The value of an operating electricity production unit

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Abstract

In this paper we develop an equilibrium-based net present value model of an operating electricity production unit whose supply is given by a stochastic, mean-reverting process. The price process for electricity is derived from an underlying pair of stochastic, aggregate supply and demand processes that are also mean-reverting, while the instantaneous supply and demand functions are iso-elastic. The model is illustrated by a set of experimental data.

1 Introduction

1.1 Electricity as a peculiar commodity

In recent years, electricity markets all over the world have been deregulated. The regulating bodies aim for more efficient production and pricing of electricity by allowing competition among producers, while still controlling the natural monopolies of distributors. New challenges are introduced along with this process. In deregulated electricity markets, price uncertainty is typically high, and valuation of real and derivative assets and risk management are therefore important topics, as opposed to the situation when prices were regulated. A deep understanding of price behaviour is necessary to facilitate such activities.

Electricity differs in several respects from many other commodities. Some important features are[1]:

- **Non-storability:** There is currently no technology by which electricity can be stored effectively once generated. Therefore electricity demand and supply has to be balanced continuously in a transmission network to prevent the network from collapsing.

- **Non-traded asset:** The lack of storage technology implies that electricity cannot be considered a financial asset held purely for investment purposes. The usual cash-and-carry arbitrage relationship does not apply to electricity.

- **No lower bound:** Since electricity cannot be sold short there is no lower bound on electricity prices. In fact, negative prices have occurred in several electricity markets. This may happen, as power plants have to get rid of excess output in periods when demand is low.

- **Generating technology:** Electricity may be generated from natural gas, coal, oil, nuclear fuel, water turbines and renewable sources such as wind power, solar energy and biomass. Some of these are more flexible when it comes to scaling production in the short run.

- **Transmission capacity:** After electricity is generated, it is transmitted over high-voltage power lines before being distributed to the end users. In periods of high demand, the electricity transmitted may come close to maximum capacity. Increased demand cannot be met by increased supply, and prices may jump to extreme levels for short periods of time. In some electricity markets "price spikes" are common (see Deng (2000) and Clewlow and Strickland (2000)).

[1]See Leong (1997), Kaminski et al. (2000), Deng (2000), Clewlow and Strickland (2000) and Philipović (1998) for a thorough discussion on several of these issues.

- **Seasonal patterns:** In many markets prices peak twice a year, once during winter due to demand for heating, and once in mid-summer caused by the demand for air-conditioning. Electricity markets also exhibit daily and weekly price patterns.

Some of these fundamentals, such as non-storability, are common to all electricity markets, while others, such as seasonal patterns due to changing weather conditions, generating technology and transmission capacity, are specific to each regional electricity market. Hence, we cannot expect to find a global "fit-all" stochastic representation of electricity price dynamics.

1.2 Spot- and forward based models

The answers to risk management and valuation questions in the electricity sector proposed to this date, builds heavily on theories that have been developed for other commodities or financial assets. Some authors have suggested electricity spot price models. This approach was originally developed for traditional commodities, building on the theory of storage.[2] From the list of electricity market characteristics discussed above, we see that electricity does not fit the standard storage based commodity model since the non-storability is the most salient feature of this commodity. Continuous dynamic hedging is impossible directly in the underlying asset. Still, spot price models have been investigated in the literature. In these models the spot price is treated

[2] In the commodity literature, the theory of storage developed by Kaldor (1939), Working (1948) and (1949), Telser (1958) and Brennan (1958) and (1991), is the dominant model of commodity spot and futures prices. The futures and spot price differential is equal to the cost of storage (including interest) and an implicit benefit that producers and consumers receive by holding inventories of a commodity. This benefit is termed the convenience yield. The most obvious benefit from holding inventory, is the possibility to sell at an occurring price peak.

Brennan and Schwartz (1985) coupled the theory of storage with the dynamic hedging argument of contingent claims analysis developed originally for the stock marked. The authors modelled the commodity spot price as a geometric Brownian motion. Assuming continuous trading opportunities in the commodity, a constant risk free rate and constant proportional convenience yield, they developed no-arbitrage valuation expressions and optimal managing policies for a real asset (mine). Gibson and Schwartz (1990) provide a generalisation in which the convenience yield is modelled as a mean reverting Ornstein-Uhlenbeck process. Schwartz (1997) added the interest rate as a third stochastic factor, while Hilliard and Reis (1998) generalised this three-factor model to also include jumps.

as a state variable on which derivatives are written, and for valuation purposes this state variable is adjusted for risk (making ad hoc assumptions). Examples are Kamat and Ohren (2000), Clewlow and Strickland (1999b) and (2000) and Philipović (1998). The technically most advanced of these studies is Deng (2000). He models the log of the spot price with mean reversion, regime switching, stochastic volatility and different types of jumps. The research mentioned above is rather limited when it comes to empirical estimation and testing. The reason is, of course, limited data availability in most electricity markets. As far as we know, the most comprehensive empirical study of electricity spot prices is Lucia and Schwartz (2000). They estimate several spot price models with mean reversion in levels and natural logs and different seasonal specifications in the Nordic electricity market.

The main problem with spot price based models is that forward prices are given endogenously from the parameters governing the spot price dynamics. Thus theoretical forward prices will in general not be consistent with market observed forward prices. As a response to this, a line of research has focused on modelling the evolution of the whole forward curve, taking as given the initial term structure. Examples of this research, building on the pioneering work on modelling forward interest rate by Heath et al. (1992), are Cortazar and Schwartz (1994), Clewlow and Strickland (1999a) and (2000) and Miltersen and Schwartz (1998). The advantages of working directly with forward prices are even greater for electricity than for other commodities. The model is consistent with observable forward and futures prices, and, as opposed to electricity spot price models, we are working directly with tradable assets. Previous work along these lines are Clewlow and Strickland (1999a) and Bjerksund et al. (2000) who assume a multi-factor lognormal evolution of the forward curve where the volatility term structure is given by deterministic functions of time to maturity of the forward contract. A forward price model can provide us with some of the answers we might be interested in. The drawback of this method is low liquidity in many regional markets, and fairly short contracts. If we want to calculate the value of a production unit, say, we need to consider far more distant prices.

1.3 An equilibrium approach

In the process of deregulation of the electricity industry, the ownership structures of the production units change. Typically production owned by municipalities or other governmental bodies are sold to private investors. The question we ask in this paper is: What is the correct price for an operating electricity unit? To address this question we construct an equilibrium model with stochastic supply and demand and a corresponding equilibrium spot price. We then derive a simple net present value formula for the value of a production unit. Using suggestive parameter values, we compute the theoretical value of a production unit. The qualitative behaviour of the model is investigated by numerical analysis using experimental data.

The resulting equilibrium electricity spot price is a generalisation of the one-factor model suggested by Schwartz (1997), but in our setting the electricity spot price is endogenous. Its dynamics depends on supply and demand dynamics in the market. There are several advantages of such an approach. First, it is possible to distinguish between generating technologies, as different technologies will have different underlying characteristics in terms of volatility, correlation with aggregate supply and demand variables etc. Second, as our spot price process is endogenous, its parameters can be estimated from historical time series of underlying variables. This representation is particular convenient as the history of deregulated electricity markets is very short. The empirical foundation for estimating the parameters of an exogenous price process is therefore accordingly short. Reliable data for the underlying supply and demand parameters are probably easier to find, since they to some extent are determined by independent factors, such as weather changes, population growth and aggregate economic growth. Based on long-term data series of such variables, possibly somewhat adjusted in light of observed market behaviour over the recent history with deregulated markets, a better foundation for predictive analysis may be obtained.

2 The economy

Let aggregate supply be iso-elastic, given by the function

$$M \times S^{\gamma}$$

where γ (> 0) is the (constant) price elasticity of supply, S is the energy price, and M, which is a rough measure of aggregate capacity, follows the mean-reverting diffusion

$$dM = (\beta_M - \kappa_M \ln(M))\, M dt + \sigma_M M dB_M \tag{1}$$

Likewise, let aggregate demand be iso-elastic, given by the function

$$N \times S^{-\varepsilon}$$

where ε (> 0) is the magnitude of the price elasticity of demand, and N, which is a rough measure of the aggregate market size, follows the diffusion

$$dN = \left(\beta_N - \kappa_N \ln(N)\right) N dt + \sigma_N N dB_N \tag{2}$$

The two Brownian motions are correlated with $dB_M dB_N = \rho_{MN} dt$. In equilibrium, where supply equals demand, we have

$$S = (N/M)^{\alpha}$$

where $\alpha = 1/(\varepsilon + \gamma)$ (> 0). Thus the spot price only depends on the sum of elasticities of supply (γ) and demand (ε). For some technologies (such as wind power), production can hardly be scaled at all, so $\gamma = 0$ is a reasonable assumption. Noting that $\frac{\partial S}{\partial M} = -\alpha \frac{S}{M}$, $\frac{\partial S}{\partial N} = \alpha \frac{S}{N}$, $\frac{\partial^2 S}{\partial M^2} = \alpha(\alpha+1)\frac{S}{M^2}$, $\frac{\partial^2 S}{\partial N^2} = \alpha(\alpha-1)\frac{S}{N^2}$, and $\frac{\partial^2 S}{\partial M \partial N} = \frac{\partial^2 S}{\partial N \partial M} = -\alpha^2 \frac{S}{MN}$, Ito's lemma yields:

$$\begin{aligned}
dS &= \left(\beta_M - \kappa_M \ln(M)\right) M \left(-\alpha \frac{S}{M}\right) dt + \left(-\alpha \frac{S}{M}\right) \sigma_M M dB_M \\
&\quad + \left(\beta_N - \kappa_N \ln(N)\right) N \left(\alpha \frac{S}{N}\right) dt + \left(\alpha \frac{S}{N}\right) \sigma_N N dB_N \\
&\quad + \frac{1}{2}\alpha(\alpha+1)\frac{S}{M^2}\sigma_M^2 M^2 dt + \frac{1}{2}\alpha(\alpha-1)\frac{S}{N^2}\sigma_N^2 N^2 dt \\
&\quad + \frac{1}{2}\left(-2\alpha^2 \frac{S}{MN}\right) \sigma_M \sigma_N \rho_{MN} M N dt \\
&= \left(\beta_S - \alpha\left(\kappa_N \ln(N) - \kappa_M \ln(M)\right)\right) S dt + \alpha S \left(\sigma_N dB_N - \sigma_M dB_M\right)
\end{aligned} \tag{3}$$

where

$$\beta_S = \alpha\left(\beta_N - \beta_M\right) + \frac{\alpha}{2}\left((\alpha+1)\sigma_M^2 + (\alpha-1)\sigma_N^2 - 2\alpha\sigma_M\sigma_N\rho_{MN}\right)$$

As observed from (3), the spot price volatility is proportional to α - the inverse of the sum of the elasticities of the underlying supply and demand curves. Thus the underlying spot price will be highly fluctuating if neither supply nor demand are elastic; even if the demand and supply processes themselves are not very volatile.

When valuing real assets, the spot price process is usually given as an exogenous process. In the seminal paper by Brennan and Schwartz (1986), the spot price is assumed to be geometric Brownian motion. This model has been modified by Schwartz (1997). He assumes mean reversion in the log of the price. If we set $\kappa_N = \kappa_M \equiv \kappa_S$, and note that $\ln(S) = (\alpha \ln(N) - \alpha \ln(M))$, (3) simplifies to

$$dS = (\beta_S - \kappa_S \ln(S))\, Sdt + \alpha S\, (\sigma_N dB_N - \sigma_M dB_M) \tag{4}$$

Hence, the price process is mean-reverting, with parameters determined by the underlying supply and demand processes. We can express (4) as

$$dS = (\beta_S - \kappa_S \ln(S))\, Sdt + \alpha S \left(\sigma_N dB_N - \sigma_M \rho_{MN} dB_N + \sigma_M \sqrt{1 - \rho_{MN}^2} dB \right)$$

where B and B_N are independent Brownian motions. If we set $\rho_{MN} = 1$, the expression above reduces to

$$dS = (\beta_S - \kappa_S \ln(S))\, Sdt + \sigma_S S dB_N \tag{5}$$

where $\sigma_S = \alpha\, (\sigma_N - \sigma_M)$. Setting $\kappa_S = 0$ gives the Brennan and Schwartz (1986) model, and assuming that κ_S is a positive constant, gives the one-factor model of Schwartz (1990).

3 The value of a production unit

We assume that the market is characterised by perfect competition so the supply from a single producer has no effect on price. Now, let the uncertain supply, Q, of a producer be given by

$$dQ = \left(\beta_Q - \kappa_Q \ln(Q) \right) Qdt + \sigma_Q Q dB_Q \tag{6}$$

The Brownian motion, dB_Q is correlated with dB_N and dB_M in (1) and (2) with $dB_Q dB_M = \rho_{QM} dt$ and $dB_Q dB_N = \rho_{QN} dt$ respectively. The income, I, is defined as $I \equiv QS$. Note that $\ln I = \ln Q + \ln S$. Applying Ito's formula we get

$$\begin{aligned} dI &= QdS + SdQ + dQdS \\ &= (\beta_I - \alpha\, (\kappa_N \ln(N) - \kappa_M \ln(M)) - \kappa_Q \ln(Q))\, Idt \\ &\quad + \alpha \sigma_N I dB_N - \alpha \sigma_M I dB_M + \sigma_Q I dB_Q \end{aligned} \tag{7}$$

where

$$\beta_I = \beta_S + \beta_Q + \alpha\sigma_Q \left(\sigma_N \rho_{QN} - \sigma_M \rho_{QM}\right)$$

If the spot price process follows the diffusion in (4), and we set $\kappa_S = \kappa_Q = \kappa_I$, then (7) reduces to

$$dI = (\beta_I - \kappa_I \ln I)\, Idt + \alpha\sigma_N IdB_N - \alpha\sigma_M IdB_M + \sigma_Q IdB_Q$$

We recognise the simple mean reverting process.

Let the cost be given by a fixed component, C_F, and a variable component dependent on the quantity produced, θQ. Hence we have

$$C = C_F + \theta Q$$

The earnings, A, are given by

$$A = I - C$$

In the set-up above, both price and quantity are continuous functions of time. However, if we observe price and quantity from the market place with hourly, daily, weekly or monthly intervals between observations, we find strong seasonal patterns in the data. If we instead use yearly production and yearly average prices, the processes suggested above make more sense.

Assuming that no new investments are made, we can find the value of a production unit, V, as the expected value of the future earnings, discounted by a risk adjusted discount rate k. At time $t = 0$ the value of the production unit is

$$\begin{aligned} V_0 &= E_0 \left[\int_0^T e^{-ks} A(s) ds \right] \\ &= \int_0^T e^{-ks} \left(E_0 \left[I(s) \right] - E_0 \left[C(s) \right] \right) ds \\ &= \int_0^T e^{-ks} \left(E_0 \left[I(s) \right] - C_F - \theta E_0 \left[Q(s) \right] \right) ds \end{aligned} \tag{8}$$

where T is the remaining life time of the production unit. This provides a closed form solution for the value of the production unit. We show in the appendix that the explicit closed form expression becomes

$$\begin{aligned} V_0 &= \int_0^T \left(I_0 e^{\Delta(s) - ks} - e^{-ks} C_F - \theta e^{\chi(s) - ks} \right) ds \\ &= \int_0^T I_0 e^{\Delta(s) - ks} ds - C_F \int_0^T e^{-ks} ds - \theta \int_0^T e^{\chi(s) - ks} ds \end{aligned} \tag{9}$$

where

$$\Delta(s) = s\left(\beta_I - \frac{1}{2}\xi^2\right) - \Xi(s) + \Psi(s) - \Theta(s) + \frac{1}{2}\Gamma(s)$$

$$\Xi(s) = \alpha\varphi_N s + \alpha \ln N(0)\left(1 - e^{-\kappa_N s}\right) + \frac{\alpha\varphi_N}{\kappa_N}(e^{-\kappa_N s} - 1)$$

$$\Psi(s) = \alpha\varphi_M s + \alpha \ln M(0)\left(1 - e^{-\kappa_M s}\right) + \frac{\alpha\varphi_M}{\kappa_M}(e^{-\kappa_M s} - 1)$$

$$\Theta(s) = \varphi_Q s + \ln Q(0)\left(1 - e^{-\kappa_Q s}\right) + \frac{\varphi_Q}{\kappa_Q}(e^{-\kappa_Q s} - 1)$$

$$\begin{aligned} \xi^2 &= \alpha^2\sigma_N^2 + \alpha^2\sigma_M^2 + \sigma_Q^2 - 2\alpha^2\sigma_N\rho_{NM}\sigma_M \\ &\quad +2\alpha\rho_{NQ}\sigma_N\sigma_Q - 2\alpha\rho_{MQ}\sigma_M\sigma_Q \end{aligned}$$

$$\begin{aligned} \Gamma(s) &= \frac{1}{2}\alpha^2\left(\frac{\sigma_Q^2}{\alpha^2}\frac{1 - e^{-2\kappa_Q s}}{\kappa_Q} + \sigma_N^2\frac{1 - e^{-2\kappa_N s}}{\kappa_N} + \sigma_M^2\frac{1 - e^{-2\kappa_M s}}{\kappa_M}\right) \\ &\quad +2\alpha\sigma_N\sigma_Q\rho_{NQ}\frac{1 - e^{-\left(\kappa_Q + \kappa_N\right)s}}{\kappa_Q + \kappa_N} - 2\alpha^2\sigma_N\sigma_M\rho_{NM}\frac{1 - e^{-(\kappa_M + \kappa_N)s}}{\kappa_M + \kappa_N} \\ &\quad -2\alpha\sigma_M\sigma_Q\rho_{MQ}\frac{1 - e^{-\left(\kappa_Q + \kappa_M\right)s}}{\kappa_Q + \kappa_M} \end{aligned}$$

$$\chi(s) = e^{-\kappa_j s}\ln Q_0 - \frac{\varphi_Q}{\kappa_Q}(e^{-\kappa_Q s} - 1) + \frac{\sigma_Q^2}{4\kappa_Q}\left(1 - e^{-2\kappa_Q s}\right)$$

See appendix for a proof.

In our numerical examples, we solve these integrals numerically. Of course the value of the firm depends crucially on the value of k. Since electricity is risky business, one might argue that k should be big. However from CAPM we know that only systematic risk is priced in an efficient marketplace. A basic result from this famous equilibrium theory says that

$$k = r_f + \frac{Cov\left(r_V, r_m\right)}{Var\left(r_m\right)}\left(r_m - r_f\right)$$

where r_f is the risk free rate, r_V is the return on an investment in a production facility and r_m is the return on the total market. We see that the crucial component is the covariance between r_V and r_m. A great deal of the uncertainty in the electricity industry is caused by exogenous factors such as weather and temperature. These factors have a far less direct impact on the return of the market as a whole, and thus this covariance component will in fact be relatively small. Setting this covariance to zero (which may be empirically plausible) means that k equals the risk free rate of return.

4 Numerical examples

In this section we explore the characteristics of the model by some numerical experiments, pertubating exogenous parameters around a reference data set. Considering the low price elasticities of supply and demand in electricity markets, we assume $\varepsilon + \gamma = 0.5$ as reference parameters. The aggregate demand and supply processes are symmetric in exogenous variables: $\sigma_M = \sigma_N = 0.15$, $\beta_M = \beta_N = 5$, and $\kappa_M = \kappa_N = 1$. This implies strong mean reversion. We assume that aggregate supply and demand are correlated by setting $\rho_{MN} = 0.5$. This could be a proxy for assumed capacity adjustments made possible e.g. in hydroelectric power production by storing water. It could also be a proxy for adjusting imports and exports along transmission lines with restricted capacity, assuming that the domestic spot price fluctuates around an international, fixed (or less volatile) spot price.

The volatility of the firm-specific production process is set to $\sigma_Q = 0.20$, as the supply of the firm may be more volatile than aggregate supply; for example, this is likely for hydroelectric power production which depends on rainfalls. The drift parameters are set to $\beta_Q = \kappa_Q = 1$. Initial values for all stochastic variables are set to the mean-reverting value ($M_0 = e^{(\beta_M/\kappa_M)}$, $N_0 = e^{(\beta_N/\kappa_N)}$, $Q_0 = e^{(\beta_Q/\kappa_Q)}$); the discount factor is 5 percent ($k = 0.05$), and the time horizon is infinite. For simplicity, all costs are set to zero.

Table 1 shows the net present value of the firm under these assumptions, and under two alternative assumptions for most exogenous variables. The net present value of income with the reference data equals 54.26. This is slightly lower than the deterministic value ($\sigma_M = \sigma_N = \sigma_Q = 0$), which yields 54.37. The corresponding geometric Brownian motion of demand and supply ($\kappa_M = \kappa_N = \kappa_Q = \beta_M = \beta_N = \beta_Q = 0$) is 77.67, so the effect of mean reversion is strong. The first row of results shows that the value of the firm

	Low		Reference data		High	
	Parameter	V_0	Parameter	V_0	Parameter	V_0
α	1.00	53.76	2.00	54.26	4.00	57.11
ρ_{MN}	0.00	55.46	0.50	54.26	1.00	53.08
ρ_{QM}	0.00	55.06	0.50	54.26	1.00	53.47
ρ_{QN}	-1.00	52.69	0.00	54.26	1.00	55.87
σ_M	0.00	53.89	0.15	54.26	0.30	59.58
σ_N	0.00	55.43	0.15	54.26	0.30	53.16
σ_Q	0.00	55.57	0.20	54.26	0.40	52.00
β_M & κ_M	0.5 & 0.1	70.69	5.0 & 1.0	54.26	50.0 & 10.0	53.77
β_N & κ_N	0.5 & 0.1	54.94	5.0 & 1.0	54.26	50.0 & 10.0	55.21
β_Q & κ_Q	0.5 & 0.1	51.36	5.0 & 1.0	54.26	50.0 & 10.0	55.37

$M_0 = exp(\beta_M/\kappa_M) = 148.4$, $N_0 = exp(\beta_N/\kappa_N) = 148.4$,
$Q_0 = exp(\beta_Q/\kappa_Q) = 2.72$

Table 1: Value of a production unit.

increases when demand or supply becomes less elastic (i.e., when α increases). This is intuitive, as low price elasticities combined with uncertainty makes price peaks more likely. The explanation of the second row is somewhat similar: The firm benefits from low correlation between supply and demand as this also makes price peaks more likely.

A small firm also benefits from having a production pattern that is not correlated with aggregate supply; i.e., ρ_{QM} ought to be low. This leads to higher production during periods when other firms do not produce as much; i.e., when prices are high. Likewise, a high ρ_{QN} is obviously preferred as this leads to high production when demand is high.

Effects of uncertainty depend on where it appears. High uncertainty in aggregate supply is beneficial, apparently because it can bring periods with low supply and extremely high prices. Since the log-function is concave, the downward speed when production is above the long-term mean-reverting value is lower than the upward speed on the lower side. Therefore increased uncertainty tends to increase the long-term expected value of production. The same effect works in the opposite direction for aggregate supply, so aggregate supply uncertainty harms the firm, while firm-specific uncertainty of supply is beneficial.

The next three rows show that the results do not change much when scaling the mean-reverting speed up or down by a factor of 10 in either direction, except for weak mean reversion in aggregate supply. Low speed is best for the firm for aggregate supply (β_M, κ_M), while high speed is preferred for aggregate demand (β_N, κ_N) and firm-specific supply (β_Q, κ_Q). Low mean reversion in aggregate supply ($\kappa_M = 0.1$) leads to a net present value of 70.69, which is surprisingly close to the geometric Brownian case (77.67). Other investigations indicate that this is highly related to the elasticity of supply and demand. With inelastic markets, slow mean reversion in aggregate supply tends to imply long periods of low supply when uncertainty is governed by a geometric diffusion term. It would be very interesting to test general validity of such a result by further empirical analyses.

5 Concluding remarks

We have developed an equilibrium-based net present value model of an operating electricity production unit. It should be noted that the model is general, so it may also apply to other markets with similar supply and demand characteristics. One nice feature of the model lies in its explicit formulation of basic supply and demand relationships. This may simplify further empirical work, which is the obvious next step, because some required variables appear to be highly related to phenomena such as temperature changes and other natural phenomena for which long-term data exist. Considering the total number of variables in the model, a satisfactory empirical analysis does not appear to be trivial. In particular, ongoing market liberalisation in many countries is coupled with capacity adjustments as well as increased trade opportunities along transmission lines with restricted capacity. Thus both production as well as transmission capacities are changing more or less continuously, and it is not straightforward to determine the extent of virtually any market. If such technical difficulties can be overcome, one may hope that the model can be used to estimate the market value of actual production units. Up until now, such valuations seem to be based on less scientific approaches.

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6 Appendix

6.1 Some useful results

The mean-reverting Ornstein-Uhlenbeck equation is given by

$$dX = (u_X - k_X X)dt + \sigma_X dB_t \tag{10}$$

where κ and m are real constants and B_t is standard Brownian motion.

Proposition 1 *The mean-reverting Ornstein-Uhlenbeck process is given by*

$$X_t = e^{-k_X t} X_0 - \frac{u_X}{k_X}(e^{-k_X t} - 1) + \sigma_X e^{-kt} \int_0^t e^{ks} dB(s) \tag{11}$$

Proof.

See eg. Øksendal (1995).

■

Let a stochastic variable, Y, be governed by the stochastic differential equation (SDE)

$$dY = (u_Y - k_Y \ln(Y))\, Y dt + \sigma_Y Y dB_t \tag{12}$$

where u_Y and k_Y are real constants and B_t is standard Brownian motion.

Proposition 2 *The natural log of Y is governed by the mean-reverting Ornstein-Uhlenbeck process:*

$$d \ln Y = (g_Y - k_Y \ln(Y))\, dt + \sigma_Y dB \tag{13}$$

where

$$g_Y = u_Y - \frac{1}{2}\sigma_Y^2$$

Proof.

Apply Ito's formula on $\ln Y$.

■

Corollary 3 *The solution of the integral $\ln Y$ is given by*

$$\begin{aligned} \int_0^t \ln(Y)(s) ds &= \frac{g_Y t}{k_Y} + \frac{\ln(Y_0)}{k_Y}\left(1 - e^{-k_Y t}\right) + \frac{g_Y}{k_Y^2}(e^{-k_Y t} - 1) \\ &+ \frac{\sigma_Y}{k_Y}\int_0^t dB(s) - \frac{\sigma_Y}{k_Y} e^{-k_Y t} \int_0^t e^{k_Y s} dB(s) \end{aligned} \tag{14}$$

Proof.

The integral form of (13) is

$$\int_0^t d\ln Y(s) = \int_0^t (g_Y - k_Y \ln(Y)(s))\, ds + \sigma_Y \int_0^t dB(s) \tag{15}$$

We also have the obvious relationship

$$\ln Y(t) - \ln Y(0) = \int_0^t d\ln Y(s) \tag{16}$$

Combining (15) and (16) gives

$$\begin{aligned}
\ln Y(t) - \ln Y(0) &= \int_0^t d\ln Y(s) \\
&= \int_0^t (g_Y - k_Y \ln Y(s))\, ds + \sigma_Y \int_0^t dB(s) \\
&= g_Y t - k_Y \int_0^t \ln Y(s) ds + \sigma_Y \int_0^t dB(s) \\
k_Y \int_0^t \ln Y(s) ds &= g_Y t + \sigma_Y \int_0^t dB(s) + \ln Y(0) - \ln Y(t) \\
\int_0^t \ln Y(s) ds &= \frac{g_Y t}{k_Y} + \frac{\sigma_Y}{k_Y} \int_0^t dB(s) + \frac{1}{k_Y} (\ln Y(0) - \ln Y(t)) \quad (17)
\end{aligned}$$

From (11) and (13) we know that the solution to $\ln Y(t)$ is

$$\ln Y(t) = e^{-k_Y t} \ln Y_0 - \frac{g_Y}{k_Y}(e^{-k_Y t} - 1) + \sigma_Y e^{-k_Y t} \int_0^t e^{k_Y s} dB(s) \tag{18}$$

Inserting (18) into (17) yields the desired result.

■

6.2 The net present value of a production unit

Now we set out to prove (9). Let the SDE governing the stochastic income, I, be given by (7). Then, by Ito's formula, the natural log of income is governed by the following SDE:

$$\begin{aligned}
d\ln I &= (\beta_I - \alpha (\kappa_N \ln(N) - \kappa_M \ln(M)) - \kappa_Q \ln (Q))\, dt \\
&\quad -\frac{1}{2}\xi^2 dt + \alpha\sigma_N dB_N - \alpha\sigma_M dB_M + \sigma_Q dB_Q
\end{aligned}$$

where

$$\begin{aligned}\xi^2 dt &= (\alpha\sigma_N dB_N - \alpha\sigma_M dB_M + \sigma_Q dB_Q)^2 \\ &= \begin{pmatrix} \alpha^2\sigma_N^2 + \alpha^2\sigma_M^2 + \sigma_Q^2 - 2\alpha^2\sigma_N\rho_{NM}\sigma_M \\ +2\alpha\rho_{NQ}\sigma_N\sigma_Q - 2\alpha\rho_{MQ}\sigma_M\sigma_Q \end{pmatrix} dt\end{aligned}$$

The integral form of $\ln I$ is

$$\begin{aligned}\int_0^t d\ln I(s)ds &= \int_0^t \left(\beta_I - \frac{1}{2}\xi^2\right) ds - \alpha\kappa_N \int_0^t \ln(N(s))\, ds \\ &+\alpha\kappa_M \int_0^t \ln(M(s))\, ds - \kappa_Q \int_0^t \ln(Q(s))\, ds \\ &+\alpha\sigma_N \int_0^t dB_N - \alpha\sigma_M \int_0^t dB_M + \sigma_Q \int_0^t dB_Q \qquad (19)\end{aligned}$$

Remembering (11) and (13), and using (14) we have the following

$$\begin{aligned}-\alpha\kappa_N \int_0^t \ln(N(s))\, ds &= -\Xi(t) - \alpha\sigma_N \int_0^t dB_N(s) \\ &+\alpha\sigma_N e^{-\kappa_N t} \int_0^t e^{\kappa_N s} dB_N(s) \qquad (20)\end{aligned}$$

$$\begin{aligned}\alpha\kappa_M \int_0^t \ln(M(s))\, ds &= \Psi(t) + \alpha\sigma_M \int_0^t dB_M(s) \\ &-\alpha\sigma_M e^{-\kappa_M t} \int_0^t e^{\kappa_M s} dB_M(s) \qquad (21)\end{aligned}$$

$$\begin{aligned}-\kappa_Q \int_0^t \ln(Q(s))\, ds &= -\Theta(t) - \sigma_Q \int_0^t dB_Q(s) \\ &+\sigma_Q e^{-\kappa_Q t} \int_0^t e^{\kappa_Q s} dB_Q(s) \qquad (22)\end{aligned}$$

where

$$\Xi(t) = \alpha\varphi_N t + \alpha \ln N(0)\left(1 - e^{-\kappa_N t}\right) + \frac{\alpha\varphi_N}{\kappa_N}(e^{-\kappa_N t} - 1)$$

$$\Psi(t) = \alpha\varphi_M t + \alpha \ln M(0)\left(1 - e^{-\kappa_M t}\right) + \frac{\alpha\varphi_M}{\kappa_M}(e^{-\kappa_M t} - 1)$$

$$\Theta(t) = \varphi_Q t + \ln Q(0)\left(1 - e^{-\kappa_Q t}\right) + \frac{\varphi_Q}{\kappa_Q}(e^{-\kappa_Q t} - 1)$$

and

$$\varphi_j = \beta_j - \frac{1}{2}\sigma_j^2$$

for $j = N$, M and Q respectively. Inserting (20), (21) and (22) into (19) and collecting drift and diffusion terms gives

$$\begin{aligned}
\ln I(t) - \ln I(0) &= \int_0^t \left(\beta_I - \frac{1}{2}\xi^2\right) ds \\
&\quad -\Xi(t) - \alpha\sigma_N \int_0^t dB_N(s) + \alpha\sigma_N e^{-\kappa_N t}\int_0^t e^{\kappa_N s} dB_N(s) \\
&\quad +\Psi(t) + \alpha\sigma_M \int_0^t dB_M(s) - \alpha\sigma_M e^{-\kappa_M t}\int_0^t e^{\kappa_M s} dB_M(s) \\
&\quad -\Theta(t) - \sigma_Q \int_0^t dB_Q(s) + \sigma_Q e^{-\kappa_Q t}\int_0^t e^{\kappa_Q s} dB_Q(s) \\
&\quad +\alpha\sigma_N \int_0^t dB_N - \alpha\sigma_M \int_0^t dB_M + \sigma_Q \int_0^t dB_Q \\
&= \int_0^t \left(\beta_I - \frac{1}{2}\xi^2\right) ds - \Xi(t) + \Psi(t) - \Theta(t) \\
&\quad +\alpha\sigma_N e^{-\kappa_N t}\int_0^t e^{\kappa_N s} dB_N(s) - \alpha\sigma_M e^{-\kappa_M t}\int_0^t e^{\kappa_M s} dB_M(s) \\
&\quad +\sigma_Q e^{-\kappa_Q t}\int_0^t e^{\kappa_Q s} dB_Q(s)
\end{aligned}$$

The natural log of income, $\ln I(t)$, is normally distributed with

$$\ln I(t) \sim \mathcal{N}\left(\ln I(0) + \Delta(t), \Gamma(t)\right)$$

where

$$\Delta(t) = \int_0^t \left(\beta_I - \frac{1}{2}\xi^2\right) ds - \Xi(t) + \Psi(t) + \Theta(t)$$

and

$$\Gamma(t) = \frac{1}{2}\alpha^2\left(\frac{\sigma_Q^2}{\alpha^2}\frac{1 - e^{-2\kappa_Q t}}{\kappa_Q} + \sigma_N^2\frac{1 - e^{-2\kappa_N t}}{\kappa_N} + \sigma_M^2\frac{1 - e^{-2\kappa_M t}}{\kappa_M}\right)$$

$$+2\alpha\sigma_N\sigma_Q\rho_{NQ}\frac{1-e^{-\left(\kappa_Q+\kappa_N\right)t}}{\kappa_Q+\kappa_N}-2\alpha^2\sigma_N\sigma_M\rho_{NM}\frac{1-e^{-(\kappa_M+\kappa_N)t}}{\kappa_M+\kappa_N}$$
$$-2\alpha\sigma_M\sigma_Q\rho_{MQ}\frac{1-e^{-\left(\kappa_Q+\kappa_M\right)t}}{\kappa_Q+\kappa_M}$$

By the property of the lognormal distribution we have that

$$\begin{aligned}
E\left[I(t)\right] &= \exp\left(\ln I_0+\Delta(t)+\frac{1}{2}\Gamma(t)\right) \\
&= I_0\exp\left(\int_0^t\left(\beta_I-\frac{1}{2}\xi^2\right)ds-\Xi(t)+\Psi(t)+\Theta(t)+\frac{1}{2}\Gamma(t)\right)
\end{aligned}$$

which proves the first integral in (9). The second integral is obvious. We know from (13) and (11) that $\ln Q_t$ is normally distributed with

$$\ln Q_t\sim\mathcal{N}\left(e^{-\kappa_Q t}Q_0-\frac{\varphi_Q}{\kappa_Q}(e^{-\kappa_Q t}-1),\frac{\sigma_Q^2}{2\kappa_Q}\left(1-e^{-2\kappa_Q t}\right)\right)$$

By the property of the lognormal distribution we know that

$$E\left[Q_t\right]=\exp\left(e^{-\kappa_Q t}Q_0-\frac{\varphi_Q}{\kappa_Q}(e^{-\kappa_Q t}-1)+\frac{\sigma_Q^2}{4\kappa_Q}\left(1-e^{-2\kappa_Q t}\right)\right)$$

which concludes the proof.

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