Market Imperfections, Investment Optionality and Default Spreads[1]

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Abstract

This paper develops a structural model that determines default spreads on risky debt. In contrast to previous research, the value of the debt's collateral is endogenously determined by the borrower's investment choice, as well as by a market demand variable that has permanent as well as temporary components. The model also considers market imperfections that limit the borrower's ability to contract to undertake the value-maximizing investment choice, and which may in addition limit the borrower's ability to raise external capital. The model is calibrated with data on commercial mortgages, and based on our calibration, we present numerical simulations that quantify the extent to which investment flexibility, incentive problems and credit constraints affect default spreads.

Starting with the seminal work of Black and Scholes (1973) [4] and Merton (1974) [33], researchers have developed contingent claims models to value risky debt. A subset of these models, known as structural models,[2] assume that markets are perfect and that the value of the collateral of the debt can be viewed as exogenous. This approach to pricing debt is in sharp contrast to the theoretical capital structure literature, which examine settings with market imperfections where endogenous collateral values are influenced by financing choices.

There is an emerging literature that addresses both pricing and capital structure issues by introducing market imperfections into contingent claims models. The goals of these models are to enrich the pricing models and to quantify some of the predictions that have arisen from the theoretical capital structure literature.[3] While progress has been made on both fronts, the models are still highly stylized and have not been calibrated to actual data.

This paper contributes to this literature in a number of ways. First, we extend the pricing literature by developing a model that values debt as a contingent claim on an asset whose value is endogenously determined by market conditions and an investment choice, which is also endogenous. Second, within the context of this model we examine how investment flexibility affects default spreads in settings both with and without perfect contracting. Finally, by calibrating the model's parameters to actual data, we are able to quantify the magnitude of default spreads as well as the costs associated with imperfect

[2] The literature on debt pricing has taken two very different paths. The models on the first path, based on the work of Black and Scholes (1973) [4] and Merton (1974) [33], are generally referred to as structural models since default probabilities and recovery rates are determined endogenously. Applications of this framework include Titman and Torous (1989) [41], Kau et al (1990) [21] and Longstaff and Schwartz (1995) [27], who solve a model where borrowers can optimally default anytime prior to or at the maturity date. The second path consists of reduced form valuation models that include exogenous default probabilities and recovery rates. Examples include Jarrow and Turnbull (1995) [19], and Duffie and Singleton (1995) [9]. Structural models are used in industry to value debt instruments like corporate bonds and commercial mortgages, while reduced form models, which are less numerically intensive since they do not solve for the optimal default strategy, have been used to price more complicated instruments such as credit swaps and credit derivatives.

[3] Quantitative models that consider capital structure and debt valuation issues within settings with market imperfections include Mello and Parsons (1992) [31], Leland (1994, 1998) [24, 25], Leland and Toft (1996) [26], Mauer and Ott (1999) [28], Parrino and Weisbach (1999) [36], Mauer and Ott (1999)[28], Moyen (2000)[34], and Goldstein et al. (2001) [17].

contracting.

In theory, investment flexibility can have an important effect on default spreads even without market imperfections. This is because the option to increase and decrease the rate of investment induces skewness in the distribution of future cash flows which, in turn, increases default probabilities for any given loan-to-value ratio. Intuitively, investment flexibility adds value to a project by increasing cash flows in the more favorable future states of the economy when the firm is unlikely to default. Hence, if loan-to-value ratios are held constant, an increase in flexibility increases spreads since the debt holders do not benefit from the higher cash flows in the favorable future states of the economy, and are hurt by the fact that investment flexibility also tends to decrease collateral value in the unfavorable states in which the firm defaults. Because of this "real options effect," the payout and volatility — which are the only characteristics considered in previous models — of a loan's collateral fluctuate stochastically, and hence initial or expected values of these parameters are not sufficient to determine the loan's default risk.

The endogeneity of the investment choice has additional effects on credit spreads that can arise because of market imperfections that were previously discussed in the corporate finance literature. The market imperfections we consider include both the underinvestment problem, first described by Myers (1977) [35], and the credit-rationing problem described by Stiglitz and Weiss (1981) [40]. Specifically, borrowers may, at times, choose to pass up positive NPV investments that benefit lenders at the expense of borrowers and, at other times, may be unable to obtain external capital because of adverse selection and agency reasons. As we show, because of these market imperfections, credit spreads are determined by characteristics of the borrower and the contracting environment as well as by characteristics of the collateral.

To quantify the effect of these market imperfections we calculate credit spreads for three types of borrowers. The first type, which we call a *restricted borrower*, is contractually obligated to follow the investment strategy that would be followed by an unlevered owner of the project. The second type, which we call an *unrestricted borrower with deep pockets*, is assumed to maximize the value of its equity, and thus underinvests relative to the restricted borrower. The final borrower, which we call an *unrestricted borrower with empty pockets*, may underinvest and default too soon because of its inability to obtain external capital.

Because of underinvestment, the credit spread for the unrestricted borrower with deep pockets must be greater than the credit spread for the restricted borrower. However, since there are a number of competing effects that influence the difference in borrowing rates between the unrestricted borrower with deep pockets and the unrestricted borrower with empty pockets, one cannot predict, á priori, which will be able to borrow at more attractive rates. On the one hand, the fact that the borrower with empty pockets defaults suboptimally tends to allow it to borrow at more attractive terms, since the lender benefits from the borrower's suboptimal choices. On the other hand, because of its credit constraints, the empty pockets borrower may be less able to invest when its cash flows are low, which could increase the spread.

To illustrate the applicability of our model and to quantify the various determinants of default spreads and agency costs we present comparative statics that are calculated numerically. Our numerical simulations use parameters that are consistent with data on commercial office buildings and their mortgages.[4] Specifically, we select parameters that allow us to roughly match the volatitility of property returns, the term structure of interest rates, the average investment to net operating income ratio and the average payout rates of the properties. As we show, with these parameters the model generates default spreads, default rates, and recovery rates that are consistent with the observed values.

Using parameters from this calibration as our base case, we present comparative statics that allow us to quantify the effect of market imperfections and investment flexibility on default spreads and agency costs. Our comparative statics indicate that investment flexibility has a very important effect on default spreads even with perfect contracting.

[4] Our model can be directly applied to the valuation of debt obligations that are collateralized by a variety of assets. Examples might include oil rigs, power plants, ships or any other asset that is typically financed with non-recourse debt, i.e., when the lender has no recourse on the borrower's other assets. Our analysis of commercial mortgages is motivated by data availability.

The incentive problems that arise because of imperfect contracting add significantly to default spreads, but it turns out that the associated agency costs are not particularly large given the parameters from our calibration.

The paper is organized as follows: Section 1 presents the model and discusses the formulation for the case of borrowing with perfect contracting. Section 2 describes the calibration of the model parameters to the case of office buildings. Section 3 presents numerical simulations and comparative statics for credit spreads for the case of borrowing with perfect contracting, as well as for the case of the deep pocket borrower and the empty pocket borrower. Section 4 compares spreads, default probabilities, and recovery rates computed from our model to ones observed from data, and compares the model of this paper to the one described in Titman and Torous (1989)[41]. Section 5 summarizes and concludes the paper.

1 Description of the Model

The borrower in our model initially borrows an exogenous amount to finance its business, which we will refer to as the "project".[5] The debt is assumed to be a coupon bond or mortgage that has a required payment each period as well as a final balloon payment on the maturity date. The project generates cash each period, which is used to meet the periodic debt obligation and can be invested to maintain the project's quality. If there is excess cash flow, it is paid out to the borrower. If there is insufficient cash flow to meet debt payments and to fund investment needs, the borrower can raise additional equity capital if it is not credit constrained.

Our model can be described by the following timeline:

— At time 0

[5] The exogenous debt structure is chosen to match the actual debt contracts that are the basis of our calibration exercise. We do not solve for the optimal debt contract or the optimal debt level. However, the assumed debt structure is consistent with the contract one might observe in a setting where a risk neutral and effort-averse entrepreneur requires external capital. See for example, Gale and Hellwig (1985) [14].

- the borrower takes out a loan, which it uses to finance the project.

— Each subsequent period

- the quality of the project depreciates,
- the project generates a random cash flow,
- after the cash flow is realized, the borrower decides whether to default on the loan and relinquish control of the project to the lender, or make the coupon payment,
- once the coupon payment is made, the borrower decides on the amount to be invested in the maintenance/upgrade of the quality of the project. Both the coupon payment and investment are financed either with the cash flow generated by the project or by issuing equity,[6]
- any remaining cash flow is paid out to the borrower.

— At the maturity of the loan

- the borrower decides whether to make the balloon payment on the loan, or default and relinquish control of the project to the lender.

The borrower is assumed to default on the loan optimally, i.e. when the market value of the firm's equity becomes zero.[7] For borrowers that are not restricted to follow a predetermined investment strategy, the investment choice maximizes the value of the firm's equity and, as in Myers (1977) [35], the levered firm will underinvest relative to an all equity firm. Lenders account for the borrowers' incentive to underinvest and default when they price the debt. In the event of default the lender is assumed to take over the project and make optimal investment decisions as an all-equity owner.[8]

[6] Equity can only be raised to cover coupon payments or investment needs.

[7] In reality, if default and agency costs are sufficiently large, the lender and borrower may attempt to renegotiate the loan prior to the default date. Depending on the costs of renegotiation and bankruptcy, the ability to renegotiate can potentially either increase or decrease spreads. This is an interesting issue, but is beyond the scope of our analysis. See Anderson and Sundaresan (1996) [1], Fan and Sundaresan (1997) [11], and Mella-Barral and Perraudin (1997) [30], for models that consider strategic debt service and renegotiation.

[8] We make the assumption that the project can be sold at the maturity date to all-equity buyers who

1.1 The Interest Rate Process

The short-term interest rate used to discount the project's cash flows follows a mean-reverting square root stochastic diffusion process, described by the one factor Cox, Ingersoll and Ross (1985)[7] model:

$$dr_t = \kappa_r(r^* - r_t)dt + \sigma_r\sqrt{r_t}dW_r \tag{1}$$

where

$\kappa_r \equiv$ mean-reversion rate

$r^* \equiv$ long-term level to which the short-term rate reverts to

$\sigma_r \equiv$ instantaneous volatility for the short-term rate

$W_r \equiv$ a standard Wiener process under the risk-neutral measure.

1.2 The Project's Cash Flow and Value

Because we will later be using information pertaining to commercial real estate and commercial mortgages, we will describe the exogenous state variable corresponding to the net income after non-discretionary expenses, as a market lease rate for a hypothetical project in perfect condition. We want the lease rate process to be homogeneous, allow for a non-flat term structure of lease rates, and have decreasing, but non-vanishing volatility for longer term lease contracts. We can satisfy these requirements with a two factor model where the market lease rate, l_t, is described by a mean-reverting stochastic process

$$dl_t = \kappa_l(L_t - l_t)dt + \sigma_l l_t dW_l \tag{2}$$

will not be subject to the agency problems and credit constraints for reasons of computational tractability. Our numerical simulations, which we present in Section 3, suggest that agency and credit constraint costs are relatively small, indicating that this assumption has only a marginal effect on our results.

with the long-term lease level to which the short-term lease rate reverts to, L_t,[9] described by geometric Brownian motion

(3) $$dL_t = L_t \mu_L dt + \sigma_L L_t dW_L$$

The parameters of the process are

$\kappa_l \equiv$ mean-reversion rate for the lease rate

$\sigma_l, \sigma_L \equiv$ instantaneous volatilities of the lease rate and the long-term lease level

$\mu_L \equiv$ growth rate of the long-term lease level

$W_l, W_L \equiv$ standard Wiener processes under the risk-neutral measure

The Wiener processes W_r, W_l, W_L are correlated, with correlation coefficients equal to $\rho_{r,l}, \rho_{l,L}, \rho_{L,r}$.

In addition to the market lease rate, the cash flow for a specific project is determined by the quality of the project q. The quality is normalized between 0% and 100%, and we assume that the lease rate for a project with quality q is given by the product $q \times l$, where l is the lease rate for a project in perfect condition.[10]

We assume that the quality of a project is a strictly concave and increasing function of the stock of maintenance, M:

(4) $$q(M) = 1 - e^{-\alpha M}$$

where α is the rate of incremental improvement per unit of investment in the quality of a project with zero initial quality level. The functional form of the quality function implies

[9] The long-term lease level is not the same as the level of lease rates for long term contracts, but rather the level towards which the short term lease rate level reverts.

[10] Quality in our model can be understood to mean more than just the state of the physical asset. For example, firms can invest in training to improve the quality of their employees' human capital, or advertise to increase their future market share. Basically, anything that costs something today and increases revenues in the future is applicable. In our discussions of this model with individuals in the real estate business we have heard anecdotes that suggest that managers of distressed properties often rent to less reliable tenants, who are more likely to damage the property or default on their rent. This example of underinvestment in tenant quality is very much consistent with the spirit of our model.

that a project in perfect condition, with a stock of maintenance M that tends to infinity, has quality equal to 100%, whereas a project with a stock of maintenance that equals zero has quality equal to 0%.

We assume that the stock of maintenance M depreciates at a constant rate γ, so that the change of M is given by

$$dM_t = -\gamma M_t d_t + m_t dt \tag{5}$$

where m_t is a choice variable that corresponds to the rate of investment in maintenance at time t. The rate of investment, m_t, is assumed to be non-negative. To account for situations where large instantaneous improvements in quality are optimal we allow m_t to be unbounded.

From the point of view of an all-equity owner, the value of a project equals the expected present value, under the risk-neutral measure, of the discounted future cash flows net of investment expenditures. Specifically, the project's value is given by the solution to the stochastic control problem

(6)

$$E^{(u)}(r, l, L, M) = \max_{m \geq 0} \left\{ \mathbb{E}_Q \left[\int_0^\infty (l_t q(M_t) - m_t)\, e^{-\int_0^t r_s ds} dt : r_0 = r, l_0 = l, L_0 = L, M_0 = M \right] \right\}$$

where $\mathbb{E}_Q$ is the expectation under the risk-neutral measure Q.

In Appendix A.1 we present the Hamilton-Jacobi-Bellman equation that corresponds to problem 6 and in Appendix B we discuss how it can be solved numerically.

1.3 Investment Flexibility and Credit Spreads: The Case with Perfect Contracting

To value the equity and debt claims on the project we will initially assume that the borrower is restricted to follow the investment strategy of the all-equity owner, described in the previous subsection. While the borrower is limited with respect to its investment strategy, it is free to default optimally.

The value of the equity, in this case, is the greater of zero and the expected discounted cash flows from the project, net of investment and interest costs. Since the borrower may default, the value of the equity $E^{(r)}$ depends on the default strategy and is given by

(7)

$$E^{(r)}(r,l,L,M,t) = \max_{\tau} \Big(0, \mathbb{E}_Q\Big(\int_t^{\tau} (l_s q(M_s) - m_s^* - c) e^{-\int_t^s r_y dy} ds + \delta(T-\tau) e^{-\int_t^T r_s ds} \max(0, E^{(u)}(r_T, l_T, L_T, M_T) - F) : \mathrm{IC}_t\Big)\Big)$$

where the stopping time τ either corresponds to the time of default, if $\tau < T$, or is equal to $\tau = T$. The function δ is given by $\delta(x) = 0$, if $x \neq 0$ and $\delta(0) = 1$. The optimal investment strategy of the all-equity owner, m^*, is given by the solution to the stochastic control problem (6). The initial conditions, IC_t at time t are

$$\mathrm{IC}_t \equiv \{r_t = r, l_t = l, L_t = L, M_t = M\}$$

The remaining parameters in (7) are:

T: the maturity of the loan,

c: the coupon rate of the loan, and

F: the balloon payment, due at time T.

Given the optimal default strategy of the borrower, the value of the debt, D, for a given coupon and maturity, can be determined numerically using dynamic programming. The value of the debt at maturity is equal to its face value, if no default occurs. If default occurs, the value of the debt, at the time of default, is equal to the value of the collateral. At maturity we price the debt for each possible state variable value and each level of quality and then move backward one instant and again price the debt for each state variable. To obtain the value of the debt, we repeat this procedure until we reach the starting date. In Appendix A.2.1 we describe the Hamilton-Jacobi-Bellman equation corresponding to the stochastic control problem (7). To determine the credit spread, we iterate over different coupon rates until the debt is priced at par, i.e. the value of the debt is equal to the debt principal.

In the absence of bankrupcy costs, the value of the unlevered collateral, $E^{(u)}$, the equity value of the borrower restricted to follow the investment strategy of the all-equity owner, $E^{(r)}$, and the value of the debt, D, satisfy

$$E^{(u)} = E^{(r)} + D$$

We have also considered the case with default, or bankruptcy costs, where a percentage of the project value is lost at default. These costs are incurred by the lender upon recovery of the project, and, therefore, given the terms of the loan, do not influence the investment and default decisions of the borrower.

2 Model Parameters for the Case of Commercial Mortgages

To evaluate the model described in the previous section, we numerically solve the model using parameter values that roughly match cashflow characteristics of commercial properties, in particular office buildings, in the interest rate environment of January 1998. Given the information available for commercial properties, it is possible to estimate approximate values for model parameters. By focusing on a particular type of property we are able to determine whether quantitative measures calculated from our model can be matched against observed property characteristics. In this section we present and motivate our parameter choices. In Section 3 we present results from a comparative static analysis of credit spreads to changes in parameter values.

We estimate the model parameters using information from three sources: the Treasury yield curve; time series of cash flows and investment rates provided by the National Council of Real Estate Investment Fiduciaries (NCREIF); and a dataset on individual commercial mortgages. Certain parameters are chosen to match observed values, while others are chosen indirectly, by determining their effect on other, observed quantities, such as financial ratios and the volatility of property values for office buildings. The parameters that are chosen to match observed values include the parameters for the interest rate process, the volatility of the lease rate and the correlations between the different stochastic factors. The depreciation rate, mean reversion rate for the lease rate process, volatility of the long term lease rate level and the term structure of lease rates are chosen indirectly.

2.1 Parameters Chosen Based on Direct Evidence

The interest rate parameters are chosen to match the term structure of interest rates as of January 1998. To choose the parameter values, we minimize the sum of absolute deviations between the bond prices implied by the Cox-Ingersoll-Ross model and the observable prices of zero coupon bonds with maturities ranging from 3 months to 30 years. The value for the mean-reversion rate is set at $\kappa_r = 0.17$ per year, the value of the instantaneous volatility at $\sigma_r = 4.8\%$ annualized, and the long-term interest rate level at $r^* = 6\%$ per year. The initial short-term rate was 5.31%. By examining the sign and magnitude of the pricing errors we verified that there is no systematic relationship between the pricing errors and the maturity of the bonds.

Other model parameters are chosen to roughly match observations from various NCREIF indices, which are widely used, appraisal-based indices for property values and returns of real estate. These indices measure the performance of real estate in the United States and provide information on capital returns, income and capital investments across different regions and commercial property types. NCREIF indices typically include hundreds of properties and provide quarterly observations between January 1978 and January 2000. There are certain drawbacks in trying to estimate parameters for individual properties using an appraisal-based index that have been extensively discussed in the literature. Geltner and Goetzmann (1988)[16] and Clayton, Geltner and Hamilton (2001)[6] point out that using appraised, rather than sales based, property values leads to artificially smoothing the index and consequently to lower volatility of property values. Additional smoothing results from the diversification implicit in the construction of the index. For example, while the volatility of property values for the NCREIF index for all commercial properties across the United States is less than 6%, the volatility of property values for the NCREIF subindex of office buildings located in Phoenix, Arizona, is 11.3%.[11] We will rely on the index for all office buildings in the United States for providing estimates of the correlations between income, property value and interest rates, where the effect of diversification should be minor. For the estimation of volatilities, on the other hand, we will use information from the subindex of office buildings in Phoenix, since it includes a relatively small number of properties (less than 20) whose values are likely to be highly correlated.

Our estimate of the model volatility of lease rates relies on estimates of the volatility of net operating income (NOI)[12] for office buildings. This volatility corresponds to the volatility of short term lease rates, σ_l, in our model. We choose 16%, which matches the volatility of income for office buildings in Phoenix.

The estimation of the correlation coefficients between the changes in the lease rate, the long term lease level and the interest rate is based on historical information from a time-series of NCREIF indices. Specifically, we estimate three correlations: 1) the correlation between the growth rate of NOI and the changes in the risk free rate, 2) the correlation between property capital returns and the changes in the risk free rate and 3) the correlation between capital returns and the growth rate of NOI.

The correlation between changes in NOI and changes in the risk free rate roughly corresponds to the correlation between the lease rate and the interest rate, $\rho_{r,l}$, in our model. From our observations, this correlation for the case of office buildings is equal to 29.6%. The correlation between changes in the long term lease rate level and changes in the interest rate, $\rho_{L,r}$ is matched to the observed correlation between capital returns and changes in interest rates, since changes in capital returns are largely determined by the long term lease level. From the data this correlation is equal to 4.0%. Similarly, the correlation between capital returns and NOI roughly corresponds to the correlation

[11] Assuming that quarterly returns are well approximated by a normal distribution, a rough estimate of instantaneous volatility, annualized, can be obtained as the largest change in property value over one quarter, over a five year period.

[12] The NOI is defined as the gross annual revenues less maintenance and other operational expenses but before taxes, depreciation and capital investments.

between unexpected changes of permanent and temporary components in the lease rates, $\rho_{l,L}$. From the data, this correlation is equal to 6.2%.

For our computations we select correlation values that roughly match the observations for the case of office buildings: $\rho_{r,l} = 30\%, \rho_{l,L} = 6\%, \rho_{L,r} = 4\%$.

2.2 Parameters Chosen Based on Indirect Evidence

Some of the parameters of our model are difficult to estimate directly from the data. These parameters include the volatility of the long term lease level, the mean reversion rate of the lease rate process, the depreciation rate, and the slope of the term structure of lease rates. The values for these parameters are chosen so that model generated cash flow ratios match observed values.

The volatility of the long term lease level and the mean reversion rate of lease rates are chosen so that the model generated volatility of property values matches the observed volatility of property values. Using the subindex of office buildings in Phoenix, the volatility of property values is 11.3%. We can match this volatility of property values by choosing the volatility of long term lease rates to be 9%, and the mean reversion rate to be 0.20 per year.[13]

The values of the depreciation rate γ, and the slope of the term structure of lease rates, while difficult to measure directly, have an important effect on observed financial ratios. In particular, the depreciation rate largely determines the percentage of NOI spent on investment, and, indirectly, the payout rate of the property. On the other hand, the ratio of the long term lease level to current lease rates, L/l, directly impacts the NOI to Property Value ratio. For our base case we chose the depreciation rate γ to equal 10% per year, and the term structure of lease rates to be initially flat. For these values, the above financial ratios, numerically generated by our model, are in line with average ratios observed for office buildings. The model-generated payout rate is 5.4%, the investment

[13] We note that this choice is not unique. To uniquely determine both the volatility of the long term lease level and the mean reversion rate of lease rates, we would need additional information, for example the volatility of the lease rate of long term lease contracts.

over NOI ratio is 37.3%, the NOI over Property Value ratio is 8.6%. These values are quite close to the observed payout rate which is 5.06%, the investment over NOI ratio which is 34.1% and the NOI over Property Value ratio which is 7.61%.

2.3 Remaining Pararameters and Mortgage Structure

The remaining model parameters include the risk neutral drift of the long-term lease level μ_L, and the quality function parameter α. While it is difficult to estimate the risk neutral value of the drift its effect is similar to choosing a larger or smaller slope for the term structure of lease rates. For our numerical experiments we set μ_L to zero, which corresponds to a situation with zero inflation.[14]

The choice of the quality function parameter α is arbitrary, as it only serves to define the units in which money is measured. Assuming that properties are maintained at an "efficient" quality level, i.e. a quality level that makes an all-equity owner indifferent about marginally increasing quality, the quality level q, and the level of stock of maintenance M are endogenously determined. Given the values of all the parameters the efficient initial quality level q is 76%. The efficient quality level depends on the initial value of all the parameters; for example, it is higher for higher lease rates. After the issuance of the mortgage, the quality level is endogenously determined.

For the structure of the commercial mortgage, we use information from a dataset of mortgages on commercial properties.[15] Most of the mortgages in the dataset are fixed rate, non-amortizing, with a ten year maturity, that were locked out from prepayment for some initial period. Most of the mortgages that originated in the 1990's have loan-to-value ratios between 72% and 82%. Based on this information, for our numerical simulations, we chose a mortgage structure that is non-amortizing, priced at par, and has a balloon payment that is due at the end of 10 years. The loan-to-value ratio was set at 80%.

[14] We also note that choosing a positive value for the drift of the long term lease rate level would be inappropriate under our assumptions, since it would progressively make investment costs cheaper relative to income, since the cost of investment in our model is time independent. An extension of our model would be to allow for investment costs that increase with time, as well as for a quality function that incorporates project obsolescence, by decreasing the maximum quality level achievable through investment.

[15] The data was provided to us by Charter Research.

All the parameter values for the base case are listed in Table 1.

3 Numerical Results and Comparative Statics

In this section we present numerical results that allow us to quantify the effects of investment flexibility and market imperfections on default spreads. Our numerical results use the parameters discussed in Section 2 as a base case, and discuss the investment and default strategies of three different borrower types. We first consider the default spreads for the borrower we call restricted, who commits to choose the investment strategy of the all equity owner and who defaults only when the value of its equity is zero.

3.1 The Case with Perfect Contracting

3.1.1 Optimal Investment Choices

To understand why and how credit spreads vary, it is important to first characterize the investment choice and quantify the changes in the efficient quality level. The intuition is that an owner of a project with an initial quality level above the efficient level allows the project to depreciate for a while, until it reaches the efficient level. In this situation, the initial payout rate of the project is higher, leading to higher spreads.[16] On the other hand, an owner of a project with an initial quality level below the efficient level immediately invests an amount that brings the project to the efficient quality level, thereby increasing collateral value and decreasing the loan-to-value ratio, and decreasing credit spreads.

In Table 2 we report the efficient quality level for different values of the parameters. The table reveals that there are two major determinants of efficient quality levels: the depreciation rate of the project, and the term structure of lease rates. Efficient quality levels decrease with higher depreciation rates since, ceteris paribus, higher depreciation rates make it more expensive to maintain a given quality level. Higher current lease rates,

[16] This discussion assumes that the loan-to-value ratio for the project is kept constant. We choose to keep the loan-to-value ratio fixed was to directly compare model generated credit spreads to observed credit spreads, which are reported for fixed loan-to-value ratios.

or the expectation of higher lease rates in the future, lead to an increase in the efficient quality level, as higher quality levels allow the borrower to capture a larger percentage of the higher lease rates.[17]

It is worth noting that the volatility of lease rates has only a marginal effect on the efficient quality level. Intuitively, since investment is not completely reversible, one would expect that with higher uncertainty the owner prefers to wait longer before determining the quality level.[18] However, for the parameter values of our base case, it turns out that depreciation is fast enough that investment is largely reversible. In situations with either smaller values of the depreciation rate or larger values of volatility we found that the effect of increases in volatility on the efficient quality level is much greater.

In our simulations we examine comparative statics for credit spreads by varying parameters from their base case values. Because we do not want our results to be dominated by the payout rates from an initially inefficient quality level, our comparative statics compare projects with quality levels which are initially efficient for the unlevered owner.[19] For example, when we compare spreads for projects with high and low depreciation rates we set the initial quality of a project with a high depreciation rate lower than the initial quality of a project with a low depreciation rate. Similarly, each time we vary the parameters, we adjust the initial leverage by setting the size of the loan at 80% of the value of the unlevered project.

3.1.2 Real Options and Credit Spreads

The owner's flexibility to alter the quality level has a major effect on credit spreads. By cutting back on investment and reducing quality when the market lease rate is low and by increasing investment and increasing quality when the market lease rate is high, the

[17] Factors other than the level of lease rates and the depreciation rate have only a minor effect on the efficient quality levels. For example, increasing either the level of short-term or long-term interest rates leads to lower efficient quality levels, since higher current or future interest rates effectively increase the cost of investment.

[18] This is consistent with the intuition described in Dixit and Pindyck [8].

[19] This choice implies that the initial quality level may not be efficient for the borrowers with deep and empty pockets. We chose to start with the efficient quality level for the unlevered owner in order to be able to compare the impact of the contracting environment on the same project.

owner induces skewness in the future cash flows and value of the project. Due to this real options effect, borrowing rates are substantially higher for projects that depreciate and that can be improved by investment.

To quantify this effect we first consider a case where the quality of the project does not depreciate and the owner cannot enhance project value through investment. We find that for the base case set of parameters described in Section 2, with the depreciation rate and the investment rate set to zero, the borrowing rate is 30 basis points above the Cox, Ingersoll and Ross risk-free rate for a coupon bond traded at par of the same maturity. Increasing the depreciation rate to 10% per year increases the borrowing rate to 109 basis points over the Treasury rate. In other words, investment flexibility more than triples the spread. As we show in Table 3, if we look across efficiently maintained projects with different depreciation rates, borrowing rates are higher for projects with higher depreciation rates.

3.1.3 Comparative Statics for the Restricted Borrower

We conducted numerical experiments on the effect of different model parameters on the credit spread charged to the restricted borrower, reported in Table 3. Based on the information in the table, we were able to determine that, in addition to the depreciation rate of the project, the slope of the term structure of lease rates, and the slope of the term structure of interest rates also affect credit spreads. For instance, in situations with an upward sloping term structure, future lease rates are expected to rise, making it easier for the borrower to cover interest payments, leading to lower credit spreads. An upward sloping interest rate term structure suggests that in the future financing costs are expected to be higher, implying higher hurdle rates, and hence less investment and higher payouts, leading to higher credit spreads.

The effect of other variables on the magnitude of the credit spreads is comparatively small. For example, higher volatility of lease rates increases the value of the borrower's default option, leading to higher spreads.

3.2 Agency Conflicts and Credit Spreads

The previous section considered the case where the borrower is assumed to follow the investment strategy of an all-equity owner. However, in the absence of enforceable covenants we need to consider the effect of the borrower's flexibility to select investment levels that maximize the levered equity value. We now consider a borrower who chooses its investment to maximize its equity value, rather than the value of the unlevered project, and is able to raise equity to cover coupon payments and investment costs without incurring additional costs. We will call such a borrower an *unrestricted borrower with deep pockets.*

The value of the equity $E^{(dp)}$ owned by this type of borrower is given by the solution to the stochastic control problem

(8)

$$E^{(dp)}(r, l, L, M, t) = \max_{\tau, m \geq 0} \big(0, \mathbb{E}_Q\big(\int_t^\tau (l_s q(M_s) - m_s - c) e^{-\int_t^s r_y dy} ds + \delta(T - \tau) e^{-\int_t^T r_s ds} \max(0, E^{(u)}(r_T, l_T, L_T, M_T) - F) : \mathrm{IC}_t \big)\big)$$

where the borrower is free to choose both the time of default, τ, and the rate of investment at any time t, m_t. Since the borrower relinquishes all cash flows generated by the project to the lender upon default, the borrower will tend to underinvest relative to the all-equity owner.

3.2.1 Comparative Statics for the Deep Pockets Borrower

We quantify the importance of underinvestment in two separate ways. First, we calculate the difference between the borrowing rate for an unrestricted, deep pockets borrower and the borrowing rate of a borrower who commits to the investment strategy of an all-equity owner. This difference is a premium charged by the lender to compensate for potential underinvestment and higher probability of default, and will be called the *agency spread.* Second, we calculate the percentage difference between the value of the unlevered project and the sum of the values of the debt and equity for the levered project. We will call this difference the *agency cost*, since it reflects the loss in value due to the borrower's inability to commit to the strategy of the unlevered owner.

We find that for our base case the agency spread is 37 basis points, which represents 25% of the total spread for the deep pockets borrower. In contrast to the agency spread, which is economically significant, the agency cost is only 0.64%. The significance of the agency spread and the insignificance of the agency costs suggests that even when the deadweight costs associated with underinvestment are small, the wealth transfer between debtholders and equity holders is potentially large.

The comparative statics, shown in Table 4, indicate that both agency spreads and agency costs are affected by variables related to investment flexibility. Overall, higher investment flexibility leads to higher agency spreads and agency costs. The most significant variables are the depreciation rate of the project and the level of short-term volatility. Intuitively, higher depreciation rates allow the borrower more opportunities to reduce the project's quality, and are associated with higher agency spreads and costs.

The effect of volatility on agency spreads and agency costs is more complicated. While, as one would expect, increases in the volatility of the permanent component of cash flows leads to increases in both agency spreads and costs, our numerical results indicate that an increase in the volatility of the temporary component of the cash flows leads to a *decrease* in the agency spread and the agency cost for the range of parameters we report. Intuitively, a volatility increase should decrease recovery rates, since higher volatility increases the option value of equity holders, inducing them to meet their interest payments (but not to maintain their project) even when collateral value falls significantly below the face value of their loan. On the other hand, both levered and unlevered owners wait longer before investing when the volatility is higher, leading to a possible decrease of the agency spread. Overall it is not intuitively clear, *ex ante*, which effect should dominate.

The introduction of default costs, which are borne by the lender upon default of the borrower, has a larger effect on the deep pockets borrower than on the restricted borrower, since the deep pockets borrower defaults more often. This additional probability of default leads to larger expected losses and higher agency spreads and costs. [20]

[20] It should also be noted that realistic changes in the initial short-term interest rate do not seem to have a significant effect on the agency spread. However, long-term interest rate increases lead to increases in the agency spreads.

3.3 Credit Constraints and Credit Spreads

Up to this point we have assumed that the borrower has deep pockets and can access capital to meet its debt payments when the payments exceed the lease revenues. In reality, borrowers are often credit constrained and are forced to default because of cash shortfalls in situations where an unconstrained borrower would choose to not default.

To understand the effect of credit constraints on credit spreads we consider a borrower that is limited in its capacity to issue additional equity. We consider the extreme case where he is never able to issue equity to finance investments, and can only invest when the income from the project is higher than the coupon payment. Additionally, when the income is less than the coupon payment, the borrower is forced to default and surrender the project to the lender if the value of the unlevered project is below the face value of the debt. Only when the value of the unlevered project is more than the face value of the debt, and the income from the project is below the coupon payment, is the borrower allowed to raise enough funds from issuing equity to cover the coupon payment. We will call such a borrower a *borrower with empty pockets.* Even though the borrower is restricted with respect to equity issuance, it retains some flexibility regarding default and investment.

The value of the equity $E^{(ep)}$ of the empty pockets borrower is given by

$$\begin{aligned} E^{(ep)}(r,l,L,M,t) = \max_{\tau, 0 \le m \le \max(0, q(M)l - c)} \Big(0, \mathbb{E}_Q\Big(\int_t^{\tau} (l_s q(M_s) - m_s - c) e^{-\int_t^s r_y dy} ds \\ + \delta(T-\tau) e^{-\int_t^T r_s ds} \max(0, E^{(u)}(r_T, l_T, L_T, M_T) - F) : \mathrm{IC}_t \Big)\Big) \end{aligned} \tag{9}$$

where the time of default τ can be chosen by the borrower as long as

$$\tau \le \min\{t : q(M_t)l_t - c < 0 \text{ and } E^{(u)}(r_t, l_t, L_t, M_t) - F < 0\}$$

3.3.1 Comparative Statics for the Empty Pockets Borrower

We quantify the effect of credit constraints in two ways. First, we compare the borrowing rates of the unrestricted, deep pockets borrower to those of the unrestricted, empty pockets borrower. We will call the difference the *credit constraint spread.* Second, we calculate *credit constraint costs*, defined similar to agency costs, as the percentage difference between the value of the unlevered project and the value of the debt plus equity for the levered project managed by a credit constrained borrower.[21]

Our simulations show that for the cases considered in Table 5 the credit constraint spreads are positive, or, equivalently, that the borrowing rates for the empty pockets borrower are lower than the borrowing rates for the deep pockets borrower, unless default costs are high. From the results in Table 5 we note that, qualitatively, credit constraint costs move in the same way as agency costs for most cases. In our examples credit constraint costs are lower than agency costs except for those cases when an additional cost is incurred by the lender in the event of default.

The sign and magnitude of credit constraint spreads, presented in Table 5, can be intuitively understood by comparing the investment strategy of the empty pockets borrower with the investment strategy of the deep pockets borrower. A priori, it is unclear whether the borrowing rates for the empty pockets, credit constrained borrower would be higher or lower than the borrowing rates for the deep pockets borrower. On the one hand, since the empty pockets borrower defaults suboptimally it would enjoy lower spreads, since the lender would recover a larger percentage of the face value of the loan upon default. On the other hand, since the empty pockets borrower anticipates its suboptimal default, it may invest less than the deep pockets borrower, which would lead to increased spreads. Our simulations show that in most of the cases we consider, the first effect is stonger than the second, and that the empty pockets borrower enjoys lower spreads than the deep pockets borrower.[22]

In situations where the empty pockets borrower is likely to generate plenty of cash from the project it is likely to follow an investment strategy similar to that of a deep

[21] The definition of the credit constraint spread allows us to isolate the effect of the credit constraint on the borrowing rate. The definition of the credit constraint cost, on the other hand, corresponds to the deadweight loss in the economic value of the project for the credit constrained borrower, due to both agency problems and credit constraints.

[22] A weakness of the model is that we do not allow the empty pockets borrower to build up a cash reserve with the excess cash flows from the project. The existence of a cash account would mitigate suboptimal default in many situations, leading to spreads similar to those of the deep pockets borrower. Adding a cash account to the model, however, would substantially increase the complexity of the numerical solution.

pockets borrower, leading to small credit constraint spreads. Such a situation arises when depreciation rates, and thus investment needs, are low.

4 Empirical Observations and Comparisons

In Section 2 we calibrated our model to be roughly consistent with observed financial ratios of office buildings. In this section we provide support for our model by comparing model generated spreads, default probabilities, and recovery rates to those either observed in our data or in other research. In addition, we compare spreads generated by our model to spreads generated by the Titman and Torous (1989) [41] model when both models are calibrated to the same financial ratios. Based on our analysis we argue that in contrast to the Titman and Torous model, which significantly underestimates credit spreads, our calibrated model generates credit spreads that approximate those observed in the data.

4.1 Model Generated vs. Empirical Spreads, Default Probabilities and Recovery Rates

4.1.1 Credit Spreads

As we report in Table 7, the average spread in January 1998 for non-prepayable office building mortgages, with 80% loan-to-value ratio was 166 basis points. In comparison, for our base case, our model generates a spread of 109 basis points for the restricted borrower, 146 basis points for the deep pocket borrower, and 131 basis points for the empty pockets borrower. While these spreads are somewhat lower than the observed spread of 166 basis points, it should be noted that we priced the mortgages relative to Treasury bonds, which have much lower yields than AAA non-government debt, which has negligible default rates but is less liquid and may be more highly taxed than Treasury bonds (see Huang and Huang (2000)[18], and Elton et al. (2001) [10]). When we add the 40 to 50 basis point difference between the Treasury rates and the AAA rates to our model-generated rates we get spreads that are comparable to the ones observed.

4.1.2 Default Probabilities

Another way to evaluate our model is to compare default probabilities and recovery rates generated by the model, with those observed. Note that default probabilities generated directly from our model are calculated under the risk neutral measure, which makes them difficult to interpret.[23]

To calculate the default probabilities under the real measure we choose the drift of the long-term lease level to match the observed capital returns for office buildings. From the NCREIF subindex for office buildings, the average capital return is 4% per year, which can be roughly matched in our model by choosing a real drift for the long term lease level of 3%.[24] Under this drift, the model-generated cumulative default probability for the restricted borrower is 13%, for the deep pockets borrower 24%, and for the empty pockets borrower 34%. Actual cumulative default probabilities for commercial mortgages are reported in a number of research papers. Snyderman (1991, 1994) [38, 39] tracked more than 10 thousand mortgages originated from 1972 and 1986, through 1991, and found a cumulative default rate of 13.8% (through 1991) and projected a lifetime default rate of 18.3%. Based on the observation of almost 10 thousand mortgages on multifamily properties, Archer et al (1999) [2] report that the overall default rate during the period between 1991 and 1996 was 17.5%. In another study, the rating agency Fitch (1996) [12] tracked almost two thousand mortgages originated between 1984 and 1987 through the end of 1991 and estimated that the cumulative default rate over this period was 14%. They argue that the actual cumulative default rate is even higher, perhaps up to 30%, since some loans in the pool were not counted as in default because they were either sold

[23] To compute the default probabilities under the real measure we first determine the default boundary, as described in Appendix A.2. We then adjust the drift of the long-term lease rate level so that capital returns generated by the model match observed ones, and compute the probability that the adjusted stochastic process will reach the default boundary. The calculation of the default probability involves solving a partial differential equation, similar to equation (19) in Appendix A.3 used for the valuation of the debt, without the discounting term rD, and with the boundary condition that the probability is set equal to 1 at the default boundary and to zero at the mortgage maturity. Kau et al. (1994) [20] follow a similar procedure to compute real default probabilities in a two factor model.

[24] The expected capital return can be approximated by the sum of $\mu_L + \sigma_P^2/2$, where σ_P is the volatility of property values. For the base case parameters in our model, the volatility of property values is equal to 11.3%.

or restructured prior to an actual default.[25] These observations are summarized in Table 8.

4.1.3 Recovery rates

The recovery rates generated by our model can also be compared with observed recovery rates. Our simulations indicate that the average recovery rate, defined as the ratio of the collateral value at default over the discounted value of the remaining cash flows of the debt, is not particularly sensitive to changes in the parameter values. Since the empty pockets borrower is often forced to default, while the deep pockets borrower has the most flexibility in the management of the project, the recovery rates are highest for the empty pockets borrower and lowest for the deep pockets borrower. The restricted borrower recovery rates are between the recovery rates for the other two borrower types. For the base case studied in this paper, the average recovery rate for the restricted borrower is 79%, for the deep pocket borrower 77% and for the empty pockets borrower 84%. Actual recovery rates for commercial mortgages, described in Gichon (1995)[15], range between 68% and 82%, depending on the property type that collaterizes the mortgage.

4.2 Comparison with the Titman-Torous (1989) Model

To determine whether including endogenous investment in our model leads to material differences in the determination of credit spreads over existing models in the literature, we compare our model to the one developed by Titman and Torous (1989) [41]. The Titman and Torous model postulates a stochastic process for the collateral value B^{TT}, instead of endogenously determining the value process from the more primitive cash flow process. Specifically, Titman and Torous assume that the collateral value is generated by the process:

$$dB^{TT} = (r - b^{TT})B^{TT}dt + \sigma_{B^{TT}} B^{TT} dW_{B^{TT}} \quad (10)$$

[25] Quigg (1998) [37] summarized some of these results.

where b^{TT} is the payout rate of the project, $\sigma_{B^{TT}}$ is the instantaneous volatility of the value of the unlevered project, and $W_{B^{TT}}$ is a standard Wiener process under the risk-neutral measure.

The short-term interest rate in the Titman and Torous model follows the same mean-reverting square root stochastic diffusion process as in the model described in this paper

$$dr_t = \kappa_r(r^* - r_t)dt + \sigma_r\sqrt{r_t}dW_r \tag{11}$$

where the correlation between the Wiener processes $W_{B^{TT}}$ and W_r is assumed to be constant, and equal to $\rho_{r,B^{TT}}$. Thus, to completely specify the Titman and Torous model one needs to determine the values of three parameters b^{TT}, $\sigma_{B^{TT}}$ and $\rho_{r,B^{TT}}$, as well as the values of the parameters for the interest rate process.

To compare the implications of our model with those of the Titman and Torous model we perform the following numerical experiment: starting with sets of parameter values for the model described in this paper, we compute the stochastic process for the unlevered collateral value. Given the collateral value we extract the initial volatility and payout rate, as well as the correlation between the value process and the short-term interest rate and use the values of these parameters as inputs for the Titman and Torous model. These parameters can be computed analytically using Itô's formula:[26]

(12)

$$
\begin{aligned}
(\sigma_{B^{TT}})^2 &= \left[\frac{1}{E^{(u)}}\frac{dE^{(u)}(r,l,L,M)}{dt}\right]^2 \\
&= \frac{1}{(E^{(u)})^2}\Big[\sigma_r^2 r(E_r^{(u)})^2 + \sigma_L^2 L^2(E_L^{(u)})^2 + \sigma_l^2 l^2(E_l^{(u)})^2 \\
&\qquad + 2\rho_{r,l}\sigma_l\sigma_r l\sqrt{r}E_l^{(u)}E_r^{(u)} + 2\rho_{l,L}\sigma_l\sigma_L Ll E_L^{(u)}E_l^{(u)} + 2\rho_{r,L}\sigma_L\sigma_r L\sqrt{r}E_L^{(u)}E_r^{(u)}\Big] \\
b^{TT} &= \frac{lq(M)-\gamma M}{E^{(u)}(r,l,L,M)}, \\
\rho_{r,B^{TT}} &= \frac{1}{E^{(u)}\sigma_{B^{TT}}}\left[\sigma_r\sqrt{r}E_r^{(u)} + \rho_{r,l}\sigma_l l E_l^{(u)} + \rho_{r,L}\sigma_L L E_L^{(u)}\right]
\end{aligned}
$$

where subscripts denote partial derivatives.

Using these parameter values for the value process and the base case parameters, given in Table 1, for the interest rate process, we then calculate spreads under the Titman and Torous model, for loans identical to the ones considered in our model. The results of this experiment are given in Table 9. In addition to credit spreads, the table provides comparative statics for the parameters of the Titman and Torous model in terms of the parameters of the model described in this paper.

The comparative statics for the parameters of the Titman and Torous model are in line with the intuition developed in this paper. In particular, the payout rate parameter is high for downward sloping lease rate term structures. An increase in the depreciation rate, on the other hand, leads to higher investment rates and lower payout rates, although the effect is relatively smaller. The Titman and Torous volatility, σ_{TT} is also influenced by the term structure of lease rates, the depreciation rate and the levels of short-term and long-term volatility. It is interesting to point out that an increase in the depreciation rate, with its associated expectation of bigger fluctuations in cash flows, leads to higher volatility in the Titman and Torous collateral value.

[26] The Titman and Torous model uses constant parameter values for the volatility and payout rate of the collateral value, while our model allows for these parameters to fluctuate endogenously. To determine credit spreads with the Titman and Torous model, we set the volatility and payout rate of the collateral value, as well as the correlation between the collateral value and the short-term interest rates, to the initial values used in our model. Other choices, such as setting the Titman and Torous parameters to the one year averages of the parameters of the collateral value generated by our model, are possible.

The credit spreads generated by the Titman and Torous model turn out to be approximately two thirds of the ones generated by our model of the restricted borrower.[27] Titman and Torous were able to generate sizable spreads only by assuming a volatility of asset value of 17% and a payout rate of 8.5%, values which are inconsistent with empirical observations.

5 Conclusions

This paper develops a model that applies insights from the literature on real options and optimal capital structure to price risky debt instruments. Previous work considered these issues within stylized models that would be difficult to calibrate to actual data. Our model allows us to develop intuition regarding the importance of investment flexibility and incentives within a model that is roughly calibrated to observable data. In particular we find that with realistic parameter values our model generates credit spreads that are consistent with observed spreads on commercial mortgages. With these parameters, which are consistent with the historical returns, payout, and default rates on commercial property, we provide comparative statics that explore how investment flexibility affects spreads. Our results indicate that investment flexibility, by altering the volatility as well as the skewness of future collateral values, substantially increases credit spreads even in the absence of incentive problems. Incentive problems that lead the borrower to underinvest relative to an all-equity owner further increase these spreads.

In addition to pricing debt, our model is used to quantify issues that were previously explored in the corporate finance literature. In particular, we confirm the Parrino and Weisbach (1999) [36] result that indicates that in most cases, very little value is destroyed by the Myers (1977)[35] underinvestment problem. Specifically, in most of the cases we examine, the loss in the value of the project due to potential underinvestment is less than 1% of its value assuming optimal investment. This suggests that the underinvestment

[27] The Titman and Torous model consistently underestimates the spreads, with the difference ranging from 20 to 120 basis points. For the base case, the Titman and Torous spread is 59 basis points, while the spread for the restricted borrower of our model is 109 basis points.

problem should have only a minor effect on the capital structure choice, despite the fact that it contributes significantly to credit spreads. This result is intuitive since credit spreads reflect the entire expected wealth transfer from debtholders to shareholders due to underinvestment, while agency costs measure the net efficiency loss or the difference between the debtholder's expected loss and the equityholder's expected gain. Our results indicate that the efficiency loss can be relatively small even when the magnitude of the transfer can be large.

We believe that our model can be used to address a number of other issues. For example, the model can be extended to consider various covenants that either place constraints on the amount that the borrower must invest or, alternatively, on the amount of cash from the investment that can be distributed. In addition, our model can be applied to assess the gains associated with risk management, as well as the costs associated with risk shifting. In particular, the model can be used to assess the interdependence between risk choices and the underinvestment problem.

To apply our model more generally to the pricing of corporate debt we would need to relax our assumption that the borrower can neither increase nor decrease the face amount of its debt obligation over time. While this assumption is reasonable for a model of project debt, the capital structures of most corporations evolve in ways that potentially affect the value of their debt. Determining optimal dynamic capital structure policies and pricing debt in such a setting is the subject of future work.

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A Valuation of Collateralized Debt

To value collateralized debt we need to determine the default boundary, the investment strategy, and the value of the collateral at the default boundary. The procedure for determining the value of the debt has three steps:

(i) we compute the optimal investment strategy and the collateral value without the debt, i.e. for a 100% equity owner,

(ii) we use the values computed for the collateral without debt to deduce the boundary conditions at the maturity of the debt. Given these boundary conditions, we compute the investment strategy and the default boundary in the presence of the debt,[28]

(iii) using the location of the default boundary and the value of the collateral without debt we compute the value of the debt.

A.1 Valuation of the Unlevered Collateral

Given the value of the short-term interest rate r, the lease rate l, the long-term lease level L and the level of the stock of maintenance M, the equity value of a project without debt $E^{(u)}(r, l, L, M)$ is independent of time and can be uniquely determined by maximizing the expected value of the equity $E^{(u)}$ under the risk neutral measure Q, for all possible

[28] A complication that arises in determining the optimal investment strategy is that in order to decide the amount to invest in maintenance, the owner must know the collateral, or project value after investment. However, the project value will depend on future investment choices, thus introducing feedback in the valuation procedure. This complication can be avoided if we consider the level of the stock of maintenance as an additional state variable, increasing the dimensionality of the problem by one. The idea of adding a state variable also appears in other examples in option pricing such as the pricing of arithmetic average options in the foreign exchange markets or swing options in the energy markets. To illustrate how the extra state variable resolves the feedback problem, consider the 100% equity owner. It has to make the decision for the amount to be invested in maintenance in a manner that maximizes its, unlevered, equity value. If the project value is known at all future times for all possible quality levels, then the choice is simple: it is the one that maximizes the project value when investment costs are taken into account.

investment choices

$$E^{(u)}(r(t), l(t), L(t), M(t)) =$$
$$\max_{m \geq 0} \big(lq - m + e^{-r(t)dt}\mathbb{E}_Q\big(E^{(u)}(r(t+dt), l(t+dt), L(t+dt), M(t)(1-\gamma dt) + mdt)\big)\big)$$

The value of the equity $E^{(u)}$ is given as the solution to the Hamilton-Jacobi-Bellman equation

(13)
$$\frac{\sigma_r^2 r}{2} E_{rr}^{(u)} + \frac{\sigma_L^2 L^2}{2} E_{LL}^{(u)} + \frac{\sigma_l^2 l^2}{2} E_{ll}^{(u)} + \rho_{r,l}\sigma_l\sigma_r l\sqrt{r} E_{rl}^{(u)} + \rho_{l,L}\sigma_l\sigma_L Ll E_{lL}^{(u)} + \rho_{r,L}\sigma_L\sigma_r L\sqrt{r} E_{rL}^{(u)}$$
$$+ \kappa_r(r^* - r)E_r^{(u)} + \kappa_l(L - l)E_l^{(u)} + \mu_L L E_L^{(u)} - \gamma M E_M^{(u)} + lq(M) - rE^{(u)}$$
$$+ \max_{m \geq 0} \big(mE_M^{(u)} - m\big) = 0$$

where subscripts denote partial derivatives.

The optimal investment choice depends on the value of the derivative of the value of the equity with respective to the level of the stock of maintenance, $E_M^{(u)}$. If the marginal increase in the value of the equity, for a \$1 investment, is greater than \$1, then the optimal investment choice is to invest until the marginal increase in the value of the equity is equal to the amount of the investment. Conversely, if the marginal increase in the value of the equity from a \$1 investment is less than \$1, then the optimal investment choice is to not invest at all. Thus, equation (13) can be rewritten without the max term, with the additional free boundary condition

$$E_M^{(u)} \leq 1 \quad (14)$$

A.2 Valuation of the Borrower's Equity

At the debt maturity date T, the value of the borrower's equity, E, is the greater of zero and the difference between the value of the unlevered project and the balloon payment

$$E(r, l, L, M, T) = \max(E^{(u)}(r, l, L, M) - F, 0)$$

where F is the value of the balloon payment for the debt, and $E^{(u)}$ the project value that solves the stochastic control problem (13).

Since different borrower types, (i.e., restricted, unrestricted with deep pockets, or unrestricted with empty pockets), face different constraints, their investment and default choices as well as their equity values may differ prior to maturity. In the following subsections we specify the optimization problems for each type.

A.2.1 Restricted Borrower

The restricted borrower has contracted to follow the investment strategy prescribed by the solution to the optimal control problem in equation (13), but may still default. The value of its equity $E^{(r)}$ is a function of r, l, L, M, t and, for $0 \leq t \leq T$, satisfies the partial differential equation

$$
\begin{aligned}
&\frac{\sigma_r^2 r}{2} E_{rr}^{(r)} + \frac{\sigma_L^2 L^2}{2} E_{LL}^{(r)} + \frac{\sigma_l^2 l^2}{2} E_{ll}^{(r)} \\
&\qquad + \rho_{r,l}\sigma_l\sigma_r l\sqrt{r} E_{rl}^{(r)} + \rho_{l,L}\sigma_l\sigma_L L l E_{lL}^{(r)} + \rho_{r,L}\sigma_L\sigma_r L\sqrt{r} E_{rL}^{(r)} \\
&\qquad + \kappa_r(r^* - r)E_r^{(r)} + \kappa_l(L - l)E_l^{(r)} + \mu_L L E_L^{(r)} \\
&\qquad + E_t^{(r)} + (-\gamma M + m^*(r, l, L, M))E_M^{(r)} + lq(M) \\
&\qquad - m^*(r, l, L, M) - c - rE^{(r)} = 0
\end{aligned}
\tag{15}
$$

where $m^*(r, l, L, M)$ is the investment strategy that maximizes the unlevered equity value for lease rate l, long-term lease level L, stock of maintenance level M, and short-term rate r. The parameter c is the continuous coupon rate. We also need to impose the free boundary condition that the equity value is greater or equal to zero, since when the value of the equity becomes zero, the borrower defaults

$$E^{(r)} \geq 0.$$

A.2.2 Unrestricted Borrower with Deep Pockets

At any time t prior to maturity, the unrestricted borrower with deep pockets chooses the maintenance level m that maximizes the value of the immediate net operating income after coupon payments $(lq - c - m)dt$ plus the expected value of the equity, under the risk neutral measure Q, at time $t + dt$

$$
\begin{aligned}
E^{(dp)}(r(t), l(t), L(t), M(t), t) = \max_{m \geq 0} \Big(\big(l(t)q(M(t)) - c - m \big) dt \\
+ e^{-r(t)dt} \mathbb{E}_Q [E^{(dp)}(l(t+dt), L(t+dt), r(t+dt), M(t)(1-\gamma dt) + mdt, t+dt)] \Big)
\end{aligned} \tag{16}
$$

If the value of the equity $E^{(dp)}$ becomes zero, the borrower defaults. The equity value $E^{(dp)}$ satisfies the Hamilton-Jacobi-Bellman equation

(17)

$$
\begin{aligned}
&\frac{\sigma_r r}{2} E^{(dp)}_{rr} + \frac{\sigma_L^2 L^2}{2} E^{(dp)}_{LL} + \frac{\sigma_l^2 l^2}{2} E^{(dp)}_{ll} + \rho_{r,l}\sigma_l \sigma_L l \sqrt{r} E^{(dp)}_{rl} + \rho_{l,L}\sigma_l \sigma_L L l E^{(dp)}_{lL} + \rho_{r,L}\sigma_L \sigma_r L \sqrt{r} E^{(dp)}_{rL} \\
&\quad + \kappa_r (r^* - r) E^{(dp)}_r + \kappa_l (L - l) E^{(dp)}_l + \mu_L L E^{(dp)}_L + E^{(dp)}_t - \gamma M E^{(dp)}_M + lq(M) - c - rE^{(dp)} \\
&\quad + \max_{m \geq 0} \big(m E^{(dp)}_M - m \big) = 0
\end{aligned}
$$

subject to the constraint

$$E^{(dp)} \geq 0.$$

A.2.3 Unrestricted Borrower with Empty Pockets

Compared to the unrestricted borrower with deep pockets, the unrestricted borrower with empty pockets is constrained with respect to the amount of equity it can issue. Its equity value satisfies equation (17), with the additional constraint that it must default if the cash flow rate from the project is below the coupon payment and the value of the unlevered project is below the balloon payment

$$E^{(ep)}(r, l, L, M, t) = 0, \text{if} \quad q(M) \times l < c \text{ and } E^{(u)}(r, l, L, M) < F.$$

Moreover, investment can only be financed by the cash flow generated by the project, after the coupon payment has been made, i.e.

$$0 \leq m \leq \max(0, lq(M) - c)$$

A.3 Debt Valuation

To obtain the value of the debt D, we need to consider the default strategy of the borrower and the value of the collateral at default. We assume that in the event of default the lender takes over the project and operates it optimally according to the investment strategy followed by an unlevered owner. At maturity T the debt value is given by the minimum of the balloon payment F and the value of the project without debt $E^{(u)}$

$$D(r,l,L,M,T) = \min(E^{(u)}(r,l,L,M), F) \tag{18}$$

Prior to maturity, the debt value satisfies the partial differential equation

$$\begin{aligned} &\frac{\sigma_r^2 r}{2} D_{rr} + \frac{\sigma_L^2 L^2}{2} D_{LL} + \frac{\sigma_l^2 l^2}{2} D_{ll} + \rho_{r,l}\sigma_l\sigma_r l\sqrt{r} D_{rl} + \rho_{l,L}\sigma_l\sigma_L Ll D_{lL} \\ &\quad + \rho_{r,L}\sigma_L\sigma_r L\sqrt{r} D_{rL} + \kappa_r(r^* - r)D_r + \kappa_l(L-l)D_l \\ &\quad + \mu_L L D_L + D_t + (-\gamma M + m^\dagger(r,l,L,M,t))D_M + c - rD = 0 \end{aligned} \tag{19}$$

where $m^\dagger$ is the corresponding borrower's investment strategy. To calculate the value of debt for the different types of borrowers we need to substitute the appropriate investment strategy and solve equation (19).

There is an additional boundary condition for the value of the debt when the borrower defaults, i.e. when the value of its equity is equal to zero:

$$D(r,l,L,M,t) = E^{(u)}(r,l,L,M), \text{ if } \; E(r,l,L,M,t) = 0 \tag{20}$$

where E may correspond to the equity value of the restricted, unrestricted with deep pockets, and unrestricted with empty pockets borrowers.

We also consider the effect of additional costs, proportional to the unlevered collateral value, imposed in the case of default. In that case, the borrowers' investment behavior would be unaltered, but the value of the debt would decrease. The value of the debt satisfies the following boundary conditions

$$\begin{aligned} D(r,l,L,M,T) &= \begin{cases} F & \text{if } E^{(u)}(r,l,L,M) \geq F, \\ (1 - \text{Default Cost})E^{(u)}(r,l,L,M) & \text{otherwise} \end{cases} \\ D(r,l,L,M,t) &= (1 - \text{Default Cost})E^{(u)}(r,l,L,M), \text{ if } E(r,l,L,M,t) = 0 \end{aligned} \tag{21}$$

B Numerical Algorithm

Below we describe the numerical algorithm used to solve the stochastic control problems for the case of the all-equity owner formulated in (13). The algorithm is similar for the other cases.

The algorithm is based on the finite-difference method augmented by a "policy iteration".[29] The calculation of the project value $E^{(u)}$ is complicated by the fact that the formulation of the problem is time independent. We reformulate the problem with a finite horizon approximation.[30] This reformulation introduces a time derivative $E_t^{(u)}$ to the left hand side in equation (13). We start the procedure with initial values for all the necessary functions in each node at a terminal time. The errors that result from the approximation of functions at the terminal time can be reduced by increasing the length of the horizon of the problem and iterating until the derivative $E_t^{(u)}$ is indistinguishable from zero, at a certain level of accuracy, for each node on the grid.

For each problem we use a discrete grid for the state space and a discrete time step Δt. The state space (r, l, L, M) is discretized using a four-dimensional grid $N_r \times N_l \times N_L \times N_M$ with corresponding spacing $\Delta r, \Delta l, \Delta L$, and ΔM.[31] In each node on the grid (r, l, L, M) the partial derivatives are approximated using first differences.[32]

The values of the all-equity project at each node of the terminal approximation time are set to the values of the expected cash flows assuming that the quality level is kept

[29] See, for example, Kushner and Dupuis (1992) [22], Barraquand and Martineau (1995) [3] and Langetieg (1986) [23] for the theory of numerical methods for stochastic control problems.

[30] Flam and Wets (1987) [13] and Mercenier and Michel (1994) [32] also discuss the approximation of infinite horizon problems in deterministic dynamic programming models.

[31] The grid step in each state variable is chosen so that the numerical algorithm is stable.

[32] For example, the first, second and cross derivatives of the equity value with respect to l and L are

$$E_l(r,l,L,M) = \frac{E(r,l+\Delta l,L,M) - E(r,l-\Delta l,L,M)}{2\Delta l},$$
$$E_{ll}(r,l,L,M) = \frac{E(r,l+\Delta l,L,M) - 2E(r,l,L,M) + E(r,l-\Delta l,L,M)}{\Delta l \Delta l},$$
$$E_{lL}(r,l,L,M) = \frac{E(r,l+\Delta l,L+\Delta L,M) - E(r,l-\Delta l,L+\Delta L,M) - E(r,l+\Delta l,L-\Delta L,M) + E(r,l-\Delta l,L-\Delta L,M}{4\Delta l \Delta L}$$

with appropriate modifications at the grid boundaries, so that we only use points within the domain of integration.

constant and does not depreciate, i.e. $E^{(u)}(r,l,L,M) = \mathbb{E}_Q \int_0^\infty l_t q(M) e^{-\int_t^\infty r_s ds} dt$, where $\mathbb{E}_Q$ is the expectation under the risk neutral measure Q. This approximation tends to overvalue projects with high initial quality and to undervalue projects with low initial quality. However, any initial errors in the approximation are "smoothed" away after a few iterations due to discounting. Iterating backward in time for each node on the grid according to the explicit finite-difference scheme and taking into account the optimal investment decision, the value of the all-equity firm $E^{(u)}_{(t-\Delta t)}$ at each node (r,l,L,M) at the next iteration, corresponding to time $t-\Delta t$ is determined as follows:

$$\text{(B1)}\quad E^{(u)}_{(t-\Delta t)}(r,l,L,M) = \max_{m\geq 0}\Big[[lq(M)-m]\Delta t + e^{-r\Delta t}\mathbb{E}_Q[E^{(u)}_{(t)}]\Big]$$
$$= \max_{m\geq 0}\Big[[lq(M)-m]\Delta t + E^{(u)}_{(t)}(r,l,L,M) + \Delta t\mathcal{L}[E^{(u)}_{(t)}(r,l,L,M-\gamma M\Delta t + m\Delta t)]\Big]$$

where $\mathcal{L}[E^{(u)}_{(t)}(r,l,L,M-\gamma M\Delta t+m\Delta t)]$ is the difference operator applied to $E^{(u)}_{(t)}$ for the node $(r,l,L,M-\gamma M\Delta t+m\Delta t)$

$$\begin{aligned}\mathcal{L}[Z] \;=\; & \frac{1}{2}\sigma_l^2 l^2 Z_{ll} + \frac{1}{2}\sigma_L^2 L^2 Z_{LL} + \frac{1}{2}\sigma_r^2 r Z_{rr} + \rho_{lL}\sigma_L\sigma_l Z_{lL} + \rho_{rl}\sigma_r\sigma_l Z_{rl} + \rho_{rL}\sigma_L\sigma_r Z_{rL} \\ & + \kappa_l(L-l)Z_l + \mu L Z + \kappa_r(r-\overline{r})Z_r + (-\gamma M + m)Z_M + rZ\end{aligned}$$

where all the derivatives are calculated using first differences.

The maximization over all possible investment choices $m \geq 0$ determines the optimal investment strategy m.[33] At the boundaries where l, L, r are equal to zero, no modification of the algorithm is necessary, since all second order derivatives are multiplied by zero and the calculation may proceed using only points within the domain of integration. On the other hand, at the boundaries where $l = N_l \times \Delta l, L = N_L \times \Delta L, r = N_r \times \Delta r$, we have modified the discretization of the second derivative so that we only use points within the domain of integration.

[33] If the maintenance level does not fall on a grid point for the stock of maintenance grid, we perform an additional, linear, interpolation.

At the boundary $M = N_M \Delta M$, $(q(M) \approx 1)$ the owner does not invest i.e., $m = 0$ and the value of the unlevered project satisfies the following partial differential equation:

(22)

$$\frac{\sigma_r^2 r}{2} E_{rr}^{(u)} + \frac{\sigma_L^2 L^2}{2} E_{LL}^{(u)} + \frac{\sigma_l^2 l^2}{2} E_{ll}^{(u)} + \rho_{r,l} \sigma_l \sigma_r l \sqrt{r} E_{rl}^{(u)} + \rho_{l,L} \sigma_l \sigma_L L l E_{lL}^{(u)} + \rho_{r,L} \sigma_L \sigma_r L \sqrt{r} E_{rL}^{(u)}$$
$$+ \kappa_r (r^* - r) E_r^{(u)} + \kappa_l (L - l) E_l^{(u)} + \mu_L L E_L^{(u)} + l - r E^{(u)} = 0$$

We iterate until the changes in function values for each node on the grid is small enough, i.e., until $\max_{(l,L,M,r)} |E_{(t)}^{(u)}(r,l,L,M) - E_{(t-\Delta t)}^{(u)}(r,l,L,M)| < \varepsilon$, where ε is the predetermined accuracy level.[34] We have found this procedure to be robust to the choice of the values at the terminal time.[35] We have also checked that the solution is accurate for the grids chosen by performing the calculation in grids with twice as many points in each state variable and obtaining credit spreads that do not differ by more than 5 basis points.[36]

The computation of the equity values of the restricted, unrestricted and credit constrained borrowers, as well as for the value of the debt, are performed in a similar manner.

[34] Given initial guesses for the values on the "terminal grid", the procedure for the valuation of the unlevered firm converges in about 2000 time steps where each time step $dt = 0.1$ year.

[35] As a test, we checked that for different "reasonable" guesses of the values at the terminal time this procedure converges to the same values, although the number of iterations may be different.

[36] The grid we used for the calculation of the spreads reported in the tables of this paper had $N_l = 20, N_L = 20, N_M = 20, N_r = 10$. The time required to price the debt for a single set of parameter values on a 300 MHz Pentium II was approximately 20 minutes. Between 8 and 10 iterations were necessary to compute the coupon rate for which the debt was priced at par.

Table 1: Parameter Values for the base case.

Slope of lease term structure L/l	100%
Quality q	76%
Short rate r	5.31% annualized
Long Term Interest Rate Level r^*	6% annualized
Loan-to-Value Ratio	80%
Mortgage Maturity T	10 years
Default Costs	0%
Depreciation Rate γ	10% annualized
Drift Rate for Long Term Leasing Rate Level μ_L	0%
Leasing Rate Mean Reversion Rate κ_l	20% annualized
Volatility of Lease Rate σ_l	16% annualized
Volatility of Long Term Lease Level σ_L	9% annualized
Volatility of Short Rate σ_r	4.8% annualized
Interest Rate Mean Reversion Rate κ_r	17% annualized
Correlation between Lease Rate and Interest Rate $\rho_{l,r}$	4%
Correlation between Lease Rate and Long Term Lease Rate Level $\rho_{L,l}$	6%
Correlation between Long Term Lease Rate and Interest Rate $\rho_{L,r}$	30%

Table 2: Comparative statics for efficient quality levels for an all equity owner. The efficient quality level is the quality level for which the all-equity owner is indifferent between investing and not investing. Initial values of lease rates l and long term lease rate levels L are expressed as a percentage of their base case values. All other rates are annualized. γ is the depreciation rate, l the lease rate, L the long term lease level, r the short rate, r^* the long term level of interest rates, σ_l the instantaneous volatility of the lease rate, σ_L the instantaneous volatility of the long term lease level, $\rho_{l,L}$ the correlation between the lease rate and the long term level of lease rates, κ_l the mean reversion rate for lease rates.

γ	4 %	6 %	8 %	10 %	12 %	14 %	16 %
Efficient quality level	84 %	82 %	78 %	76 %	72 %	69 %	65 %
l	40 %	60 %	80 %	100 %	120 %	140 %	160 %
Efficient quality level	51 %	64 %	71 %	76 %	79 %	82 %	83 %
L	40 %	60 %	80 %	100 %	120 %	140 %	160 %
Efficient quality level	66 %	70 %	73 %	76 %	77 %	78 %	79 %
r	0.00 %	1.77 %	3.54 %	5.31 %	7.08 %	8.85 %	10.62 %
Efficient quality level	83 %	81 %	78 %	76 %	73 %	70 %	68 %
r^*	3 %	4 %	5 %	6 %	7 %	8 %	9 %
Efficient quality level	77 %	77 %	76 %	76 %	75 %	75 %	74 %
σ_l	7 %	10 %	13 %	16 %	19 %	22 %	25 %
Efficient quality level	76 %	76 %	76 %	76 %	75 %	75 %	75 %
σ_L	3 %	5 %	7 %	9 %	11 %	13 %	15 %
Efficient quality level	76 %	76 %	76 %	76 %	75 %	75 %	75 %
$\rho_{l,L}$	-9 %	-4 %	1 %	6 %	11 %	16 %	21 %
Efficient quality level	76 %	76 %	76 %	76 %	76 %	76 %	76 %
κ_l	5 %	10 %	15 %	20 %	25 %	30 %	35 %
Efficient quality level	75 %	75 %	76 %	76 %	76 %	76 %	76 %

Table 3: Comparative statics for credit spreads for a borrower that is restricted to follow the investment strategy of an all equity owner. Parameter values are the same as in Table 1. Initial values of lease rates l and long term lease rate levels L are expressed as a percentage of their base case values. All other rates are annualized. γ is the depreciation rate, l the lease rate, L the long term lease level, r the short rate, r^* the long term level of interest rates, σ_l the instantaneous volatility of the lease rate, σ_L the instantaneous volatility of the long term lease level, $\rho_{l,L}$ the correlation between the lease rate and the long term level of lease rates, κ_l the mean reversion rate for lease rates. Default costs are the costs, expressed as a percentage of value, incurred by the lender upon default of the borrower. Initial quality for all projects is set at the efficient level for the all equity owner, and loans are adjusted so that the loan-to-value ratio stays at 80%, while priced at par. Credit spreads are expressed in basis points.

γ	4 %	6 %	8 %	10 %	12 %	14 %	16 %
Credit Spread	62	78	93	109	128	145	162
l	40 %	60 %	80 %	100 %	120 %	140 %	160 %
Credit Spread	58	72	87	109	133	155	180
L	40 %	60 %	80 %	100 %	120 %	140 %	160 %
Credit Spread	531	279	165	109	81	63	53
r	0.00 %	1.77 %	3.54 %	5.31 %	7.08 %	8.85 %	10.62 %
Credit Spread	183	156	130	109	94	82	73
r^*	3 %	4 %	5 %	6 %	7 %	8 %	9 %
Credit Spread	71	83	96	109	124	137	153
σ_l	7 %	10 %	13 %	16 %	19 %	22 %	25 %
Credit Spread	97	102	105	109	115	121	127
σ_L	3 %	5 %	7 %	9 %	11 %	13 %	15 %
Credit Spread	37	55	83	109	139	167	198
$\rho_{l,L}$	-9 %	-4 %	1 %	6 %	11 %	16 %	21 %
Credit Spread	102	106	108	109	112	115	116
κ_l	5 %	10 %	15 %	20 %	25 %	30 %	35 %
Credit Spread	105	107	109	109	112	115	118
Default Costs	0 %	5 %	7.5 %	10 %	12.5 %	15 %	20 %
Credit Spread	109	134	147	161	172	190	223

Table 4: Comparative statics for agency spreads and agency costs. Parameter values are the same as in Table 1. The agency spread is the difference between the borrowing rate for an unrestricted, deep pockets borrower and the borrowing rate for a borrower who commits to the maintenance strategy of an all equity owner. Agency costs are the percentage difference between the value of the unlevered project and the sum of the values of the debt and equity for the levered project. Initial values of lease rates l and long term lease rate levels L are expressed as a percentage of their base case values. All other rates are annualized. γ is the depreciation rate, l the lease rate, L the long term lease level, r the short rate, r^* the long term level of interest rates, σ_l the instantaneous volatility of the lease rate, σ_L the instantaneous volatility of the long term lease level, $\rho_{l,L}$ the correlation between the lease rate and the long term level of lease rates, κ_l the mean reversion rate for lease rates. Default costs are the costs, expressed as a percentage of value, incurred by the lender upon default of the borrower. Initial quality for all projects is set at the efficient level for the all equity owner, and loans are adjusted so that the loan-to-value ratio stays at 80%. Mortgages are priced at par. Agency spreads and costs are expressed in basis points.

γ	4 %	6 %	8 %	10 %	12 %	14 %	16 %
Agency Spread	13	21	29	37	44	53	63
Agency Cost	25	40	54	64	74	83	90
l	40 %	60 %	80 %	100 %	120 %	140 %	160 %
Agency Spread	22	28	35	37	40	43	44
Agency Cost	52	55	61	64	66	66	67
L	40 %	60 %	80 %	100 %	120 %	140 %	160 %
Agency Spread	100	80	55	37	28	22	18
Agency Cost	71	70	69	64	51	41	35
r	0.00 %	1.77 %	3.54 %	5.31 %	7.08 %	8.85 %	10.62 %
Agency Spread	49	44	41	37	32	30	28
Agency Cost	77	73	70	64	59	58	57
r^*	3 %	4 %	5 %	6 %	7 %	8 %	9 %
Agency Spread	15	22	28	37	45	51	55
Agency Cost	38	41	54	64	70	72	73
σ_l	7 %	10 %	13 %	16 %	19 %	22 %	25 %
Agency Spread	44	41	40	37	34	31	29
Agency Cost	76	69	65	64	62	59	59
σ_L	3 %	5 %	7 %	9 %	11 %	13 %	15 %
Agency Spread	32	37	35	37	35	37	37
Agency Cost	63	63	64	64	64	65	65
$\rho_{l,L}$	-9 %	-4 %	1 %	6 %	11 %	16 %	21 %
Agency Spread	38	37	37	37	35	35	35
Agency Cost	65	65	64	64	63	62	62
κ_l	5 %	10 %	15 %	20 %	25 %	30 %	35 %
Agency Spread	40	38	37	37	35	34	34
Agency Cost	54	59	61	64	65	66	69
Default Costs	0 %	5 %	8 %	10 %	13 %	15 %	20 %
Agency Spread	37	38	39	40	42	46	58
Agency Cost	64	104	119	123	129	145	153

Table 5: Comparative statics for credit constraint spreads and credit constraint costs. The credit constraint spread is the difference between the borrowing rates of the unrestricted, deep pockets borrower and the unrestricted, empty pockets borrower. Credit constraint costs are the percentage difference between the value of the unlevered project and the value of the debt plus equity for the levered project managed by a credit constrained borrower. Initial values of lease rates l and long term lease rate levels L are expressed as a percentage of their base case values. All other rates are annualized. γ is the depreciation rate, l the lease rate, L the long term lease level, r the short rate, r^* the long term level of interest rates, σ_l the instantaneous volatility of the lease rate, σ_L the instantaneous volatility of the long term lease level, $\rho_{l,L}$ the correlation between the lease rate and the long term level of lease rates, κ_l the mean reversion rate for lease rates. Default costs are the costs, expressed as a percentage of value, incurred by the lender upon default of the borrower. Initial quality for all projects is set at the efficient level for the all equity owner, and loans are adjusted so that the loan-to-value ratio stays at 80%. Mortgages are priced at par. Credit constraint spreads and credit constraint costs are expressed in basis points.

γ	4 %	6 %	8 %	10 %	12 %	14 %	16 %
Credit Constraint Spread	4	8	10	15	19	22	28
Credit Constraint Cost	20	29	41	50	60	72	78
l	40 %	60 %	80 %	100 %	120 %	140 %	160 %
Credit Constraint Spread	10	10	12	15	18	18	20
Credit Constraint Cost	36	42	47	50	53	54	57
L	40 %	60 %	80 %	100 %	120 %	140 %	160 %
Credit Constraint Spread	4	14	15	15	15	16	18
Credit Constraint Cost	59	56	55	50	34	18	15
r	0.00 %	1.77 %	3.54 %	5.31 %	7.08 %	8.85 %	10.62 %
Credit Constraint Spread	18	17	16	15	14	10	9
Credit Constraint Cost	57	52	51	50	45	45	43
r^*	3 %	4 %	5 %	6 %	7 %	8 %	9 %
Credit Constraint Spread	4	8	12	15	15	15	15
Credit Constraint Cost	24	26	33	50	56	57	61
σ_l	7 %	10 %	13 %	16 %	19 %	22 %	25 %
Credit Constraint Spread	11	12	13	15	18	19	23
Credit Constraint Cost	61	54	52	50	45	41	30
σ_L	3 %	5 %	7 %	9 %	11 %	13 %	15 %
Credit Constraint Spread	10	9	10	15	19	19	24
Credit Constraint Cost	44	47	49	50	51	52	52
$\rho_{l,L}$	-9 %	-4 %	1 %	6 %	11 %	16 %	21 %
Credit Constraint Spread	13	14	15	15	15	16	16
Credit Constraint Cost	52	52	51	50	49	48	47
κ_l	5 %	10 %	15 %	20 %	25 %	30 %	35 %
Credit Constraint Spread	20	18	17	15	15	14	14
Credit Constraint Cost	23	40	47	50	51	53	57
Default Costs	0 %	5 %	8 %	10 %	13 %	15 %	20 %
Credit Constraint Spread	15	6	1	-4	-7	-11	-13
Credit Constraint Cost	50	152	186	186	200	241	271

Table 6: Qualitative comparative statics for efficient quality level, spread over treasury rates for the owner that is restricted to follow the investment strategy of an all equity owner, agency spread, agency cost, credit constraint spread and credit constraint cost. The efficient quality level is the level that makes the all equity owner indifferent between marginally increasing quality and not investing. The agency spread is the difference between the borrowing rate for an unrestricted, deep pockets borrower and the borrowing rate for a borrower who commits to the maintenance strategy of an all equity owner. Agency costs are the percentage difference between the value of the unlevered project and the sum of the values of the debt and equity for the levered project. The credit constraint spread (C.C. spread) is the difference between the borrowing rates of the unrestricted, deep pockets borrower and the unrestricted, empty pockets borrower. The credit constraint cost (C.C. cost) is the percentage difference between the value of the unlevered project and the value of the debt plus equity for the levered project managed by a credit constrained borrower. γ is the depreciation rate, l the lease rate, L the long term lease level, r the short rate, r^* the long term level of interest rates, σ_l the instantaneous volatility of the lease rate, σ_L the instantaneous volatility of the long term lease level, $\rho_{l,L}$ the correlation between the lease rate and the long term level of lease rates, κ_l the mean reversion rate for lease rates. Default costs are the costs, expressed as a percentage of value, incurred by the lender upon default of the borrower. For the columns reporting spread, agency spread, agency cost, credit constraint spread and credit constraint cost, the initial quality is set at the efficient level for the all equity owner, and loans are adjusted so that the loan-to-value ratio stays at 80% and mortgages are priced at par.

↑	Efficient quality level	Spread	Agency spread	Agency cost	C.C. spread	C.C. cost
γ	↓	↑	↑	↑	↑	↑
l	↑	↑	↑	↑	↑	↑
L	↑	↓	↓	↓	↑	↓
r	↓	↓	↓	↓	↓	↓
r^*	↓	↑	↑	↑	↑	↑
σ_l	↓	↑	↓	↓	↑	↓
σ_L	↓	↑	↔	↔	↑	↑
$\rho_{l,L}$	↔	↑	↔	↔	↔	↔
κ_l	↔	↑	↓	↑	↓	↑
Default cost	↔	↑	↑	↑	↓	↑

Table 7: Average spread for commercial mortgages on office buildings. All mortgages are balloon, non-amortizing, locked out from prepayment, with loan-to-value ratio equal to 80%, maturity equal to 10 years, originated in January 1998. The spreads are measured in basis points (b.p.) and are calculated as the difference between the mortgage rate and the 10 year Treasury yield. The numerically calculated spreads are for the different borrower types for the base case set of parameters.

Property Type	Average Spread
Office	166 b.p.
Borrower Type	Spread
Restricted	109 b.p.
Deep Pocket	146 b.p.
Empty Pocket	131 b.p.

Table 8: This table summarizes results in the literature of cumulative default probabilities of commercial mortgages. The last two columns indicate the origination dates and the number of mortgages considered in each study.

Property Type	Cumulative Default Probabilities	Source	Origination Dates	# of Mortgages
All Types	13.8%-18.3%	Snyderman (1994)	1972 - 1986	10,955
All Types	14%-30%	Fitch (1996)	1984 - 1987	1,524
Multifamily	17.5%	Archer et al (1999)	1971 - 1986	9,637

Table 9: Credit spreads for the Titman-Torous (1989) model. Parameters of the Titman-Torous (TT) model are calculated from the model presented in this paper. The volatility, correlation and payout rate in the Titman-Torous model refer to the value process of the project, and not to the cashflow process. Parameter values for the model presented in this paper (called TTT in the Table), other than the ones reported in the table, are the same as in Table 1. Initial values of lease rates l and long term lease rate levels L are expressed as a percentage of their base case values. All other rates are annualized. γ is the depreciation rate, l the lease rate, L the long term lease level, r the short rate, r^* the long term level of interest rates, σ_l the instantaneous volatility of the lease rate, σ_L the instantaneous volatility of the long term lease level. Initial quality for all projects is set at the efficient level. Unless otherwise noted, loans are adjusted so that the loan-to-value ratio remains at 80% and mortgages are priced at par. Spreads are expressed in basis points.

γ	4 %	6 %	8 %	10 %	12 %	14 %	16 %
TT volatility	9.6 %	10.1 %	10.7 %	11.3 %	11.9 %	12.6 %	13.2 %
TT correlation	-32.5 %	-29.4 %	-26.6 %	-24.1 %	-21.7 %	-19.6 %	-17.8 %
Payout rate	5.6 %	5.6 %	5.5 %	5.4 %	5.3 %	5.2 %	5.1 %
TT Credit Spread	37	44	52	59	68	78	88
TTT Credit Spread	62	78	93	109	128	145	162
l	40 %	60 %	80 %	100 %	120 %	140 %	160 %
TT volatility	14.3 %	12.8 %	11.9 %	11.3 %	10.9 %	10.7 %	10.6 %
TT correlation	-29.7 %	-28.2 %	-26.3 %	-24.1 %	-21.8 %	-19.5 %	-17.3 %
Payout rate	1.0 %	2.6 %	4.1 %	5.4 %	6.6 %	7.7 %	8.7 %
TT Credit Spread	19	25	38	59	86	118	153
TTT Credit Spread	58	72	87	109	133	155	180
L	40 %	60 %	80 %	100 %	120 %	140 %	160 %
TT volatility	11.1 %	11.2 %	11.2 %	11.3 %	11.3 %	11.4 %	11.5 %
TT correlation	-1.8 %	-12.1 %	-19.2 %	-24.1 %	-27.4 %	-29.8 %	-31.5 %
Payout rate	12.8 %	9.2 %	6.9 %	5.4 %	4.4 %	3.7 %	3.1 %
TT Credit Spread	417	201	105	59	37	25	18
TTT Credit Spread	531	279	165	109	81	63	53
r^*	3 %	4 %	5 %	6 %	7 %	8 %	9 %
TT volatility	12.5 %	11.9 %	11.5 %	11.3 %	11.0 %	10.8 %	10.7 %
TT correlation	-31.0 %	-29.0 %	-26.5 %	-24.1 %	-21.9 %	-19.9 %	-17.9 %
Payout rate	3.3 %	3.9 %	4.7 %	5.4 %	6.1 %	6.7 %	7.2 %
TT Credit Spread	49	51	54	59	64	67	70
TTT Credit Spread	71	83	96	109	124	137	153
σ_l	7 %	10 %	13 %	16 %	19 %	22 %	25 %
TT volatility	10.5 %	10.7 %	10.9 %	11.3 %	11.7 %	12.1 %	12.6 %
TT correlation	-33.7 %	-30.6 %	-27.3 %	-24.1 %	-20.9 %	-17.9 %	-15.0 %
Payout rate	5.4 %	5.4 %	5.4 %	5.4 %	5.4 %	5.4 %	5.4 %
TT Credit Spread	47	50	55	59	65	71	80
TTT Credit Spread	97	102	105	109	115	121	127
σ_L	3 %	5 %	7 %	9 %	11 %	13 %	15 %
TT volatility	6.7 %	8.0 %	9.5 %	11.3 %	13.1 %	14.9 %	17.2 %
TT correlation	-43.0 %	-35.3 %	-28.9 %	-24.1 %	-20.5 %	-17.7 %	-15.4 %
Payout rate	5.5 %	5.5 %	5.5 %	5.4 %	5.4 %	5.3 %	5.1 %
TT Credit Spread	12	22	38	59	83	111	147
TTT Credit Spread	37	55	83	109	139	167	198
Loan to Value	65 %	70 %	75 %	80 %	85 %	90 %	95 %
TT volatility	11.3 %	11.3 %	11.3 %	11.3 %	11.3 %	11.3 %	11.3 %
TT correlation	-24.1 %	-24.1 %	-24.1 %	-24.1 %	-24.1 %	-24.1 %	-24.1 %
Payout rate	5.4 %	5.4 %	5.4 %	5.4 %	5.4 %	5.4 %	5.4 %
TT Credit Spread	21	34	46	59	94	142	212
TTT Credit Spread	52	68	87	109	139	186	257