" MARKET AND PROCESS RISKS IN PRODUCTION OPPORTUNITIES: DEMAND AND YIELD UNCERTAINTY "

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(Conference Version: February, 2003)
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ABSTRACT

By adopting a real options framework, we develop and analyze a production based valuation model that jointly incorporates process and market risks. Given this setting, techniques of contingent claims analysis and stochastic control theory are employed to obtain value maximizing operating policies in a constrained capacity environment. In our analysis, adjustments to operating policies are analogously modeled as a sequence of complex (real) options whose optimal exercise maximizes their inherent flexibility value.

(I) INTRODUCTION

Contingent Claims methods have now become an industry standard for valuing financial claims. Effectively, these techniques accommodate valuation of a claim whose future payoffs depend on the uncertain prices (or cash flows) of other assets. Contingent Claims Analysis (CCA) methods may can also be adapted for evaluation and analysis of real claims such as production and manufacturing, agricultural, real estate, mining and natural resource based investment projects. For instance, CCA methods can be applied to estimate the flexibility value arising from a multipurpose production facility when demand variability and product substitutability are important factors to the analysis. CCA techniques can be used to characterize the implicit operating risks resulting from exchange rate fluctuations in multinational manufacturing ventures.

In this context, review of the real options literature indicates that the majority of CCA applications implicate production efforts is mining or extraction based projects with output prices as the typical source of uncertainty. Other measures of market risk including exchange rate or demand uncertainty have, though to a lesser extent, also found their way into the current (real options) literature. However, a notable void in this literature concerns the omission of process risk as typified by reliability issues, lead-time uncertainty, system breakdowns or output yield variability. In addition, the application of CCA methodology to a broader set of production-based problems is another missing component of the current efforts in this arena. Specifically, analysis of manufacturing or other production related projects including but not limited to chemical and electronics industries and in the presence of both market and process uncertainty is of tremendous interest. In this vein, an objective is to broaden the scope of their applicability.

In this paper, we consider the problem of valuing production (primarily manufacturing and mining) projects characterized by "market" and "process" uncertainty. In this environment, techniques of Contingent Claims Analysis (CCA) and Stochastic Control theory are used to properly account for the underlying project risk structure and to adequately establish production polices in a manner consistent with a value maximization objective. In our analysis, market uncertainty is defined by demand variability, which in many ways also generalizes the scope of applications[1]. The notion of "process" or "operating" risk is captured by yield uncertainty: which is defined as a random multiplier to the output

[1] Since the output is not a traded commodity and in that sense, it's market risk cannot spanning the market.

quantity, reflecting the usable portion of the output levels which are then sold in competitive markets. Here, variability in the output yield is introduced to allow for the inevitable variations that can arise in the pattern of output quantities typically due to quality or processing reasons. By incorporating output yield as an uncertain factor in our analysis, we can also explicitly allow for the inherent operating options that maybe available to the managers in the more severe cases of yield variability. For example, in a manufacturing setting such an option may be manifest as trigger for system or technology choice alternatives: upgrading, or new facility acquisitions, etc.

As recent literature in this arena reveals, most applications of CCA involve projects with well-defined risk characteristics. Essentially this involves the class of projects whose costs or revenues directly depend on can be linked to the prices of traded assets or commodities, so that data for quantifying their risk is, at least partially, available. For these and other similar type projects, CCA methods can be applied to obtain an arbitrage free valuation model where financial risks maybe fully eliminated through proper hedging in the futures market. This arbitrage valuation framework is attractive since in the absence of priced risk elements the model's complexity in terms of parameter estimates and discount rate derivations is substantially reduced. The use of CCA these methods together with techniques of stochastic control theory for the purposes of performing valuations has a further advantage in that the combined technology also results in optimal production policies.

The application and advantages of a CCA approach to the analysis of real options has been well cited in literature. Brennan and Schwartz (1985) consider production flexibility issues in mining projects with multiple options to open, close and to subsequently abandon the project. In their paper, the notion of market risk is captured through output prices, which are assumed random in nature. Furthermore, the output is also taken to be homogenous in its composition and therefore, not subject to yield variability. In some ways, our paper offers a generalization to their elegant findings, as we account for the random yield (heterogeneity) inherent in the output levels. The general solution, to the classical "duration" problem of the optimal control of a long-term renewable resource is offered by Morck, Schwartz and Stangeland (1990). In their production control model, uncertainty is captured by price as well as the level of inventories. Audreou (1990) provides a model for valuing flexible plant capacity when demand conditions are uncertain whereas He and Pindyck (1992) consider an investment model of flexible production capacity. More recently, Kamrad and Lele (1998) consider the notion of price uncertainty and system failure risk and develop an optimal production and maintenance expenditure policy in light of a warranty on shared failure repair costs. Exchange rate uncertainty and production mode adjustment dynamics are addressed by Kouvelis and Sinha (1994). Related work in the presence of exchange rate risk includes Dasu and Li (1993) who develop optimal operating policies, Huchezermeier and Cohen (1996) addressing operational flexibility concerns for the purposes of strategic global manufacturing, and Kogut and Kulatilaka (1994) who consider production shifts among plants in a network of manufacturing centers. Kamrad and Siddique (2003) also consider supply contract valuation problems in the presence of multiple exchange rates and adjustment options.

Through adopting a CCA framework, we develop a production control model for analysis of manufacturing and mining projects typified by market and process risk. In what follows, market risk is characterized by output demand uncertainty. That is, the uncertainty in the pattern of demand for the output. Here, we derive a model in a general equilibrium context that parallels the findings of Constantinides (1979) and McDonald and Siegel (1985). Our intent for depicting market risk by demand uncertainty is partially triggered by the fact that for a large class of outputs, typically manufactured items, price uncertainty is not a serious risk issue. In addition, this choice also addresses the general non-tradability concern pertinent to manufactured outputs. Process uncertainty, on the other hand, is characterized by output yield and defined as a random multiplier to the output quantity, reflecting the refined or the usable output portions. By incorporating the yield factor into the analysis, we can take into account that in the case of mining projects, some reserves may be less accessible and more costly to extract (i.e. the resource to be exploited is non-homogeneous), therefore, inducing an abandonment option consideration in the more severe situations. In a manufacturing environment, however, yield variability may induce a system replacement or repair an overhaul option if system calibration fails to regulate the yield problem. Through vastly different from the more traditional models of yield variability encountered in the operations and manufacturing literature, our approach maintains the similarity that the yield variable is modeled as an independent multiplier to the output quantity.[2] In this light, we formulate a production control model maximizing the value of the operations in an environment typified by operating options. For this purpose, techniques of stochastic control theory are employed to optimally adjust the rate of production in a manner consistent with a value-maximizing criterion. Given appropriate yet straightforward modifications, the yield variability problem may also be modeled as uncertainty in the input quality (or usability) of the locally supplied inputs in a broader context of a supply chain problem.

[2] See for instance, Gerchak, Parlar and Vickson (1988), Porteus (1986), Lee and Yano (1985).

The paper is organized as follows. In the next section we define the notation, state the necessary assumptions and develop an arbitrage based production control valuation model resulting in a Bellman equation subject to appropriate boundary conditions. We assume that the downside risk of the output's yield is not priced, and therefore does not induce an additional premium. The uncertainty for the output's demand and the uncertainty governing the yield factor are defined as a stochastic processes Since the Bellman valuation equation does not yield an analytic solution, it must be solved numerically to obtain results. Nonetheless, closed form solutions for the optimal production policies are obtained.

In the next section , by invoking the Feynmann-Kac results we numerically solve for the solution along with the resulting optimal production policy to be followed. To that end, a multinomial lattice method offers the needed basis for approximating the stochastic evolution of the state variable, where a dynamic programming approach obtains the solutions numerically. This recursive procedure is then numerically illustrated through a stylized example. Section V captures concluding results.

The contributions of this paper are as follows. First, it introduces a framework for the analysis of production based projects characterized by *both* market and process uncertainty. Second, the paper further extends the current literature findings to a much broader class of production problems. In particular, to the analysis of manufacturing related production control problems where process outputs reflect non-traded assets. In that capacity, this paper also extends an opportunity for further future research in production based industries using a CCA approach. Third, and in light of quite robust numerical results, the models presented in this paper are sufficiently flexible to allow for capturing other sources of market or process uncertainty. Given appropriate adjustments, we can characterize market uncertainty by exchange risk and the yield variability by demand uncertainty.

(II) <u>Assumptions and Model Development</u>

Let $Z_D(t) \in \mathbb{R}$ and $Z_Y(t) \in \mathbb{R}$ *define* standard Brownian motions that are martingales with respect to the probability space, $(\Omega, \text{ö}, \text{Á}, \text{-})$. The filtered probability space, $(\Omega, \text{ö}, \text{Á}, \text{-})$ is defined over the pre-established time interval $[0,t]$ where the augmented filtration, $\text{Á} = \{\text{ö}_t : t \in [0,t]\}$, is right-continuous and increasing. In general, let the process depicting uncertainty be defined by $\{X(t) : t \geq 0\}$, where its sample path is posited by an Ito differential equation of the form:

$$dX(t) = M(X,t)dt + S(X,t)dZ_X(t) \qquad (1)$$

The drift function, $M(X,t)$ denotes the instantaneous change in *X(t).* The volatility function, $S(X,t)$ denotes the standard deviation of the growth rate, and $dZ_X(t)$ is an instantaneous increment to the Brownian motion; $Z(t) \in \mathbb{R}$, defined above.

We now specify the model assumptions, using (1) as a general form. In particular, the output's demand process is depicted by $\{D(t) : t \geq 0\}$ with the specific form of (1):

$$dD(t) = D(t)\{a_D dt + s_D dZ_D(t)\} \qquad D(0) = D_0 > 0 \qquad (2)$$

In the above expression, the constant drift parameter a_D represents the instantaneous expected growth rate in demand; the constant per unit variance of the growth rate is s_D^2 and the Brownian increment is $dZ_D(t)$ which was defined earlier. The demand process captured by (2) implies that the conditional distribution of *D(t)* given *D(s)*, $t > s \in [0,t]$, is lognormal and that *D(t)* > 0 for all $t \in [0,t]$, if $D_0 > 0$. In addition, we assume demand substitutability is not alternative and that there is no back logging. Furthermore, the producer does not stockpile finished products and hence there are no finished goods inventory concerns. This implies that given the available production capacity, the producer aims to meet as much of the demand as possible. To reduce the potential for the overage costs, we implicitly impose a penalty constraint to that effect. We also assume that the producer's actions do not affect the market demand for the output and that the producer is a value maximizer. In the current context, producer's actions are depicted by the rate of production, $q(D, I, t) \equiv q(t),\ t \in [0,t]$ with $q(t) \in (0, Q)$ and where Q defines the current available production capacity. In our set up, $q = \{q(t) : t \in [0,t]\}$ is an adapted positive real-valued process. The flexibility afforded by having the option to revise operating policies in reaction to both market (i.e.,demand) and process (i.e.,yield) uncertainty is value additive and as such is viewed as a sequence of (real) nested options.

Given that there are no finished good inventories in meeting the demand for the output, the producer simply produces at rate $q(t)$ in a manner that maximizes the operating profits. The producer, however, maintains an inventory of needed raw materials from which finished goods are produced. Let $I(t),\ t \in [0,t]$ define time t level of input inventory. Supposing $I(0) = I_0 > 0$ defines the initial known level of the resource, we have,

$$\frac{dI(t)}{dt} = f(q(t)) - q(t) \qquad (3)$$

The function $f(.)$ is used to determine whether or not the resource in question is renewable. For instance, consider the simple functional form, $f(q(t)) = x\,q(t)$ with $0 \leq x \leq 1.0$. In the case where $x = 0$, the resource considered reflects a non-renewable resource and the RHS of equation (3) simply reduces to $-q(t)$. When $x = 1.0$, the situation reflects the case of an instantaneously renewable resource or in other words, an infinite resource case. In all other cases (i.e. $0 < x < 1$), the situation considered represents a partially renewable resource with the RHS of equation (3) becoming $(x-1)q(t)$. In this case, the rate of depletion or extraction is faster than the rate of replenishment. Though not used in the context of this paper, it is also possible for $x > 1$, implying that rate of inventory replenishment is faster than the depletion rate and thereby resulting in excess inventory. While other functional forms for $f(.)$ may be also appropriate, for modeling purposes we adopt the current one: i.e., $f(q(t)) = x\,q(t)$. The production cost function, $K(q(t))$ is assumed to be non-linear, depicting increasing or decreasing marginal cost of producing an additional unit of the output.

The net usable output resulting from production at rate $q(t)$ is defined as $q(t)Y(t)$. The yield variable $\{Y(t),\ t \geq 0\}$ is conceptualized as an independent multiplier to the output rate and is assumed to follow a stochastic process that is also characterized by an Ito differential equation:

$$dY(t) = m(Y,t)\,dt + s_Y(Y,t)\,dZ_Y(t) \qquad Y(0) = Y_0 > 0 \tag{4}$$

Expression (4) fully characterizes the process depending on the choice of the drift function, $m(.)$ and the volatility function, $s_Y(.)$. Furthermore, in light of specific functional forms for $m(.)$ and $s_Y(.)$, and conditional on time $s \in [0,t]$ information, it may be possible to define the probability distribution for $Y(t)$ with $Y(s)$ given, $s < t \in [0,t]$. We defer specifying functional forms for $m(.)$ and $s_Y(.)$ and address this concern in our results' section and in light of a contextually meaningful distribution for $Y(t)$ to follow. We assume the Brownian increments defined earlier are orthogonal. That is,

$$r_{DY}\,dt = E\left(dZ_D(t) \cdot dZ_Y(t)\right) = 0 \tag{5}$$

Let $p(t),\ t \in [0,t]$ define the deterministic output price so that the yield-affected revenue resulting from producing at rate $q(t)$ at time t, is $q(t)Y(t)p(t)$. To develop the valuation model, let $V(D,Y,I,t;q)$ represent the production value at time t given that the demand is $D(t)$, the yield factor is $Y(t)$, the level of input inventory (or remaining untapped resource level) is $I(t)$, and where the production rate is set at $q(t)$. The function $V(D,Y,I,t;q)$ is taken to be Ito differentiable.

As a preliminary to developing our valuation model, we allow for the existence of a financial security that has the same covariance with market return as does demand[1]. Suppose $W(t)$ depicts the price of this traded security at time $t \in [0,t]$ and that the equilibrium growth rate on this financial asset is a_W. We

[1] We will use this condition to arrive at equations (8 and 10).

assume that the instantaneous change in the price level of this security is characterized by the following stochastic differential equation,

$$dW(t) = W(t)a_W dt + W(t)s_W dZ_W(t) \tag{6}$$

Expression (6) defines a geometric Brownian motion, with a_W and s_W reflecting the constant drift (expected rate of return) and volatility (standard deviation of rate of return) parameters. Here, a_W represents the equilibrium rate of return on a financial security having the same covariance with the market return as does the demand. Let a_M and s_M define the instantaneous expected rate of return (drift) and the standard deviation of the rate of return on the market, respectively. The unexpected rate of return component defined by, $s_M dZ_M(t)$, with $Z_M(t) \in \mathbb{R}$ as a standard Brownian motion that is also a martingale with respect to the probability space, $(\Omega, ö, Á, -)$. The constant and riskless rate of return is depicted by r. Employing Merton's (1973) Intertemporal Capital Asset Pricing Model, the market premium on this financial security is given by $l r_{WM} s_W$, which for valuation purposes is equivalent to $l r_{DM} s_D$. Here, r_{WM} and r_{DM} define the instantaneous correlation on returns between the financial security and the market and that of the demand and the market, respectively[2]. By definition,

$$l = \frac{a_M - r}{s_M} \tag{7}$$

Furthermore, let the rate of return shortfall be defined by $y = a_W - a_D$, with y unrestricted in sign. By employing an intertemporal CAPM approach, the equilibrium rate of return on the financial security must reflect an adjustment for systematic risk. In this context, we have:

$$a_W = r + l r_{DM} s_D \tag{8}$$

Recall, by definition, $l r_{WM} s_W = l r_{DM} s_D$[3]. Given this setup, we can obtain $V(.)$ using a replicating portfolio approach. In particular, consider portfolio $G(t)$ consisting of a long position in $V(.)$ together with a short position of d units in security, $W(t)$. The instantaneous change in the value of this portfolio, in light of the necessary cash flow adjustment is,

$$dG(t) = dV(t) - d\,dW(t) + \left\{q(t)Y(t)p(t) - K\left(q(t)\right)\right\}dt \tag{9}$$

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[2] By definition $r_{DM} s_M s_D = r_{WM} s_M s_W$, since by assumption the financial security has the same return covariance with the market as does the demand. Therefore, $r_{DM} s_D = r_{WM} s_W$. Stated more precisely $E(dZ_M(t) \cdot dZ_D(t))s_D = E(dZ_M(t) \cdot dZ_W(t))s_W$. We use this identity to arrive at expression (8) using a CAPM framework.

[3] See also Constantinides (1979) and, McDonald and Siegel (1985). Equation (8) is consistent with the findings of Constantinides (1979) and, McDonald and Siegel (1985).

To ensure the existence of only diversifiable risks, set $d = (D/W)\cdot(\partial V(\cdot)/\partial D(t))$. Absent arbitrage opportunities, this implies that the expected return on this portfolio, $E\left(dG(t)\right)$ should be the risk less rate so that,

$$E\left(dG(t)\right) = rV(t)dt \tag{10}$$

Through applying Ito's lemma to the right hand side of equation (8), taking the resulting expectations, and equating it to expression (9) we obtain the desired Bellman valuation equation. It follows without loss of generality that[3],

$$\underset{q\in[0,Q]}{Max}\left\{\frac{\partial V}{\partial D}D(r-y)+\frac{\partial V}{\partial Y}m(Y,t)+\frac{\partial V}{\partial t}+\frac{1}{2}\frac{\partial^2 V}{\partial D^2}D^2 s_D^2+\frac{1}{2}\frac{\partial^2 V}{\partial Y^2}s_Y^2(Y,t)+q[Yp-\frac{\partial V}{\partial I}(1-x)]-K(q)-rV\right\}=0 \tag{11}$$

s.t.

$$\lim_{Y\to 0} V(D, Y, I, t; q) = 0 \tag{12a}$$

$$\lim_{D\to 0} V(D, Y, I, t; q) = 0 \tag{12b}$$

$$\lim_{D\to\infty} \frac{V(D, Y, I, t; q)}{D} < \infty \tag{12c}$$

$$q(t) \in [0, Q] \tag{12d}$$

$$V(D, Y, 0, t; q) = 0 \tag{12e}$$

$$V(D, Y, I, t; q) = C(I, t) \tag{12f}$$

$$V(D, Y, I, t; q) + P(t) \geq 0 \tag{12g}$$

$$V(D, Y, I, t^*; q) = 0 \text{ where } t^* = inf\left(t : Y(t) = L,\ t \in [0, t]\right) \tag{12h}$$

Equations (12a-h) characterize the constraints to the above Bellman equation (11). The terminal value at the close of the project is defined by function $C(I,t)$ via equation (12f). To account for abandonment as a flexibility option, equation (12g) ensures that the operating value of the production effort exceeds the corresponding abandonment option cost, as reflected by $P(t)$. Several issues regarding equation (11) are noteworthy. In particular, consider the first term in equation (11) reflecting the quantity $(r-y)$. This quantity has effectively replaced the original drift term of the demand process as a result of a replicating strategy barring arbitrage opportunities (see equations (9) and (10)) . In the current context, $(r-y)$ is an equivalent martingale representing the average growth rate for the demand process obtained as a direct consequence of financial risk elimination. Specifically, $(r-y)$ characterizes the "market" adjusted instantaneous growth rate of demand with y unrestricted in its sign and where $y = a_W - a_D$. Recall that a_W is the expected rate of return on a financial asset having the same (financial) risk as the demand variable. When $y < 0$, it implies that the expected growth rate of demand is greater than the equilibrium rate of return on a security (here, proxied by *W*) that has the same financial risk in the market as the demand. In contrast, when $y > 0$, the expected growth rate of demand is less than the aforementioned

[3] Note that changes in the yield uncertainty are assumed independent of the overall economy and therefore, not priced.

equilibrium rate of return[4]. The above discussion and findings are consistent with results obtained by Constantinides (1978) and McDonald and Siegel (1985) whereas in the latter a more detailed and intuitive discussion is also furnished[5]. In light of the above setup and results furnished specifically by equation (11), for valuation purposes the demand process can be depicted by,

$$dD(t) = D(t)\{(r - y)dt + s_D dZ_D(t)\} \qquad D(0) = D_0 > 0 \tag{13}$$

In the following section, closed form results regarding the optimal policies are provided. We note, however, that the value of the project, as characterized by equation (11), must be obtained numerically since the Bellman valuation equation does not yield a closed form expression for obtaining the project's value. In section III.2, we solve for the optimal value using numerical techniques.

(III) Results

This section provides closed form results for the optimal operating policies. Later, numerically obtained results for the project's value are addressed and reviewed. To this end, we assume that the production cost function $K(q(t))$ is non-decreasing (monotone) in the rate of production, $q(t)$. In particular, we assume that the production cost function is quadratic, having the functional form:

$$K(q(t)) = k_0 + k_1 q(t) + k_2 q^2(t) \tag{14}$$

where the monoticity conditions imply that $k_1, k_2 \geq 0$. We further assume that the drift and volatility functions to the yield process are defined by,

$$m(Y, t) = m$$
$$s_Y(Y, t) = s_Y$$

where m and s are constant parameters. That is,

$$dY(t) = m\,dt + s_Y dZ_Y(t) \qquad Y(0) = Y_0 > 0 \tag{15}$$

thereby implying that the distribution of the yield variable $Y(t)$, given $Y(s)$ with $t > s$, and $t, s \in [0, t]$ is normal and that for $Y(0) = Y_0 > 0$, zero is a natural absorbing barrier for $Y(t)$.

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[5] Demand uncertainty can also be depicted via price uncertainty as an alternative. In the current context, if instead of demand uncertainty we were to represent market uncertainty via stochastic prices for the output and if the output is a commodity for which futures contracts are traded then, y would represent the commodity's convenience yield, taken as a constant proportion to the spot price of the commodity. See Brennan and Schwartz (1985).

[6] See also Majd and Pindyck (1987). See also Trigeorgis(1993).

(III.1) Optimal Production Policies

Theorem 1: *Assume that for all* $t \in [0, t]$, *the inventory level* $I(t) > Q$ *is finite. Also assume that the production cost function,* $K(q(t))$ *is an increasing convex function in the production rate,* $q(t)$ *.The optimal production policy* $\{q^*(t),\ q^* \in [0,\ Q]\},\ t \in [0, t]$ *is given by:*

$$q^*(t) = \begin{cases} Q & if \quad Y(t) \geq \overline{Y}(t) \\ \tilde{q}(t) & if \quad Y^*(t) < Y(t) < \overline{Y}(t) \\ 0 & if \quad Y(t) \leq Y^*(t) \end{cases} \tag{16}$$

with

$$\tilde{q}(t) = \frac{p(t)Y(t) - (k_1 + \frac{\partial V(\cdot)}{\partial I(t)}(1-x))}{2k_2} \tag{17}$$

$$\overline{Y}(t) = \frac{2k_2 Q + (k_1 + \frac{\partial V(\cdot)}{\partial I(t)}(1-x))}{p(t)} \tag{18}$$

$$Y^*(t) = \frac{(k_1 + \frac{\partial V(\cdot)}{\partial I(t)}(1-x)) + 2(k_0 k_2)^{1/2}}{p(t)} \tag{19}$$

Proof: See Appendix A

Corollary 1: *Let* $\int_0^t q(t)dt \leq t\,Q < I_0$. *Assume further that* $x = 1$. *In this case, the optimal production policy* $\{q^*(t),\ q^* \in [0, Q]\},\ t \in [0, t]$ *is given by*

$$q^*(t) = \begin{cases} Q & if \quad Y(t) \geq \overline{Y}(t) \\ \tilde{q}(t) & if \quad Y^*(t) < Y(t) < \overline{Y}(t) \\ 0 & if \quad Y(t) \leq Y^*(t) \end{cases} \tag{20}$$

$$\tilde{q}(t) = \frac{pY(t) - k_1}{2k_2} \tag{21}$$

$$\overline{Y}(t) = \frac{2k_2 Q + k_1}{p(t)} \tag{22}$$

$$Y^*(t) = \frac{k_1 + 2(k_0 k_2)^{1/2}}{p(t)} \tag{23}$$

Proof: See Appendix A

The above Corollary applies more appropriately to mining production opportunities wherein the finite resource levels cannot be fully depleted during the venture's life due to typically limited production capacities. As such, the problem may be analogously viewed and managed as an infinite resource case.

<u>Theorem 2:</u> *Let* $k_2 = 0$, and *assume the inventory level,* $I(t),\ t \in [0,t]$ *is finite with* $I(t) > Q$. *Then, the optimal production policy* $\{q^*(t),\ q^* \in [0,\ Q]\},\ t \in [0,t]$ *is given by:*

$$q^*(t) = \begin{cases} Q & \text{if} \quad Y(t) \geq Y^*(t) \\ 0 & \text{if} \quad Y(t) < Y^*(t) \end{cases} \tag{24}$$

with the critical yield factor,

$$Y^* = \frac{\dfrac{\partial V(\cdot)}{\partial I(t)}(1-x) + k_1}{p(t)} \tag{25}$$

<u>Proof:</u> See Appendix A

Equations (24) and (25) characterize a "bang-bang" production policy in that we produce at the maximum feasible rate of production, Q only if the yield factor exceeds the "profit adjusted" variable cost of raw material inventory and production. To offer additional insight, recall that low realizations of the yield factor reflect higher profit losses when compared to higher yield realizations. In effect, when yield adjusted revenues exceed the variable cost of inventory and production, as shown by the numerator of equation (24), it becomes profitable enough to produce at the maximum rate; otherwise, a no production mode is the optimal policy to follow. In the current context, the level of the raw material inventory is considered finite and therefore, the variable inventory cost or more appropriately "shadow price", $\partial V(\cdot)/\partial I(t)$ has a direct bearing on the optimal production policy. As $I(t)$ defines the level of a renewable resource, its shadow price in light of this framework can be interpreted as the holding (or carry) cost rate of the raw materials' inventory, excluding the opportunity cost of capital. The opportunity cost of capital has been indirectly incorporated into the analysis when considering the return shortfall rate, $y = a_W - a_D$.

<u>Corollary 2:</u> *Let* $\int_0^t q(t)dt \leq t\,Q < I_0$. *Assume further that* $x = 1$. *Then, the optimal production policy* $\{q^*(t),\ q^* \in [0,\ Q]\},\ t \in [0,t]$ *is given by*

$$q^*(t) = \begin{cases} Q & \text{if} \quad Y(t) \geq Y^*(t) \\ 0 & \text{if} \quad Y(t) < Y^*(t) \end{cases} \tag{26}$$

with the critical yield factor,

$$Y^*(t) = \frac{k_1}{p(t)} \tag{27}$$

<u>Proof:</u> Follows the proof of Corollary 1 and Theorem 2.

<u>Theorem 3:</u> *Let* $n^* = \underset{q}{Max}[V(D,\ Y,\ I,\ t;\ q)]$ *with* $q(t) \in (0,Q)$. n^* *is unique.*

<u>Proof:</u> See Appendix B

(III.2) Numerical Results

To obtain solutions numerically, one can select from a rich menu of alternatives. For our purposes we adopt a multinomial lattice approach to approximate the stochastic evolution of the demand and the yield processes. To that end, while the stochastic process defining the demand implies that the demand is lognormally distributed , the yield process implies a normal distribution for the yield factor. To employ the intented lattice approach appropriately one of the two processes must be converted to the other so that a multinomial lattice can approximate their joint stochastic evolution over time. A simple log-transformation is the required adjustment and the relationship between these two processes is well established in Karlin and Taylor (1981). For our numerical results, two types of multinomial lattices may be used. Here, we use the 4-jump model of Boyle, Evnine and Gibbs (1989). An alternative approach is the 5-jump lattice model by Kamrad Ritchken (1990). As in section III.1, a backward recursion is used to dynamically superimpose our production control problem on 4-jumps lattice.[7] We control for the upper and lower bound values on the yield process by establishing appropriate barriers at zero and one. Table (2) below depicts the case parameters and functional forms where t = 1.0 year and n = 5 production periods.

Table (2) : Base Case Parameters and Function Coefficients

Production Cost	$K(q)$	k_0 = 100.00; k_1 = 15.00; k_2 = 5.00
Price	p	p = \$300.00 per unit
Initial Inventory	I	I_0 = 20 units
Initial Demand	D_0	D_0 = 10 units
Initial Yield	Y_0	Y_0 = 0.70 per annum
Demand Volatility	s_D	s_D = 0.30 per annum
Yield Volatility	s_Y	s_Y = 0.2 per annum
Interest Rate	r	r = 0.08 per annum
Average Output Yield	m	m = 0.10 per annum
Adjustments to drift shortfall	y	y = 0.03 per annum
Production Capacity	Q	Q = 5 units
Renewable Resource constant	x	$x = 0$
Switching Cost Function	$(q_i - q_{i-1})^2$	
Penalty Cost Function	$5.0(D_i - q_i)^2$	
Salvage Cost Function: $C(I, t_n) = 0.50 I_n Y_n p_n$ given the state at time t_n		

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[7] Due to space limitation the details of lattice set up are omitted.

The effect of increasing production capacity on the value function is shown in Figure-1 which is precisely the anticipated results. Increase in the average yield and its' impact on the project's value is depicted by Figure-2. This obviates the need for further discussion.

[Figure-1] and [Figure-2]

Of interest is the case where increased volatility in the yield process has corresponding increase in the value function. As s_Y increases, so will the upside potential while the downside risks are truncated through a "no-production" option. Thus, on an average basis the project's value improves as the revenues are improved. However, when the volatility of the demand is increased, the project's value diminishes. This too is logical in light of the situation at hand. Note that as s_D increases, it becomes harder to satisfy demand due to limited production capacity and the non-existence of finished goods inventory. Beyond this capacity, any discrepancy between demand realization and production levels are also penalized as constrained in Table (2).

[Figure-3] and [Figure-4]

In Figure-5, the effect of y on the value function is shown. Here, as y increase, the implied average growth rate on the demand process drops. All things being equal, and light of our capacity constraint, it becomes that much easier to meet demand. This reduction in the total operating costs is manifested by a corresponding increase in the project's value.

[Figure-5]

V. CONCLUSIONS

In this paper we have developed an option theoretic (or CCA) model for evaluation and analysis of production efforts characterized by both market and process uncertainty. In this vein, we have modeled the production rate as an adapted positive real-valued process leading to a stochastic control problem embodied by a Bellman optimization equation. Extension of the basic model to a more general setting was also considered where in all cases closed form solutions to the optimal operating policies were provided. In other cases, stylized numerical results provided additional insights to model behavior.

The contributions of this paper are multifold. First, it provides a framework for the analysis of production based projects characterized by *both* market and process uncertainty. In that vein, the initial part of the paper offers a generalization to the existing literature by accounting for the inherent heterogeneity in the output levels. Second, the paper further extends its initial findings to a much broader class of production problems. In particular, to the analysis of manufacturing related production control problems where process outputs reflect non-traded assets. In that capacity, this paper also extends an opportunity for future research in production based industries using a CCA technology. Third, and in light of quite robust numerical results, the models presented in this paper are sufficiently flexible to allow for capturing other sources of market or process uncertainty. Given appropriate adjustments, we can characterize market uncertainty by exchange risk and the yield variability by demand uncertainty to address other production concerns.

Appendix A

Proof of Theorem 1:

We have from equation (11),

$$\frac{\partial V}{\partial D} D(r-y) + \frac{\partial V}{\partial Y} m + \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial D^2} D^2 s_D^2 + \frac{1}{2}\frac{\partial^2 V}{\partial Y^2} s_Y^2 + q(pY - \frac{\partial V}{\partial I}(1-x)) - K(q) = rV \tag{A1}$$

From the above equation (A1) the necessary and sufficient conditions imply that for maximization purposes,

$$p\,Y(t) - k_1 - 2k_2 q - \frac{\partial V(.)}{\partial I(t)}(1-x) = 0 \tag{A2}$$

Soling for the equation's zero results,

$$\tilde{q}(t) = \frac{p\,Y(t) - (k_1 + \frac{\partial V(\cdot)}{\partial I(t)}(1-x))}{2k_2} \tag{A3}$$

at $q(t) = Q$, (A2) obtains,

$$2k_2 Q = p\,Y(t) - k_1 - \frac{\partial V(.)}{\partial I(t)}(1-x) \tag{A4}$$

implying that the minimum yield level to induce production at capacity is,

$$\overline{Y}(t) = \frac{2k_2 Q + (k_1 + \frac{\partial V(\cdot)}{\partial I}(1-x)}{p} \tag{A5}$$

Solving (A1) with $\tilde{q}(t)$ results in

$$Y^*(t) = \frac{(k_1 + \frac{\partial V(\cdot)}{\partial I}(1-x)) + 2(k_0 k_2)^{1/2}}{p} \tag{A6}$$

This completes the proof.

Proof of Corollary 1:

The proof follows from the above in a straightforward manner. Specifically,

$$\int_0^t q(t)dt \le t\,Q < I_0 = \infty \;\Rightarrow\; \frac{\partial V(\cdot)}{\partial I(t)} = 0 \tag{A7}$$

Substituting equation (A7) into equations (17)-(19) obtains the desired results.

Proof of Theorem 2:

From the Bellman equation (11), it follows that $V(D, Y, I, t: q)$ is maximized if $q(t)$ is either zero or at maximum Q since by assumption the production cost function, $K(q(t))$ is linear. Specifically, $q^*(t) = Q$ so long as:

$$p\,Y(t) \geq k_1 + \frac{\partial V(.)}{\partial I(t)}(1-x) \tag{A8}$$

which results in equation (25). However, if the marginal operating revenues are less than the corresponding operating costs then, $q^*(t) = 0$.

<u>APPENDIX B</u>

To prove n^* is unique, we will prove the concavity of the value function $V(\cdot)$ in q. Differentiating Bellman equation (11) successively with respect to $q(t)$ we obtain,

$$r\frac{\partial V(\cdot)}{\partial q} = p\,Y - (\frac{\partial K(q)}{\partial q} + \frac{\partial V(\cdot)}{\partial I}(1-x)) \tag{B1}$$

$$r\frac{\partial^2 V(\cdot)}{\partial q^2} = -\frac{\partial^2 K(q)}{\partial q^2} \tag{B2}$$

The second derivative is negative by definition of $K(q(t))$. Consider (B1) where in perfect competition,

$$p\,Y = \frac{\partial K(q)}{\partial q} + \frac{\partial V(\cdot)}{\partial I}(1-x)$$

and in the case of monopoly,

$$p\,Y > \frac{\partial K(q)}{\partial q} + \frac{\partial V(\cdot)}{\partial I}(1-x)$$

Therefore, the RHS of (B1) is at least zero. Therefore, it can have at most one real maximum, that is n^*.

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Figure - 1
Effect of Average Yield on Project Value

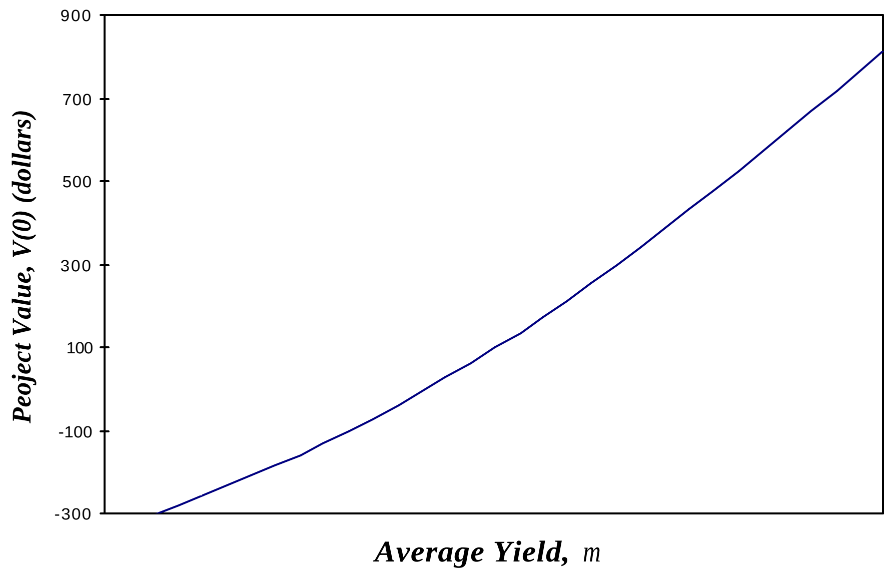
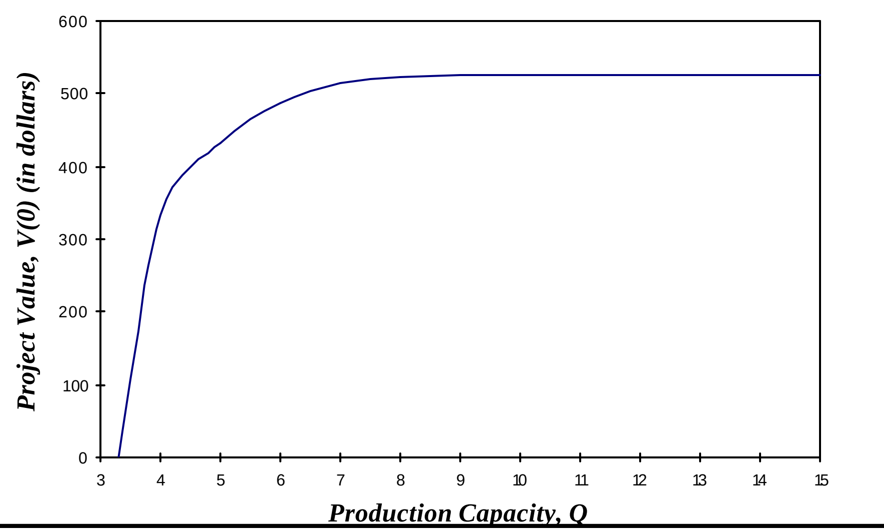


Figure - 2
Effect of Production Capacity on Project Value



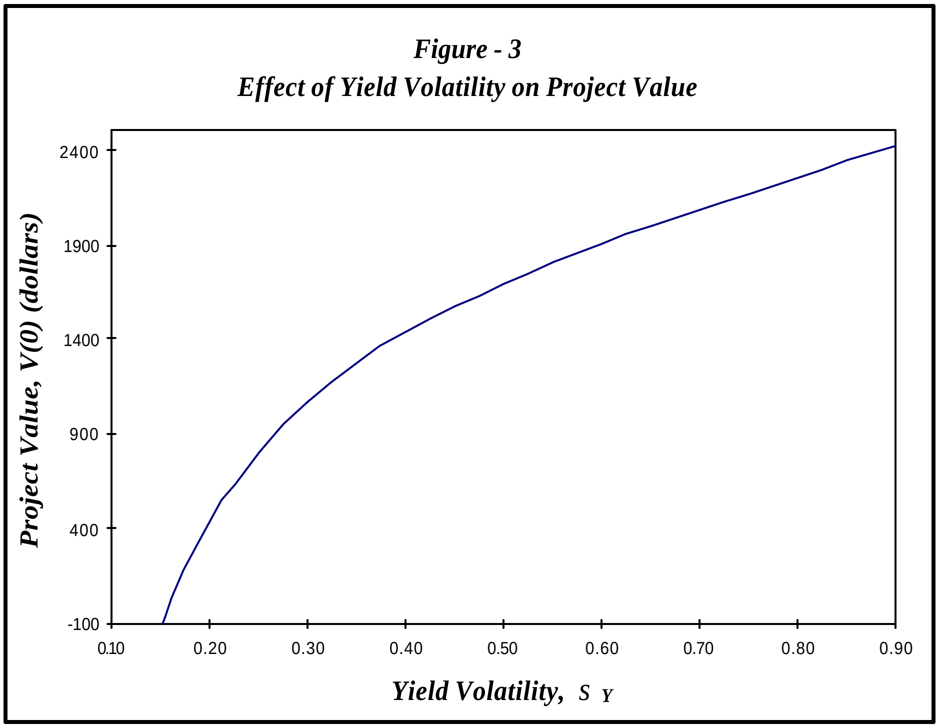


Figure - 3
Effect of Yield Volatility on Project Value
Project Value, V(0) (dollars)
2400
1900
1400
900
400
-100
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
Yield Volatility, S Y

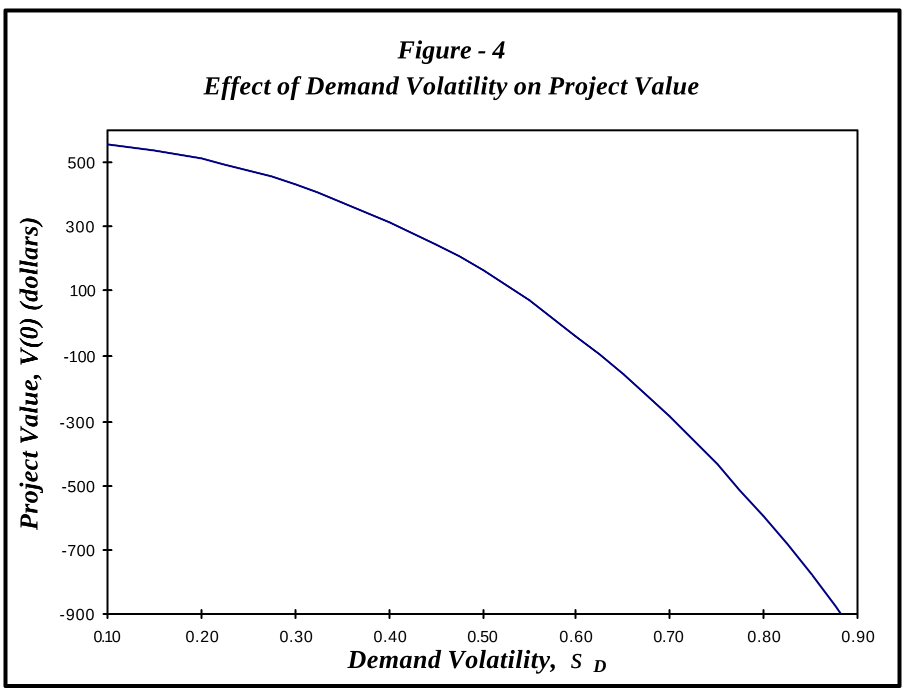


Figure - 4
Effect of Demand Volatility on Project Value
Project Value, V(0) (dollars)
500
300
100
-100
-300
-500
-700
-900
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
Demand Volatility, S D

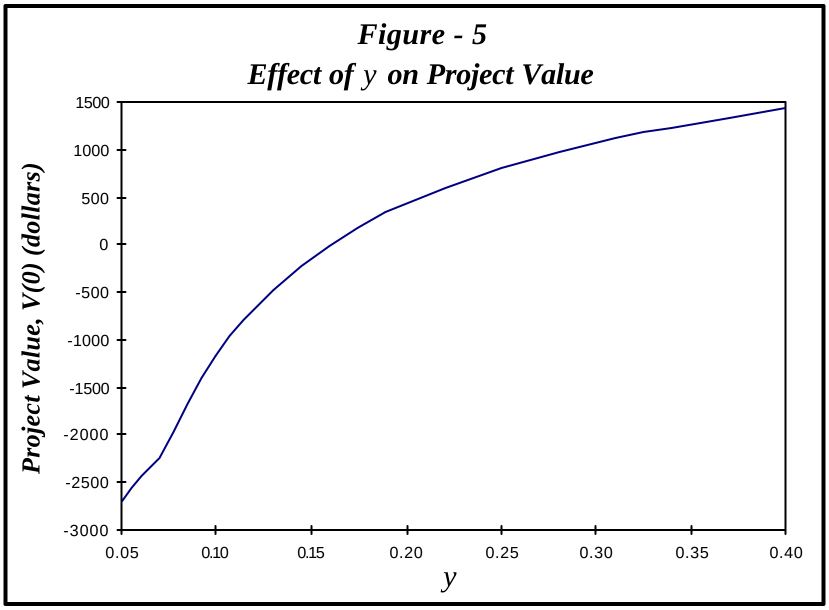


Figure - 5
Effect of y on Project Value
Project Value, V(0) (dollars)
1500
1000
500
0
-500
-1000
-1500
-2000
-2500
-3000
0.05
0.10
0.15
0.20
0.25
0.30
0.35
0.40
y