BARRIERS AND OPTIMAL INVESTMENT RULES.[1]

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Abstract

This paper revisits the simplest stochastic investment decision: when to incur a sunk cost in exchange for a random payoff. It shows that the standard real options approach typically yields incorrect decision rules except for reflecting or unattracting barriers. Optimal investment rules are derived for different barriers and illustrated for common stochastic processes. An explicit solution for the perpetual call option with a lower absorbing barrier is also obtained; it shows that the standard perpetual call option overestimates the corresponding investment threshold when uncertainty is high enough. These results have implications for all stochastic investment problems in continuous time.

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1. INTRODUCTION

Barriers are often assumed away in the stochastic investment literature, yet intuitively they should matter. This paper fills this gap for simple investment problems by making three contributions. First, we show how to derive optimal investments rules, based on elementary considerations, when the underlying payoff follows an autonomous diffusion process constrained by a barrier. Second, we derive an analytical solution for the perpetual call option with a limiting lower absorbing barrier. A comparative statics analysis shows that the resulting investment threshold may not increase monotonically with uncertainty and converges to twice the value of the initial investment as uncertainty increases. Finally, we prove that the standard real options approach usually yield incorrect solutions, except in the presence of reflecting or unattracting barriers. These results can easily be extended to more complex investment problems, and they have implications for stochastic problems in continuous time.

When to pay a constant (sunk) amount I for a payoff X that follows an autonomous diffusion process is probably the most basic investment problem. As such, it has already received a lot of attention (e.g., see McDonald and Siegel 1986, Dixit and Pindyck 1994, or Dixit, Pindyck, and Sødal 1999, and the references herein). The conventional wisdom in this context is that increasing uncertainty delays investment and relatively little attention is paid to the presence of a lower barrier. Yet, we intuitively expect to invest more conservatively in the presence of a lower absorbing barrier, which, if reached, makes investing permanently unattractive, than if a lower reflecting barrier allows the investment payoff to rebound and grow larger with volatility.[1]

An absorbing barrier could result, for example, from demand shifts following innovations by competitors (in electronics, pharmaceuticals…), from gradual changes in tastes, from bankruptcy if the investment opportunity consists in exercising a call option to purchase another firm, or from the disappearance of a natural resource (overfishing may permanently depress a fish stock, for example).

Conversely, a reflecting barrier may arise from government imposed price floors (as for some agricultural commodities), or when a resource has residual value from alternate uses. For example, the owner of a vacant urban plot of land can either erect a commercial building if the economy is booming, or build a temporary parking lot if the real estate market is depressed.

A third possibility is a barrier unreachable in finite time (an unattainable barrier). This is the case for barriers at infinity, but finite barriers can also be unattainable; an example is 0 with the geometric Brownian motion (GBM) for the perpetual call option. Unattainable barriers are popular because they tend to simplify the solution of stochastic investment problems. Yet, as we will see, this apparent simplicity is deceptive.

The standard real options approach for tackling this type of simple investment problems is to treat the investment opportunity as a perpetual call option. To find the investment threshold x^*, the value of X at which the investment should be made, three steps are required. First, a Bellman equation for the option value, $F(X)$, is derived, and when possible, a general solution for this second order ordinary differential equation is obtained. Second, a lower boundary condition (typically at 0 for the Geometric Brownian Motion) is applied, so $F(X)$ is known within a constant. This constant is calculated simultaneously with x^* by applying the value matching and smooth pasting conditions. This approach has been extended to different contexts and it is now widely applied in economics (see Dixit and Pindyck 1994 for details and illustrations).

By contrast, we propose an intuitive framework, based on stochastic discount factors, that relies only on elementary considerations. Indeed, the standard real options approach hinges on the smooth-pasting condition, for which the underlying theory is hardly accessible to most economists, as noted in Sødal (1998). In the presence of a lower absorbing barrier, our results imply that the option term does not verify the Bellman equation typically written for it because the Bellman equation does not explicitly account for changes in the probability that the option to invest vanishes if the lower barrier is reached before the investment threshold. Moreover, we argue that an unattainable barrier should be seen as the limit of an attainable barrier, so different solutions are possible depending on the nature of the latter (e.g., it could be reflecting or absorbing). This seems to have been overlooked in the literature and we show that it has important implications for real options, and more generally for the theory of investment under uncertainty in continuous time.

This paper is organized as follows. Section 2 introduces our framework. Section 3 presents results for two widely used stochastic processes. In Section 4, we prove that the standard real options approach does not deal correctly with absorbing or attracting but unattainable barriers. Section 5 concludes.

2. INVESTING WITH BARRIERS

Consider again the simple framework introduced above and suppose that the net present value of the investment (the payoff), X, follows the autonomous diffusion process

$$dX = \mu(X)dt + \sigma(X)dz. \qquad (1)$$

Equation (1) is valid on the open interval (L,R) where $-\infty \leq L < R \leq +\infty$.[2] For convenience, $\mu(.)$, the infinitesimal trend of X, and $\sigma(.)$, the infinitesimal standard deviation, are assumed continuously differentiable on (L,R). Moreover, $\sigma(x)$ is strictly positive on (L,R), and dz is an increment of a standard Wiener process (Dixit and Pindyck 1994). The investor's objective is to maximize the expected present value of net benefits. For simplicity, these benefits are discounted using a constant discount rate ρ.

We first recall some important concepts characterizing barriers that have not received the attention they deserve in economics. We then examine three common possibilities for a lower barrier $\ell \in [L,R)$: first, ℓ could be reflecting, so that X simply rebounds upon reaching ℓ; second, ℓ could be absorbing, which means that X remains stuck at ℓ as soon as it hits ℓ; finally, ℓ cannot be reached in finite time but it may "attract" X. Although our focus is on lower barriers, it is straightforward to generalize our results to upper barriers.

2.1 *Key Concepts*

We present here some essential definitions and properties of barriers without proofs; a formal treatment can be found in Karlin and Taylor (1981, Chapter 15).

Definition 1. A lower barrier $\ell \in [L,R)$ is said to be attracting if there is a non-zero probability that X reaches ℓ before any interior point x. We denote by $p_{x;\ell|y}$ the probability that X reaches ℓ before x starting from y. Conversely, if ℓ is non-attracting, then X is certain to reach any interior point x before ℓ, and thus $p_{\ell;x|y} = 1$.

It is important to note that the attracting property of a barrier holds for all interior points as a result of the requirement that the function $\sigma(.)$ be strictly positive on (L,R) and the definition of $p_{\ell;x|y}$. Indeed, Karlin and Taylor (1981) show that

$$p_{\ell;x|y} = \frac{S(\ell, y)}{S(\ell, x)}, \tag{2}$$

where, for $L<x_1< x_2<R$,

$$S(x_1, x_2) = \int_{x_1}^{x_2} \exp\left[\int_{c_1}^{\xi} \frac{-2\mu(\zeta)}{\sigma^2(\zeta)} d\zeta\right] d\xi. \tag{3}$$

In (3), $c_1 \in (L,R)$ is an arbitrary constant with no influence on the value of $p_{\ell;x|y}$: indeed, changing c_1 is akin to multiplying the numerator and the denominator of (2) by the same number. From (2) and (3), we see that ℓ is attracting if and only if $\lim_{\xi \to \ell+} S(\xi, z)$ is finite for $z \in (\ell, R)$.

Definition 2. For $x \in (\ell, R)$ and $y \in (\ell, x)$, let $E\left(T_{\ell,x|y}\right)$ denote the expected time it takes X to reach either ℓ or x starting from y. A lower barrier ℓ is said to be attainable if and only if $E\left(T_{\ell,x|y}\right) < \infty$. If ℓ is not attainable, it is unattainable.

From Appendix A,

$$E\left(T_{\ell,x|y}\right) = 2\left\{\frac{S(\ell, y)}{S(\ell, x)}\int_{y}^{x} S(\xi, x) m(\xi) d\xi + \frac{S(y, x)}{S(\ell, x)}\int_{\ell}^{y} S(\ell, \xi) m(\xi) d\xi\right\}, \tag{4}$$

where $S(.,.)$ is defined by (3) and

$$m(\xi) = \frac{1}{\sigma^2(\xi)} \exp\left[\int_{c_1}^{\xi} \frac{2\mu(\zeta)}{\sigma^2(\zeta)} d\zeta \right].$$

The constant c_1 also appears in the definition of *S*(.,.), so it does not affect the value of $E\left(T_{\ell,x|y}\right)$. From (4), we see that the choice of $x \in (\ell, R)$ does not determine whether ℓ is attracting or not.

It can be shown that unattainable barriers may or may not be attracting. However, all attainable barriers are attracting. There are therefore three types of barriers: 1) attainable and attracting, which include reflecting and absorbing (also called exit) barriers; 2) unattainable but attracting, such as +∞ for a Brownian motion with a positive trend; and 3) unattainable and unattracting. Let us now examine specific types of barriers in the context of a simple investment problem.

2.2 *Reflecting Barrier*

Let us first suppose that $\ell \in (L, R)$ is reflecting. To capture the impact of the lower barrier on the decision to invest, we rely on stochastic discount factors. While this approach is not new (see for example MacDonald and Siegel 1986 or Dixit, Pyndick, and Sødal 1999), our contribution here is to use stochastic discount factors to formulate simple investment problems in the presence of different types of barriers.[3] Let us thus write the investment problem as:

$$\underset{x}{Max}\, D_{x|y}(x - I), \tag{5}$$

where y=$X(0)$ is the value of X at time 0; $D_{x|y} \equiv E\left(e^{-\rho T_{x|y}}\right)$ is the expected value of the discount factor; and $T_{x|y}$ is the random duration between time 0 and the moment where X first hits x. For future reference, it is important to note that (5) implicitly assumes that x^* is attainable.

It is well known (e.g., see Karlin and Taylor 1981) that $W(y) \equiv D_{x|y}$ verifies the linear, second-order, ordinary differential equation

$$\frac{\sigma^2(y)}{2}\frac{d^2W(y)}{dy^2} + \mu(y)\frac{dW(y)}{dy} - \rho W(y) = 0. \tag{6}$$

We thus need two conditions to fully define $D_{x|y}$. By construction,

$$D_{x|x} = 1. \tag{7}$$

The other condition is linked to the presence of a reflecting barrier at ℓ. To derive it, let us suppose that, at time 0, $X=\ell$. In the neighborhood of ℓ, X behaves as a Brownian motion with infinitesimal mean $\mu(\ell)$ and variance $\sigma^2(\ell)$. Now consider a discrete approximation of the Brownian motion, as in Dixit (1993). By construction, X cannot take a value lower than ℓ, so after a small time increment Δt, X moves up from ℓ by a small increment $\Delta\ell>0$ (i.e., $X(\Delta t)=\ell+\Delta\ell$), where $\Delta\ell \approx \sqrt{\Delta t} \gg \Delta t$. Then, for $x\geq\ell$,

$$D_{x|\ell} = E_0\{\exp(-\rho\int_0^{T_{x|\ell}} d\tau)\} = E_0\{\exp(-\rho\int_0^{\Delta t} d\tau)\exp(-\rho\int_{\Delta t}^{T_{x|\ell}} d\tau)\}$$

$$= [1\text{-}\rho\Delta\text{t}+\text{o}(\Delta\text{t})]E_0\{\exp(-\rho\int_0^{T_{x|\ell+\Delta\ell}} d\tau)\} = [1\text{-}\rho\Delta\text{t}+\text{o}(\Delta\text{t})]D_{x|\ell+\Delta\ell}$$

$$= [1\text{-}\rho\Delta\text{t}+\text{o}(\Delta\text{t})][D_{x|\ell} + \Delta\ell\frac{dD_{x|\ell}}{dx} + o(\Delta\ell)].$$

The transition from the first to the second line above relies on the law of total probability and the

Markov property (Karlin and Taylor 1981). Simplifying, dividing by $\Delta\ell$, and taking $\Delta\ell$ to 0, gives

$$\frac{dD_{x|\ell}}{dx}\Big|_{x=\ell} = 0. \tag{8}$$

Equation (8) generalizes a result derived by Dixit (page 26, 1993) for a GBM.

Let $W_0(y)$ and $W_1(y)$ be two independent solutions of (6). Combining the two boundary conditions (7) and (8) leads to

$$D_{x|y} = \frac{W_1'(\ell)W_0(y) - W_0'(\ell)W_1(y)}{W_1'(\ell)W_0(x) - W_0'(\ell)W_1(x)}. \tag{9}$$

The first order necessary condition for our problem is obtained by writing the first derivative of the objective function at $x=y=x^*$ to avoid any time inconsistency, so that

$$\frac{\partial D_{x|y}}{\partial x}\Big|_{x=y=x^*} (x^* - I) + 1 = 0, \tag{10}$$

since $D_{x^*|x^*} = 1$. We see that, at the optimum, the sum of two marginal changes in the value of the investment equal zero: one comes from the change in the discount factor, and the other results from the change in the net payoff (it equals unity here).

2.3 *Absorbing Barrier*

Let us now assume instead that X remain constant at $\ell<I$ as soon as it hits ℓ, which makes investing permanently uninteresting. The decision maker's objective function should thus reflect the possibility that, starting from y (where $L<\ell<y<x<R$), X may either first reach ℓ, where the investment possibility disappears, or first reach x^* where the investment should take place. With this in mind, to find the investment threshold we need to solve

$$\underset{x}{Max}\, D_{\ell,x|y}\, p_{\ell;x|y}[x-I], \tag{11}$$

where y=$X(0)$ is the value of X at time 0; $D_{\ell,x|y} \equiv E\left(e^{-\rho T_{\ell,x|y}}\right)$ is the expected value of the discount factor; $T_{\ell,x|y}$ denotes the elapsed time between now (X=y) and the moment where X hits either ℓ or x for the first time; and $p_{\ell;x|y}$ is the probability that X first hits x before ℓ starting from y. Along with the presence of $p_{\ell;x|y}$ in Equation (11), the expression of $D_{\ell,x|y}$ reflects a fundamental difference between a stochastic problem with an absorbing barrier and its deterministic counterpart: whereas in the latter we know with certainty whether a variable will reach a threshold, in the former a variable may reach a threshold in some cases and never in others.

It is easy to show that, for $y \in (\ell, x)$, $W(y) = D_{\ell,x|y}$ also verifies (6). By construction, $D_{\ell,x|\ell} = 1$ and $D_{\ell,x|x} = 1$, so the two conditions needed to fully define $W(y)$ are simply

$$W(\ell) = 1 \text{ and } W(x) = 1. \tag{12}$$

Then, if $W_0(y)$ and $W_1(y)$ are two independent solutions of (6) defined over (ℓ,x),

$$D_{\ell,x|y} = \frac{\left[W_1(x) - W_1(\ell)\right]W_0(y) - \left[W_0(x) - W_0(\ell)\right]W_1(y)}{W_1(x)W_0(\ell) - W_0(x)W_1(\ell)}. \tag{13}$$

As for the reflecting case, we write the first order condition at x=y=x^* in order to ensure time consistency. We get

$$\frac{\partial p_{\ell;x|y}}{\partial x}\Big|_{x=y=x^*} (x^* - I) + \frac{\partial D_{\ell,x|y}}{\partial x}\Big|_{x=y=x^*} (x^* - I) + 1 = 0, \tag{14}$$

In addition to the marginal changes in the discount factor and in the value of the net payoff, Equation (14) shows that we also need to account for the marginal change in the probability that the investment opportunity vanishes (if X hits a before x^*); the importance of this point for optimum investment decisions will be illustrated shortly.

2.4 *Unattainable Barriers*

In economics, unattainable barriers often result from simplifying assumptions; a typical example is a barrier at zero for the geometric Brownian motion. It is safer, however, to see an unattainable barrier ℓ as the limit of an attainable barrier.

Indeed, let us first suppose that ℓ is unattracting (and thus unattainable) and that interior points of (ℓ,R) are attainable. We start from an interior barrier (reflecting or attracting, it does not matter here), solve for the discount factor, and take the lower limit towards ℓ. Typically, one of the two independent solutions of (6) goes to infinity at ℓ while the other has a finite limit. Let us suppose here that $W_1(\ell) = \infty$. Then, the first order condition (10) holds with $D_{x|y}$ given by

$$D_{x|y} = \frac{W_1(y)}{W_1(x)}. \tag{15}$$

Conversely, if ℓ is unattainable but attracting and x^* is attainable, the correct first order condition is (14), with $D_{x|y}$ above replacing $D_{\ell,x|y}$.

3. ILLUSTRATIONS

3.1 *Brownian Motion*

To start with, let us assume that X follows the Brownian motion with infinitesimal trend μ>0 and variance σ^2:

$$dX = \mu dt + \sigma dz. \tag{16}$$

Without loss of generality (by simply shifting X) we set a barrier at l=0. The derivation of $E(T_{0,x|y})$ shows that l=0 is attainable and therefore also attracting (see Appendix B). To simplify our notation, it is useful to introduce the dimensionless parameters

$$\lambda \equiv \frac{\mu}{\sigma^2},\ \delta = \frac{\rho}{\mu}.$$

First, let us suppose that l=0 is reflecting. Two independent solutions of (6) are $W_0(x) = e^{x\omega^+}$ and $W_1(x) = e^{x\omega^-}$, where

$$\omega^{\pm} = -\lambda \pm \sqrt{\lambda^2 + 2\lambda\delta}. \tag{17}$$

Clearly, $\omega^+ > 0$ and $\omega^- < 0$. Given $0<y<x$, the expected discount factor here is (see (9))

$$D_{x|y} = \frac{\omega^- e^{y\omega^+} - \omega^+ e^{y\omega^-}}{\omega^- e^{x\omega^+} - \omega^+ e^{x\omega^-}}.$$

As expected, $D_{x|y}$ increases with y (we are closer to the target x) and decreases with x (the target is farther away). Unfortunately, x_r^* (the investment threshold with a reflecting barrier at l=0) cannot be found explicitly, but it can be approximated for small or large values of σ.

When σ is close to 0,

$$x_r^* = I + \frac{1}{\omega^+}\left[1 + e^{-2\lambda x_0^*} + o(e^{-2\lambda x_0^*})\right], \tag{18}$$

where $I + \frac{1}{\omega^+} = x_0^* + \frac{1}{2\lambda} + o(\frac{1}{2\lambda})$ and $x_0^* = I + \frac{1}{\delta}$ is the investment threshold under certainty. x_r^* is thus larger than x_0^* and it increases locally with the square of the volatility parameter since $\frac{1}{2\lambda} = \frac{\sigma^2}{2\mu} > 0$: a little bit of uncertainty helps reach x_r^* faster so it is worth waiting for a slightly higher value of the investment threshold.

When σ is large, x_r^* becomes independent of *I* and it increases linearly with σ:

$$x_r^* \approx \frac{\tilde{z}}{\sqrt{2\lambda\delta}} = \frac{\tilde{z}\sigma}{\sqrt{2\rho}}, \tag{19}$$

where $\tilde{z} \approx 1.20$ is the unique root of $\frac{e^{2z}+1}{e^{2z}-1} - z = 0$.

Now suppose instead that *l=0* is absorbing. From (2), the probability that *X* reaches *x* before 0, starting from $y \in (0, x)$, is

$$p_{0;x|y} = \frac{1-e^{-2\lambda y}}{1-e^{-2\lambda x}},$$

and from (13), the expected discount factor equals

$$D_{0,x|y} = \frac{(e^{x\omega^-}-1)e^{y\omega^+} - (e^{x\omega^+}-1)e^{y\omega^-}}{e^{x\omega^-} - e^{x\omega^+}}.$$

There is again no explicit expression for x_a^*, the investment threshold with an absorbing lower barrier. When σ is small, x_a^* can be approximated by

$$x_a^* = I + \frac{1}{\omega^+}\left[1 - \frac{2\lambda}{\delta} e^{-(\delta+2\lambda)x_0^*} + o(\frac{\lambda}{\delta} e^{-2\lambda x_0^*})\right]. \tag{20}$$

Comparing (18) and (20), $x_a^* \le x_r^*$, but the difference between x_r^* and x_a^* vanishes with the risk that the investment opportunity disappears. When σ is large, however, the impact of the lower absorbing barrier on the decision to invest is clearly apparent since

$$x_a^* \approx \left(\frac{I}{\lambda\delta}\right)^{\frac{1}{3}} = \left(\frac{I\sigma^2}{\rho}\right)^{\frac{1}{3}}. \tag{21}$$

Thus, x_r^* increases with σ much faster than x_a^* (Equation (19)) because a high volatility with a lower absorbing barrier increases the risk that the investment opportunity disappears.

For intermediate values of σ, x_r^* and x_a^* have to be compared numerically (see Figure 1). As expected, it is optimal to invest sooner with an absorbing than with a reflecting lower barrier because of the risk of loosing the opportunity to invest. Moreover, a higher uncertainty magnifies the difference in expected net profits, π. Assume, for example, that I=\$1 and that the initial value of X equals y=0.5. For $\delta = \frac{\rho}{\mu} = 2$, when uncertainty varies from $\frac{1}{\gamma} = \frac{\sigma^2}{\mu} = 0.1$ to $\gamma^{-1} = 100$, π changes from \$0.08 to \$2.40 with a reflecting barrier and from \$0.08 to only \$0.43 for an absorbing barrier. This spread increases when the discount rate decreases, as the present value of future net revenues goes up. Thus, for $\delta = 0.5$, when γ^{-1} varies between 0.1 and 100, π increases from \$0.59 to \$4.86 for a reflecting barrier, but it decreases from \$0.59 to \$0.51 for an absorbing barrier.

3.2 *Geometric Brownian Motion*

Let us now consider the case where X follows the geometric Brownian motion

$$dX = \mu X dt + \sigma X dz, \tag{22}$$

where $\mu > 0$ and σ^2 are respectively the infinitesimal trend and variance parameters.

For convenience, we define the dimensionless parameters

$$\kappa \equiv 1 - \frac{2\mu}{\sigma^2},\ \delta = \frac{\rho}{\mu}.$$

As σ increases from 0^+ to $+\infty$ (holding μ constant), κ varies from $-\infty$ to 1. We require that $\delta > 1$ ($\mu < \rho$) to guarantee the existence of a finite investment threshold.

We focus here is on a barrier at $\ell = 0$, a common assumption in the economics literature. The derivation of $E(T_{\ell,x|y})$ (see Appendix B) shows that while $\ell > 0$ is attainable, $\ell = 0$ is not. To find out if $\ell = 0$ is attracting, we take the limit of $p_{\ell;x|y}$ when $\ell \to 0$ (see (B.8)) and obtain

$$p_{0;x|y} = \begin{cases} 1, & \text{if } \kappa \le 0 \text{ (i.e., if } \sigma \le \sqrt{2\mu}\text{)}, \\ \left(\dfrac{y}{x}\right)^{\kappa}, & \text{if } \kappa \in (0,1) \text{ (i.e., if } \sigma > \sqrt{2\mu}\text{)}. \end{cases} \tag{23}$$

Thus, $\ell = 0$ is only attracting when σ is large enough (i.e., when $\sigma > \sqrt{2\mu}$). This result should not be surprising: indeed, we know from Ito's lemma that $Ln(X)$ follows a Brownian motion with infinitesimal trend $\mu - \frac{\sigma^2}{2}$, so a high enough value of σ^2 makes the lower barrier $-\infty$ attracting (but unattainable of course). This property has implications for the timing of investing in our simple framework.

Let us first suppose that the barrier at 0 is the limit of a reflecting barrier. Taking the limit of $D_{\ell,x|y}$ (see (B.9)) when $\ell \to 0$ leads to

$$D_{x|y} = \left(\frac{y}{x}\right)^{\theta^+}, \tag{24}$$

where

$$\theta^+ = \frac{\kappa}{2} + \sqrt{\frac{\kappa^2}{4} + \delta(1-\kappa)} > 0\,.$$

When we insert (24) into the first order condition (10) and solve for x_r^*, we obtain

$$x_r^* = \frac{\theta^+}{\theta^+ - 1} I. \tag{25}$$

This expression is familiar (c.f. Dixit and Pindyck 1994): whereas the neoclassical investment theory tells us that a firm should invest as soon as $X \geq I$, Equation (25) shows that firms should invest only above a threshold proportional to the cost of investing. A simple comparative statics analysis shows that the ratio $\frac{\theta^+}{\theta^+ - 1}$ increases monotonically to $+\infty$ with σ.

In fact, this result depends on the presence of a reflecting barrier at $\ell = 0+$. Indeed, let us assume instead that $\ell = 0$ is the limit of an absorbing barrier. A simple calculation shows that the expected discount factor is still given by (24). If we introduce the expressions of $p_{0;x|y}$ (Equation (23)) and $D_{x|y}$ (Equation (24)) into the first order condition (Equation (14)), we get (25) for $\kappa < 0$, but for $\kappa \in (0,1)$ (i.e., for $\sigma > \sqrt{2\mu}$), we find instead

$$x_a^* = \frac{\theta^+ + \kappa}{\theta^+ + \kappa - 1} I. \tag{26}$$

A comparative statics analysis shows that, unlike $\frac{\theta^+}{\theta^+ - 1}$, the multiplier $M(\kappa) \equiv \frac{\theta^+(\kappa) + \kappa}{\theta^+(\kappa) + \kappa - 1}$ for δ fixed, does not increase to infinity with σ. In fact, it may not even change monotonically with σ. We need to distinguish three cases.

For $\delta \in (1,2)$, as σ increases, M decreases from $\frac{\sqrt{\delta}}{\sqrt{\delta} - 1} > 1$ to 2. For $\delta \in (2,9)$, M first decreases from $\frac{\sqrt{\delta}}{\sqrt{\delta} - 1} > 1$ to $M(2\delta - 1.5\sqrt{2\delta(\delta - 1)}$ and then increases towards 2. Finally, for $\delta \geq 9$, M increases monotonically with σ towards 2.

The variations of M stem from the opposite behavior of $p_{0;x|y} = \left(\frac{y}{x}\right)^{\kappa}$ and $D_{x|y} = \left(\frac{y}{x}\right)^{\theta^+}$ as functions of σ. Indeed, for $0 < y < x$ fixed, when σ increases $p_{0;x|y}$ decreases because the risk that X hits ℓ=0 goes up, whereas $D_{x|y} = \left(\frac{y}{x}\right)^{\theta^+}$ increases ($\frac{d\theta^+}{d\sigma} = \frac{\theta^+ - \delta}{2\theta^+ - \kappa} \frac{d\kappa}{d\sigma} < 0$) because x can be reached faster. For low values of δ (for a relatively low discount rate), $D_{x|y}$ does not go up as much as $p_{0;x|y}$ goes down, so x_a^* decreases with σ; the risk of loosing the investment opportunity dominates. Conversely, for high values of δ (for a relatively high discount rate), $D_{x|y}$ goes up faster than $p_{0;x|y}$ goes down when σ increases, so x_a^* is now an increasing function of σ; the discount factor effect dominates. Finally, for intermediate values of δ (i.e. for

$\delta \in (2,9)$), we get a bit of both effects. Calculated values of x_r^* and x_a^* for different values of κ and δ are shown on Figure 2.

As for the Brownian motion, the nature of the lower barrier and the level of uncertainty determine expected net profits. Consider again a unit investment (I=\$1) with an initial value for X of y=0.5. For $\delta = \frac{\rho}{\mu} = 10$, when $\kappa = 1 - \frac{2\mu}{\sigma^2}$ increases from 0.0 to 0.98, π varies from \$0.016 to \$0.286 with a reflecting barrier, and from \$0.016 to \$0.052 with an absorbing barrier. For $\delta = 1.5$, when κ varies from 0.0 to 0.98, π increases from \$0.239 to \$0.470 for a reflecting barrier, but it decreases from \$0.239 to \$0.063 for an absorbing barrier.

The drastic change of behavior of the investment threshold for the GBM in the presence of an attracting absorbing barrier has implications for testing empirically the theory of investment under uncertainty. It also shows that an absorbing barrier impacts the investment threshold differently than a jump process that brings the expected payoff (i.e., X here) to zero. Indeed, we recall from McDonald and Siegel (1986) that for a Poisson jump process, we simply need to augment the discount rate by the rate of arrival of jumps.

Finally, these results illustrate that unattainable but attracting barriers can play a pivotal role in the solution of stochastic investment problems. Some results for mean-reverting processes can be found in Saphores (2002).

4. LINKS WITH THE STANDARD REAL OPTIONS APPROACH

Let us now examine how the stochastic discount factor approach described above relates to the standard real options approach. As above, $W_0(y)$ and $W_1(y)$ denote two independent solutions of Equation (6) defined over (ℓ, R).

Proposition 1. With an absorbing barrier at ℓ, the standard real options approach usually does not give the correct stopping value x_a^* because it does not properly account for the possibility that x_a^* may never be reached.

Proof. From Dixit and Pindyck (1994), the value of the option to invest I to get x, denoted by $F(x)$, verifies the Bellman equation (6). It can thus be written

$$F(x) = A_0 W_0(x) + A_1 W_1(x),$$

where A_0 and A_1 are two unknown constants to be determined simultaneously with the investment threshold; we denote it here by $\tilde{x}_a$ to distinguish it from x_a^*, which verifies the first order condition (14). Since ℓ is an absorbing barrier, the option to invest at ℓ is 0 so that

$$A_0 W_0(\ell) + A_1 W_1(\ell) = 0. \quad (27)$$

Moreover, the continuity and smooth pasting conditions at $\tilde{x}_a$ are respectively

$$A_0 W_0(\tilde{x}_a) + A_1 W_1(\tilde{x}_a) = \tilde{x}_a - I, \quad (28)$$

$$A_0 W_0'(\tilde{x}_a) + A_1 W_1'(\tilde{x}_a) = 1. \quad (29)$$

From (27) and (28), we can solve for A_0 and A_1. If we assume that both $W_0(.)$ and $W_1(.)$ intervene in the expression of $F(x)$, $W_0(\ell)W_1(\tilde{x}_a)-W_1(\ell)W_0(\tilde{x}_a)\neq 0$. Inserting the expressions of A_0 and A_1 in the smooth-pasting condition (Equation (29)), we get

$$-\frac{W_0(\ell)W_1^{'}(\tilde{x}_a)-W_1(\ell)W_0^{'}(\tilde{x}_a)}{W_0(\ell)W_1(\tilde{x}_a)-W_1(\ell)W_0(\tilde{x}_a)}(\tilde{x}_a-I)+1=0. \tag{30}$$

By contrast, expanding the first order condition for a lower absorbing barrier (Equation (14)) leads to

$$\begin{aligned}&-\frac{W_0(\ell)W_1^{'}(x_a^*)-W_1(\ell)W_0^{'}(x_a^*)}{W_0(\ell)W_1(x_a^*)-W_1(\ell)W_0(x_a^*)}(x_a^*-I)+1=\\&\left[\frac{W_0^{'}(x_a^*)W_1(x_a^*)-W_1^{'}(x_a^*)W_0(x_a^*)}{W_0(\ell)W_1(x_a^*)-W_1(\ell)W_0(x_a^*)}-\frac{\partial p_{\ell;x|x_0}}{\partial x}|_{x=x_0=x_a^*}\right](x_a^*-I).\end{aligned} \tag{31}$$

Equations (30) and (31) have identical left sides, but their right sides differ. Their solutions are thus different in general. □

To investigate the difference between the two approaches, let us reconsider the Brownian motion illustration above. From the standard real options approach (Equation (30)), the investment threshold, $\tilde{x}_a$, when ℓ=0 is attracting, verifies

$$-\frac{\omega^- e^{\tilde{x}_a\omega^-}-\omega^+ e^{\tilde{x}_a\omega^+}}{e^{\tilde{x}_a\omega^-}-e^{\tilde{x}_a\omega^+}}(\tilde{x}_a-I)+1=0. \tag{32}$$

This equation admits no explicit solution in general but a numerical investigation shows that $\tilde{x}_a \geq x_a^*$. When uncertainty is small, $\tilde{x}_a$ and x_a^* have the same approximation (Equation (20)); when uncertainty is large, however, $\tilde{x}_a$ exceeds x_a^* by the factor $1.5^{1/3}\approx 1.14$ since

$$\tilde{x}_a \approx \left(\frac{3I}{2\lambda\delta}\right)^{\frac{1}{3}} = \left(\frac{3I\sigma^2}{2\rho}\right)^{\frac{1}{3}}. \tag{33}$$

The difference between the two approaches is even clearer when ℓ is attracting but unattainable. As discussed above, let us assume that $\lim_{x\to\ell} W_1(x) = \infty$ while $W_0(\ell)$ is well defined and finite (as for the geometric Brownian motion). Then, the stopping rule with the standard real options approach (Equation (30)) becomes

$$-\frac{W_0^{'}(\tilde{x}_a)}{W_0(\tilde{x}_a)}(\tilde{x}_a - I) + 1 = 0, \tag{34}$$

whereas the first order condition with an absorbing lower barrier (Equation (31)) simplifies to

$$\left[\frac{\partial p_{\ell;x|x_0}}{\partial x}\Big|_{x=x_0=x_a^*} - \frac{W_0^{'}(x_a^*)}{W_0(x_a^*)}\right](x_a^* - I) + 1 = 0. \tag{35}$$

We see that Equations (34) and (35) have the same solution only if ℓ is not attracting. This situation is illustrated by the GBM case, for which ℓ =0 is an unattainable barrier. When $\sigma \le \sqrt{2\mu}$ (or equivalently, when $\kappa \le 0$), the standard real options approach leads to the correct threshold (Equation (25)). However, when uncertainty is large (i.e., for $0 < \kappa < 1$ or equivalently, for $\sigma > \sqrt{2\mu}$), ℓ=0 is attracting and the difference between the real options investment threshold (Equation (25)) and the correct solution (Equation (26)) grows unbounded as σ increases. Hence, we have:

Corollary. When the barrier ℓ is unattainable, the standard real options approach yields the correct investment threshold when ℓ is unattracting, but not when ℓ is attracting.

The difference between $\tilde{x}_a$ and x_a^* thus depends on the stochastic process followed by X and on the level of uncertainty.

Intuitively, the standard real options approach yields an incorrect solution because, unlike the formulation proposed in this paper, the Bellman equation upon which it relies does not explicitly account for the risk that the investment opportunity disappears if the lower barrier is reached before the investment threshold x^*; this possibility is dealt with only through the boundary conditions verified by the option term. In fact, a small change in X also changes non-trivially the probability of reaching an attracting barrier. The option term thus does not verify the Bellman Equation (6) here. This result can be readily extended to the case where a payoff is received upon reaching the lower barrier. Examples include entry/exit problems or the loss of existence value resulting from extinction in resource economics.[4] More generally, the standard real options approach is unlikely to be correct when any payoff is received at the "other end" of the investment threshold.

For a reflecting barrier, however, both approaches give the same result.

Proposition 2. With a reflecting barrier at ℓ, the standard real options approach gives the same investment threshold x_r^* as the stochastic discount factor approach.

Proof. With the standard real options approach, the value of the option to invest, $F(x)$, verifies the Bellman Equation (6). Since ℓ is reflecting, we can use the logic we followed to derive the second boundary condition for a reflecting barrier (Equation (8)) to show that $F^{'}(\ell)=0$. Hence,

$$A_0 W_0^{'}(\ell) + A_1 W_1^{'}(\ell) = 0. \tag{36}$$

The continuity and smooth-pasting conditions (Equations (28) and (29)) are unchanged. Combining (28) and (36), we get

$$A_0 = \frac{-W_1^{'}(\ell)(x_r^* - I)}{W_0^{'}(\ell)W_1(x_r^*) - W_1^{'}(\ell)W_0(x_r^*)},\ A_1 = \frac{W_0^{'}(\ell)(x_r^* - I)}{W_0^{'}(\ell)W_1(x_r^*) - W_1^{'}(\ell)W_0(x_r^*)}.$$

Inserting these expressions into the smooth-pasting condition (29) leads to

$$-\frac{W_0^{'}(\ell)W_1^{'}(x_r^*) - W_1^{'}(\ell)W_0^{'}(x_r^*)}{W_0^{'}(\ell)W_1(x_r^*) - W_1^{'}(\ell)W_0(x_r^*)}(x_r^* - I) + 1 = 0. \tag{37}$$

Alternatively, inserting the expression of the discount term (Equation (9)) into the first order condition (Equation (10)) and rearranging terms also leads to (37). □

Proposition 2 obtains from the formal analogy between the option term $F(y)$ and $D_{x_r^*|y}(x_r^* - I)$. More generally, these results show that, in our simplified framework, the value of the option to invest is simply the net present value of the investment at $y \in (l, x^*)$. When ℓ is reflecting or unattracting, the option term is $D_{x_r^*|y}(x_r^* - I)$; when ℓ is absorbing, the option term is $D_{\ell, x_r^*|y}\, p_{\ell; x_r^*|y}(x_r^* - I)$; and finally, when ℓ is attracting but unattainable, the option term is

$D_{x_r^*|y} p_{\ell;x_r^*|y} (x_r^* - I)$. As expected intuitively, in this context the investor seeks x^* that maximize the value of the option to invest.

5. CONCLUSIONS

While barriers are often assumed away in stochastic investment problems, this paper shows that barriers matter. We provide an intuitive methodology based on stochastic discount factors to derive simple investments rules for autonomous diffusion process in the presence of common types of barriers. Using tools provided in Karlin and Taylor (1981, Chapter 15), this approach can easily be extended to many other investment problems, including for example barriers with more complex payoffs or investments that modify a monetary flow.

By contrast, we prove that, while the standard real options approach works for reflecting or unattracting barriers, it does not yield correct investment thresholds for absorbing or for attracting but unattainable barriers, and it is unlikely to work for more complex barriers. These results are important given the increasing popularity of real options to tackle decision-making problems under uncertainty.

An illustration of our approach shows that investment rules based on the perpetual call option may overestimate the investment threshold when uncertainty is “high enough” because the perpetual call option is the limiting case of (at least) two different investment problems; in fact, with a lower absorbing barrier, the investment threshold may not be a monotonic function of uncertainty and it tends to twice the investment cost as uncertainty goes to infinity.

Future work should consider the impact of barriers for investment opportunities with time limits, and revisit the pricing formulas of financial options.

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APPENDIX A

For $y \in (\ell, x)$, consider calculating the functional

$$W(y) = E_0\left[f\left(\int_0^{T_{\ell,x|y}} g(X(\tau))d\tau \right)\right], \tag{A.1}$$

where $f(.)$ is C^2, $g(.)$ is continuous and bounded, and E_0 denotes the expectation with respect to X given the information available at time t=0.[5] Knowing $W(.)$ enables the calculation of $D_{\ell,x|y}$ or $D_{x|y}$, but also of $E_0\left(T_{\ell,x|y}\right)$ and $Var_0\left(T_{\ell,x|y}\right)$, which can be useful for management purposes.

Using a Taylor expansion, the law of total probability, the Markov property of X, and Equation (1), Karlin and Taylor (1981, pages 202-203) show that $W(.)$ verifies

$$\frac{v^2(y)}{2}\frac{d^2W(y)}{dy^2} + m(y)\frac{dW(y)}{dy} + g(y)E\left[f'\left(\int_0^{T_{\ell,x|y}} g(X(\tau))d\tau \right)\right] = 0, \tag{A.2}$$

with boundary conditions

$$W(\ell) = W(x) = f(0). \tag{A.3}$$

If $f(y)=y$ and $g(y)=1$, then $W(y) = E_0\left(T_{\ell,x|y}\right)$. Integrating the solution twice with $W(\ell)=W(x)=0$ gives (4). Alternatively, choosing $f(y)=y^2$ and $g(y)=1$, helps calculate $Var_0\left(T_{\ell,x|y}\right)$.

Instead, if $f(y)=e^{-\rho y}$ and $g(y)=1$, Equation (A.2) becomes Equation (6). We get $D_{x|y} = E\left(e^{-\rho T_{x|y}}\right)$ with boundary conditions (7) and (8) if ℓ is reflecting, or $D_{\ell,x|y} = E\left(e^{-\rho T_{\ell,x|y}}\right)$ with boundary conditions (12) if ℓ is absorbing.

APPENDIX B

Brownian Motion

Let us first analyze the nature of the barrier at l=0. Let $0 < y \le x$. From (4), the expected time for X to reach either 0 or x for the first time, starting from y, is

$$E(T_{0,x|y}) = \frac{x - y + ye^{-2\lambda x} - xe^{-2\lambda y}}{(1-e^{-2\lambda x})\mu}. \tag{B.1}$$

For $0 \le y \le x$, let $f(y) = E(T_{0,x|y})$. We have: $f(0) = f(x) = 0$, $f^{''}(y) = \frac{-4\lambda^2 xe^{-2\lambda y}}{(1-e^{-2\lambda x})\mu} < 0$, with $f^{'}(0) > 0$ and $f^{'}(x) < 0$. As a result, $f(y)$ is non-negative and finite, so l=*0* is attainable and therefore also attracting.

Properties of x_r^*

Here, the first order necessary condition for x_r^* (Equation (10)) is

$$\frac{\omega^- e^{\omega^+ x} - \omega^+ e^{\omega^- 1x}}{\omega^+ \omega^- (e^{\omega^+ x} - e^{\omega^- x})} = x - I. \tag{B.2}$$

When σ is small, $\omega^{\pm}$ (see Equation (17)) can be approximated by

$$\begin{cases} \omega^+ = \dfrac{\rho}{\mu}\left[1 - \dfrac{\rho}{2\mu^2}\sigma^2 + o(\sigma^3)\right] = \delta\left[1 - \dfrac{\delta}{2\lambda} + o(\sigma^3)\right], \\ \omega^- = \dfrac{-2\mu}{\sigma^2} - \dfrac{\rho}{\mu} + o(\sigma) = -2\lambda - \delta + o(\sigma), \end{cases} \tag{B.3}$$

and we expect x_r^* to be close to $I + \frac{\mu}{\rho}$, the solution of the corresponding deterministic problem.

To find an approximate expression for x_r^*, we insert (B.3) into (B.2), compare the magnitude of different terms, and simplify to find (18).

When σ is large, $\omega^{\pm}$ (see again Equation (17)) is approximately equal to

$$\omega^{\pm} = \pm\frac{\sqrt{2\rho}}{\sigma}\left[1 \mp \frac{\mu}{\sqrt{2\rho}\sigma} + \frac{\mu^2}{4\rho\sigma^2} + o(\frac{1}{\sigma^2})\right]. \tag{B.4}$$

We then assume that $x_r^* = a\sigma + o(\sigma)$, where a is an unknown constant. We plug this approximation and (B.4) into (B.2), take σ to +∞, and find (19).

Properties of x_a^*

The first order necessary condition for an interior solution for x_a^* (Equation (14)) is

$$\frac{(\omega^+ + \omega^-)e^{(\omega^+ + \omega^-)x}}{1 - e^{(\omega^+ + \omega^-)x}}(x - I) + \frac{(\omega^- - \omega^+)e^{(\omega^+ + \omega^-)x} - \omega^- e^{\omega^- x} + \omega^+ e^{\omega^+ x}}{e^{\omega^+ x} - e^{\omega^- x}}(x - I) + 1 = 0. \tag{B.5}$$

When σ is small, we proceed as above: we insert (B.3) into (B.5), compare the magnitude of different terms, and simplify to find (20).

When σ is large, we suppose that x_a^* does not grow at fast as σ, so when we introduce (B.4) into (B.5), compare the magnitude of all the terms and simplify, we obtain (21). Alternatively, we could also derive (21) by verifying that $\sigma^{\frac{2}{3}}$ times a constant verifies (B.5).

Geometric Brownian Motion

We first inquire about the nature of the barrier $\ell \geq 0$. Let $0 < \ell < y \leq x$. From the definition of $E\left(T_{\ell,x|y}\right)$ (Equation (4)),

$$E(T_{\ell,x|y}) = \frac{2}{\kappa^2\sigma^2}\left\{\frac{y^\kappa - \ell^\kappa}{x^\kappa - \ell^\kappa}\left[\left(\frac{x}{y}\right)^\kappa - 1 - \kappa\ln\left(\frac{x}{y}\right)\right] + \frac{x^\kappa - y^\kappa}{x^\kappa - \ell^\kappa}\left[\kappa\ln\left(\frac{y}{l}\right) + \left(\frac{\ell}{y}\right)^\kappa - 1\right]\right\}, \quad \text{(B.6)}$$

for $\kappa \neq 0$, and for $\kappa = 0$,

$$E(T_{\ell,x|y}) = \frac{1}{\sigma^2}\left\{\frac{\ln(y) - \ln(\ell)}{\ln(x) - \ln(\ell)}\ln^2\left(\frac{x}{y}\right) + \frac{\ln(x) - \ln(y)}{\ln(x) - \ln(\ell)}\ln^2\left(\frac{y}{\ell}\right)\right\}. \quad \text{(B.7)}$$

From (B.7), $E\left(T_{\ell,x|y}\right)$ is clearly positive and finite. For $\kappa \neq 0$ (Equation (B.6)), we need to do a bit more work. For $y \in (\ell, x)$, let $g(y) = \left(\frac{x}{y}\right)^\kappa - 1 - \kappa\ln\left(\frac{x}{y}\right)$. We have that $g(x) = 0$ and $g^{'}(y) < 0$ so $g(y) > 0$ on (ℓ, x). Moreover, for $y \in (\ell, x)$, let $h(y) = \kappa\ln\left(\frac{y}{l}\right) + \left(\frac{\ell}{y}\right)^\kappa - 1$. Clearly, $h(\ell) = 0$ and $h^{'}(y) > 0$, so $h(y) > 0$ on (ℓ, x). As a result, $E\left(T_{\ell,x|y}\right)$ for $\kappa \neq 0$ is positive and finite, and $\ell>0$ is attainable. When $\ell \to 0$, however, $E(T_{\ell,x|y}) \to +\infty$ so $\ell=0$ is not attainable.

From Equation (2), we calculate $p_{\ell;x|y}$ to see if $\ell=0$ is attracting. We find

$$p_{\ell;x|y} = \begin{cases} \dfrac{y^\kappa - \ell^\kappa}{x^\kappa - \ell^\kappa}, & \text{if } \kappa \equiv 1 - \dfrac{2\mu}{\sigma^2} \neq 0, \\ \dfrac{Ln(y) - Ln(\ell)}{Ln(x) - Ln(\ell)} & \text{if } \kappa \equiv 1 - \dfrac{2\mu}{\sigma^2} = 0. \end{cases} \quad \text{(B.8)}$$

Taking the limit of $p_{\ell;x|y}$ when $\ell \to 0$ gives (23), so $\ell=0$ is attracting if and only if $\kappa \in (0,1)$.

Let us now derive the discount factors for a barrier at ℓ>0. Two independent solutions of Equation (6) are $W_0(x) = x^{\theta^+}$ and $W_1(x) = x^{\theta^-}$, where

$$\theta^{\pm} = \frac{\kappa}{2} \pm \sqrt{\frac{\kappa^2}{4} + \delta(1-\kappa)}.$$

Clearly, θ^+>0 and θ^-<0. From Equation (9), the discount factor for a reflecting barrier is thus

$$D_{x|y} = \frac{\theta^- \ell^{\theta^-} y^{\theta^+} - \theta^+ \ell^{\theta^+} y^{\theta^-}}{\theta^- \ell^{\theta^-} x^{\theta^+} - \theta^+ \ell^{\theta^+} x^{\theta^-}}. \tag{B.9}$$

If instead ℓ>0 is absorbing, we find (from (13)),

$$D_{\ell,x|y} = \frac{(x^{\theta^-} - \ell^{\theta^-}) y^{\theta^+} - (x^{\theta^+} - \ell^{\theta^+}) y^{\theta^-}}{\ell^{\theta^+} x^{\theta^-} - \ell^{\theta^-} x^{\theta^+}}. \tag{B.10}$$

When we take ℓ to zero in either (B.9) or (B.10), we obtain the expected discount factor in (24).

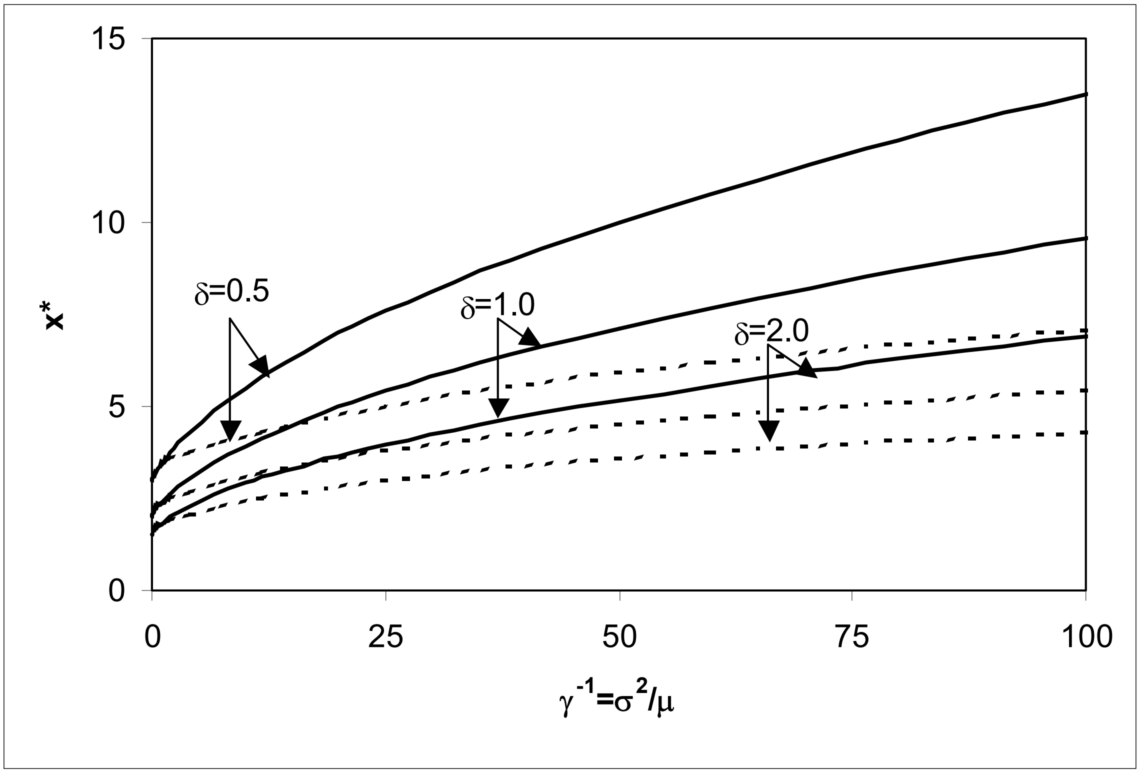

FIGURE 1: x_a^* **and** x_r^* **versus** $\frac{1}{\gamma}=\frac{\sigma^2}{\mu}$ **for the Brownian motion**.

Notes: Solid lines correspond to reflecting barriers and dotted lines to absorbing barriers. $\delta=\frac{\rho}{\mu}$, where ρ is the discount factor and μ is the infinitesimal trend of X. In the definition of γ, σ^2 is the infinitesimal variance of X.

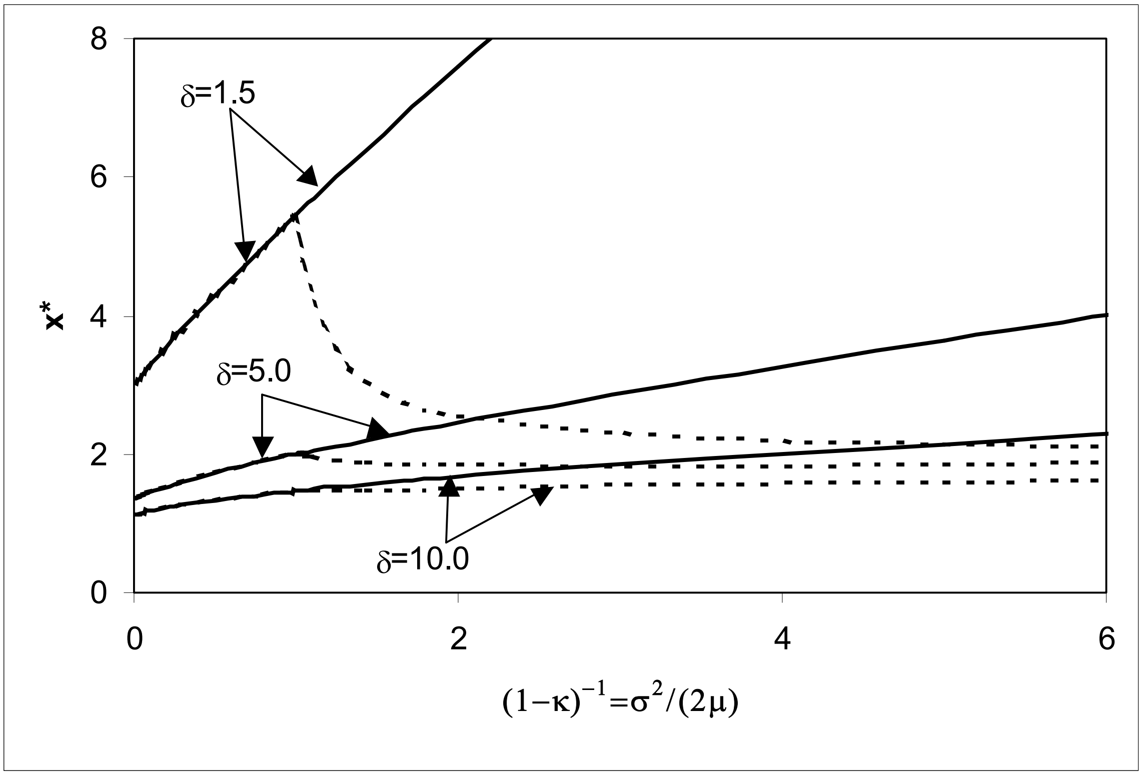

FIGURE 2: x_a^* **and** x_r^* **versus** $\frac{1}{1-\kappa}=\frac{\sigma^2}{2\mu}$ **for the geometric Brownian motion**.

Notes: Solid lines correspond to reflecting barriers and dotted lines to absorbing barriers. $\delta = \frac{\rho}{\mu}$, where ρ is the discount factor and μ is the infinitesimal trend parameter of X. In the definition of κ, σ^2 is the infinitesimal variance parameter of X.

[1] Although these types of barriers are most common in economics, there are other also other types of barriers (e.g., see Dumas 1991 or Dixit 1993).

[2] A parenthesis means that an interval is open at that end, while a square bracket means that it is closed. Thus (a, b] includes b but not a.

[3] Indeed, Dixit, Pyndick, and Sødal (1999) do not discuss the potential nature of barriers on the decision to invest and they impose that $D_{x|y} \to 0$ as x-y→∞. We show in this paper that different conditions apply in general.

[4] With a payoff L at the lower boundary, the decision maker's objective becomes $\underset{x}{Max}\, D_{\ell,x|y}\left(p_{\ell;x|y}[x-I]+p_{x;\ell|y}L\right)$. A first order necessary condition is easily derived.

[5] A C^2 function is twice differentiable and its second derivative is continuous.