Carrots or Sticks? Optimal Compensation for Firm Managers

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Abstract

We investigate the existence of and explicitly characterize compensation structures that eliminate agency conflicts between a leveraged firm (or its shareholders) and the manager due to managerial asset substitution within a continuous time framework. The manager may dynamically switch between two strategies with different risks and expected returns after debt is in place. We show that when the strategies satisfy a specific condition that (roughly) ensures that the difference in their drifts is not large compared with the difference in their volatilities, a periodic compensation structure that completely aligns the manager's interests with those of the firm (or its shareholders) is one where the manager's payoff is proportional to the firm's operating cash flows, but subject to a *floor* and a *ceiling*. This result explains the prevalence of compensation schemes where firm managers obtain shares of firm profits subject to floors and ceilings apart from the usual components of cash, stock, and options. We also investigate conditions under which convex and concave compensation structures are optimal. We show that a *concave* compensation structure where the manager obtains a proportion of firm cash flows subject to a ceiling is optimal when the higher volatility strategy also has a higher expected return. On the other hand, a *convex* compensation structure where the manager obtains a proportion of firm cash flows subject to a floor is optimal when the higher volatility strategy has a lower expected return. Our theoretical analysis therefore offers insights into features of compensation contracts that mitigate agency conflicts due to managerial asset substitution.

Key Words: Optimal Compensation, Asset Substitution, Agency Costs

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Introduction

The importance of agency theory in understanding the investment and financing decisions of firms has been understood and accepted at least since the seminal work of Jensen and Meckling [1976]. They argue persuasively that equity holders of a leveraged firm could make use of *asset substitution* to extract value from bondholders after debt is in place. This phenomenon creates *agency costs* that must be controlled for thereby forging an inextricable link between the capital structure of the firm and its investment decisions.

Leland [1998] proposes a realistic unified framework in continuous time to quantify the significance of the agency costs due to *shareholder asset substitution*. Specifically, he investigates a model where the firm's shareholders may dynamically alter the volatility of the firm's cash flows after debt is in place. He finds that the resulting agency costs are insignificant when compared with the tax advantages of debt. However, as he emphasizes in his conclusions, Leland's [1998] analysis relies on an important assumption: the manager of the firm behaves in the interests of shareholders. In other words, he assumes that the manager is compensated in such a way as to completely align his incentives with those of shareholders.

Our primary objective in this paper is to investigate and characterize compensation structures that completely align the manager's interests with those of the firm or its shareholders when the strategies available to the firm are asset substitution strategies. In reality, the considerations that determine compensation schemes for firm managers are far more complex than the objective to minimize agency conflicts due to asset substitution.[1] Therefore, we must necessarily adopt a normative, rather than a positive approach in this paper. Although the scope of the analysis is

[1] See Abowd and Kaplan [1999] for a survey of some of the issues in this literature.

restricted to the asset substitution scenario, we believe that it is nevertheless useful since it allows us to focus on the features of general compensation contracts that would serve to mitigate, if not eliminate, the agency costs associated with managerial asset substitution.

We consider a continuous time framework similar to that of Leland [1998] where the manager of a leveraged firm with long-term debt in place may dynamically choose between two *marketed* strategies: a high volatility strategy and a low volatility strategy. These strategies may well have different drifts or expected return. As in Leland [1998], if the firm's cash flows are lower than the required coupon payment, shareholders inject capital to service debt. The control of the firm is transferred to bondholders when the value of equity is zero and the shareholders declare bankruptcy. We assume that the manager continues to operate the firm after bankruptcy since our focus in this paper is on the role of *explicit incentives* in the form of compensation structures rather than *implicit incentives* in the form of career concerns (that is, the risk of getting fired after bankruptcy) in aligning the manager's interests with those of the firm (shareholders).

The risk-neutral manager is compensated periodically. We restrict the analysis to compensation structures that are *piecewise linear* in the firm's cash flows since commonly observed compensation schemes have this form. We determine conditions on the available strategies for the existence of non-renegotiable compensation structures that align the manager's incentives with those of the firm (shareholders). More precisely, we determine conditions under which there exist compensation structures such that the manager's optimal dynamic asset substitution policy would be identical to that of either the firm or its shareholders when managerial behavior could hypothetically be contracted for ex ante, that is, before debt is in place. Therefore, ours is a continuous time principal-agent framework where the principal is either the firm or its shareholders and the manager is the agent. For expositional convenience, we refer to compensation structures that align managerial incentives with those of the firm or its shareholders as *first best* and *second best* respectively. We then explicitly characterize (not necessarily unique) first and second best compensation structures.

We first investigate the scenario wherein the high volatility strategy also has a higher expected return. When the strategies satisfy a specific condition that ensures that the difference in their drifts is not large compared with the difference in their volatilities, we establish that both first and second best compensation structures (not necessarily unique) provide the manager with a payoff that is proportional to the firm's cash flows, but subject to a *ceiling* (the manager's payoff is a *covered call* on the firm's cash flows). The ceilings differ, in general, in the first and second best compensation schemes. Hence, the manager is penalized for under performing a threshold rather than being rewarded for outperforming it.

The intuition behind this result is as follows. The goal of the firm (shareholders) is to maximize the *market value* of the firm (equity), that is, to maximize the discounted expected after-tax cash flows to the firm (shareholders) under the *risk-neutral* measure. However, the marketed strategies have the same expected return under the risk-neutral measure. In this situation, Leland [1998] shows that the optimal policy for the firm (shareholders) is to choose the high volatility strategy whenever the firm's cash flows are below a threshold and the low volatility strategy when the firm's cash flows exceed the threshold. The primary cause of the agency conflict between the sub-optimally diversified manager and the firm (shareholders) is that the manager is an expected utility maximizer, that is, the manager maximizes the discounted expected value of his compensation where the expectation is under the *real* or *actual* probability measure.

A manager compensated with a payoff that is a covered call on firm cash flows receives a constant payoff when the firm's cash flows exceed a threshold. This induces him to choose the low volatility-low expected return strategy when the firm's cash flows exceed this threshold to reduce the probability that the firm's performance declines to the level where he is penalized for under performance. However, when the firm's performance is significantly below this threshold, the manager faces a payoff structure that is "almost linear" in firm cash flows. In this case, it is optimal for him to choose the high volatility-high expected return strategy since his expected compensation is higher. Therefore, the manager chooses high volatility when the firm is performing poorly and low volatility when it is performing well, similar to the optimal policies for the firm or its shareholders. The ceiling can be chosen so that the optimal switching point for the manager is identical to that of the firm (shareholders) so that the corresponding contract is first best (second best).

We then investigate the scenario wherein the higher volatility strategy has lower expected return. We show that when the strategies satisfy a condition that ensures that the difference in their drifts is not large compared with the difference in their volatilities, there exist first and second best compensation structures that provide the manager with a payoff that is proportional to the firm's cash flows, but subject to a *floor* (the manager's payoff is a *call option* on the firm's cash flows). Therefore, the manager receives a constant payoff when the firm's performance is below a threshold and is rewarded for outperforming it.

The intuition for this result is the following. When the firm's performance far exceeds the threshold, the manager's compensation is "almost linear". Therefore, it is optimal for him to choose the higher expected return strategy, but this strategy has lower volatility. When the firm's performance is below the threshold, the manager's payoff is constant. The fact that the compensation structure rewards the manager for outperforming a threshold induces him to choose the strategy that increases the probability that the firm's performance exceeds the threshold. When the drifts and volatilities of the strategies satisfy the condition for the optimality of this compensation structure, it is optimal for the manager to choose the high volatility strategy even though it has lower expected return. The functional form of this condition between the strategies is identical to that for the optimality of the covered call compensation structure when the higher volatility strategy also has higher drift.

Finally, we investigate the scenario wherein the two strategies have equal expected returns, that is, they differ only in unsystematic risk. In this case, we show that a first-best (second best) compensation structure provides the manager with a payoff that is proportional to the firm's cash flows, but subject to a floor as well as a ceiling. The intuition for the result of the proposition is the following. When the firm's cash flows exceed the ceiling, the manager prefers the lower volatility strategy since it lowers the probability that the firm's cash flows will fall to a level below the ceiling where the manager is penalized for under performance. On the other hand, when the firm's cash flows are lower than the floor, the manager prefers the higher volatility strategy since it increases the probability that the cash flows will increase above the floor where the manager is rewarded for out performance.

An analysis of the proof of this result as well as the intuition behind it reveals that this compensation structure can achieve first-best (second-best) even when the drifts of the strategies are not equal to each other. It turns out that when the drifts and volatilities of the available strategies satisfy a condition functionally identical to that derived for the optimality of the covered call (call option) compensation structure when the higher volatility strategy has higher (lower) drift, the compensation structure described above can achieve first best (second best). This leads us to the most general result of the paper: when the available asset substitution strategies satisfy a condition that ensures that the difference in their drifts is not large compared with the difference in their volatilities, a compensation structure that achieves first best (second best) provides the manager with a proportion of the firm's cash flows, but subject to a floor as well as a ceiling.

Murphy [1999] reports that typical compensation packages of firm managers include, apart from a base salary, equity, and options, a portion that provides managers with shares of firm profits subject to floors and ceilings. The result above therefore suggests that this component of managerial compensation packages could serve to mitigate agency conflicts due to asset substitution by the manager.

The issue of determining compensation schemes that minimize agency conflicts due to asset substitution has been investigated in previous literature. In a seminal paper, Green [1984] examines the potential for convertible bonds and warrants to mitigate agency conflicts between *shareholders and the firm* due to asset substitution. In a single-period framework where the firm has a choice between two asset substitution strategies that have the same mean, but different variances, he shows that the ability of these instruments to reverse the convex shape of levered equity when the firm is performing well can, under certain conditions, eliminate agency costs associated with asset substitution by shareholders. Hennessy [2002] extends the framework of Green [1984] to a continuous-time infinite horizon setting and shows that, under limited liability, warrants cannot eliminate the asset substitution problem, but serve to mitigate it. This impossibility result is similar to that obtained by Eberhart and Senbet [1993], who consider convertible bonds as devices for mitigating the asset substitution problem in a single period model with volatility fixed at the start of the project. They find that convertibles cannot eliminate the agency problem when the firm is near bankruptcy at the start of the period, but show that a simple deviation from the Absolute Priority Rule can induce first-best when coupled with convertibles. Hennessy's [2002] numerical simulations indicate that deviations from APR coupled with warrants mitigate risk taking for bad states, but exacerbate the problem for intermediate states.

All the papers above assume that the manager behaves in the interests of shareholders. Hence, we differ significantly from them in that we focus on agency conflicts between the manager and the firm or its shareholders due to asset substitution. We focus on determining first and second best compensation schemes for the manager within a continuous time framework where asset substitution may occur dynamically.

Another significant stream of the literature examines optimal compensation contracts in continuous time principal agent problems where the moral hazard arises from the agent's choice of *effort* that is not directly contractible (see, for example, Holmstrom and Milgrom 1987, Schattler and Sung 1993, Sung 1995) or the costs the agent faces in implementing different strategies (for example, Ou-Yang 2002).[2] Under differing assumptions, the optimal contracts in these frameworks are typically *symmetric* about a performance measure, that is, the agent's reward for out performing a benchmark by a certain amount is equal in magnitude to his penalties for under performing a benchmark by the same amount. The present paper differs significantly from this stream of the literature in that the moral hazard in our principal-agent framework arises from the manager's dynamic asset substitution. This important difference in the framework leads to the optimal contracts being, in general, *asymmetric* in the agent's performance.

Several authors have pointed out that agency problems can be reduced or eliminated through the use of managerial incentive schemes. Brander and Poitevin [1992] propose a model where they show how the terms of the compensation contract offered to management by shareholders can reduce the agency costs of debt finance. They derive a managerial compensation contract that is first best and leads to a local irrelevance result for financial structure. John and John [1993] propose a two-period framework to study the interrelationship between top-management compensation and the

[2] See Garen [1994], Haubrich [1994] for other principal-agent frameworks where managerial compensation is endogenously derived.

design and mix of external claims issued by a firm. The optimal managerial compensation structures depend on not only the agency relationship between shareholders and management, but also the conflict of interests that arise in other contracting relationships within the firm. They consider the possibility of asset substitution by the manager. However, within the two-period framework they consider, the manager can only make a static choice of strategies at a single date. We investigate the existence of and derive managerial compensation structures that eliminate the agency costs of debt due to dynamic asset substitution by the manager within a continuous time framework.

The plan for the paper is the following. **Section 1** presents the model. In **Section 2**, we investigate the existence of and explicitly characterize first-best (second-best) compensation structures for the manager for various possible choices of available asset substitution strategies. In **Section 3**, we numerically derive optimal compensation contracts for various possible choices of the available strategies. **Section 4** concludes the paper. All detailed proofs are relegated to the Appendix.

1. The Model

Throughout the paper, we consider a filtered probability space (Ω, F, P, F_t) with the filtration F_t (completed and augmented) generated by two independent Brownian motions B_1, B_2. A firm has a certain amount of long-term debt in place that is completely amortized, i.e. the firm is liable for an interest (coupon) payment of J per unit time over an infinite time horizon.

The manager of the firm is undiversified. His goal is to choose firm strategies so as to maximize the (discounted) expected utility of cash flows comprising his compensation. We assume that the firm does not retire its debt or restructure it at intermediate times. The manager of the firm is risk-neutral and his discount factor or opportunity cost of capital b is greater than the risk-free rate.

$P(.)$ is the process for a state variable that determines the cash flows from the firm's operations. The cash flows (per unit time) arising from the firm's operations (before interest and taxes) are equal to the value $P(t)$ of the state variable at time t. $P(.)$ is the price process of a traded asset that has a cash payout ratio of d per unit time. As long as the firm's cash flows exceed the required interest payments, they are used to service debt. We assume, as in Leland (1998), that if the cash flows are lower than the required interest payments, the shareholders of the firm inject capital to service debt as long until the endogenous level p_b at which the value of equity falls to zero. The creditors then obtain control of the firm. The bankruptcy costs are proportional.

The Manager's Compensation Structures

The goal of the paper is to examine conditions for the existence of *first-best* and *second-best* compensation structures for the manager, that is, non-renogotiable compensation structures that will completely align his incentives with those of the firm and its shareholders respectively. We restrict our consideration to periodic compensation schemes that are *piecewise linear* and *monotonically increasing* in the firm's cash flows since this is a good approximation for compensation structures observed in reality. Since the manager is compensated periodically, we make the standard continuous time approximation that the manager is compensated at the *rate* $C(t)$ at time t. We define the manager's compensation to be piecewise linear if there exists a finite number n and $0 = w_0 < w_1 < ... < w_n = \infty$ and $0 \leq r_0, r_1, ..., r_{n-1} << 1$ such that

(1) $$C(t) = \sum_{i=1}^{n} 1_{w_{i-1} \leq P(t) \leq w_i} [r_{i-1}(P(t) - w_{i-1}) + \sum_{j=0}^{i-2} r_j (w_{j+1} - w_j)]$$

We examine compensation structures where the manager continues to run the firm after bankruptcy. For analytical simplicity, we also assume that the manager's compensation is a negligible portion of the firm's cash flows. This is expressed by the condition

$0 \le r_0, r_1, ..., r_{n-1} << 1$. Recall that the cash flows per unit time $P(.)$ are before interest and taxes, but after all employees including the manager are paid.

Available Strategies for the Manager

At any instant of time, the manager of the firm can switch between two strategies *without cost*. The state variable $P(.)$ evolves in the *real world* as follows under the two strategies[3]:

(2)
$$\begin{aligned} dP(t) &= P(t)[(m_1{'} - d)dt + s_{11}dB_1(t) + s_{12}dB_2(t)] \quad \text{Strategy 1} \\ dP(t) &= P(t)[(m_2{'} - d)dt + s_{21}dB_1(t) + s_{22}dB_2(t)] \quad \text{Strategy 2} \end{aligned}$$

Therefore, we may rewrite equations (1a) as follows :

(3)
$$\begin{aligned} dP(t) &= P(t)[m_1 dt + s_1 dB_1^*(t)] \quad \text{Strategy 1} \\ dP(t) &= P(t)[m_2 dt + s_2 dB_2^*(t)] \quad \text{Strategy 2} \end{aligned}$$

where B_1^*, B_2^* are (not necessarily perfectly correlated) F_t − Brownian motions with $s_1 = \sqrt{s_{11}{}^2 + s_{12}{}^2}, m_1 = m_1{'} - d, s_2 = \sqrt{s_{21}{}^2 + s_{22}{}^2}, m_2 = m_2{'} - d$. Thus, if the manager has initially chosen strategy 1 and switches to strategy 2 at time t^*, then the evolution of the state variable $P(.)$ for times $t > t^*$ is described by the drift and volatility parameters (m_2, s_2) until it switches back to strategy 1 in which case the evolution is governed by the drift and volatility parameters (m_1, s_1). Therefore, the state variable process $P(.)$ is always continuous. We assume that m_1, m_2, s_1, s_2 are constants, $b > |m_1|, |m_2|$[4] and $s_1 > s_2$, but don't make any further assumptions on their values. Therefore, the manager's policies Γ may be described as follows:

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[3] The state variable processes under the two strategies need not be perfectly correlated with each other.

[4] It is easy to see that if $b < m_1$, the value function for the manager's optimization problem (3) is infinite.

(4) $$\Gamma \equiv \{t_1, t_2, t_3, ...\}$$

where t_i are increasing F_t – stopping times (reflecting the fact that the manager's decisions cannot anticipate the future) representing the instants where the manager switches strategies. The goal of the manager is to choose his policy to maximize his expected discounted compensation that is given by

(5) $$U_\Gamma(p) = E[\int_0^\infty \exp(-bt)C(t)dt].$$

If $u(.)$ is the value function of the dynamic optimization problem (1.5), then we can use traditional dynamic programming arguments (see e.g. Oksendal 1998) to write down the following formal Hamilton-Jacobi-Bellman equation for u:

(6) $$\sup_{i=1,2}[-bu + m_i p u_p + \frac{1}{2}s_i^2 p^2 u_{pp}] + C(p) = 0, \, p \in (0,\infty) \text{ or}$$
$$\sup_{i=1,2}[L^i(u)] + C(p) = 0$$

where $L^i(u) = -bu + m_i p u_p + \frac{1}{2}s_i^2 p^2 u_{pp}$,

In the dynamic programming framework, the variable p above represents the value of the state variable $P(.)$ so that the term $C(p)$ is the instantaneous rate of compensation of the manager. We shall now state without proof the following standard verification result.

Proposition 1

If $u:[0,\infty) \to R_+$ is a continuous function that is twice differentiable on $(0,\infty)$ satisfying the HJB equation (6) and $\lim_{p\to\infty} \frac{u(p)}{p} < \infty$ (no bubbles condition), then u is the value function of the manager's optimization problem (3).

Proof. See, for example, Karatzas and Shreve [1998].

The Firm's (Shareholders') Objective

The firm's (shareholders') objective is to maximize the *market value* of the firm (equity). The first-best (second-best) compensation scheme for the manager would therefore be such that the optimal policy the manager would adopt in response would be exactly the policy that maximizes the market value of the firm (equity), that is, the policy the manager would adopt if he always behaved in the interests of the firm (shareholders). In order to determine the conditions under which first-best (second-best) compensation schemes exist and determine them, we therefore first discuss the optimal policies for the firm (shareholders) assuming that the manager always behaves in the firm's (shareholders') interests. Let the effective corporate tax rate be denoted by t . The cash flows to the firm and its shareholders prior to bankruptcy are therefore given by

$$
\begin{aligned}
&D[P(t)] = (1-t)P(t) + tJ;\ P(t) \geq J \quad \text{for the firm} \\
&D[P(t)] = P(t), P(t) < J \quad \text{for the firm} \\
&D[P(t)] = (1-t)(P(t) - J);\ \text{for shareholders} \\
&D[P(t)] = P(t) - J \quad \text{for the firm}
\end{aligned}
\tag{7}
$$

In (7), we have assumed partial loss of tax shields when the firm's cash flows are below the required coupon payment (Leland 1998).

Since the objective of the firm (shareholders) is to maximize the *market value* of the firm (equity), we work under the *risk neutral measure* under which the drifts of both strategies are equal to $r-d$ where r is the risk free rate and d is the payout rate for the state variable $P(.)$ that we have assumed to be the price process of some traded asset in the market. Therefore, the state variable $P(.)$ evolves as follows under the *risk neutral measure*:

$$
\begin{aligned}
&dP(t) = (r-d)P(t)dt + s_1 P(t) dB_1^{**}(t);\ \text{Strategy 1} \\
&dP(t) = (r-d)P(t)dt + s_2 P(t) dB_2^{**}(t);\ \text{Strategy 2}
\end{aligned}
\tag{8}
$$

The goal of the firm (shareholders) is to maximize the market value of the firm (equity) that is given by

(9) $$V_{\Gamma}(p) = E_{\Gamma}[\int_{0}^{t_b} \exp(-rt) D(P(t)) dt]$$ under the switching policy Γ.

In the above, t_b is the random time at which the value of equity is zero and bankruptcy occurs. We assume that control of the firm is transferred to creditors after bankruptcy after the firm bears some proportional costs and the manager continues to operate the firm. After bankruptcy, it is easy to see that the firm is indifferent between either of the available strategies since both are marketed.

Leland's Result on The Firm's (Shareholders') Optimal Policy

Leland (1998) showed that the optimal policy for the firm (shareholders) is stationary and can be described as follows. There exists a level $p^*(p_*)$ such that it is optimal for the firm (shareholders) to choose the *high-volatility* strategy, that is, strategy 1, whenever $P(.) \leq p^*(p_*)$ and the *low-volatility* strategy, that is, strategy 2, whenever $P(.) > p^*(p_*)$. There may be scenarios where *either* $p^*(p_*) = \infty$ in which case it is optimal for the firm (shareholders) to always choose the high volatility strategy *or* $p^*(p_*) = p_b$ in which case it is optimal for the firm (shareholders) to always choose the low volatility strategy. Therefore, the first-best (second-best) compensation contract for the manager, if it exists, should be such that the policy described by Leland's result solves the manager's dynamic optimization problem. *After bankruptcy,* it is easy to see that since the available strategies are marketed, the owners of the firm, that is, the bondholders, are indifferent to the manager's choice of strategy. Since the optimal policies for the firm and the shareholders are *qualitatively* similar, the first and second best compensation contracts are also qualitatively similar.

For expositional simplicity, we therefore consider only first best compensation contracts in the rest of the paper. This completes the formulation of the model.

2. Optimal Compensation Contracts

We begin by introducing two quadratic equations that are the characteristic equations of the generators L^1, L^2 defined in (5) and (6).

(10)
$$\begin{aligned} &\frac{1}{2}s_1^{\,2}x^2+(m_1-\frac{1}{2}s_1^{\,2})x-b=0 \\ &\frac{1}{2}s_2^{\,2}x^2+(m_2-\frac{1}{2}s_2^{\,2})x-b=0 \end{aligned}$$

Each of the equations above has two real roots, one of which is strictly positive and the other strictly negative. Let us denote the positive and negative roots of the equations above by h_1^+, h_1^- and h_2^+, h_2^- respectively. The managers' value functions are expressed in terms of these roots. Throughout the paper, we shall assume that $h_1^+ \neq h_2^+, h_1^- \neq h_2^-$, i.e. the available strategies are such that the roots of equations (10) are all distinct[5]. The following lemma collects properties of these roots that we use frequently.

Lemma 1

1. $1 < h_i^+$
2. *If* $m_1 \geq m_2$ *then* $h_1^+ < h_2^+$
3. *If* $m_1 \leq m_2$ *then* $h_2^- < h_1^-$

Proof. In the Appendix.

A. Higher Volatility Strategy has Strictly Higher Expected Return

We first consider the situation where $m_1 > m_2, s_1 > s_2$, that is, the higher volatility strategy has higher expected return. The following proposition provides a precise necessary and sufficient

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[5] This assumption avoids unnecessarily complicating the statements of several propositions.

condition on the available strategies for the existence of a piecewise linear first-best compensation contract, that is, one that completely aligns managerial incentives with those of the firm.

Proposition 2

If the firm's optimal policy involves the choice of the low volatility strategy when the firm's cash flows exceed a finite threshold, then a first best compensation contract exists for the manager if and only if $h_1^- > h_2^-$.

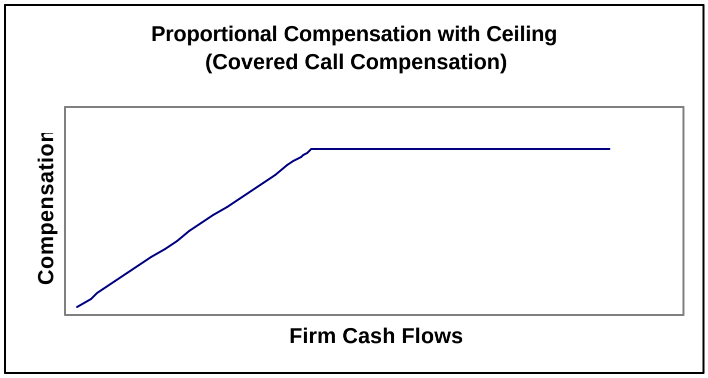
Proof. In the Appendix.

We may see that $h_1^- < h_2^-$ when the difference between the drifts of the strategies $m_1 - m_2$ is much greater than the difference in their volatilities $s_1 - s_2$. The intuition for the result of the above proposition is that since the manager's compensation is piecewise linear and periodic, when the firm is performing extremely well so that its cash flows far exceed required debt payments, the manager prefers to choose strategy 1 even though it has higher risk, since its expected growth rate far exceeds that of strategy 2. Therefore, there exists a level p^{**} such that for $P(.) > p^{**}$, the manager chooses strategy 1. However, as shown by Leland (1998), the firm's optimal policy is to choose the low-volatility strategy, that is, strategy 2, when the firm is performing extremely well. Therefore, the no contract can achieve first-best.

The following result shows that if $h_1^- > h_2^-$, a first-best compensation contract for the manager is one where the manager obtains a payoff that is proportional to firm cash flows subject to a *ceiling*. In other words, there exists $K > 0$, such that the following contract achieves first-best, that is, completely aligns managerial incentives with those of the firm.

$$
\begin{aligned}
C(t) &= rP(t); P(t) \le K \\
&= rK; P(t) > K
\end{aligned}
\tag{11}
$$

where r is chosen so that the manager's reservation utility is met. We note that the condition $h_1^- > h_2^-$ is satisfied when (roughly) the difference in the drifts of the available strategies is not large compared with the difference in the volatilities.



Proposition 3

The condition $h_1^- > h_2^-$ *is both necessary and sufficient for the existence of* K *so that the compensation contract defined by (11) is first best.*

Proof. In the Appendix.

We emphasize that the first-best contract described by (11) and **Proposition 3** is not necessarily unique. We would like to emphasize here that if the optimal policy for the firm is to always choose the high volatility strategy, then a first-best compensation contract for the manager is given by (11) with $K = \infty$. From (11), it is easy to see that the manager's compensation is linear in the firm's cash flows in this scenario. On the other hand, if the optimal policy for the firm is to always choose the low volatility strategy, then we can choose the value of K so that the optimal policy for the manager is to switch from strategy 1 to strategy 2 at $p^* < p_b$ where p_b is the endogenous bankruptcy point. Therefore, when the firm is solvent, the manager always chooses the low volatility strategy that coincides with that of the firm.

Suppose the firm's optimal policy involves switching from high to low volatility at some finite threshold. Then the following proposition establishes that *any* first-best contract for the manager when the strategy 1 has strictly greater drift than strategy 2 must be such that the manager obtains a constant payoff when the firm's performance exceeds a threshold.

Proposition 4

Suppose the optimal policy for the firm is to switch from high to low volatility at some finite threshold. Suppose further that $m_1 > m_2$ *and a compensation contract described by (1) is first best for some n. Then we must have* $r_{n-1} = 0$. *That is, the manager receives a constant payoff when the firm's cash flows are beyond a threshold.*

Proof. In the Appendix.

In particular, the result of the above proposition implies that a convex compensation structure can never achieve first-best when $m_1 > m_2$. This follows from the fact that any convex, monotonically increasing, piecewise linear compensation contract must be strictly increasing in the firm's cash flows when it is performing sufficiently well.

We now explain the intuition behind these results. Since the higher volatility strategy has the higher expected return, the manager, with a piecewise linear compensation structure that is strictly increasing when the firm performs extremely well, always has the incentive to choose the high volatility-high expected return strategy in regions where the firm is performing extremely well. This policy conflicts with the incentive for the firm to lower volatility in this region and explains why any compensation structure that is first best must necessarily provide a constant payoff to the manager when the firm's cash flows exceed a threshold. A manager compensated with a payoff that is proportional to firm cash flows subject to a ceiling, receives a constant payoff when the firm's cash flows exceed the ceiling. When the difference in the drifts of the strategies is not large compared with the difference in their volatilities, that is, $h_1^- > h_2^-$, the manager prefers the low volatility-low expected return strategy when the firm's cash flows exceed this ceiling to reduce the probability that the firm's performance declines to the level where he is penalized for under performance. However, when the firm's performance is significantly below the ceiling, the manager faces a payoff structure that is "almost linear" in firm cash flows. In this case, it is optimal for him to choose the high volatility-high expected return strategy since his expected compensation is higher. Therefore, the manager chooses high volatility when the firm is performing poorly and low volatility when it is performing well, similar to the optimal policy for the firm. For a specific ceiling level, the optimal policy for the manager is identical to that of the firm so that the contract is first best.

B. Higher Volatility Strategy has Strictly Lower Expected Return

In the previous sub-section, we showed that when the available asset substitution strategies are such that the higher volatility strategy also has higher expected return, convex compensation cannot achieve first-best. We now investigate the scenario wherein the higher volatility strategy has lower expected return, that is,

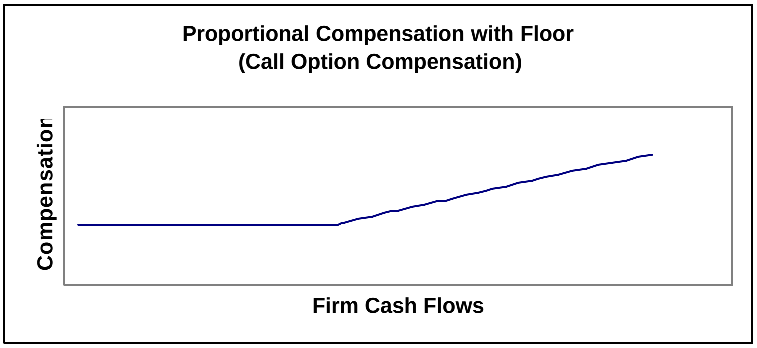
(12) $m_1 < m_2, s_1 > s_2$

By the result of part 3) of **Lemma 1,** this implies that

(13) $h_2^- < h_1^-$

where h_1^-, h_2^- are the negative roots of the quadratic equations (10). In this case, we show that under a condition on the available strategies, a compensation structure where the manager derives a payoff that is proportional to the firm's cash flows subject to a *floor* can achieve first best:

(14) $$\begin{aligned} C(t) &= C_0 + r(P(t) - K); P(t) \geq K \\ &= C_0; P(t) < K \end{aligned} \quad \text{where } C_0 \geq 0, r > 0$$



The following proposition establishes a condition on the available strategies for the compensation structure defined by (14) to achieve first best for a certain value K of the floor.

Proposition 5

If $m_1 < m_2$, *then the condition* $h_1^+ < h_2^+$ *where* h_1^+, h_2^+ *are the positive roots of the quadratic equations (10), is both necessary and sufficient for the existence of* C_0, r, K *such that a compensation contract described by (14) is first best.*

Proof. In the Appendix.

The intuition for this result is the following. When the firm's performance far exceeds the threshold, the manager's compensation is "almost linear". Therefore, it is optimal for him to choose the higher expected return strategy, but this strategy has lower volatility. When the firm's performance is below the threshold, the manager's payoff is constant. The fact that the compensation structure (14) rewards the manager for outperforming a threshold induces him to choose the strategy that increases the probability that the firm's performance exceeds the threshold. When the drifts and volatilities of the strategies satisfy the condition for the optimality of the compensation structure (14), it is optimal for the manager to choose the high volatility strategy even though it has lower expected return.

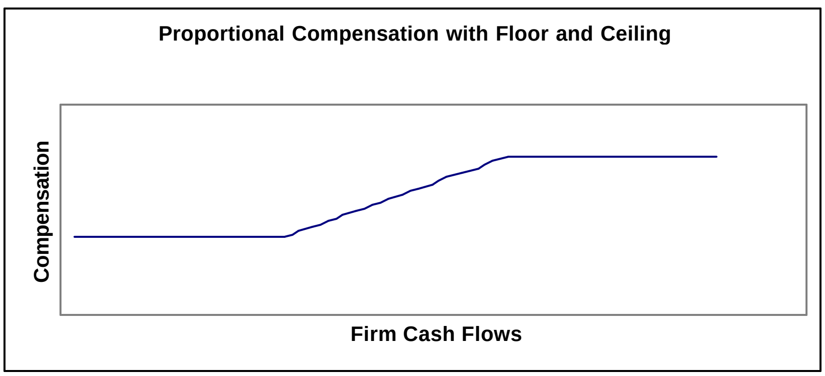
C. Strategies have Equal Expected Returns

We now examine the remaining scenario wherein the available strategies have equal expected returns, that is, the strategies differ only in *unsystematic risk*. In this case, we show that a compensation structure that achieves first best provides the manager with a payoff that is linear in the firm's cash flows, but with a *floor* and a *ceiling*. This compensation structure is described as follows:

$$C(t) = C_l; P(t) \le K$$

(15) $$C(t) = C_l + r(P(t) - K); K < P(t) < gK; g > 1$$

$$C(t) = C_h; P(t) \ge gK$$



where $C_h = C_l + rg(K-1)$. The following proposition establishes that a compensation structure of the form (15) can achieve first-best (second-best)

Proposition 6

For any $g > 1$, there exist parameters C_l, C_h, r, K such that the compensation structure described by (15) is first best..

Proof. In the Appendix.

It is interesting to note that the result of the above proposition implies that there exists a first-best compensation structure for any $g > 1$. In other words, the floor and the ceiling in the compensation structure (15) may be arbitrarily close to each other. The intuition for the result of the proposition is the following. When the firm's cash flows exceed gK, the manager prefers strategy 2 since it lowers the probability that the firm's cash flows will fall to a level lower than gK where the manager is penalized for under performance. On the other hand, when the firm's cash flows are lower than K, the manager prefers strategy 1 since it increases the probability that the cash flows will increase above K where the manager is rewarded for out performance. Hence, there is an intermediate level of performance where the manager switches from high to low volatility, a policy that is qualitatively identical to that of the firm. The floor and ceiling can be chosen so that the manager's optimal switching point is identical to that of the firm so that the corresponding contract is first best.

A close examination of the proof of the proposition reveals that its result only depends on the following relation between the roots of the quadratic equations (10) that is satisfied, in particular, when $m_1 = m_2$:

(16) $h_2^- < h_1^- < h_1^+ < h_2^+$.

Condition (16) holds whenever the difference in the drifts is not large compared with the difference in their volatilities. In other words, the compensation structure (15) achieves first best for any pair of asset substitution strategies characterized by drift-volatility parameters such that (16) is satisfied. From the results of **Lemma 1, Proposition 3** and **Proposition 5,** we see that first best compensation structures were derived in the scenario where $m_1 \neq m_2$ exactly when condition (16) is satisfied. This leads us to the following result:

Corollary 1

If the pair of available asset substitution strategies are such that condition (16) is satisfied, then, for any $g > 1$, there exist parameters C_l, C_h, r, K such that the compensation structure described by (15) is first best.. Moreover, condition (16) is necessary for the compensation structure (15) to be first best for some set of parameters.

Thus, under condition (16), we see that a compensation structure that provides the manager with a proportion of the firm's cash flows subject to a floor and a ceiling is first best. Murphy [1999] reports that typical compensation packages of firm managers consist of a share of the firms' profits with floors and ceilings, stock and options. The result of Corollary 1 suggests that this compensation structure could serve the role of mitigating the agency costs due to managerial asset substitution.

3. Numerical Derivation of Optimal Contracts

In the previous section, we established conditions for the existence of and explicitly characterized first best compensation contracts for the manager of a leveraged firm. In this section, we explicitly derive these contracts for various choices of underlying parameter values. In the process, we explore the various comparative static relationships between the optimal contracts and the underlying parameter values.

We first derive the optimal policies for the firm, that is, we determine the endogenous bankruptcy level p_b and the optimal switching threshold p^* such that the optimal policy for the firm is to choose the high-volatility strategy below the threshold and the low-volatility strategy above the threshold. Depending on the sign of the difference in the drifts of the available strategies, we determine the first best contract for the manager. Since the first best compensation contracts are not unique, for concreteness, we restrict our consideration to compensation contracts that are defined as follows:

Proportional Compensation with Ceiling

$$\begin{aligned} C_K(t) &= P(t); P(t) \leq K \\ &= K; P(t) > K \end{aligned}$$

Proportional Compensation with Floor

$$\begin{aligned} C_{K'}(t) &= K'; P(t) \leq K' \\ &= P(t); P(t) > K' \end{aligned}$$

Proportional Compensation with Floor and Ceiling

$$\begin{aligned} C_{K,K'}(t) &= K';P(t) \leq K' \\ &= P(t);K' < P(t) \leq K \\ &= K;P(t) > K \end{aligned}$$

The subscripts on the compensation function indicate the explicit dependence on the respective thresholds K, K'. For various possible choices of available asset substitution strategies, we determine the “optimal” thresholds when either a covered call compensation structure or a call option compensation structure is first best.

Tables 1, 2, and **3** display first best compensation contracts in the scenarios where the higher volatility strategy has higher drift, lower drift, and when the strategies have equal drifts respectively. The initial value of the firm’s cash flows $p0$ is normalized to 1 and the other parameters are as displayed in the figure. We also display the endogenous bankruptcy point, the optimal switching point for the firm, the value of the firm, and the value of equity.

TABLE 1 : First-Best Compensation Structures/Higher Volatility Strategy has Higher Drift

The following table displays the first-best compensation structure for the manager when the higher volatility strategy has higher drift for varying choices of the volatilities of the strategies. We display the value of the ceiling that defines the first-best compensation structure for the manager. We also display the endogenous bankruptcy point, the optimal switching point, the firm value, and the value of equity. The other parameters are as follows:

p0 = 1, q = 0.5, r = 0.05, delta = 0.02, alpha = 0.1, tau = 0.2

mu1	mu2	sigma1	sigma2	Bankruptcy Point	Switching Point	Firm Value	Equity Value	Ceiling
0.20	0.10	0.25	0.16	0.14	0.18	41.97	31.99	0.20
0.20	0.10	0.31	0.20	0.12	0.17	41.90	32.02	0.20
0.20	0.10	0.27	0.18	0.13	0.18	41.94	32.00	0.20
0.20	0.10	0.29	0.20	0.12	0.18	41.90	32.02	0.20
0.20	0.10	0.27	0.14	0.14	0.19	41.99	31.99	0.21
0.20	0.10	0.33	0.18	0.12	0.18	41.94	32.00	0.21
0.20	0.10	0.29	0.16	0.13	0.19	41.97	31.99	0.22
0.20	0.10	0.25	0.18	0.13	0.20	41.94	32.00	0.22
0.20	0.10	0.25	0.14	0.14	0.20	41.99	31.99	0.22
0.20	0.10	0.31	0.18	0.12	0.19	41.94	32.00	0.22
0.20	0.10	0.25	0.10	0.16	0.20	42.00	32.00	0.22
0.20	0.10	0.27	0.16	0.13	0.20	41.97	31.99	0.22
0.20	0.10	0.27	0.10	0.16	0.20	42.00	32.00	0.22
0.20	0.10	0.33	0.14	0.13	0.19	41.99	31.99	0.23
0.20	0.10	0.33	0.16	0.12	0.19	41.97	31.99	0.23
0.20	0.10	0.33	0.20	0.11	0.19	41.90	32.03	0.23
0.20	0.10	0.29	0.18	0.12	0.20	41.94	32.00	0.23
0.20	0.10	0.29	0.10	0.15	0.20	42.00	32.00	0.23
0.20	0.10	0.29	0.12	0.14	0.20	42.00	32.00	0.23
0.20	0.10	0.29	0.14	0.13	0.20	41.99	31.99	0.23
0.20	0.10	0.25	0.12	0.14	0.21	42.00	32.00	0.23
0.20	0.10	0.31	0.10	0.15	0.20	42.00	32.00	0.23
0.20	0.10	0.31	0.14	0.13	0.20	41.99	31.99	0.23
0.20	0.10	0.31	0.16	0.12	0.20	41.97	31.99	0.23
0.20	0.10	0.33	0.10	0.15	0.20	42.00	32.00	0.24
0.20	0.10	0.27	0.12	0.14	0.21	42.00	32.00	0.24
0.20	0.10	0.31	0.12	0.13	0.21	42.00	32.00	0.24
0.20	0.10	0.33	0.12	0.13	0.21	42.00	32.00	0.25

TABLE 2 : First-Best Compensation Structures/Higher Volatility Strategy has Lower Drift

The following table displays the first-best compensation structure for the manager when the higher volatility strategy has lower drift for varying choices of the volatilities of the strategies. We display the value of the floor that defines the first-best compensation structure for the manager. We also display the endogenous bankruptcy point, the optimal switching point, the firm value, and the value of equity. The other parameters are as follows:

p0 = 1, q = 0.5, r = 0.05, delta = 0.02, alpha = 0.1, tau = 0.2

mu1	mu2	sigma1	sigma2	Bankruptcy Point	Switching Point	Firm Value	Equity Value	Floor
0.10	0.18	0.33	0.10	0.15	0.20	42.00	32.00	0.19
0.10	0.16	0.33	0.18	0.12	0.18	41.94	32.00	0.17
0.10	0.16	0.33	0.14	0.13	0.19	41.99	31.99	0.17
0.10	0.16	0.33	0.10	0.15	0.20	42.00	32.00	0.17
0.10	0.16	0.33	0.16	0.12	0.19	41.97	31.99	0.17
0.12	0.18	0.33	0.14	0.13	0.19	41.99	31.99	0.18
0.10	0.16	0.31	0.10	0.15	0.20	42.00	32.00	0.18
0.12	0.18	0.33	0.10	0.15	0.20	42.00	32.00	0.18
0.12	0.18	0.33	0.16	0.12	0.19	41.97	31.99	0.18
0.10	0.16	0.33	0.12	0.13	0.21	42.00	32.00	0.18
0.10	0.16	0.31	0.14	0.13	0.20	41.99	31.99	0.18
0.14	0.20	0.33	0.14	0.13	0.19	41.99	31.99	0.18
0.10	0.16	0.29	0.10	0.15	0.20	42.00	32.00	0.19
0.12	0.18	0.31	0.10	0.15	0.20	42.00	32.00	0.19
0.10	0.16	0.31	0.16	0.12	0.20	41.97	31.99	0.19
0.10	0.16	0.29	0.12	0.14	0.20	42.00	32.00	0.19
0.14	0.20	0.33	0.10	0.15	0.20	42.00	32.00	0.19
0.10	0.16	0.31	0.12	0.13	0.21	42.00	32.00	0.19
0.12	0.18	0.33	0.12	0.13	0.21	42.00	32.00	0.19
0.10	0.16	0.27	0.10	0.16	0.20	42.00	32.00	0.19
0.12	0.18	0.29	0.10	0.15	0.20	42.00	32.00	0.19
0.14	0.20	0.31	0.10	0.15	0.20	42.00	32.00	0.20
0.12	0.18	0.31	0.12	0.13	0.21	42.00	32.00	0.20
0.14	0.20	0.33	0.12	0.13	0.21	42.00	32.00	0.20

TABLE 3 : First-Best Compensation Structures/Both Strategies have the Same Drift

The following table displays the first-best compensation structure for the manager when both strategies have the same drift for varying choices of the volatilities of the strategies. We display the values of the floor and the ceiling that define the first-best compensation structure for the manager. We also display the endogenous bankruptcy point, the optimal switching point, the firm value, and the value of equity. The other parameters are as follows:

p0 = 1, q = 0.5, r = 0.05, delta = 0.02, alpha = 0.1, tau = 0.2

mu1	mu2	sigma1	sigma2	Bankruptcy Point	Switching Point	Firm Value	Equity Value	Floor	Ceiling
0.10	0.10	0.31	0.20	0.12	0.17	41.90	32.02	0.12	0.24
0.10	0.10	0.33	0.18	0.12	0.18	41.94	32.00	0.13	0.25
0.12	0.12	0.31	0.20	0.12	0.17	41.90	32.02	0.13	0.25
0.10	0.10	0.25	0.20	0.13	0.17	41.89	32.01	0.13	0.26
0.10	0.10	0.33	0.14	0.13	0.19	41.99	31.99	0.13	0.26
0.12	0.12	0.33	0.18	0.12	0.18	41.94	32.00	0.13	0.26
0.14	0.14	0.31	0.20	0.12	0.17	41.90	32.02	0.13	0.26
0.10	0.10	0.33	0.16	0.12	0.19	41.97	31.99	0.13	0.26
0.10	0.10	0.29	0.20	0.12	0.18	41.90	32.02	0.13	0.26
0.10	0.10	0.27	0.18	0.13	0.18	41.94	32.00	0.13	0.26
0.10	0.10	0.25	0.16	0.14	0.18	41.97	31.99	0.13	0.26
0.12	0.12	0.25	0.20	0.13	0.17	41.89	32.01	0.13	0.27
0.10	0.10	0.33	0.10	0.15	0.20	42.00	32.00	0.13	0.27
0.10	0.10	0.31	0.18	0.12	0.19	41.94	32.00	0.13	0.27
0.10	0.10	0.33	0.20	0.11	0.19	41.90	32.03	0.13	0.27
0.14	0.14	0.33	0.18	0.12	0.18	41.94	32.00	0.13	0.27
0.10	0.10	0.29	0.16	0.13	0.19	41.97	31.99	0.13	0.27
0.12	0.12	0.33	0.14	0.13	0.19	41.99	31.99	0.13	0.27
0.16	0.16	0.31	0.20	0.12	0.17	41.90	32.02	0.13	0.27
0.10	0.10	0.31	0.10	0.15	0.20	42.00	32.00	0.14	0.27
0.12	0.12	0.29	0.20	0.12	0.18	41.90	32.02	0.14	0.27
0.10	0.10	0.27	0.14	0.14	0.19	41.99	31.99	0.14	0.27
0.12	0.12	0.33	0.16	0.12	0.19	41.97	31.99	0.14	0.27
0.12	0.12	0.27	0.18	0.13	0.18	41.94	32.00	0.14	0.27
0.10	0.10	0.29	0.10	0.15	0.20	42.00	32.00	0.14	0.27
0.14	0.14	0.25	0.20	0.13	0.17	41.89	32.01	0.14	0.27
0.12	0.12	0.33	0.20	0.11	0.19	41.90	32.03	0.14	0.28

Conclusions

We investigate the existence of and explicitly characterize compensation structures that eliminate agency conflicts between a leveraged firm (or its shareholders) and the manager due to managerial asset substitution within a continuous time framework. The manager may dynamically switch between two strategies with different risks and expected returns after debt is in place. We show that when the strategies satisfy a specific condition that (roughly) ensures that the difference in their drifts is not large compared with the difference in their volatilities, a periodic compensation structure that completely aligns the manager's interests with those of the firm or its shareholders is one where the manager obtains a payoff that is proportional to the firm's operating cash flows, but subject to a floor and a ceiling. This result explains the prevalence of compensation schemes where firm managers obtain shares of firm profits subject to floors and ceilings.

In the scenario wherein the higher volatility strategy also has higher drift, we show that a concave compensation structure that provides the manager with a payoff that is proportional to firm cash flows subject to a ceiling is first best. We also show that a necessary condition for a first best compensation structure is that it provides the manager with a constant payoff beyond a threshold level of firm cash flows. In particular, therefore, a convex compensation structure cannot be first best in this scenario. In the scenario wherein the higher volatility strategy has lower drift, we show that a convex compensation structure that provides the manager with a payoff that is proportional to firm cash flows subject to a floor is first best.

The analysis in the paper therefore precisely identifies the scenarios wherein concave and convex compensation structures may eliminate agency conflicts due to managerial asset substitution. When the difference between the drifts of the available strategies may be positive, negative, or zero, the first best compensation structure is neither convex nor concave. In future research, it would be interesting and important to examine optimal compensation contracts for managers when the strategy space is not merely restricted to asset substitution strategies. It would be particularly important to examine scenarios wherein there may also be positive and negative NPV strategies available. This would allow us to address the design of compensation schemes that minimize the agency costs due to under investment or over investment in addition to those due to dynamic asset substitution.

Appendix

Proof of Lemma 1

1. We note that

$$\frac{1}{2}s_i^{\,2}(1)^2+(m_i-\frac{1}{2}s_i^{\,2})(1)-b=m_i-b<0 \text{ by hypothesis.}$$

From the above, 1 must lie between the roots of the equations (10). Therefore, $1<h_i^+$.

2. We note that

$$[\frac{1}{2}s_1^{\,2}(h_2^+)^2+(m_1-\frac{1}{2}s_1^{\,2})h_2^+-b]-[\frac{1}{2}s_2^{\,2}(h_2^+)^2+(m_2-\frac{1}{2}s_2^{\,2})h_2^+-b]$$
$$=\frac{1}{2}(s_1^{\,2}-s_2^{\,2})(h_2^{+2}-h_2^+)+(m_1-m_2)h_2^+>0$$

where the last inequality follows from the fact that $s_1>s_2$, $1<h_2^+$ by the result of part 1., $m_1\geq m_2$ by hypothesis. Since h_2^+ is a root of the second quadratic equation in (10), the above implies that

$$\frac{1}{2}s_1^{\,2}(h_2^+)^2+(m_1-\frac{1}{2}s_1^{\,2})h_2^+-b>0$$

Therefore, we must have $h_1^+<h_2^+$.

3. We note that

$$[\frac{1}{2}s_1^{\ 2}(h_2^-)^2 + (m_1 - \frac{1}{2}s_1^{\ 2})h_2^- - b] - [\frac{1}{2}s_2^{\ 2}(h_2^-)^2 + (m_2 - \frac{1}{2}s_2^{\ 2})h_2^- - b]$$
$$= \frac{1}{2}(s_1^{\ 2} - s_2^{\ 2})(h_2^{-2} - h_2^-) + (m_1 - m_2)h_2^- > 0$$

where the last inequality follows from the fact that $h_2^- < 0, s_1 > s_2$, and $m_1 \leq m_2$ by hypothesis.

Since h_2^- is a root of the second quadratic equation in (10), it follows that

$$\frac{1}{2}s_1^{\ 2}(h_2^-)^2 + (m_1 - \frac{1}{2}s_1^{\ 2})h_2^- - b > 0$$

Therefore, we must have $h_2^- < h_1^-$. This completes the proof. ♦

Proof of Proposition 2

The proof proceeds by showing that the optimal policy for the manager can never coincide with the optimal policy for shareholders. Suppose the manager is compensated as in (1.1). We consider two separate cases:

Case 1: $r_{n-1} > 0$

From (1.1) we may easily check that the above condition implies that the manager's compensation is strictly increasing in the value of the state variable $P(.)$ on the interval (w_{n-1}, ∞). Since the optimal policy for shareholders is to choose strategy 2, that is, the low volatility strategy beyond a threshold p^*, it suffices to show that this policy is strictly sub-optimal for the manager. The proof proceeds by contradiction. Suppose the policy is optimal for the manager. Then, by the result of Proposition 1, if u is the value function of this policy, we must have

(A1) $$\begin{matrix} L^1(u) + C(p) \leq 0 \\ L^2(u) + C(p) = 0 \end{matrix} \quad \text{for } p > p^*$$

From the functional form for $C(p)$ described by (1), we may see that for $p > \max(p^*, w_{n-1})$, the manager's value u must have the functional form

(A2) $$u(p) = Ap^{h_2^-} + \frac{\sum_{j=1}^{n-2} r_{j-1}(w_{j+1} - w_j)}{b} + \frac{r_{n-1} p}{b - m_2} - \frac{r_{n-1} w_{n-1}}{b} \text{ for } p > \max(p^*, w_{n-1}),$$

where A is a constant. From (A2), we see that

(A3) $$L^1(u) + C(p) = A(\frac{1}{2} s_1^{\ 2} (h_2^-)^2 + (m_1 - \frac{1}{2} s_1^{\ 2}) h_2^- - b) p^{h_2^-} + \frac{r_{n-1}(m_1 - m_2) p}{b - m_2}$$

Since $m_1 > m_2$ and $h_2^- < 0$, we see from the (A3) that

(A4) $$\lim_{p \to \infty} L^1(u) + C(p) = \infty$$

Therefore, (A1) cannot be satisfied and the hypothesized policy cannot be optimal for the manager.

Case 2: $r_{n-1} = 0$

This represents the case where the manager obtains a fixed wage when $P(.) \geq w_{n-1}$. The manager's value function is given by (A2) with $r_{n-1} = 0$. We must have $A < 0$ in (A2) since the manager's value function cannot exceed $\frac{\sum_{j=1}^{n-2} r_{j-1}(w_{j+1} - w_j)}{b}$ that is the value if he were to always obtain the fixed wage $\sum_{j=1}^{n-2} r_{j-1}(w_{j+1} - w_j)$. From (A3) and the fact that h_1^+, h_1^- are the roots of the first quadratic equation (10), we see that

(A5) $$L^1(u) + C(p) = A(\frac{1}{2} s_1^{\ 2} (h_2^- - h_1^-)(h_2^- - h_1^+)) p^{h_2^-} > 0$$

where the inequality follows from the fact that $h_1^- < h_2^- < h_1^+$ by the hypothesis of the proposition. Therefore, the hypothesized policy cannot be optimal. Hence, the shareholders' optimal policy is sub-optimal for the manager. This completes the proof. ♦

Proof of Proposition 3

We begin by fixing $K > 0$ and showing that there exists $0 < e < 1$ such that it is optimal for the manager to choose strategy 1, that is, the high volatility strategy for $P(.) \leq eK$ and strategy 2, that is, the low volatility strategy for $P(.) > eK$. Since the firm's (shareholders') optimal policy is to choose strategy 1 for $P(.) \leq p^*(p_*)$ and strategy 2 for $P(.) > p^*(p_*)$, it follows that the optimal policy for the manager coincides with that for shareholders if $K(K') = \frac{p^*}{e}(\frac{p_*}{e})$.

Step 1

We consider the class of policies for the manager indexed by the parameter $q \leq K$ where the manager chooses strategy 1 for $P(.) \leq q$ and strategy 2 for $P(.) > q$. For notational simplicity, we set $r = 1$ in (11). The value function u_q of the manager must have the following functional form:

$$
\begin{aligned}
u_q(p) &= A_q p^{h_2^-} + \frac{K}{b};\ p \geq K \\
&= B_q p^{h_2^+} + C_q p^{h_2^-} + \frac{p}{b - m_2};\ q \leq p < K \qquad \text{(A6)} \\
&= D_q p^{h_1^+} + \frac{p}{b - m_1};\ p < q
\end{aligned}
$$

We have indicated the explicit dependence of the coefficients on the switching level q. We now note that for any $q \leq K$, we must have $A_q < 0$ since the value function of the manager cannot exceed the value of always obtaining the fixed wage K. It follows that

$$
L^1(u_q) + C(p) = A_q(\frac{1}{2}s_1{}^2(h_2^- - h_1^-)(h_2^- - h_1^+))p^{h_2^-} < 0 \quad \text{for } p > K \qquad \text{(A7)}
$$

since $h_2^- < h_1^- < h_1^+$ by hypothesis.

Step 2

We now show that there exists a value $q^* < K$ such that

(A8) $$L^1(u_{q^*}) + C(p)|_{p=q^*+} = 0$$

From (A7), we see that

(A9) $$L^1(u_K) + C(p)|_{p=K+} < 0\,.$$

We establish the existence of q^* by first proving that

(A10) $$L^1(u_0) + C(p)|_{p=0+} > 0$$

where u_0 is the value function of the policy of always choosing strategy 2. We may see that u_0 must have the following functional form:

$$u_0(p) = A_0 p^{h_2^-} + \frac{K}{b}; p \geq K$$

$$= D_0 p^{h_2^+} + \frac{p}{b - m_2}; p < K$$

Therefore, we have

(A11) $$L^1(u_0) + C(p) = D_0 \frac{1}{2} s_1{}^2 (h_2^+ - h_1^-)(h_2^+ - h_1^+) p^{h_2^+} + \frac{(m_1 - m_2) p}{b - m_2}; p < K$$

Since $m_1 > m_2$ and $h_2^+ > 1$, we see from (A11) that, regardless of the value of D_0,

(A12) $$\lim_{p \to 0} L^1(u_0) + C(p) > 0$$

(A12) is equivalent to (A10). Since $L^1(u_q) + C(p)|_{p=q+}$ is a continuous function of q, (A9) and (A10) together imply the existence of $q^* < K$ such that (A8) holds.

Step 3

By the definition of u_{q^*}, we see that

(A13)
$$L^1(u_{q^*}) + C(p)|_{p=q^*-} = 0$$
$$L^2(u_{q^*}) + C(p)|_{p=q^*+} = 0$$

Since $C(.)$ is continuous, it follows that u_{q^*} is twice differentiable at $p = q^*$ and that

(A14)
$$L^1(u_{q^*}) + C(p)|_{p=q^*} = L^2(u_{q^*}) + C(p)|_{p=q^*} = 0$$

Step 4

We now show that u_{q^*} is the optimal value function for the manager, that is, the policy of choosing strategy 1 for $p \le q^*$ and strategy 2 for $p > q^*$ is optimal for the manager. By the result of **Proposition 1,** we need to show that

(A15)
$$L^2(u_{q^*}) + C(p) \le 0; p < q^*$$
$$L^1(u_{q^*}) + C(p) \le 0; p \ge q^*$$

By (A6), we may express u_{q^*} as follows:

(A16)
$$\begin{aligned} u_{q^*}(p) &= A_{q^*} p^{h_2^-} + \frac{K}{b};\ \ p \ge K \\ &= B_{q^*} p^{h_2^+} + C_{q^*} p^{h_2^-} + \frac{p}{b - m_2};\ \ q^* \le p < K \\ &= D_{q^*} p^{h_1^+} + \frac{p}{b - m_1};\ \ p < q^* \end{aligned}$$

By the discussion following (A6), $A_{q^*} < 0$. Therefore,

(A17)
$$L^1(u_{q^*}) + C(p) = A_{q^*} \frac{1}{2} s_1{}^2 (h_2^- - h_1^-)(h_2^- - h_1^+) p^{h_2^-} < 0; p \ge K$$

since $h_2^- < h_1^- < h_1^+$ by hypothesis. We now note that

(A18)
$$L^2(u_{q^*}) + C(p) = D_{q^*} \frac{1}{2} s_2{}^2 (h_1^+ - h_2^-)(h_1^+ - h_2^+) p^{h_1^+} + \frac{(m_2 - m_1)p}{b - m_1}; p < q^*$$

By the definition of q^*,

(A19) $$L^2(u_{q^*}) + C(p)|_{p=q^*} = 0\,.$$

Since $m_2 < m_1$ and $h_2^- < h_1^+ < h_2^+$, (A18) implies that (A19) is true only if

(A20) $$D_{q^*} < 0\,.$$

This implies that the right hand side of (A18) is less than zero for $0 < p < q^*$. Hence,

(A21) $$L^2(u_{q^*}) + C(p) < 0; p < q^*$$

It remains to show that

(A22) $$L^1(u_{q^*}) + C(p) \le 0; q^* \le p < K$$

From (A16), we therefore need to show that

(A22) $$L^1(u_{q^*}) + C(p) = B_{q^*}\frac{1}{2}s_1{}^2(h_2^+ - h_1^-)(h_2^+ - h_1^+)p^{h_2^+} + C_{q^*}\frac{1}{2}s_1{}^2(h_2^- - h_1^-)(h_2^- - h_1^+)p^{h_2^-} + \frac{(m_1 - m_2)p}{b - m_2};\ q^* \le p < K$$

Using the result of *Step 3* that u_{q^*} is twice differentiable at $p = q^*$ and some tedious algebra, we can show that

$$B_{q^*} = \frac{(h_1^+ - h_2^-)(h_1^+ - 1)}{(h_2^+ - h_2^-)(h_2^+ - 1)} D_{q^*}(q^*)^{h_1^+ - h_2^+}$$

$$C_{q^*} = \frac{(h_1^+ - h_2^+)(h_1^+ - 1)}{(h_2^- - h_2^+)(h_2^- - 1)} D_{q^*}(q^*)^{h_1^+ - h_2^-}$$

Since $h_2^- < h_1^- < 1 < h_1^+ < h_2^+$ by hypothesis and $D_{q^*} < 0$ from (A20), the above implies that

(A23) $$B_{q^*} < 0, C_{q^*} > 0$$

(A17) and the definition of q^* together imply that

(A24)
$$L^1(u_{q^*}) + C(p)|_{p=K} < 0$$
$$L^1(u_{q^*}) + C(p)|_{p=q^*+} = 0$$

(A23), (A24) and the functional form of the right hand side of (A22) together imply that

(A25) $$L^1(u_{q^*}) + C(p) \le 0; q^* \le p < K$$

Therefore, the value function u_{q^*} satisfies all the hypotheses of **Proposition 1.** Therefore, the policy of choosing strategy 1 for $p \le q^*$ and strategy 2 for $p > q^*$ is optimal for the manager.

It is not difficult to check that the manager's optimization problem is "scale-independent", that is, for any value of K, the corresponding optimal switching point q^* is such that $\frac{q^*}{K}$ is constant. It is therefore easy to see that we may choose K such that the manager's optimal switching point coincides with that of the firm and the corresponding contract is therefore first best. This completes the proof. ♦

Proof of Proposition 4

The proof proceeds by contradiction. Suppose, to the contrary, that a contract described by (1) is first best (second best) for some finite number n and $0 = w_0 < w_1 < ... < w_n = \infty$ and $0 \le r_0, r_1, ..., r_{n-1} << 1$. Suppose $r_{n-1} > 0$ and the optimal policy for the manager with such a contract is to choose strategy 2, that is, the low-volatility strategy, when the firm's cash flows exceed a threshold p^*. Then, from (1), for $p > \max(w_{n-1}, p^*)$, the manager's value function $u(p)$ must have the following functional form:

(A26) $$u(p) = \frac{A}{b} + Bp^{h_2^-} + \frac{r_{n-1}p}{b - m_2}; p > \max(w_{n-1}, p^*),\ A, B \text{ are constants}$$

By the result of **Proposition 1** and standard dynamic programming arguments, a necessary condition for the manager's value function to be his optimal value function is that

(A27) $L^1(u) + C(p) \leq 0; p > \max(w_{n-1}, p^*)$

From (A26), we may see that

(A28) $L^1(u) + C(p) = B(\frac{1}{2}s_1{}^2(h_2^-)^2 + (m_1 - \frac{1}{2}s_1{}^2)h_2^- - b)p^{h_2^-} + \frac{r_{n-1}(m_1 - m_2)p}{b - m_2}; p > \max(w_{n-1}, p^*)$

Since $m_1 > m_2$ by hypothesis and $h_2^- < 0$, we see from (A28) that

(A29) $\lim_{p \to \infty} L^1(u) + C(p) = \infty$

This implies that (A27) cannot be satisfied. Therefore, $u(p)$ cannot be the optimal value function of the manager. This contradiction completes the proof of the proposition. ♦

Proof of Proposition 5

We begin by fixing $K > 0$ and showing that there exists $e > 1$ such that it is optimal for the manager to choose strategy 1, that is, the high volatility strategy for $P(.) \leq eK$ and strategy 2, that is, the low volatility strategy for $P(.) > eK$. Since the firm's (shareholders') optimal policy is to choose strategy 1 for $P(.) \leq p^*(p_*)$ and strategy 2 for $P(.) > p^*(p_*)$, it follows that the optimal policy for the manager coincides with that for shareholders if $K(K') = \frac{p^*}{e}(\frac{p_*}{e})$.

Step 1

We consider the class of policies for the manager indexed by the parameter $q \geq K$ where the manager chooses strategy 1 for $P(.) \leq q$ and strategy 2 for $P(.) > q$. For notational simplicity, we set $r = 1$ in (14). The value function u_q of the manager must have the following functional form:

(A30)
$$\begin{aligned} u_q(p) &= A_q p^{h_2^-} + \frac{C_0 - K}{b} + \frac{p}{b - m_2};\ p \geq q \\ &= B_q p^{h_1^+} + C_q p^{h_1^-} + \frac{C_0 - K}{b} + \frac{p}{b - m_1};\ K \leq p < q \\ &= D_q p^{h_1^+} + \frac{C_0 - K}{b};\ p < K \end{aligned}$$

We have indicated the explicit dependence of the coefficients on the switching level q. We now note that for any $q \geq K$, we must have $D_q > 0$ since the value function of the manager is strictly greater than the value of obtaining the fixed wage $C_0 - K$. It follows that

(A31) $$L^2(u_q) + C(p) = D_q(\frac{1}{2}s_1{}^2(h_1^+ - h_2^+)(h_1^+ - h_2^-))p^{h_1^+} < 0 \quad \text{for} \quad p < K$$

since $h_2^- < h_1^- < h_1^+ < h_2^+$ by hypothesis. From (A31), we see that

(A32) $$L^2(u_K) + C(p)|_{p=K-} < 0.$$

By definition,

(A33) $$L^2(u_K) + C(p)|_{p=K+} = 0$$

and

(A34) $$L^1(u_K) + C(p)|_{p=K-} = 0$$

Subtracting (A32) from (A33) and using the fact that u_K is differentiable everywhere, we have

(A35) $$\frac{1}{2}s_2{}^2K^2[\frac{d^2u_K}{dp^2}|_{p=K+} - \frac{d^2u_K}{dp^2}|_{p=K-}] > 0$$

and therefore,

(A36) $$\frac{1}{2}s_1{}^2K^2[\frac{d^2u_K}{dp^2}|_{p=K+} - \frac{d^2u_K}{dp^2}|_{p=K-}] > 0$$

(A34) and (A36) now imply that

(A37) $$L^1(u_K) + C(p)|_{p=K+} > 0$$

Next, we consider the value function u_∞ corresponding to the policy of always choosing strategy 1. This must have the following functional form:

$$u_\infty(p) = A_\infty p^{h_1^-} + \frac{C_0 - K}{b} + \frac{p}{b - m_1};\ p \geq K$$
$$= D_\infty p^{h_1^+} + \frac{C_0 - K}{b};\ p < K$$

We now note that

(A38)
$$\lim_{p\to\infty} L^2(u_\infty) + C(p) =$$
$$\lim_{p\to\infty} A_\infty\left(\frac{1}{2}s_2{}^2(h_1^-)^2 + (m_2 - \frac{1}{2}s_2{}^2)h_1^- - b\right)p^{h_1^-} + \frac{(m_2 - m_1)p}{b - m_1} = \infty$$

since $h_1^- < 0$ and $m_2 > m_1$. Among the class of policies indexed by the parameter q where the manager chooses strategy 1 for $P(.) \leq q$ and strategy 2 for $P(.) > q$, (A37) and **Proposition 1** imply that the optimal policy cannot be that of choosing strategy 1 for $P(.) \leq K$ and strategy 2 for $P(.) > K$. Similarly, (A38) and **Proposition 1** imply that the optimal policy also cannot be that of choosing strategy 1 for all values of $P(.)$. By the continuity of the value function u_q in the parameter q, it follows that there exists $q^* > K$ such that the value function u_{q^*} of switching strategies at q^* is optimal among the sub-class of policies where the manager switches from strategy 1 to strategy 2 at some $q \geq K$. Our goal is to show that this policy is in fact optimal among the entire class of feasible policies for the manager and the value function u_{q^*} is in fact his optimal value function.

Step 3

We now show that by the definition of the threshold q^*, we must have

(A39)
$$L^2(u_{q^*}) + C(p)|_{p=q^*-} = 0$$
$$L^1(u_{q^*}) + C(p)|_{p=q^*+} = 0$$

By the definition of the value function u_{q^*}, we know that

(A40)
$$L^1(u_{q^*}) + C(p)|_{p=q^*-} = 0$$
$$L^2(u_{q^*}) + C(p)|_{p=q^*+} = 0$$

Suppose, to the contrary that $L^2(u_{q^*}) + C(p)|_{p=q^*-} > 0$. Then we may use Ito's lemma to show that there exists $\bar{q} < q^*$ such that the policy of switching strategies at $\bar{q}$ instead of q^* has strictly greater value contradicting the choice of q^*. On the other hand, suppose that $L^2(u_{q^*}) + C(p)|_{p=q^*-} < 0$. The differentiability of u_{q^*} and (A40) implies that $L^1(u_{q^*}) + C(p)|_{p=q^*+} > 0$. We may use to Ito's lemma to show that there exists $\bar{\bar{q}} > q^*$ such that the policy of switching strategies at $\bar{\bar{q}}$ instead of q^* has strictly greater value again contradicting the choice of q^*. It follows that we must have $L^2(u_{q^*}) + C(p)|_{p=q^*-} = 0$. We may use similar arguments to show that $L^1(u_{q^*}) + C(p)|_{p=q^*+} = 0$ thereby establishing (A40).

Since $C(.)$ is continuous, it may be seen that that (A40) implies that u_{q^*} is twice differentiable at $p = q^*$ and that

(A41)
$$L^1(u_{q^*}) + C(p)|_{p=q^*} = L^2(u_{q^*}) + C(p)|_{p=q^*} = 0$$

Step 4

We now show that u_{q^*} is the optimal value function for the manager, that is, the policy of choosing strategy 1 for $p \le q^*$ and strategy 2 for $p > q^*$ is optimal for the manager among his entire set of feasible policies. By the result of **Proposition 1,** we need to show that

(A42)
$$L^2(u_{q^*}) + C(p) \le 0; p < q^*$$
$$L^1(u_{q^*}) + C(p) \le 0; p \ge q^*$$

By (A30), we may express u_{q^*} as follows:

(A43)
$$\begin{aligned} u_{q^*}(p) &= A_{q^*} p^{h_2^-} + \frac{C_0 - K}{b} + \frac{p}{b - m_2};\; p \ge q^* \\ &= B_{q^*} p^{h_1^+} + C_{q^*} p^{h_1^-} + \frac{C_0 - K}{b} + \frac{p}{b - m_1};\; K \le p < q^* \\ &= D_{q^*} p^{h_1^+} + \frac{C_0 - K}{b};\; p < K \end{aligned}$$

By the discussion following (A6), $D_{q^*} > 0$. Therefore,

(A44)
$$L^2(u_{q^*}) + C(p) = D_{q^*} \frac{1}{2} s_2^{\,2} (h_1^+ - h_2^-)(h_1^+ - h_2^+) p^{h_1^+} < 0; p \le K$$

since $h_2^- < h_1^- < h_1^+ < h_2^+$ by hypothesis. We now note that

(A45)
$$L^1(u_{q^*}) + C(p) = A_{q^*} \frac{1}{2} s_1^{\,2} (h_2^- - h_1^-)(h_2^- - h_1^+) p^{h_2^-} + \frac{(m_1 - m_2)p}{b - m_2}; p > q^*$$

By the definition of q^*,

(A46)
$$L^1(u_{q^*}) + C(p)|_{p=q^*} = 0 .$$

Since $m_2 > m_1$ and $h_2^- < h_1^- < h_1^+ < h_2^+$, (A45) implies that (A46) is true only if

(A47)
$$A_{q^*} > 0 .$$

This implies that the right hand side of (A45) is a *decreasing* function of p for $p > q^*$. Hence, (A41) implies that

(A48) $$L^1(u_{q^*}) + C(p) < 0; p > q^*$$

It remains to show that

(A49) $$L^2(u_{q^*}) + C(p) \le 0; K \le p < q^*$$

From (A43), we therefore need to show that

(A50) $$\begin{aligned} L^2(u_{q^*}) + C(p) = B_{q^*}\frac{1}{2}s_2{}^2(h_1^+ - h_2^-)(h_1^+ - h_2^+)p^{h_1^+} + \\ C_{q^*}\frac{1}{2}s_2{}^2(h_1^- - h_2^-)(h_1^- - h_2^+)p^{h_1^-} + \frac{(m_2 - m_1)p}{b - m_1};\ K \le p < q^* \end{aligned}$$

Using the result of *Step 3* that u_{q^*} is twice differentiable at $p = q^*$ and some tedious algebra, we can show that

$$B_{q^*} = \frac{(h_2^- - h_1^-)(h_2^- - 1)}{(h_1^+ - h_1^-)(h_1^+ - 1)} A_{q^*}(q^*)^{h_2^- - h_1^+}$$
$$C_{q^*} = \frac{(h_2^- - h_1^+)(h_2^- - 1)}{(h_1^- - h_1^+)(h_1^- - 1)} A_{q^*}(q^*)^{h_2^- - h_1^-}$$

Since $h_2^- < h_1^- < 1 < h_1^+ < h_2^+$ by hypothesis and $A_{q^*} > 0$ from (A47), the above implies that

(A51) $$B_{q^*} > 0, C_{q^*} > 0$$

(A44) and the definition of q^* together imply that

(A52) $$\begin{aligned} L^2(u_{q^*}) + C(p)|_{p=K+} < 0 \\ L^2(u_{q^*}) + C(p)|_{p=q^*-} = 0 \end{aligned}$$

(A51), (A52) and the functional form of the right hand side of (A50) together imply that

(A53) $$L^2(u_{q^*}) + C(p) \le 0; K \le p < q^*$$

Therefore, the value function u_{q^*} satisfies all the hypotheses of **Proposition 1.** Therefore, the policy of choosing strategy 1 for $p \le q^*$ and strategy 2 for $p > q^*$ is optimal for the manager. It is not difficult to check that the manager's optimization problem is "scale-independent", that is, for any value of K, the corresponding optimal switching point q^* is such that $\frac{q^*}{K}$ is constant. It is therefore easy to see that we may choose K such that the manager's optimal switching point coincides with that of the firm and the corresponding contract is therefore first-best. This completes the proof.

♦

Proof of Proposition 6

By the result of **Lemma 1**, the fact that $m_1 = m_2$ implies that

(A54) $$h_2^- < h_1^- < h_1^+ < h_2^+$$

Fix some $K > 0$. For concreteness and notational convenience, we show that a compensation structure of the following form can achieve first-best (second-best).

(A55) $$\begin{aligned} C(t) &= K; P(t) < K \\ &= P(t) - K; K \le P(t) < gK \\ &= gK; P(t) \ge gK \end{aligned}$$

We consider the set of policies indexed by the parameter q with $K \le q \le gK$ such that the manager chooses the high volatility strategy, that is, strategy 1 for $P(t) \le q$ and the low volatility strategy, that is, strategy 2, for $P(t) > q$. We then show that there exists q^* with $K < q^* < gK$ such the policy of switching at q^* is optimal for the manager.

Step 1. The value function u_q corresponding to the policy of switching at q has the following functional form:

$$(A56)\quad \begin{aligned} u_q(p) &= A_q p^{h_1^+} + \frac{K}{b}; p \le K \\ &= B_q p^{h_1^+} + C_q p^{h_1^-} + \frac{p}{b - m_1} - \frac{K}{b}; K < p \le q \\ &= D_q p^{h_2^+} + E_q p^{h_2^-} + \frac{p}{b - m_2} - \frac{K}{b}; q < p \le gK \\ &= F_q p^{h_2^-} + \frac{gK}{b} \end{aligned}$$

where the coefficients are determined by continuity and differentiability conditions at K, q, gK .

Step 2: The nature of the manager's compensation structure (A55) implies that the value function $u_q(p)$ must lie between $\frac{K}{b}$ that is the value of obtaining the fixed wage K and $\frac{gK}{b}$ that is the value of obtaining the fixed wage $\frac{gK}{b}$. It follows that in (A56), we must have

$$(A57)\quad A_q > 0, F_q < 0$$

We now note that

$$(A58)\quad L^2(u_q) + C(t) = A_q \frac{1}{2} s_2^{\,2} (h_1^+ - h_2^-)(h_1^+ - h_2^+) p^{h_1^+} < 0; p \le K$$

since $A_q > 0$ from (A57) and $h_2^- < h_1^- < h_1^+ < h_2^+$ from (A54). Similarly,

$$(A59)\quad L^1(u_q) + C(t) = F_q \frac{1}{2} s_1^{\,2} (h_2^- - h_1^-)(h_2^- - h_1^+) p^{h_2^-} < 0; p > gK$$

since $F_q < 0$ from (A57) and $h_2^- < h_1^- < h_1^+ < h_2^+$ from (A54). We emphasize that (A58) and (A59) hold for *any* q such that $K \le q \le gK$.

Step 2: We now establish the existence of q^* with $K < q^* < gK$ such that

$$(A60)\quad L^2(u_{q^*}) + C(t)|_{p=q^*-} = L^1(u_{q^*}) + C(t)|_{p=q^*+} = 0$$

By (A58), we see that

(A61) $L^2(u_K) + C(t)|_{p=K-} < 0$.

By (A59),

(A62) $L^1(u_{gK}) + C(t)|_{p=gK+} < 0$.

However, by the definition (A56) of the value function u_{gK}, we see that

(A63) $L^1(u_{gK}) + C(t)|_{p=gK-} = 0$ and

(A64) $L^2(u_{gK}) + C(t)|_{p=gK+} = 0$

Subtracting (A63) from (A62) and using the fact that u_{gK} is differentiable at $p = gK$, we see that

(A65) $$\frac{d^2}{dp^2}u_{gK}|_{p=gK+} - \frac{d^2}{dp^2}u_{gK}|_{p=gK-} < 0$$

(A64) and (A65) now imply that

(A66) $L^2(u_{gK}) + C(t)|_{p=gK-} > 0$

Now, (A61), (A66) , the intermediate value theorem and the continuity of the expression $L^2(u_q) + C(t)|_{p=q-}$ as a function of q imply the existence of q^* satisfying $L^2(u_{q^*}) + C(t)|_{p=q^*-}$. It is easy to show that this implies that $L^1(u_{q^*}) + C(t)|_{p=q^*+} = 0$ thus establishing (A60).

Step 3: Since $L^2(u_{q^*}) + C(t)|_{p=q^*+} = L^1(u_{q^*}) + C(t)|_{p=q^*-} = 0$ by the definition (A56) of the value function u_{q^*}, (A60) implies that u_{q^*} is twice differentiable at $p = q^*$. We now show that q^* is the required "optimal switching point" for the manager. From (A58), (A59) and the result of **Proposition 1**, it suffices to show that

(A67) $$\begin{aligned} &L^2(u_{q^*}) + C(p) \le 0; K < p \le q^* \\ &L^1(u_{q^*}) + C(p) \le 0; q^* < p \le gK \end{aligned}$$

From (A56) and using the fact that $m_1 = m_2$, we see that

(A68)
$$L^2(u_{q^*}) + C(p) = B_{q^*}(\frac{1}{2}s_2{}^2(h_1^+)^2 + (m_2 - \frac{1}{2}s_2{}^2)h_1^+ - b)p^{h_1^+}$$
$$+ C_{q^*}(\frac{1}{2}s_2{}^2(h_1^-)^2 + (m_2 - \frac{1}{2}s_2{}^2)h_1^- - b)p^{h_1^-}; K < p \le q^*$$

From (A58) and (A60), we note that

(A69) $$L^2(u_{q^*}) + C(p)|_{p=K} < 0; L^2(u_{q^*}) + C(p)|_{p=q^*} = 0$$

(A69) and the functional form of the right hand side of (A68) imply that

(A70) $$L^2(u_{q^*}) + C(p) < 0; K < p < q^*$$

We can use similar arguments to show that $L^1(u_{q^*}) + C(p) < 0; q^* < p < gK$. We have therefore shown that the function u_{q^*} satisfies all the hypotheses of **Proposition 1.** It is therefore the optimal value function of the manager and the policy of switching strategies at q^* is optimal. It is not difficult to check that the manager's optimization problem is "scale-independent", that is, for any value of K, the corresponding optimal switching point q^* is such that $\frac{q^*}{K}$ is constant. It is therefore easy to see that we may choose K such that the manager's optimal switching point coincides with that of the firm and the corresponding contract is therefore first best. ♦

References

Abowd, J. and Kaplan, D. [1999]. "Executive Compensation: Six Questions that Need Answering," *The Journal of Economic Perspectives,* **13**(4), 145-168.

Barnea, A., Haugen, R., and Senbet, L. [1985]. *Agency Problems and Financial Contracting*. Prentice-Hall, Englewood Cliffs, N.J.

Brander, J. and Poitevin, M. [1992]. "Managerial Compensation and the Agency Costs of Debt Finance," *Managerial and Decision Economics,* **13**, 55-64.

Eberhart, A. and Senbet, L. [1993]. "Absolute Priority Rule Violations and Risk Incentives for Financially Distressed Firms," *Financial Management* (Autumn), 101-116.

Garen, J. [1994]. "Executive Compensation and Principal-Agent Theory," *Journal of Political Economy,* **102**(6), 1175-1199.

Green, R. [1984]. "Investment Incentives, Debt, and Warrants," *Journal of Financial Economics,* **13,** 115-136.

Haubrich, J. [1994]. "Risk Aversion, Performance Pay, and the Principal-Agent Problem," *Journal of Political Economy,* **102**(2), 258-276.

Hennessy, C. [2001]. "Debt, Warrants, Priority, and the Asset Substitution Problem," *Working Paper.*

John, K. and John, T. [1993]. "Top-Management Compensation and Capital Structure," *Journal of Finance*, **48**(3), 949-974.

Leland, H. [1998]. "Agency Costs, Risk Management, and Capital Structure," *Journal of Finance,* **49**, 1213-1252.

Holmstrom, B. and Milgrom, P. [1987]. "Aggregation and Linearity in the Provision of Intertemporal Incentives," *Econometrica,* **55,** 303-328.

Karatzas, I. and Shreve, S. [1998]. *Methods of Mathematical Finance*, Springer-Verlag.

Murphy, K.J. [1999]. "Executive Compensation," *Handbook of Labor Economics*, Ashenfelter, Orley and Card (Editors), Volume 3.

Ou-Yang, H. [2002]. "Optimal Contracts in a Continuous-Time Delegated Portfolio Management Problem," forthcoming in the *Review of Financial Studies.*

Schattler, H., and Sung, J. [1993]. "The First-Order Approach to the Continuous-Time Principal Agent Problem with Exponential Utility," *Journal of Economic Theory,* **61**, 331-371.