Option Pricing in the Real World: A Generalized Binomial Model with Applications to Real Options[1]

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Abstract

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JEL Classification: A23, G13.

Keywords: Binomial Option Pricing, Real Options.

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Abstract

We extend a popular binomial model to allow for option pricing using real-world rather than risk-neutral world probabilities. There are three benefits. First, our model allows direct inference about relevant real-world probabilities (e.g., of success in a real-option project, of default on a corporate bond, or of an American-style option finishing in the money). Second, practitioners using our model for corporate real-option applications completely avoid the managerial anxiety that competing risk-neutral models generate when they use risk-free discount rates for risky cash flows. Third, our model simplifies option pricing when higher moments (e.g., skewness and kurtosis) appear in asset pricing models.

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I Introduction

We show how to price options in the "real world" rather than in a risk-neutral world. That is, we demonstrate option pricing without using the change of probability measure required to price in the risk-neutral world. Our method is appealing to researchers and practitioners faced with real-option valuation problems where risk-neutral pricing may only serve to create practical problems on the one hand or conceptual problems on the other. Practical problems with risk-neutral pricing arise when inferred option pricing parameters do not apply to the real world. For example, the probability of success of a real-option project, the probability of default on a corporate bond, the probability that an American-style option will finish in the money, and the likelihood of a jump in a jump process are each different in the real and risk-neutral worlds.[1,2] Similarly, if higher moments (e.g., skewness and kurtosis) play a part in the asset pricing model, then practical problems arise because the variance and higher moments can differ between the real and risk-neutral worlds. On the other hand, conceptual problems arise because it is difficult to understand why we need event probabilities from an economy that does not compensate risk bearing, even though we are pricing assets from a real-world economy that does compensate risk bearing. By performing the real-option analysis using the probability distributions of the real-world economy, we avoid these difficulties—the final answer is, of course, the same (Cox et al. (1985)).

In Section II we derive the model (with details relegated to Appendix A). Section III discusses implications of the model. Section IV discusses risk-neutral versus real-world option pricing when higher moments (e.g., skewness and kurtosis) appear in the asset pricing model. We give a numerical example of a real-option application in Section V. Section V also includes a pedagogical by-product of our model—a simple illustration of why non-option based NPV rules are difficult to implement in real-option settings. Section VI concludes with a summary of our findings and potential future research topics. Appendix A contains mathematical derivations and an extended numerical example.

II The Model

A continuous-time option pricing model under the real-world probability measure requires a stochastic risk-adjusted discount rate; no single risk-adjusted discount rate will do the job.[3] Black and Scholes recognize this with their "instantaneous CAPM" approach to deriving the Black-Scholes PDE (Black and Scholes (1973)). However, the Black-Scholes model that results is difficult to interpret with respect to the real world because the real-world probability measure parameters fall out. Our model is a discretized version of the original Black-Scholes instantaneous CAPM derivation that allows for changing risk-adjusted discount rates. The discretization allows us to infer real-world parameters—an inference not explicitly available in the continuous time limit of the model.

Our model—in both its one-period and multi-period forms—is a direct generalization of the Cox, Ross, and Rubinstein (CRR) binomial option pricing model (Cox et al. (1979)). CRR do not give enough information to price options in the real world. Cox and Rubinstein (1985), however, do give enough information to deduce real-world option pricing (see discussion in our Appendix A.3), but the information is not used explicitly for that purpose. Rather, they present the information to help evaluate option performance in a portfolio theory context (Cox and Rubinstein (1985), p. 185). We take their analysis one step further and generalize their model in the sense that options are priced under any discount rate. Using the risk-free rate employs the probability measure for the risk-neutral economy that yields the CRR model; using the underlying security's actual discount rate employs the probability measure for the actual economy (i.e., the "real world").[4] Like the CRR model, our generalized model prices both European- and American-style options.

Inferring probabilistic information from option prices is not new. Like us, Stutzer (1996) also infers "subjective" (i.e., real world) probability densities from options data. He differs from us, however, in that he uses diffusions instead of binomial trees, he uses historical data which we do not need, and he uses the subjective density to estimate the risk-neutral density for risk-neutral pricing (the focus of his paper), whereas we price in the real-world. Like us, Jackwerth and Rubinstein (1996) infer probabilities from option prices using binomial trees. They differ from us because they use risk-neutral probabilities. They also use nonparametric techniques that require large data sets, whereas our methods require very little data. Ait-Sahalia and Lo (1998) and Jackwerth (2000) also infer probabilities densities from option prices. They differ from us in that they use diffusions rather than binomial trees, they infer risk-neutral densities not real-world ones, and they use nonparametric techniques.

Our model is derived in three different ways in Appendix A: a general proof in Appendix A.1; a certainty equivalent argument related to but slightly different from that in Constantinides (1978) and Bogue and Roll (1974) in Appendix A.2; and a CAPM-based proof using the fact that the Sharpe ratio of a security and the Sharpe ratio of a call option on the security are the same in a one-period binomial model in Appendix A.3 (c.f. Cox and Ross (1976), Equation (15)).[5] A similar argument can be given using Treynor measures (also in Appendix A.3).

Let S_0 and S_T be the underlying asset price at time $t = 0$, and time $t = T$, respectively. Assume the asset pays no dividends,[6] then $R_S \equiv \frac{S_T}{S_0}$ is the total (or "gross") discretely-compounded return on the asset from time 0 to time T (e.g., a realized value of $R_S = 1.15$ indicates 15 percent growth). R_F is similarly the total risk-free rate from time 0 to time T (so $R_F = 1.10$ represents 10 percent growth). Let $SD(\cdot)$ denote standard deviation, $E(\cdot)$ denote expectation under real-world probability, and $E^*(\cdot)$ denote expectation under the risk-neutral measure. Following Cox and Rubinstein (1985), we use "ν" to denote standard deviation of total discretely-compounded return to distinguish it from σ, which we reserve for standard deviation of continuously-compounded returns. Let r_F be the annualized continuously-compounded risk-free rate [so that $e^{r_F T} = R_F$], k_S be the annualized continuously-compounded expected return on the stock (so that $e^{k_S T} = E(R_S)$), and use $u = e^{\sigma\sqrt{T}}$ and $d = e^{-\sigma\sqrt{T}}$ as the one-period multiplicative stock price growth factors as per the CRR specification.[7] With these definitions, Appendices A.1 through A.3 derive the one-period option pricing formulae in Equations (1) and (2).

$$V_0 = \frac{1}{R_F}\left[E(V_T) - \left(\frac{V_u - V_d}{u - d}\right)(E(R_S) - R_F)\right] \tag{1}$$

$$V_0 = e^{-r_F T}\left[E(V_T) - \left(\frac{V_u - V_d}{e^{\sigma\sqrt{T}} - e^{-\sigma\sqrt{T}}}\right)\left(e^{k_S T} - e^{r_F T}\right)\right]. \tag{2}$$

Note that although Equations (1) and (2) involve discounting at the risk-free rate, this is *not* risk-neutral pricing. There is no change of probability measure. The expected cash flow $E(V_T)$ is in the real world, not a risk-neutral world, and it is not directly discounted at the risk-free rate. Rather, the risk-adjusted expected cash flow (i.e., the certainly equivalent) is discounted at the risk-free rate. This risk-adjusted expected cash flow is the real-world expected cash flow less a risk premium.

The relationships we use to derive our option pricing formula hold only for the one-period case of the binomial option pricing model. A multi-stage binomial tree is a set of iterative single-period binomial models. Thus, we may apply our generalized one-period option pricing model (GOPOP) in an iterative manner to create a multi-stage binomial tree that prices American- and European-style options.[8]

The expectation operator $E(\cdot)$ in all of the equations is evaluated under the probability measure that exists in the real-world economy, assuming we have the correct discount rate for the underlying security. As will be seen shortly, the probability measure—and consequently the expectation operator—are dependent on the discount rate assigned to the underlying asset. Technically, many different discount rates produce the correct option price because the probability measure changes endogenously to adjust for the discount rate. If the discount rate is set to the risk-free rate then the model reduces to a risk-neutral option pricing model [set R_S to R_F and k_S to r_F in Equations (1) and (2), and replace $E(\cdot)$ by $E^*(\cdot)$]. As long as the discount rate for the underlying security is the real-world discount rate, we are using the real-world probability measure to value options. Any other discount rate changes the probability measure to that of a different economy (i.e., an economy in which the agents have different risk preferences). However, the option price is correct no matter which discount rate (and its related economy) we are using. We can assume a risk-neutral world to price options correctly but we find it difficult to make inferences about the real-world economy based on the probability density function of prices and returns in this risk-neutral world. Further, we may find it difficult to explain the risk-neutral pricing logic to non-academics.

We now apply the GOPOP model in a multi-stage CRR binomial model form, creating a generalized multi-period option pricing model (GMPOP). The main difference between GMPOP and CRR is the use of the real-world underlying security discount rate in place of the risk-free rate in the assessment of the two probability measures that allow the underlying security price to increase and decrease at a given stage in the binomial tree.[9]

Given all of the assumptions of the CRR model, the real-world probability of a price increase in the one-step binomial model is $p \equiv \left(\frac{e^{k_S T}-d}{u-d}\right)$ where k_S is the continuously-compounded annualized risk-adjusted expected rate of return for the underlying security, T is the proportion of a year for one stage of the binomial tree, and $u = e^{\sigma\sqrt{T}}$ and $d = e^{-\sigma\sqrt{T}}$ are the multiplicative growth factors for one stage of the binomial tree. Allowing S to denote the current price of the underlying security, we develop a five-stage binomial tree in Table I. To value an option on this security, we go to the possible underlying security prices in the terminal period and determine the option value at each one of these security prices. We can consider the option payoffs to be V_{uuuuu}, V_{duuuu}, V_{dduuu}, V_{ddduu}, V_{ddddu}, and V_{ddddd} (where the subscript denotes what has occurred to the security price over the five periods without respect to ordering). We can calculate the option prices in period four recursively. For example, we can find V_{uuuu} using p, $1-p$, R_F, u, d, k, V_{uuuuu}, and V_{duuuu} in the GOPOP model.

$$\begin{aligned} V_{uuuu} &= e^{-r_F T}\Bigg\{ [pV_{uuuuu} + (1-p)V_{duuuu}] - \\ &\left(\frac{V_{uuuuu} - V_{duuuu}}{e^{\sigma\sqrt{T}} - e^{-\sigma\sqrt{T}}}\right)\left(e^{k_S T} - e^{r_F T}\right)\Bigg\}. \end{aligned} \tag{3}$$

More formally, we let "i" be the number of upward price movements and "j" be the number of downward price movements. Then for stage "$i+j$" (where $i+j$ is less than the terminal stage), the option price $V(i,j)$ follows the recursive scheme given in Equation (4).

$$\begin{aligned} V(i,j) &= e^{-r_F T}\Bigg\{ [pV(i+1,j) + (1-p)V(i,j+1)] \\ &- \left(\frac{V(i+1,j) - V(i,j+1)}{e^{\sigma\sqrt{T}} - e^{-\sigma\sqrt{T}}}\right)\left(e^{k_S T} - e^{r_F T}\right)\Bigg\}. \end{aligned} \tag{4}$$

By using the GOPOP model iteratively we generate the GMPOP model shown in Table II.

Again, the probabilities generated for the price movements are from the actual economy and not a risk-neutral economy. If we generate the model using the risk-free rate r_F instead of the underlying security's discount rate, k_S, the GMPOP model becomes risk neutral and is the same as the CRR model.

The GMPOP model in Equation (4) is set up to price a European-style option. However, if at each node we take the maximum between the GOPOP solution $V(i, j)$ and the option's immediate exercise value, the GMPOP model can price American-style options. This added condition at each node is the same condition for pricing American-style options under the CRR model. It follows that our model can give traders real-time, real-world probabilities that individual American-style options will finish in the money. On a Bloomberg terminal, for example, our model builds on Bloomberg's BETA function to get the expected return on the underlying (you can define your own market index proxy and confidence intervals on the estimated beta then flow through to confidence intervals on option value and on probability of success).

We conclude this section with a numerical example (adapted from Hull (1997), pp. 346–347) of a European put and an American put using the GMPOP model. Let the current price for a non-dividend paying stock be \$50, the continuously-compounded annualized risk-free rate be $r_F = 0.10$, the stock's annualized continuously-compounded expected return be $k_S = 0.15$, and the annualized volatility of continuously compounded stock returns be $\sigma = 0.40$. We value the five-month put option with strike price \$50 using a five-stage tree. The European put appears in Table III and the American put is in Table IV. Table VIII is an extended version of Table III with further details of the calculation (see discussion in Appendix A.4). Let us remind the reader that the tree for the underlying is the same whether we use real-world or risk-neutral world valuation. The probabilities and discount rates are, of course, different.

III Some Implications

Note the difference between the real-world economy probability measure of future events and the risk-neutral economy probability measure of future events in Tables III and IV. Given that the underlying security's discount rate is greater than the risk-free rate (i.e., a positive risk premium), risk-neutral valuation takes probability from higher value states and redistributes the probability to lower value states (where "higher" and "lower" refer to the price of the underlying security). The proof of this implication is seen in the construction of the probability of a price increase in the underlying security in a single period of the binomial tree. Compare the price increase probability p under the GOPOP model using the real-world security return to the price increase probability q under the GOPOP model using the risk-free return (i.e., the CRR model).

$$k > r_F \Rightarrow p = \left(\frac{e^{kT} - d}{u - d}\right) > \left(\frac{e^{r_F T} - d}{u - d}\right) = q. \tag{5}$$

The inequality in Equation (5) is reversed if the underlying security's risk premium is negative.

This means that research concerned with parameters inferred from option prices such as tail probabilities (e.g., "value at risk"), jump models, skewness in return distributions, and kurtosis in return distributions is susceptible to error if a risk-neutralized option pricing model is employed (assuming these parameters/measures are desired for the real-world economy). The amount of error is a function of the absolute value of the risk premium for the underlying security.[10]

IV Beyond Mean-Variance

We have discussed several benefits of pricing options under the real-world probability measure. When the goal is merely to price the option, however, then it may seem that the GOPOP/GMPOP model provides no benefit beyond those already provided by risk-neutral valuation, because the pricing is the same. In fact, there is an additional benefit to pricing options under the real-world measure, but it does not become apparent until we move beyond the traditional mean-variance framework used for asset returns in continuous time (e.g., beyond the Black-Scholes world).

If we are working in a Black-Scholes framework (i.e., geometric Brownian motion with constant diffusion coefficient) then the instantaneous variance takes the same value in both the real and risk-neutral worlds and no higher moments matter, so risk-neutral pricing is not difficult to implement. If the only goal is to price the option, then, in this case, there is little incentive to modeling risk premia using the real-world probability measure.

For more than a decade, however, continuous-time option pricing models have incorporated higher-order moments, such as skewness and kurtosis, in the underlying data generating process via stochastic volatility with or without jumps (Hull and White (1987); Scott (1987); Wiggins (1987); Bates (1996a); Bakshi and Chen (1997); Scott (1997)). In this case, the instantaneous variance is not necessarily the same in the real and risk-neutral worlds, and higher order moments can also differ (Cont (1997), Section 6; Madan et al. (1998), Footnote 14).[11] Risk-neutralization of these models is more problematic than in the Black-Scholes world. It depends upon the selection of a utility function rather than incorporating a model for risk premiums for the underlying asset and the option (Bates (1996b), Section 2.1; Jackwerth (2000)).[12]

A second problem with risk-neutral pricing when higher-order moments matter is that the risk-neutral moments must be inferred from, rather than matched to, the associated real-world moments and might depend upon the utility function used. In other words, the risk-neutralized variance, skewness, and kurtosis cannot be calculated directly from real-world returns. This issue is not new.

There is now, however, a movement toward the use of statistical moments beyond the second moment in models of asset returns (e.g., Harvey and Siddique (2000); Leland (1999)). The GOPOP/GMPOP framework allows such asset pricing models to be incorporated directly into the pricing of options. This provides a rich context for option pricing and has distinct advantages over risk-neutral pricing. The primary advantages are the ability to use real-world moments from which our data are generated and to avoid the use of utility functions.

The GOPOP/GMPOP model relies upon only two conditions (the same two conditions it always requires) to be valid with three or more moments in the asset return distribution: (1) the underlying discount rate correctly incorporates compensation for the moments, and (2) the asset pricing process can be generated by a binomial tree (the tree need not necessarily recombine). This result follows from the existence of certainty equivalents and the form of the elasticity of the option price relative to the underlying price in the one-period binomial framework (see Appendix A.1). In other words, given the two conditions mentioned, Equations (1) and (2) hold. It follows that the GOPOP model and, iteratively, the GMPOP model, are both valid.

There are asset pricing models that incorporate skewness (mentioned earlier in this section) and there is a risk-neutral binomial option pricing model that incorporates skewness (Johnson, Pawlukiewicz, and Mehta (1997)). Blending these models under the GOPOP/GMPOP framework can provide an option pricing model in the real world—assuming the real world compensates variance and skewness in the asset return. Once one moves beyond the mean-variance framework, differences between real and risk-neutral world moments imply that the GOPOP/GMPOP models are necessary to be able to incorporate directly information readily available from real-world data (Rubinstein (1998), for example, allows only for risk-neutral higher moments).

V Real Options and Why Discounted Cash Flow Methods Fail

Real-option analysis has many decision tree applications where risk-neutral pricing and risk-free discount rates are at odds with practitioners' "gut instincts." At a minimum, management prefers that a risk-adjusted rate (e.g., the cost of capital) be used as a discount rate. Management may also be interested in the probability of the success of a project being valued as a real option. A risk-neutral option pricing model does not provide the probability of success in the real-world economy. However, the GMPOP model can provide this probability. We demonstrate this using a product development decision tree example.

Suppose a firm is considering entering a particularly volatile product market. Once a product hits the market, it must recoup all of its investment in the initial year because the industry is prone to fads where the product must be different from what customers already own in order to be marketable. If the product is introduced today, it will generate \$200 million in sales, but at a production cost of \$300 million. There are also initial development and design costs to add to the production costs, so entering the market today is certain to lead to financial failure. What if the firm develops the product now, but delays going into production until the market has had the opportunity to expand? The downside to immediate development is that the developmental and design costs must be incurred now; the upside is that patents will be obtained ahead of competitors, thus establishing a toehold. The benefit to delaying production is that the uncertain dollar value of sales in the future has at least the potential to cover the future production costs.

Let us ignore taxes (or assume that all cash flows are calculated after taking account of taxes). Suppose the continuously-compounded expected growth in dollar sales available to the firm in this market is 18 percent per annum, with the standard deviation estimate for the growth rate being 60 percent per annum. The expected growth rate of sales and its associated standard deviation are the return and volatility of the "underlying" in the model. The product market payoffs are an asset available for future purchase if the appropriate development expenditure is made today. Given a 5 percent annual risk-free rate, we need to determine the value of establishing a toehold by developing the product now. If this value exceeds the developmental and design costs, then the project has a positive NPV.

The development of this product provides the right, but not the obligation, to take the product to market in the future. Suppose we restrict ourselves to considering a five-year horizon at which time the costs of going into production will be $400 million. Then development now provides us with a five-year, European-style call option on the annual sales with a strike price of $400 million (we get the sales if we spend the production costs). To value the option, we first model the sales using a five-stage binomial tree (Table V). The tree of possible annual sales levels is displayed using both the real-world and risk-neutral world probability measures.

In Table VI we determine the value of the European call option (i.e., the value of developing the product) using the GMPOP model: \$73.25 million. If development and design expenditures are less than \$73.25 million, then the firm should develop the product. The probability that the product will produce profit is 19.92 percent. Let us emphasize that the GMPOP model in this application requires only forecast growth rate of sales, forecast standard deviation of this growth rate, the risk-free rate, and future production costs—each arguably available to an experienced manager. The sales growth is the "discount rate" consistent with the actual probability measure.

Using the CRR model (and thus the risk-neutral probability measure) produces the same option value. However, the CRR model does not provide the real-world probability that the product will be profitable. By setting our GMPOP model firmly in the real world, we make real-option analysis more lucid to skeptical managers. We also gain probabilistic information about the actual economy.[13]

In addition to comparing the GMPOP model to the CRR model, we can use the GMPOP model to illustrate the advantage of real-option analysis over traditional non-option-based DCF analysis. Traditional DCF analysis generally fails in a real-option based project analysis for one of two reasons:

1. DCF does not model management's ability to exit a project. In essence, traditional DCF assumes the option must be exercised.

2. Even if the DCF analysis is rigged to capture the option-like nature of the project, the appropriate discount rate is not the cost of capital, nor is it the discount rate for the underlying security. Rather, it is the path-dependent stochastic discount rate of the option.

For us to value the product development decision using a traditional DCF, we must take a weighted average of all 32 possible future values (FVs) where negative FVs are set to zero when poor market conditions would lead us to abort production (24 such cases).[14] The discounting varies between each of the six positive FV cases, but all of the cases use single-period option discount rates. The calculation is summarized in Table VII: No particular discount rate is appropriate for each of the cases where the FV is positive. Using real-world probabilities, there does exist a single discount rate that discounts the FVs to give the correct present value, but it is a peculiar non-linear weighted average of the path-dependent discount rates.[15]

One major flaw with the traditional DCF analysis is that we had to perform the real-option analysis to generate the correct discount factors for the DCF valuation. That is, we had to find the real option's value only to re-find it using traditional DCF methods. It is this flaw that makes real-option analysis a better alternative to traditional DCF analysis in the first place. The GOPOP and GMPOP models illustrate this issue clearly.

VI Conclusion and Extensions

We develop generalized one-period and multi-period binomial option pricing models (GOPOP and GMPOP) that can employ different probability measures, including those of the real-world economy and of the risk-neutral world economy. Our model allows parameter inference from the real-world probability density function of the underlying security (e.g., real-world likelihood of success of a real-option project, real-world as opposed to risk-neutral world likelihood of default by a bond issuer, real-world likelihood of bankruptcy in a model of a venture-capital-backed startup, real-world probability that an American-style option finishes in the money). Similarly, our model allows real-world statistical information (e.g., historical or forecast volatility) to be incorporated into option pricing. This is particularly important when higher moments appear in the asset pricing model and variance and higher moments need not be the same in the real and risk-neutral worlds even in continuous time. In addition, our model allows stochastic parameters throughout the option's life, which are here omitted for clarity of presentation.

We give three proofs. The first proof is general enough that the CAPM, APT, or multifactor empirical asset pricing models (e.g., Fama and French (1996); Carhart (1997)) apply. The second proof applies in a general mean-variance framework. The third proof applies only to the CAPM.

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VIII Footnotes

1. The potential future values of a particular security's price affect the value of an option on that security. However, the probability measure of those future values does not, and it is adjustable (Baxter and Rennie (1996), p. 30). If a researcher infers the probability density of the underlying security using implied binomial trees (Rubinstein (1994); Jackwerth and Rubinstein (1996)) then the derived parameter and distributional information may apply only to a risk-neutral economy and not the real-world economy.

2. In rare cases, a simple correction already exists. For example, $N(d_2)$ of the Black-Scholes model (Black and Scholes (1973)) provides the probability that the European call option finishes in the money under the risk-neutral measure and can be adjusted directly to a real-world probability (Baz and Strong (1997))—but most traded options and most real options are American style, so their applications are limited. In other cases, the real-world and risk-neutral world parameters are the same: beta, implied volatility, and the optimal exercise boundary for an American-style option are some that come to mind—at least in

continuous-time models.

3. Other things being equal, as the stock price rises (falls), the degree of leverage implicit within a call option on that stock falls (rises), and thus the risk-adjusted discount rate for the call option also falls (rises). It follows that the risk-adjusted discount rate for the option that applies over the entire life of the option is a path-dependent *random variable* driven by the randomness of the stock price, and no single deterministic number can capture this real-world required rate of return on the option.

4. We make these assertions because, conditional upon the binomial nature of the model, the measure is uniquely determined. We note, however, that the measure is "unique" only up to the structure of the binomial model. Other binomial representations exist and each has a slightly different measure (Tian (1993)). The binomial model is an approximation to an assumed real-world diffusion or pure jump process, and our "real-world" probability measure is, similarly, just such an approximation. Different approximations give slightly different measures.

5. Simple intuition tells you that in a one-period binomial model (or in an

infinitesimal time step in continuous time) the call option as a leveraged investment in the underlying must have the same Sharpe ratio. The leverage simply moves the asset along a straight line in expected return-standard deviation space. For graphical illustration, see Figure 5-1 in Cox and Rubinstein (1985).

6. Dividends—either continuous or lump-sum—can be incorporated quite easily (Jackwerth and Rubinstein (1996); Hull (1997)).

7. Caution must be exercised in translating between discretely and continuously-compounded returns. The Sharpe ratio equality result that drives our model is exactly true only for discretely-compounded returns. If we begin with parameters based on continuously-compounded returns, then they must be translated to their discretely-compounded counterparts before employing this equality. For example, if σ is the standard deviation of annualized continuously-compounded returns on a stock, then $SD(S_T) = p(Su - E(S_T))^2 + (1-p)(Sd - E(S_T))^2$, where $E(S_T) = pSu + (1-p)Sd$, $u = e^{\sigma\sqrt{T}}$, $d = e^{-\sigma\sqrt{T}}$ and $p = \frac{e^{k_S T} - d}{u-d}$.

8. We get a closed-form formula in the limit as step size tends to zero only in the same cases that the risk-neutral version of the model leads

to a closed-form formula—because the pricing is the same. Our work is related to Rubinstein (1976) but differs on two dimensions. First, he gets exact pricing formulae assuming a geometric Brownian motion and discrete trading opportunities, whereas we provide approximations by simplifying to a binomial world. Second, his analysis is demonstrated only in the European-style option case, whereas we demonstrate our analysis for both European- and American-style options. If we restrict attention only to European-style options and let our step size go to zero, then we get the same result as Rubinstein (1976): the Black-Scholes formulae.

9. We can use a stochastic underlying stock return and/or risk-free rate at any given stage within the GMPOP model. We omit this for clarity of presentation.

10. One special property of our model is that the *pricing* is completely robust to the choice of k. That is, any k used leads to the correct derivative price because the measure adjusts endogenously. The real-world inferences, however, like probability of success, are not robust in this fashion and incorrect discount rates lead to incorrect inferences.

11. If prices are driven solely by diffusion processes (e.g., the Black-Scholes world, or a stochastic volatility model without jumps) then instantaneous volatility is the same in both the real and risk-neutral worlds. With stochastic volatility, however, the expected future path of volatility is not necessarily the same in both worlds. If the process is discontinuous, then the instantaneous variance, skewness, and kurtosis can differ between real and risk-neutral worlds. Of course, even in the CRR model the real-world and risk-neutral world variances are not the same in the one-step model. They do, however, coincide in the limit as the step sizes go to zero if the process becomes a continuous-state diffusion (Cox et al. (1979), p. 253).

12. If the jump process is a pure jump (i.e., a Poisson jump process with no diffusion coefficient) and if the jump size is non-random, then the jump can be perfectly hedged without a utility function or any risk adjustment (Cox and Ross (1976)). Similarly, if the jump process is a jump-diffusion with a diversifiable jump, then the non-jump portion of the process can be hedged, and the remainder has risk-free required return and a PDE can be solved (Merton (1976)). Otherwise, either

a utility function or an adjustment for a risk premium is needed. The former is standard practice, but the latter is not—because asset pricing models for the risk premium can create problems with risk-neutral pricing (Bates (1996), p. 571).

13. Conversely, experienced management may have a good estimate of the actual probability of success of a project being valued as a real option. The GMPOP model may be calibrated to this probability and a discount rate for the underlying may be inferred directly. A risk-neutral model has no use for such information and does not allow such "real-world" information to influence valuation.

14. A five-step binomial tree has $2^5 = 32$ paths from the initial node to the possible ending nodes. The payoff to this option is not path-dependent: the future value at a given ending node is the same no matter which path was followed to get there. However, the discount rates in the GMPOP model *are* path-dependent: they differ depending upon the path followed through the tree. Thus, a traditional DCF analysis in this setting must separate the cases path-by-path before discounting.

15. In the case of a European-style option, Cox and Rubinstein (1985,

p. 324) give a closed-form formula for the real-world expected return on the option. A related, but different concept, is Mark Garman's "fugit" of an option (Garman (1989)). This is the risk-neutral expected time until exercise.

16. Cox and Rubinstein (1985, pp. 186–187) give some useful formulae: $\nu_S = \sqrt{p(1-p)(u-d)^2}$, $\nu_C \equiv \sqrt{p(1-p)\left(\frac{C_u - C_d}{C}\right)^2}$.

17. Unfortunately, we cannot produce a closed-form solution even with Normality because of the stochastic risk premium of the option under the real-world probability measure.

18. Although the Sharpe ratio relationships are all stated for European options, the equality of Sharpe ratios in the one-period model also holds for American-style calls. Of course, if early exercise of the American call is optimal at some node and the optimal strategy is followed, then the American call's Sharpe ratio is not defined. This does not, however, have any impact on our pricing technique because we exploit the equality of Sharpe ratios for the *European-style* call when we value the American-style call.

19. Note that nonlinearity of the natural logarithm function implies that the annualized continuously-compounded expected return on an asset is not the same as the expected annualized continuously-compounded return. That is $\frac{1}{T}\ln[E(R_S)] \neq E[\frac{1}{T}\ln(R_S)]$. We use the former (denoted k_S), not the latter.

A Derivations

Appendix A.1 presents a general proof that applies to the CAPM, APT, or any multifactor asset pricing model (e.g., Fama and French (1996); Carhart (1997)). In the latter cases, the asset pricing models are neither equilibrium models nor arbitrage-free. It can be argued, however, that the Fama-French three-factor model accounts for an errors-in-variables problem in the traditional CAPM. As such, the extra factors are simply proxies for measurement error in the market index and are being used to get better estimates of the equilibrium CAPM expected return than a single factor model can give (see Ferguson and Shockley (1999)). Appendix A.2 presents a certainty equivalent proof that works in a general mean-variance framework. Appendix A.3 is a more traditional proof restricted to the CAPM framework. Appendix A.4 gives a numerical illustration.

Each proof uses the following definitions. Let S_0 and S_T be the asset price at time $t = 0$, and time $t = T$, respectively. Assume the asset pays no dividends, then $R_S \equiv \frac{S_T}{S_0}$ is the total (or "gross") discretely-compounded return on the asset from time 0 to time T (e.g., a realized value of $R_S = 1.15$ indicates 15 percent growth). R_F is similarly the total risk-free rate from time

0 to time T (so $R_F = 1.10$ represents 10 percent growth). Let $SD(\cdot)$ denote standard deviation, $E(\cdot)$ denote expectation under real-world probability, and $E^*(\cdot)$ denote expectation under the risk-neutral measure. Let $\nu_S \equiv SD(R_S)$. Define R_C, R_P, ν_C, ν_P similarly for calls and puts. Following Cox and Rubinstein (1985), we have used "ν" to denote standard deviation of total discretely-compounded return to distinguish it from σ which we shall reserve for standard deviation of continuously-compounded returns.[16]

A.1 General Proof

Let V_t be the price of the derivative at time t. Let R_V be the total discretely-compounded return on the derivative from time 0 to time T. Then, under the assumptions of whichever asset pricing model we are using, we get the following.

$$V_0 E(R_V) = E(V_T) \tag{6}$$

$$\Rightarrow V_0[R_F + E(R_V) - R_F] = E(V_T) \tag{7}$$

$$\Rightarrow V_0 = \frac{E(V_T) - V_0[E(R_V) - R_F]}{R_F}. \tag{8}$$

The price elasticity Ω for a call is given by Equation (9).

$$\Omega = \frac{S\Delta}{C} = \frac{S_0(V_u - V_d)}{V_0(u-d)S_0} = \frac{(V_u - V_d)}{V_0(u-d)}, \tag{9}$$

where $\Delta = \frac{(V_u - V_d)}{(u-d)S_0}$, and u and d are the multiplicative up and down growth factors, respectively, for the underlying. In the case of a put, $\Omega = \frac{S\Delta}{P}$, but the end result is algebraically the same as in Equation (9).

Now use the elasticity Ω from Equation (9) to substitute for the derivative's risk premium in Equation (8).

$$V_0 = \frac{E(V_T) - V_0\Omega[E(R_S) - R_F]}{R_F} \tag{10}$$

$$\Rightarrow V_0 = \frac{E(V_T) - \left(\frac{V_u - V_d}{u-d}\right)[E(R_S) - R_F]}{R_F}. \tag{11}$$

Equation (11) is the same as Equation (1) from which our model follows. Thus, given an appropriate asset pricing model and a binomial representation of asset price movement, we can always derive the GOPOP model (and, iteratively, the GMPOP model).

A.2 Certainty Equivalent Proof

This proof uses the properties of mean-variance analysis and certainty equivalents (e.g., Luenberger (1998), Equation 7.7, p. 189). Under the assumptions of whichever mean-variance asset pricing model we are using, Equation (12) holds.

$$V_0 = \frac{1}{R_F}\left[E(V_T) - \frac{\text{cov}(V_T, R_{MKT})[E(R_{MKT}) - R_F]}{\nu_{MKT}^2}\right], \tag{12}$$

where R_{MKT} is the total discretely-compounded return on the market portfolio from time $t = 0$ to time $t = T$. We may derive the following using the properties of the covariance operator:

$$\begin{aligned}
\text{cov}(V_T, R_{MKT}) &= \text{cov}(V_T/V_0, R_{MKT})V_0 = \text{cov}(R_V, R_{MKT})V_0 \\
&= \text{cov}(R_V - R_F, R_{MKT})V_0 \stackrel{*}{=} \text{cov}(\Omega(R_S - R_F), R_{MKT})V_0 \\
&= \text{cov}((R_S - R_F), R_{MKT})\Omega V_0 = \text{cov}(R_S, R_{MKT})\Omega V_0 \\
&\stackrel{**}{=} \text{cov}(R_S, R_{MKT})\left(\frac{V_u - V_d}{u - d}\right),
\end{aligned}$$

where step $*$ uses Cox and Rubinstein (1985, p. 189), and step $**$ uses our Equation (9). If we now plug the latter result into Equation (12), we get

Equation (13).

$$V_0 = \frac{1}{R_F}\left[E(V_T) - \frac{\text{cov}(R_S, R_{MKT})[E(R_{MKT}) - R_F]}{\nu_{MKT}^2}\left(\frac{V_u - V_d}{u - d}\right)\right]. \quad (13)$$

From Cox and Rubinstein (1985, p. 189) or Ingersoll (1987, p. 90) we know that

$$\frac{\text{cov}(R_S, R_{MKT})[E(R_{MKT}) - R_F]}{\nu_{MKT}^2} = E(R_S) - R_F, \quad (14)$$

and plugging this into Equation (13) yields our Equation (1) from which all our analysis may be derived.

A.3 Sharpe Ratio Proof

For this proof we assume the conditions from Ingersoll (1987, Chapter 4) for the mean-variance problem with a risk-free asset. In our binomial setting we therefore need to assume quadratic utility for the agents in the economy because our asset returns are not from an elliptical distribution. We recognize that quadratic utility has some problems (e.g., Constantinides (1978), p. 610). We might argue, however, that although strictly speaking we do need quadratic utility, the binomial model is an approximation to a continuous time model where continuously-compounded returns (which are very close to discretely-compounded ones for small time intervals) are normally distributed.[17] As such, if step sizes are small, and thus the approximation is good, our results will hold approximately for general utility functions. This assumes, of course, that the underlying genuinely follows a diffusion or pure jump process. Note that the CAPM used here assumes that the underlying can be sold short. Thus, this proof may apply only to financial options and not real options.

In a one-period binomial model, Equations (15) through (20) hold from Cox and Rubinstein (1985, Chapter 5-5) altered to our notation.

$$E(R_C - R_F) = \Omega E(R_S - R_F) \tag{15}$$

$$\nu_C = \Omega \nu_S \tag{16}$$

$$\beta_C = \Omega \beta_S \tag{17}$$

$$E(R_P - R_F) = \Omega E(R_S - R_F) \tag{18}$$

$$\nu_P = -\Omega \nu_S \tag{19}$$

$$\beta_P = -\Omega \beta_S \tag{20}$$

where Ω is the elasticity of the option price relative to the current underlying security price (Equation (9)).

The Sharpe ratio relationships for a call option and put option follow from our Equation (9) together with Equations (15) through (20).[18]

$$\frac{E(R_C) - R_F}{\nu_C} = \frac{\Omega(E(R_S) - R_F)}{\Omega \nu_S} = \frac{E(R_S) - R_F}{\nu_S} \tag{21}$$

$$\frac{E(R_P) - R_F}{\nu_C} = -\frac{\Omega(E(R_S) - R_F)}{\Omega \nu_S} = -\frac{E(R_S) - R_F}{\nu_S} \tag{22}$$

The Treynor measure relationships follow similarly.

$$\frac{E(R_C) - R_F}{\beta_C} = \frac{\Omega(E(R_S) - R_F)}{\Omega \beta_S} = \frac{E(R_S) - R_F}{\beta_S} \tag{23}$$

$$\frac{E(R_P) - R_F}{\beta_C} = -\frac{\Omega(E(R_S) - R_F)}{\Omega \beta_S} = -\frac{E(R_S) - R_F}{\beta_S} \tag{24}$$

The Sharpe ratio is usually expressed in terms of rates of return. However, for our derivation we need the Sharpe ratio expressed in terms of asset prices as in Equation (25).

$$\begin{aligned} \text{Sharpe Ratio} &= \frac{E(R_S - 1) - (R_F - 1)}{\nu_S} \\ &= \frac{E(R_S) - R_F}{SD(R_S)} \end{aligned}$$

$$\begin{aligned} &= \frac{E\left(\frac{S_T}{S_0}\right) - R_F}{SD\left(\frac{S_T}{S_0}\right)} \\ \Rightarrow \text{Sharpe Ratio} &= \frac{E(S_T) - S_0 R_F}{SD(S_T)}. \end{aligned} \tag{25}$$

We now set the call option's Sharpe ratio equal to the underlying asset's Sharpe ratio and solve for the call option price C_0 at time $t = 0$ as in Equation (26).

$$C_0 = \frac{\left\{E(C_T) - SD(C_T) \times \left[\frac{E(S_T) - S_0 R_F}{SD(S_T)}\right]\right\}}{R_F}, \tag{26}$$

where C_T is the option price at time $t = T$. In Equation (26), we may substitute for the following terms:

$$SD(C_T) = C_0 \Omega \nu_S = \frac{C_0 (C_u - C_d) \nu_S}{C_0 (u - d)} = \frac{(C_u - C_d) \nu_S}{(u - d)} \tag{27}$$

$$SD(S_T) = S_0 \nu_S. \tag{28}$$

When these are inserted into Equation (26) we get:

$$C_0 = \frac{1}{R_F}\left\{E(C_T) - \left(\frac{C_u - C_d}{u - d}\right)\left[\frac{E(S_T) - S_0 R_F}{S_0}\right]\right\} \tag{29}$$

$$= \frac{1}{R_F}\left[E(C_T) - \left(\frac{C_u - C_d}{u-d}\right)(E(R_S) - R_F)\right]. \quad (30)$$

We can convert Equation (30) to its continuous-time equivalent in Equation (31).

$$C_0 = e^{-r_F T}\left[E(C_T) - \left(\frac{C_u - C_d}{e^{\sigma\sqrt{T}} - e^{-\sigma\sqrt{T}}}\right)\left(e^{k_S T} - e^{r_F T}\right)\right]. \quad (31)$$

The derivation for a put option equates the negative of the put option's Sharpe ratio to the underlying security's Sharpe ratio. Let P denote the put option value, then we arrive at Equation (32) which is similar, but not identical, to Equation (26).

$$P_0 = \frac{\left\{E(P_T) + SD(P_T) \times \left[\frac{E(S_T) - S_0 R_F}{SD(S_T)}\right]\right\}}{R_F}. \quad (32)$$

We may take Equation (28) together with Equation (33)

$$SD(P_T) = -P_0 \Omega \nu_S = -\frac{P_0(P_u - P_d)\nu_S}{P_0(u-d)} = -\frac{(P_u - P_d)\nu_S}{(u-d)} \quad (33)$$

and substitute them into Equation (32) to get Equations (34) and (35) for a put.

$$P_0 = \frac{1}{R_F}\left[E(P_T) - \left(\frac{P_u - P_d}{u-d}\right)(E(R_S) - R_F)\right] \tag{34}$$

$$P_0 = e^{-r_F T}\left[E(P_T) - \left(\frac{P_u - P_d}{e^{\sigma\sqrt{T}} - e^{-\sigma\sqrt{T}}}\right)\left(e^{k_S T} - e^{r_F T}\right)\right]. \tag{35}$$

Equations (34) and (35) for a put are of identical functional form to Equations (30) and (31) for a call, so we rewrite these equations using V to denote the value of an option that can be either a call or a put.

$$V_0 = \frac{1}{R_F}\left[E(V_T) - \left(\frac{V_u - V_d}{u-d}\right)(E(R_S) - R_F)\right] \tag{36}$$

$$V_0 = e^{-r_F T}\left[E(V_T) - \left(\frac{V_u - V_d}{e^{\sigma\sqrt{T}} - e^{-\sigma\sqrt{T}}}\right)\left(e^{k_S T} - e^{r_F T}\right)\right]. \tag{37}$$

Equations (36) and (37) are the same as Equations (1) and (2).

We may equate Equation (36) to $E(V_T)/[1 + k_V^{\text{disc}}]$ and equate Equation (37) to $E(V_T)e^{-k_V^{\text{cont}}T}$ to deduce the option's total required risk-adjusted discretely-compounded discount rate k_V^{disc} and its annualized required risk-adjusted continuously-compounded expected discount rate k_V^{cont} in Equations (38) and (39), respectively.[19]

$$k_V^{\text{disc}} = \frac{R_F}{\left[1 - \frac{(V_u - V_d)(E(R_S) - R_F)}{(u-d)E(V_T)}\right]} \tag{38}$$

$$k_V^{\text{cont}} = r_F - \ln\left[1 - \frac{(V_u - V_d)\left(e^{k_S T} - e^{r_F T}\right)}{\left(e^{\sigma\sqrt{T}} - e^{-\sigma\sqrt{T}}\right) E(V_T)}\right] \times \left[\frac{1}{T}\right]. \tag{39}$$

The GOPOP model may also be derived using the Treynor measure. The Treynor measure for a call option is equal to the Treynor measure for the underlying security. The Treynor measure for a put is the negative of the Treynor measure for the underlying security. Thus, the implication above for the Sharpe ratio in a one-period binomial model setting is also true for the Treynor measure.

Using the Treynor measure, we can develop an equation similar to Equation (26) for a call option.

$$C_0 = \frac{E(C_T) - \frac{C_0 \beta_C}{S_0} \times \left[\frac{E(S_T) - S_0 R_F}{\beta_S}\right]}{R_F}. \tag{40}$$

When we substitute $\Omega\beta_S$ for β_C in Equation (40), we arrive at Equation (30).

A similar relationship holds for the put option as follows:

$$P_0 = \frac{E(P_T) + \frac{P_0\beta_P}{S_0} \times \left[\frac{E(S_T)-S_0R_F}{\beta_S}\right]}{R_F}. \qquad (41)$$

Again, we make a similar substitution, $-\Omega\beta_S$ for β_P, and arrive at Equation (34).

A.4 Discussion and Numerical Illustration

Our derivations are used to develop the GOPOP model and are valid only for a single step within the binomial tree. GMPOP is the iterative multi-period model that uses successive GOPOP calculations. To demonstrate the one-period nature of the GOPOP model, we expanded Table III and created Table VIII to show the one-period risk premium of the option"RP," the one-period standard deviation of the option return "SD," and the one-period Sharpe ratio "SR."

The one-period Sharpe ratio for the underlying security is 0.036429 throughout the entire binomial tree and is computed by taking the one-period expected return under the real-world probability less the one-period risk-free rate divided by the standard deviation of the one-period return under the real-world probability. We notice that the relationship for the Sharpe ratio in the GOPOP model is maintained every period whenever the option has value. Using this relationship iteratively, we produce the GMPOP model. Although the single-period Sharpe ratio is constant whenever the option has value, the components for the Sharpe ratio of the option change every period. In particular, the risk premium RP of the option changes every period with a proportionate change in the one-period return volatility SD. Thus, unlike the underlying security, the expected rate of return on the option changes every period. The GOPOP model is a single-period model because of the changing components in the Sharpe ratio of the option every period. Consequently, the GOPOP model does not use a single discount rate for the option since the discount rate changes every period. Instead, we use the one-period Sharpe ratio since it is constant every period (assuming the option has value). In contrast to the Sharpe ratio, the only way to have a constant discount rate for the option every period is to use the risk-neutral measure (the Sharpe ratio and risk premium become zero for all securities). Again, all of these points are also true using the Treynor measure.

Current	**Period 1**	**Period 2**	**Period 3**	**Period 4**	**Period 5**	**Probability**
S	S_u	S_{uu}	S_{uuu}	S_{uuuu}	S_{uuuuu}	p^5
	S_d	S_{du}	S_{duu}	S_{duuu}	S_{duuuu}	$5p^4(1-p)$
		S_{dd}	S_{ddu}	S_{dduu}	S_{dduuu}	$10p^3(1-p)^2$
			S_{ddd}	S_{dddu}	S_{ddduu}	$10p^2(1-p)^3$
				S_{dddd}	S_{ddddu}	$5p(1-p)^4$
					S_{ddddd}	$(1-p)^5$

Table I: Stock Values in Five-Stage Binomial Tree

Asset values and terminal probabilities assuming the asset price jumps by a multiplicative factor u with probability p, and by a factor d with probability $(1-p)$. Thus, $S_{ddu} = S \times d \times d \times u$, for example.

Current	**Period 1**	**Period 2**	**Period 3**	**Period 4**	**Period 5**	**Probability**
$V(0,0)$	$V(1,0)$	$V(2,0)$	$V(3,0)$	$V(4,0)$	V_{uuuuu}	p^5
	$V(0,1)$	$V(1,1)$	$V(2,1)$	$V(3,1)$	V_{duuuu}	$5p^4(1-p)$
		$V(0,2)$	$V(1,2)$	$V(2,2)$	V_{dduuu}	$10p^3(1-p)^2$
			$V(0,3)$	$V(1,3)$	V_{ddduu}	$10p^2(1-p)^3$
				$V(0,4)$	V_{ddddu}	$5p(1-p)^4$
					V_{ddddd}	$(1-p)^5$

Table II: Option Values in Five-Stage Binomial Tree

Option values and terminal probabilities assuming the asset price jumps by a multiplicative factor u with probability p, and by a factor d with probability $(1-p)$.

Current	Period 1	Period 2	Period 3	Period 4	Period 5	Real-World Probability	Risk-Neutral Probability
\$4.32	\$2.11	\$0.64	\$0.00	\$0.00	\$0.00	4.01%	3.36%
	\$6.66	\$3.67	\$1.30	\$0.00	\$0.00	18.09%	16.32%
		\$9.86	\$6.18	\$2.66	\$0.00	32.67%	31.69%
			\$13.81	\$9.90	\$5.45	29.50%	30.78%
				\$18.08	\$14.64	13.32%	14.95%
					\$21.93	2.41%	2.90%

Table III: European Put Option Value in Five-Stage Binomial Tree
$S = \$50$, $k_S = 0.15$, $\sigma = 0.40$, $X = \$50$, $T = 1/12$ (i.e., each step size is one month), $r_F = 0.10$. The underlying grows with either $u = e^{\sigma\sqrt{T}} = e^{0.40\sqrt{\frac{1}{12}}} = 1.1224$, or $d = e^{-\sigma\sqrt{T}} = e^{-0.40\sqrt{\frac{1}{12}}} = 0.8909$ over each time step of $T = 1/12$. The real-world probability of an up move is $p = \frac{e^{k_S T} - d}{u - d} = 0.52551$ at each step.

Current	Period 1	Period 2	Period 3	Period 4	Period 5	Real-World Probability	Risk-Neutral Probability
\$4.49	\$2.16	\$0.64	\$0.00	\$0.00	\$0.00	4.01%	3.36%
	\$6.96	\$3.77	\$1.30	\$0.00	\$0.00	18.09%	16.32%
		\$10.36	\$6.38	\$2.66	\$0.00	32.67%	31.69%
			\$14.64	\$10.31	\$5.45	29.50%	30.78%
				\$18.50	\$14.64	13.32%	14.95%
					\$21.93	2.41%	2.90%

Table IV: American Put Option Value in Five-Stage Binomial Tree
$S = \$50$, $k_S = 0.15$, $\sigma = 0.40$, $X = \$50$, $T = 1/12$ (i.e., each step size is one month), $r_F = 0.10$.

Current	Period 1	Period 2	Period 3	Period 4	Period 5	Real-World Probability	Risk-Neutral Probability
\$200.00	\$364.42	\$664.02	\$1209.93	\$2204.64	\$4017.11	3.42%	0.96%
	\$109.76	\$200.00	\$364.42	\$664.02	\$1209.93	16.50%	7.34%
		\$60.24	\$109.76	\$200.00	\$364.42	31.81%	22.52%
			\$33.06	\$60.24	\$109.76	30.65%	34.55%
				\$18.14	\$33.06	14.77%	26.50%
					\$9.96	2.85%	8.13%

Table V: Annual Market Sales Revenue (\$ Millions)

Annual levels of potential future sales and terminal probabilities assuming the sales levels jump by a multiplicative factor u with probability p, and by a factor d with probability $(1-p)$, where $u = e^{\sigma\sqrt{T}}$, $d = e^{-\sigma\sqrt{T}}$, $p = \frac{e^{kT}-d}{u-d}$ in the real world, and $p = \frac{e^{r_F T}-d}{u-d}$ in the risk-neutral world, where $r_F = 0.05$, $\sigma = 0.60$, $k = 0.18$, and $T = 1$ per period.

Current	Period 1	Period 2	Period 3	Period 4	Period 5	Real-World Probability	Risk-Neutral Probability
\$73.25	\$170.48	\$388.45	\$859.73	\$1824.14	\$3617.11	3.42%	0.96%
(25.686%)	(24.970%)	(24.004%)	(22.628%)	(20.511%)			
	\$16.08	\$42.84	\$114.12	\$304.02	\$809.93	16.50%	7.34%
	(30.500%)	(30.500%)	(30.500%)	(30.500%)			
		\$0.00	\$0.00	\$0.00	\$0.00	31.81%	22.52%
		(0.000%)	(0.000%)	(0.000%)			
			\$0.00	\$0.00	\$0.00	30.65%	34.55%
			(0.000%)	(0.000%)			
				\$0.00	\$0.00	14.77%	26.50%
				(0.000%)			
					\$0.00	2.85%	8.13%

Table VI: Option Value (\$ Millions) with Periodic Option Rate of Return

Option values and terminal probabilities using the GMPOP model based on the sales levels in Table V. The probability of success in the real world is 3.42 percent+16.50 percent=19.92 percent.

Panel A: Individual Present Value Calculations		
Future Value	**Discount Factor***	**PV****
\$3,617.11	$\exp(-(25.686\% + 24.970\% + 24.004\% + 22.628\% + 20.511\%))$	\$1113.70
\$809.93	$\exp(-(25.686\% + 24.970\% + 24.004\% + 22.628\% + 20.511\%))$	\$249.38
\$809.93	$\exp(-(25.686\% + 30.500\% + 30.500\% + 30.500\% + 30.500\%))$	\$184.95
\$809.93	$\exp(-(25.686\% + 24.970\% + 30.500\% + 30.500\% + 30.500\%))$	\$195.46
\$809.93	$\exp(-(25.686\% + 24.970\% + 24.004\% + 35.500\% + 35.500\%))$	\$208.58
\$809.93	$\exp(-(25.686\% + 24.970\% + 24.004\% + 22.628\% + 35.500\%))$	\$225.67
* see Table VI for the single-period discount rate inputs; exp(·) is the exponential function		
**PV= value from Column 1 multiplied by value from Column 2		
Panel B: Probability Weighted Present Value Calculations		
PV	**Probability**	**Probability*PV**
\$1113.70	3.42427%	\$38.14
\$249.38	3.30014%	\$8.23
\$184.95	3.30014%	\$6.10
\$195.46	3.30014%	\$6.45
\$208.58	3.30014%	\$6.88
\$225.67	3.30014%	\$7.45
		Sum: \$73.25

Table VII: Traditional DCF Analysis of Product Development (\$ Millions)
This table shows the calculations necessary to replicate the real-option valuation using traditional DCF analysis with path-dependent discount rates drawn from Table VI.

Current:	**Period 1**	**Period 2**	**Period 3**	**Period 4**	**Period 5**
\$4.32 RP: −1.9158% SD: 0.52590 SR: −.036429	\$2.11 RP: −2.6124% SD: 0.71712 SR: −.036429	\$0.64 RP: −3.7231% SD: 1.02202 SR: −.036429	\$0.00 RP: 0.0000% SD: 0.00000 SR: N/A	\$0.00 RP: 0.0000% SD: 0.00000 SR: N/A	\$0.00 RP: 0.0000% SD: 0.00000 SR: N/A
	\$6.66 RP: −1.6882% SD: 0.46342 SR: −.036429	\$3.67 RP: −2.4143% SD:0.66274 SR: −.036429	\$1.30 RP: −3.7231% SD: 1.02202 SR: −.036429	\$0.00 RP: 0.0000% SD: 0.00000 SR: N/A	\$0.00 RP: 0.0000% SD: 0.00000 SR: N/A
		\$9.86 RP: −1.4096% SD: 0.38695 SR: −.036429	\$6.18 RP: −2.1302% SD: 0.58476 SR: −.036429	\$2.66 RP: −3.7231% SD: 1.02202 SR: −.036429	\$0.00 RP: 0.0000% SD: 0.00000 SR: N/A
			\$13.81 RP: −1.0779% SD: 0.29589 SR: −.036429	\$9.90 RP: −1.6887% SD: 0.46355 SR: −.036429	\$5.45 RP: 0.0000% SD: 0.00000 SR: N/A
				\$18.08 RP: −0.7336% SD: 0.20139 SR: −.036429	\$14.64 RP: 0.0000% SD: 0.00000 SR: N/A
					\$21.93 RP: 0.0000% SD: 0.00000 SR: N/A

Table VIII: Expanded Calculation for European Put Option

This table shows more detail for the European put option valuation in Table III. $S = \$50$, $k_S = 0.15$, $\sigma = 0.40$, $X = \$50$, $T = 1/12$ (i.e., each step size is one month), $r_F = 0.10$. The underlying grows with either $u = e^{\sigma\sqrt{T}} = e^{0.40\sqrt{\frac{1}{12}}} = 1.1224$, or $d = e^{-\sigma\sqrt{T}} = e^{-0.40\sqrt{\frac{1}{12}}} = 0.8909$ over each time step of $T = 1/12$. The real-world probability of an up move is $p = \frac{e^{k_S T} - d}{u-d} = 0.52551$ at each step. To aid reader replication and interpretation, note that at the first step, $S = 50$, $Su = 56.1200451$, $Sd = 44.5473626$, $P_u = 2.11412203$, $P_d = 6.66278573$, and Equation (4) or (3) is used to get $P = 4.319018717$. $RP = E(R_P) - R_F = \frac{pP_u + (1-p)P_d}{P} - e^{r_F T} = 0.989210262 - 1.008368152 = -0.0191578903$. $SD = \frac{\sqrt{p(P_u - E(P_T))^2 + (1-p)(P_d - E(P_T))^2}}{P} = 0.525899514$, where $E(P_T) = pP_u + (1-p)P_d = 4.2724176357$, $SR = \frac{RP}{SD} = \frac{-0.01915789}{0.525899514} = -0.0364288040$. It can be verified that this is the negative of the Sharpe ratio for the stock: $SR = \frac{E(R_S) - R_F}{\nu_S} = \frac{e^{k_S T} - e^{r_F T}}{\left[\sqrt{p(Su - E(S_T))^2 + (1-p)(Sd - E(S_T))^2}\right]/S} = \frac{0.004210299}{0.115576107} = 0.0364288040$