4.100	6.884	20.74	1.878	2.932	9.653
$\kappa = 0.85$	17.16	23.36	43.99	4.114	5.544	11.60	1.917	2.393	5.100
$\kappa = 1.05$	17.11	21.30	34.03	4.147	5.036	8.591	1.946	2.215	3.688

Open mine, Panel (c): $\varrho = 0.80$

	$\bar{\delta} = \delta_0 = 0.01$			$\bar{\delta} = \delta_0 = 0.10$			$\bar{\delta} = \delta_0 = 0.15$		
	$\gamma = 0.05$	$\gamma = 0.10$	$\gamma = 0.15$	$\gamma = 0.05$	$\gamma = 0.10$	$\gamma = 0.15$	$\gamma = 0.05$	$\gamma = 0.10$	$\gamma = 0.15$
$\kappa = 0.65$	14.81	21.32	51.34	3.192	4.327	12.95	1.441	1.629	5.368
$\kappa = 0.85$	15.27	18.81	33.31	3.442	3.875	7.664	1.576	1.549	2.951
$\kappa = 1.05$	15.61	17.86	26.98	3.612	3.787	5.952	1.673	1.581	2.280

Table 8: The value of an *open* copper mine (in \$ millions) when the copper price and convenience yield evolve according to equations (9) and (24) respectively. In all cases, the copper price is 0.7 US\$/lb, the convenience yield price of risk is assumed zero, and the rest of the mine characteristics are as in Table 6. $\bar{\delta}$ is the long–run convenience yield level, κ is the speed of mean reversion in the convenience yield process, γ is the convenience yield volatility and ϱ is the correlation between changes in commodity prices and the convenience yield.

Closed mine, Panel (a): $\varrho = 0.20$

	$\bar{\delta} = \delta_0 = 0.01$			$\bar{\delta} = \delta_0 = 0.10$			$\bar{\delta} = \delta_0 = 0.15$		
	$\gamma = 0.05$	$\gamma = 0.10$	$\gamma = 0.15$	$\gamma = 0.05$	$\gamma = 0.10$	$\gamma = 0.15$	$\gamma = 0.05$	$\gamma = 0.10$	$\gamma = 0.15$
$\kappa = 0.65$	19.94	36.16	104.3	4.926	9.713	30.89	2.235	4.411	15.18
$\kappa = 0.85$	18.95	28.24	56.90	4.664	7.265	16.16	2.135	3.252	7.587
$\kappa = 1.05$	18.48	24.84	42.02	4.550	6.256	11.46	2.093	2.799	5.226

Closed mine, Panel (b): $\varrho = 0.50$

	$\bar{\delta} = \delta_0 = 0.01$			$\bar{\delta} = \delta_0 = 0.10$			$\bar{\delta} = \delta_0 = 0.15$		
	$\gamma = 0.05$	$\gamma = 0.10$	$\gamma = 0.15$	$\gamma = 0.05$	$\gamma = 0.10$	$\gamma = 0.15$	$\gamma = 0.05$	$\gamma = 0.10$	$\gamma = 0.15$
$\kappa = 0.65$	17.20	28.06	73.61	3.940	6.719	20.57	1.731	2.780	9.498
$\kappa = 0.85$	16.98	23.18	43.81	3.954	5.380	11.43	1.770	2.242	4.946
$\kappa = 1.05$	16.94	21.12	33.85	3.988	4.872	5.952	1.800	2.064	3.535

Closed mine, Panel (c): $\varrho = 0.80$

	$\bar{\delta} = \delta_0 = 0.01$			$\bar{\delta} = \delta_0 = 0.10$			$\bar{\delta} = \delta_0 = 0.15$		
	$\gamma = 0.05$	$\gamma = 0.10$	$\gamma = 0.15$	$\gamma = 0.05$	$\gamma = 0.10$	$\gamma = 0.15$	$\gamma = 0.05$	$\gamma = 0.10$	$\gamma = 0.15$
$\kappa = 0.65$	14.63	21.13	51.16	3.028	4.156	12.77	1.289	1.468	5.199
$\kappa = 0.85$	15.10	18.62	33.12	3.279	3.705	7.487	1.425	1.390	2.784
$\kappa = 1.05$	15.43	17.68	26.79	3.449	3.617	5.776	1.522	1.423	2.114

Table 9: The value of a *closed* copper mine (in \$ millions) when the copper price and convenience yield evolve according to equations (9) and (24) respectively. In all cases, the copper price is 0.7 US\$/lb, the convenience yield price of risk is assumed zero, and the rest of the mine characteristics are as in Table 6. $\bar{\delta}$ is the long–run convenience yield level, κ is the speed of mean reversion in the convenience yield process, γ is the convenience yield volatility and ϱ is the correlation between changes in commodity prices and the convenience yield.

values decrease only slightly most of the times as κ increases. Interestingly, there are two exceptions to this pattern. The first is that the negative relation between speed of mean reversion and project value appears more pronounce for cases of high volatility (the $\gamma = 0.15$ scenarios in all panels). The second is that this relationship actually reverses (i.e. mine value *increases* with κ) for cases of high correlation and low yield volatility (the $\gamma = 0.05$ scenarios in panel (c)). The reason behind this reversal can be better understood once one establishes the fact that under stochastic output prices and convenience yields, the total volatility of the natural resource investment is a non–linear function of σ, γ, ϱ and κ.[9] It seems that in these cases of high correlation and strong mean–reversion, the total volatility of the investment might actually increase, increasing the value of switching operating status and thus bringing the mine valuations up.

As a means of summary, it should be stressed that the ease with which the copper mine example of Brennan and Schwartz (1985) is extended in this subsection to include a stochastic convenience yield, demonstrates clearly the applicability of the *L&S* simulation–based method for corporate decision–making. Being a very transparent method, simulation can help to improve the realism of the modelling approach: features like time–dependent information, inflation, taxes, royalties and finitely lived concessions can also be introduced at no significant computational burden.

6 Conclusions

Monte Carlo simulation, despite its advantages over other traditional numerical methods, was until recently considered inappropriate for most option pricing applications of interest. A growing stream of literature has demonstrated that simulation algorithms can indeed be employed for more complex

[9]In the context of the two–factor model of Schwartz (1997), it is easy to show that the variance of futures contracts returns on the commodity is given by $\sigma_F^2 = \sigma^2 + \gamma^2 \frac{\left(1-e^{-\kappa T}\right)^2}{\kappa^2} - 2\varrho\sigma\gamma\frac{\left(1-e^{-\kappa T}\right)}{\kappa^2}$ in our notation (see for example Bjerksund (1991)). Even though here we are evaluating a commodity mine with several operating possibilities and not futures on the commodity, it is reasonable to expect that a similar dependence—at least in terms of the parameters involved—would characterise the total variance of the investment, i.e. it would depend on σ, γ, ϱ and κ.

derivatives types and several different approaches have been proposed.

In this paper we contribute to this literature by assessing the applicability of a promising approach introduced by Longstaff and Schwartz (2001). Our focus is on whether this newly proposed Monte Carlo simulation technique can provide a flexible framework for real options valuation. Initially, a step–by–step outline of the algorithm is presented before applying it to the pricing of American–style financial options written on one and two stochastic processes. When benchmarked against traditional lattice and finite difference methods, the simulation approach is found to produce accurate and fast–converging option price estimates.

We then turn to the evaluation of natural resource investments. We evaluate the hypothetical copper mine considered in Brennan and Schwartz (1985, Table 2), where the decision maker has to optimally time production of the commodity in the face of stochastic copper prices, entry and exit sunk costs, finite reserves quantity, income and property taxes, inflation, royalties and flexible production modes.

The simulation algorithm provides valuations within 1% of those reported in the original Brennan and Schwartz (1985, Table 2) paper. Unlike their numerical method however, the Monte Carlo approach can readily be extended to accommodate issues like stochastic convenience yield and interest rates, uncertain level of commodity reserves and multiple underlying prices. We demonstrate these possible extensions by valuing the same hypothetical copper mine with the extra complexity of an uncertain, mean–reverting convenience yield. To our knowledge such results have not been reported before.

Thus it seems that Monte Carlo simulation methods provide a flexible and transparent valuation tool which can accommodate the majority of issues important in a real options valuation framework.

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