Real Investment Opportunity Valuation and Timing Using a Finite-Lived American Exchange Option Methodology

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Abstract

In practice, the investment opportunities that can be delayed are more like exchange than simple call options, because there are uncertainties both in the gross project value (underlying asset) and in the investment cost (exercise price). Companies that have the option to invest at anytime until a certain date (the maturity), also often have some opportunity costs (the lost cash flows) in holding the option instead of the project. Incorporating these aspects leads to a more realistic evaluation process. In this research, we value three real investment projects as finite-lived American exchange options, correcting and applying the Carr (1988) model. We conclude that, as expected, the traditional Net Present Value method substantially undervalues projects with this kind of flexibility (excluding those that are deep in-the-money). This leads to wrong decisions about the timing of these investments. We also conclude that the results from using the corrected 1988 Carr model differ substantially from those that we obtain from using the uncorrected version. As expected, the corrected model gives results that are higher in value.

Keywords: Real Options; Investment Under Uncertainty; Deferment Option; Exchange Options.

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1. Introduction

It is widely accepted that the traditional Net Present Value (NPV) method is inadequate to value real investment opportunities in an uncertain environment (see Pindyck, 1991; Trigeorgis, 1993; Ross, 1995; among others). Although several papers focus on the value and timing of project adoption under uncertainty,[1] these papers make a simplifying but problematic assumption. Although the exercise price is fixed and known in advance (at the moment of the purchase of the option) in a typical ("vanilla") call option, such is rarely the case in a real options context. While a company may be able to make a fairly accurate estimate of the cost of current investment, there is much less precision about investment costs in the future.

As a consequence, the real option to invest in the future corresponds to an exchange option and not to a simple call option, because of its uncertain exercise price. The investment corresponds to the exchange of a risky asset, investment cost, for another, the gross project value. So, generally, when we value an investment opportunity, we are exposed to two sources of uncertainty, i.e., to two stochastic variables. Simplifying the evaluation process and assuming only one stochastic variable (the gross project value) may lead to wrong results. Therefore, the Black-Scholes (1973) model should not be used to value projects with these characteristics.

There are only a few models with the capacity to value investment opportunities with two stochastic variables. The most relevant models are by Margrabe (1978), Mcdonald and Siegel (1986) and Carr (1988 and 1995).[2] Each of these models has shortcomings and limitations.

Margrabe's model is not fully adequate because "his" exchange (European) option can only be exercised at maturity. This characteristic is unrealistic because a company owning an option to invest can, in principle, exercise that option at any time until maturity. In other words, the investment opportunities generally are American options. The Margrabe model can value American options only in the particular situation where the underlying asset does not distribute dividends. The reason is that, in the absence of dividends, an American option should never be exercised prior to maturity. In a real options context, "dividends" are the opportunity costs inherent in the decision to defer an investment (Majd and Pindyck, 1987). As in a financial options context, deferment implies the loss of the project's cash flows. Those lost cash flows must be seen as foregone "dividends", and must be taken into account.

The model of McDonald and Siegel values American options with two stochastic variables. The model has an important shortcoming since it assumes that the option, if unexercised, has an infinite maturity (that is, the option is perpetual). In practice, most investment opportunities do not continue forever, so they cannot be accurately valued using this model.

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[1] Examples include Majd and Pindyck (1987), Siegel, Smith and Paddock (1987), Dixit (1989), Trigeorgis (1991), Ingersoll and Ross (1992), Kemna (1993) and Lee (1997).

[2] Also Pindyck (1993) studies the impact of the uncertain costs on the project's value. However his work differs from ours in several ways, namely: (i) the projects take considerable time (several years) to complete (e.g., a nuclear plant, or the development of a new drug), (ii) the uncertainty is <u>over</u> the cost of completion, (iii) there are two types of uncertainty during the completion (technical uncertainty and input cost uncertainty), and (iv) finally, the value of the completed project is known with certainty and, at the end, his model is extended to incorporate uncertainties in that variable using a dynamic programming procedure.

The Carr model has the capacity to evaluate finite-lived American exchange options (AEOs). It simultaneously incorporates a limited temporal dimension, the flexibility to act (invest) at any time until maturity, and also uncertainty both in the gross project value and in the investment cost while allowing them to be correlated. In this paper, we follow, correct and apply the Carr (1988) model.

By viewing investment opportunities as options, analogous to financial options, we overcome the rigid and deterministic characteristics of the traditional NPV method, and capture some important, and contingent, sources of value, like operational flexibility and strategic interactions (Dixit and Pindyck, 1994; and Trigeorgis, 1996). We use the model to evaluate three real investment projects, and to analyze the optimal time to implement these investment projects.

We make three important contributions to the literature. Firstly, we correct an error in the Carr (1988) model that has a non-negligible impact on the option's value. Secondly, we apply the corrected version of this model to three real project evaluations. Third, we illustrate the substantial differences in the values obtained by applying the original and the corrected versions of the Carr model and the traditional NPV model.

The remainder of this paper is organised as follows. In the next section, we present the methodology and correct an error in the Carr (1988) paper.[3] In section 3, we show how the methodology can be operationalized, describing the three real investment projects and the inputs of the model. In section 4, we apply the corrected methodology to these three real investment projects, and analyse the model outputs. Section 5 concludes the paper.

2. Methodology

2.1 Valuation of Compound Exchange Options

Carr (1988) develops a model to value European compound exchange options. Carr (1988) proposes an approximate method to value American exchange options (AEOs), using a two moment extrapolation process. Carr (1995) extends the previous model to value AEOs using a three moment extrapolation process.[4]

The methodology proposed by Carr to value an AEO involves three steps. In the first step, the value of an European Exchange Option (EEO) on dividend-paying assets is determined. Assuming that V and D follow a geometric Brownian motion process given by:

$$\frac{dV}{V} = (\mu_V - \delta_V)dt + \sigma_V dZ \qquad (1a)$$

$$\frac{dD}{D} = (\mu_d - \delta_d)dt + \sigma_d dZ \qquad (1b)$$

then the value of the EEO, or *e(V,D,t)*, is given by the following equation (Mcdonald and Siegel, 1985):

[3] The Carr (1988) paper has an error, which is corrected in this paper.
[4] This paper extends the methodology of Carr (1988). While this paper uses a three moment extrapolation process, it does not correct for the error in Carr (1988).

$$e(V,D,t) = Ve^{-\delta_v t} N_1(d_1) - De^{-\delta_d t} N_1(d_2) \qquad (2)$$

where:

V is Gross Project Value;

D is Investment Cost;

μ_v and μ_d are the instantaneous expected return on V and the expected growth rate of the investment cost, respectively;

σ_v and σ_d are the volatility of V and D, respectively;

δ_v and δ_d are the "dividend-yields" on V and D, respectively;

dZ is the standard Wiener process (dZ = $\varepsilon_\tau \sqrt{dt}$, ε~(0,1));

$N_1(d)$ is the cumulative standard normal distribution;

$$d_1 = \frac{Ln(Pe^{-\delta . t}) + \frac{1}{2}\sigma^2 t}{\sqrt{\sigma^2 t}};$$

$$d_2 = d_1 - \sigma\sqrt{t};$$

$$P \equiv \frac{V}{D};$$

$\delta \equiv \delta_v - \delta_d$; and

$\sigma^2 = \sigma_v^2 + \sigma_d^2 - 2\rho\sigma_v\sigma_d$, where ρ is the correlation between changes in V and D.

In equations (1a) and (1b), the current values for V and D are known, and their future values have two components. The first component is deterministic since it corresponds to the drift, and the second component is stochastic since it corresponds to a stochastic process with variance increasing linearly with time.

In the second step, the value of a Pseudo-American Exchange Option (PAEO) is determined. Let t be the evaluation date, and T be the maturity date of the option. We can divide the time interval (T-t) into n equal periods. Let $E_n(T-t)$ be the value of the PEAO, where (T-t) is the time to maturity and the index n says that the option can be exercised at the end of any of the n periods until maturity. For n=1, the value is the same as that of the EEO, as given by the expression (2).

For n=2, we have a PAEO that can be exercised at T/2 or T. The option will <u>not</u> be exercised at T/2 if the opportunity cost of doing that exceeds the benefits; i.e., if:

$$Ve^{-\delta_v \Delta T} N_1(d_1) - De^{-\delta_d \Delta T} N_1(d_2) > \mathrm{V - D} \qquad (3)$$

where ΔT=T/2.

Instead of using two stochastic variables (V and D), we can redefine the expression by using P≡V/D. So:

$$Pe^{-\delta_v \Delta T} N_1(d_1) - e^{-\delta_d \Delta T} N_1(d_2) > P - 1 \qquad (4)$$

Let P* be the unique value of P that transforms the expression (4) into the following equation:

$$P^* e^{-\delta_v \Delta T} N_1(d_1) - e^{-\delta_d \Delta T} N_1(d_2) = P^* - 1 \qquad (5)$$

If the value of P is higher than P* at moment T/2, then the option is exercised to pay (V-D). In the other case where P<P*, the option is not exercised. It is essentially an EEO that expires at T and pays max(0, V-D). These contingent payoffs can be replicated by a portfolio with three European options, as shown in Table 1. The first option is an EEO with maturity at T. The second is also an European option with maturity at ΔT, and involves the exchange of P* units of D for one unit of V. The third option is a compound EEO that involves the exchange of (P*-1) units of D for the first EEO. Its maturity also is at ΔT.

[Please place Table 1 about here.]

The PAEO is similar to a portfolio containing the first two EEOs, with maturities corresponding to the two possible exercise dates (ΔT and T). However, if an option is exercised earlier it cannot be exercised at maturity. This is the reason for also including the short position on the compound option.

The value of a PAEO with n=2 corresponds to the value of this portfolio. The first two options can be valued using the expression (2). The third option can be valued using the following formula (Carr, 1988):

$$c(s(V,D,T),(P^*-1)D,\Delta T;\delta_V,\delta_d) = Ve^{-\delta_V . T} N_2\left(d_1^*, d_1; \rho\right) - De^{-\delta_d . T} N_2\left(d_2^*, d_2; \rho\right)$$

$$-(P^*-1)De^{-\delta_d . \Delta T} N_1\left(d_2^*\right) \qquad (6)$$

where

$$d_1^* = \frac{Ln\left(\frac{Pe^{-\delta\Delta T}}{P^*}\right) + 0{,}5\sigma^2\Delta T}{\sigma\sqrt{\Delta T}}$$

$$d_2^* = d_1^* - \sigma\sqrt{\Delta T}$$

N_2 is the bivariate cumulative standard normal distribution (see Appendix A)

$\rho = \sqrt{\frac{\Delta \mathrm{T}}{\mathrm{T}}}$; and

all the other variables are as previously defined.

By grouping all of the options, we arrive at the following value of the PAEO:

$$PAEO = Ve^{-\delta_V T} N_1(d_1) - De^{-\delta_d T} N_1(d_2) + Ve^{-\delta_V \Delta T} N_1(d_1^*) - P^* De^{-\delta_d \Delta T} N_1(d_2)$$
$$-\left[Ve^{-\delta_V.T} N_2\left(d_1^*, d_1; \rho\right) - De^{-\delta_d.T} N_2\left(d_2^*, d_2; \rho\right) - (P^* - 1)De^{-\delta_d.\Delta T} N_1\left(d_2^*\right)\right] \quad (7)$$

Using the following statistical equivalence:

$$N_1(b) - N_2(a, b; \rho) = N_2(-a, b; -\rho)$$

we can simplify (7) to obtain:

$$PAEO = Ve^{-\delta_V \Delta T} N_1(d_1^*) - De^{-\delta_d.\Delta T} N_1\left(d_2^*\right) + Ve^{-\delta_V.T} N_2\left(-d_1^*, d_1; -\rho\right)$$
$$- De^{-\delta_d.T} N_2\left(-d_2^*, d_2; -\rho\right) \quad (8)$$

In the third step, the AEO is valued using an extrapolation process. Let E_1 be the value of the EEO and E_2 the value of the PAEO. The value of the AEO can be estimated using the Richardson extrapolation method. Using the corrected version of the extrapolation formula presented by Carr (1988) gives (see Appendices B and C for more details on the derivation): [5]

$$AEO \approx E_2 + \frac{E_2 - E_1}{3} \quad (9)$$

[5] The extrapolation formula presented by Carr is: $AEO \approx E_1 + \frac{E_2 - E_1}{3}$.

Since all of the options have the same underlying assets and maturities, E_1 and E_2 can be valued using equations (2) and (8), respectively.

2.2 Application of the Methodology to the Valuation of Real Options

Because of the "now or never" characteristic of the traditional NPV method, the value of a real investment opportunity equals the difference between V_t and D_t, where V_t and D_t are the gross project value and the investment cost at moment t, respectively. According to this criterion, it is optimal to invest if $V_t > D_t$.

As pointed out by several authors, this criterion may be incorrect given uncertainty, and when project implementation can be deferred. The real options theory not only overcomes these problems but it also changes the valuation formula and the optimal investment decision (timing). The value of a real investment opportunity must include both the in-the-money value (NPV) and the value of the flexibility to postpone. Thus, the value of the project equals the traditional NPV plus the value of the Deferment Option (DO).[6]

In practice, the value of the investment opportunity (IO) is given by the maximum of the values of the AEO and the NPV; i.e., max (AEO, NPV). The value of a project is equal to its traditional NPV if, and only if, the value of the DO is zero. This may occur under the circumstances discussed by Kester (1984).

Also, it is important to know what is the optimal moment to invest. At every moment t, the company tries to maximise the value of the IO by choosing between: (i) immediate exercise of the option to invest, or (ii) the deferment of the investment option in order to obtain more information about V and D. The company defers whenever the value of the "live" option is larger than the value of the option if exercised today, i.e., when the DO's value is positive.

3. Model Implementation: Assumptions and Inputs

One of the largest Portuguese companies has the opportunity to invest abroad in three independent real investment projects (A, B and C) (for further details see Appendix D). The values for each of the valuation inputs for each of these projects are summarized in Table 2 and are discussed next.

[Please place Table 2 about here.]

3.1 Gross Project Value (V)

The Gross Project Value V corresponds to the present value of the project's appropriately discounted expected cash flows, given the information available at the evaluation date. V is the value that the firm receives by paying the exercise price (i.e., by making the investment). While the value of V at the evaluation date is known, its future values are unknown. We assume that V is a stochastic variable that follows the geometric Brownian motion process defined in (1a). According to the data provided by the said company, the values of V_0 are: 1,844,575 Euros, 2,419,106 Euros, and 1,785,776 Euros for projects A, B and C, respectively.

[6] Since our methodology calculates the value of the IO as a whole, we can obtain the value of the DO by finding the difference between the value of the AEO and the traditional NPV.

3.2 Investment Cost (D)

The investment cost D is the exercise price of the IO. It is the amount of capital that the company needs to invest "today" in the project. We do not know the value of D in the future, when the option to invest will be exercised. As for V, we assume that D follows the geometric Brownian motion process presented in (1b). According to the data provided by the said firm, the current values of D are: 1,662,000 Euros for projects A and B, and 2,622,000 Euros for project C.

3.3 Time-to-Maturity (T-t)

The company estimates that each of the projects can be deferred for about 4 years before each opportunity disappears. Thus, we adopt a 4 year maturity for each project's deferment option. Since the options are American, the IO can be exercised anytime until (or at) the maturity date.

3.4 Dividend-Yield of V (δ_v)

Let μ be the (total) expected rate of return on V and α be the expected percentage rate of changes of V. We assume that $\delta=\mu-\alpha$ so that investment before the maturity date may be optimal, as in Dixit and Pindyck (1994).

As with call options, δ corresponds to the dividend yield of the stock. The total return earned by the owner of the stock is then: $\delta+\alpha=\mu$. In the absence of dividends on the underlying stock, the optimal decision is to hold the option until maturity. Since the total return on the stock is reflected in the prices of both the underlying stock and the option, there is no opportunity cost to maintaining the option "alive". In the case of a positive δ, there is an opportunity cost in holding the option instead of the stock. This opportunity cost corresponds to the dividends paid on the stock that are foregone by option holders.

The expected return from owning the completed project also is given by μ.[7] This market-determined equilibrium rate includes an appropriate risk premium. If $\delta_v > 0$, then the (capital) gains on V will be lower than μ, so δ_v is the opportunity cost of deferring the project. If $\delta_v = 0$, no opportunity cost exists. Thus, it is never optimal to invest earlier than at maturity. For high values of δ_v (for high opportunity costs associated with holding the option), the value of the option goes to zero. This transforms the project into a "now or never" type, and makes the traditional NPV a valid assessment method.

In practice, δ_v may represent several types of opportunity costs. One such opportunity cost is the cash flows foregone. Some authors (e.g. Trigeorgis, 1996) argue that δ_v may also incorporate another type of opportunity cost. Specifically, project deferment may contribute to the early entrance of a competitor in a competitive environment, which, in turn, may have a negative impact on the value of the project. Herein, we assume that the only cost resulting from the deferment decision is the lost cash flows. Thus, the parameter δ_v is the rate of cash flow yielded by the project.

[7] Remember that the expected rate of return is irrelevant given the current asset values, as in Black-Scholes (1973).

As noted above, δ_v can be calculated as the difference between the total expected or required return on the project (i.e., the cost of capital or μ), and the expected growth rate of the project's value (α). The company estimated a value of μ of 10%. We calculate α using, $\alpha = \left(V_n / V_0\right)^{(1/n)} - 1$, where V_n is the expected value of the project in year n, and V_0 is the project's current value if completed. The α estimates for the projects A, B and C are 3.52%, 3.33% and 3.24%, respectively. Using the estimates of μ and α yields δ_v estimates of 6.48%, 6.67% and 6.76% for projects A, B and C, respectively.

3.5 Dividend-Yield of D (δ_d)

According to the assumptions of the model, the "dividend yields" are assumed to be nonnegative constants. While this is true for δ_v, δ_d is negative when carrying costs are associated with the project's capital cost. In this model, we need to assume that such costs do not exist because δ_d cannot be negative. As pointed out by McDonald and Siegel (1986), the gain from deferral may increase with larger δ_d. In our application, we assume that δ_d=0 by assuming that there are no carrying costs associated with a project's capital costs nor benefits (from the capital cost's level) from deferring the project.

3.6 Volatility of V and D (σ_v, σ_d)

We assume that the volatility of the company's stock is an adequate proxy for the volatility of V (see, for example, Davis, 1998; Paxson, 1999; and Amram and Kulatilaka, 1999). It also is necessary to assume that the volatility of V is constant during the life of the option. The σ_v is calculated based on the natural logarithm of the daily returns [Ln(n/(n-1)] of the company's quotations during the period between January 4, 1993 and June 30, 1999. The annual σ_v corresponds to the daily σ_v multiplied by the square root of the number of transaction days in a year (247). The stock's quotations are drawn from Bloomberg, and are already adjusted for dividends. This yields a σ_v of 0.3058. As to the volatility of D, and knowing that the investment costs are essentially construction costs, we assume as in Patel and Paxson (2001) that the volatility of the construction companies index (CCI) is an adequate proxy for representing the volatility of the investment cost. So σ_d is calculated from the daily returns on the CCI in the same way and over the same period as that for estimating σ_v. The annual σ_d is equal to the daily σ_d multiplied by the square root of the number of transaction days in a year (247). The data is drawn from Finibanco, and the quotations are adjusted for dividends. This yields a value of σ_d of 0.2202.

3.7 Correlation between the changes in V and D [$\rho(v,d)$]

We assume that the correlation between the changes in V and D can be approximated by the correlation between the daily returns on the company's stocks and the daily returns on CCI during the time period referred to above (as in Patel and Paxson, 2001). This yields a correlation of 0.2532.

4. Model Implementation: Empirical Outputs

4.1 The Results for the Initial Input Values

Using the methodology presented in section 2, and the input values detailed in section 3, we obtain the results reported in Table 3 for each of the three projects. The value of project A is 346,911 Euros,[8] which is over 90% higher than the value calculated using the traditional NPV method. More than 47% of the total value of this investment project is attributable to the value of the DO. Although the value obtained from the traditional NPV method is almost half of the project's real value, its positive sign suggests that the project should be adopted immediately. On the contrary, the AEO evaluation methodology used herein indicates that the project should not be undertaken immediately because of the high positive value of the DO.

[Please place Table 3 about here.]

The valuation results for project B differ from those for project A. The deferment option has no value because the project is far in-the-money, and the uncertainties about V and D are not sufficiently high to induce value in the deferment option. As a result, it is more valuable to exercise the option to invest now than to keep that option alive. In other words, since the option to invest is sufficiently in-the-money to compensate for the lost option to defer the project, the option to invest should be exercised now. As in Kester (1984), a high traditional NPV value is one of the factors that justifies a project's current implementation.

The input characteristics of project C differ substantially from those for projects A and B. Project C has a lower gross project value, and a higher implementation cost. When evaluated using the traditional NPV methodology, this project has no value given its negative NPV. However, based on the AEO evaluation methodology, this investment opportunity has a positive value of 111,320 Euros due to the high value of its deferment option of 947,544 Euros. This changes the investment decision from not adopting the project (based on the traditional NPV rule) to the deferment of its implementation (based on the AEO rule). The later decision gives the company the flexibility of "waiting to see", and leaves the company with the right to invest in the project in the future if the uncertainties are resolved in the project's favor. Interestingly, although this IO is significantly out-of-the money, this investment opportunity has a large positive value.

The values emitted by each of the two valuation methodologies and the resulting investment timing decisions are summarized in Table 4. The NPV rule not only significantly undervalues projects A and C but its implementation leads to an incorrect decision. Only for project B do the two evaluation methodologies yield the same value, and thus, emit the same investment timing signal. Thus, this vividly illustrates that the traditional NPV methodology is not adequate to value investment opportunities in an uncertain environment, particularly when projects can be deferred to a later adoption date.

[Please place Table 4 about here.]

4.2 The Results Using Simulated Input Values

In this section we perturbate some of the initial input values in order to study the impact on IO values calculated using the finite-lived American Exchange Option methodology. We fix the values for V, D, δ_d and (T-t), and vary each of the other input values in turn. We use values of 25%, 30% and 35% for σ_v, of 20%, 25% and 30% for σ_d, of 0.20, 0.25 and 0.30 for ρ, and of 5%, 7% and 9% for δ_v.

These results are presented in Tables 5, 6 and 7 for projects A, B and C, respectively. Only the shaded values signal that the project should be undertaken now. As for the remaining values, the correct timing decision is to defer implementation. Also, the value of the AEO increases with higher σ_v, higher σ_d, lower ρ between V and D, and lower δ_v.

[Please place Tables 5, 6 and 7 about here.]

5. Some Limitations

The model building and tests conducted herein have a number of potential limitations. First, as is also noted by Mcdonald and Siegel (1986), the assumption of geometric Brownian motion (GBM) is more reasonable for the project gross value V than it is for the investment cost D. Second, the maturity date of the deferment option is likely to be uncertain at the point at which an investment decision is being made. Third, the model assumes away the importance of carrying costs for the project's capital cost because it assumes that "dividend yields" are nonnegative parameters. Fourth, the model does not account for the value of other options such as the option to abandon the project. And finally, the investment opportunity is assumed to be proprietary due to, for example, the absence of competition in the market.

6. Conclusions

In this paper we propose that many investment opportunities, which have a finite time to maturity, should be valued as finite-lived American exchange options. We argue that such projects have a finite temporal dimension with uncertain gross project values and investment costs, and flexibility in implementation up to and including some future calendar date.

We correct and apply the Carr (1988) methodology, and find support for our expectation that the traditional NPV method significantly undervalues projects with the characteristics presented above. The use of the traditional NPV method may lead to incorrect investment or investment timing decisions. By evaluating investment opportunities using an American Exchange Options methodology, we find substantial changes compared to the traditional NPV method in both the value of the investment opportunities and the timing of when the project is undertaken. The model outputs for the corrected and uncorrected Carr (1988) models differ

[8] The value of the IO = Max (AEO, NPV).

substantially. As expected, the AEOs based on the corrected model are higher, and reduces type two decision errors (i.e., falsely rejecting an acceptable investment project).

APPENDIX A: The Bivariate Cumulative Normal Distribution: An Estimation Method Without Using Double Integrals

The probability density function for the stochastic variables x and y is:

$$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left\{-\frac{Q(x, y)}{2}\right\}, \qquad (A1)$$

where :

$-\infty < x < +\infty$

$-\infty < y < +\infty$

$\sigma_x > 0, \sigma_y > 0$

$|\rho| \le 1$

Q(x, y) is the quadratic form,

$$Q(x, y) = \frac{1}{1-\rho^2}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right], \qquad (A2)$$

where :

$-\infty < \mu_x < +\infty$

$-\infty < \mu_y < +\infty$

Setting:

$$a = \left(\frac{x-\mu_x}{\sigma_x}\right), \; b = \left(\frac{y-\mu_y}{\sigma_y}\right) \qquad (A3)$$

yields:

$$N_2(a, b; \rho) = \int_{-\infty}^{a}\int_{-\infty}^{b} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left(a^2 - 2\rho ab + b^2\right)\right\} da.db \qquad (A4)$$

where :

$$da.db = \frac{1}{\sigma_x\sigma_y} dx.dy$$

and N_2 (a, b; ρ) is the cumulative bivariate standard normal distribution.

Drezner (1978)[9] presents a method to calculate a reasonable approximation, of the cumulative probabilities of the bivariate standard normal distribution.[10] For values of a, b and ρ less than or equal to zero, the cumulative density function is given by the following equation:

$$N_2(a, b; \rho) = \frac{\sqrt{1-\rho^2}}{\pi} \sum_{i,j=1}^{4} A_i A_j f(B_i, B_j) \qquad (A5)$$

where:

$$f(x, y) = \exp[a'(2x - a') + b'(2y - b') + 2\rho(x - a')(y - b')]$$

$$a' = \frac{a}{\sqrt{2(1-\rho^2)}}$$

$$b' = \frac{b}{\sqrt{2(1-\rho^2)}}$$

$$A_1 = 0.3253030, A_2 = 0.4211071, A_3 = 0.1334425, A_4 = 0.006374323$$

$$B_1 = 0.1337764, B_2 = 0.6243247, B_3 = 1.3425378, B_4 = 2.2626645$$

When the multiplication of (a, b and ρ) is negative or zero, one of the following identities should be used:

$$N_2(a, b; \rho) = N_1(a) - N_2(a, -b; -\rho) \qquad (A5a)$$

$$N_2(a, b; \rho) = N_1(b) - N_2(-a, b; -\rho) \qquad (A5b)$$

$$N_2(a, b; \rho) = N_1(a) + N_1(b) - 1 + N_2(-a, -b; \rho) \qquad (A5c)$$

When the multiplication of (a, b and ρ) is positive, the following identity should be used with the results previously presented:

$$N_2(a, b; \rho) = N_2(a, 0; -\rho 1) + N_2(b, 0; -\rho 2) - \delta \qquad (A6)$$

In the previous equation:

$$\rho_1 = \frac{(\rho a - b)\operatorname{sgn}(a)}{\sqrt{a^2 - 2\rho ab + b^2}}$$

$$\rho_2 = \frac{(\rho b - a)\operatorname{sgn}(b)}{\sqrt{a^2 - 2\rho ab + b^2}}$$

$$\delta = \frac{1 - \operatorname{sgn}(a)\operatorname{sgn}(b)}{4}$$

$$\operatorname{sgn}(x) = \begin{cases} +1 \text{ if } x \geq 0 \\ -1 \text{ if } x < 0 \end{cases}$$

[9] The Drezner paper has been corrected by Hull (1997).
[10] The method is exact to four decimal places.

APPENDIX B: The Two Moments Richardson Extrapolation Formula[11]

The objective is to estimate the value of the definite integral:

$$\int_a^b f(x)dx\,, \tag{B1}$$

hereafter denoted by *I*.

Assuming that *f(x)* is continuous on *[a,b]* and an appropriate number of derivatives can be found, the estimate of the error has the form:

$$I - I_n \approx \frac{K}{n^r}\,, \tag{B2}$$

where I_n denotes the numerical integral; *K* is a constant that may vary with the function *f(x)*, the interval *[a,b]*, and the approximation method; and *r* is a real number that depends on the approximation method (e.g., *r*=2 for the Trapezoidal rule, and *r*=4 for Simpson's rule).

Let *a* be a positive integer. Replacing *n* by *an* in (B2), then:

$$I - I_{an} \approx \frac{K}{a^r n^r}\,, \tag{B3}$$

Rewriting the expression (B3) yields:

$$a^r\left(I - I_{an}\right) \approx \frac{K}{n^r}\,. \tag{B4}$$

Note that (B4) can be compared with (B2) as follows:

$$a^r\left(I - I_{an}\right) \approx I - I_n\,. \tag{B5}$$

Solving (B5) for *I* yields:

$$I \approx \frac{a^r I_{an} - I_n}{a^r - 1}\,. \tag{B6}$$

Take r=2 for the Trapezoidal rule, and let *a*=2 for the two moments extrapolation. Then:

$$I \approx \frac{2^2 I_{2n} - I_n}{2^2 - 1}\,. \tag{B7}$$

Using the paper's notation, (B7) can be rewritten as:

$$AEO \approx \frac{4E_2 - E_1}{3} = E_2 + \frac{E_2 - E_1}{3} \tag{B8}$$

[11] For more details, see, for example, Hildebrand (1956) or Atkinson (1993).

APPENDIX C: The Value of the Finite-Lived American Exchange Option for the Carr and Corrected Carr Approximations

The values of the finite-lived AEO for each project using the uncorrected and corrected Carr (1988) formula are reported in Table C1. As expected, the value estimate based on the uncorrected Carr approximation always undervalues the AEO.

[Please place Table C1 about here.]

Also, according to the Carr expression (see footnote 4 above), the value of the AEO is always lower than that of the PAEO. This cannot be possible because an option, which can be exercised at anytime, must have more value than one that can only be exercised at one of two moments in time.

APPENDIX D: A Brief Description of the Three Investment Opportunities

A company, based in Portugal, is planning to invest in three independent real projects (A, B, and C). This company acts in several markets (telecommunications, real estate, wood, tourism, and retail markets, such as malls, shopping centres, and specialized retail business), and in several countries in Europe and in Latin America.

The three projects being evaluated are in the "specialized retail business" category. Each project is in a different city. The investments consist of the acquisition of vacant land (appropriately located) as well as the installation of the facilities. Each project's NPV, when calculated using **standard** DCF techniques, is reported in table D1.

[Please place Table D1 about here.]

A major characteristic of these investment opportunities is that they can be delayed or deferred for up to four years in order to resolve the uncertainties governing each project's value. However, if the company decides to postpone a project, it faces the uncertainties associated with future investment costs.

Projects with these characteristics are similar to finite-lived American exchange options. Specifically, they have a finite maturity, they can be implement anytime before or at the maturity date, and both the present value of the projects' cash flows and the investment costs behave stochastically.

References

Amram, Martha; and Nalin Kulatilaka (1999). “Real Options: Managing Strategic Investment in an Uncertain World,” *Harvard Business School Press*, Boston, Massachusetts.

Atkinson, Kendall (1993). “Elementary Numerical Analysis,” *John Wiley & Sons, Inc.*, Second Edition, New York.

Black, Fischer; and Myron Scholes (1973). “The Pricing of Options and Corporate Liabilities,” *Journal of Political Economy* 81, 637-659.

Carr, Peter (1988). “The Valuation of Sequential Exchange Opportunities,” *The Journal of Finance* XLIII:5, 1235-1256.

Carr, Peter (1995). “The Valuation of American Exchange Options with Application to Real Options,” *in*: *Real Options in Capital Investment: Models, Strategies and Applications*, ed. by Lenos Trigeorgis, Westport, Connecticut, London, Praeger.

Davis, Graham (1998). “Estimating Volatility and Dividend Yield when Valuing Real Options to Invest or Abandon,” *Working Paper*, Division of Economics and Business, Colorado School of Mines.

Dixit, Avinash (1989). “Entry and Exit Decisions Under Uncertainty,” *Journal of Political Economy* 97: 3, 620-638.

Dixit, Avinash; and Robert Pindyck (1994). “Investment Under Uncertainty,” *Princeton University Press*, Princeton, New Jersey.

Drezner, Z. (1978). “Computation of Bivariate Normal Integral,” *Mathematics of Computation* 32, 277-279.

Hildebrand, F. B. (1956). “Introduction to Numerical Analysis,” Dover, New York.

Hull, John C. (1997). “Options, Futures and Other Derivatives,” *Prentice Hall*, 3rd Edition, Englewood Cliffs (NJ).

Ingersoll, Jonathan; and Stephen Ross (1992). “Waiting to Invest: Investment and Uncertainty,” *Journal of Business* 65: 1, 128-156.

Kemna, Angelien G Z (1993). “Case Studies on Real Options,” *Financial Management* 22: 3, Autumn 1993, 259-270.

Kester, W Carl (1984). “Today’s Options for Tomorrow’s Growth,” *Harvard Business Review*, March-April, 153-160.

Lee, Moon Hoe (1997). “Valuing Finite-Maturity Investment-Timing Options,” *Financial Management* 26: 2, Summer 1997, 58-66.

Majd, Saman; and Robert S Pindyck (1987). “Time to Build, Option Value and Investment Decisions,” *Journal of Finance Economics* 18: 1, 7-27.

Margrabe, William (1978). “The Value of an Option to Exchange One Asset for Another,” *Journal of Finance* 33: 1, 177-186.

McDonald, Robert; and Daniel Siegel (1985). “Investment and the Valuation of Firms When There is an Option to Shut Down,” *International Economic Review* 26: 2, June 1985, 331-349.

McDonald, Robert; and Daniel Siegel (1986). “The Value of Waiting to Invest,” *Quarterly Journal of Economics* 101: 4, 707-727.

Patel, Kanak; and Dean Paxson (2001). “Real Urban Development Options at Canary Wharf,” *in Real Options*, FT-Prentice Hall, London.

Paxson, Dean (1999). “R&D Strategy Using Real Option Values,” *Finance Research Seminar*, Universidade do Minho, Braga, Portugal.

Pindyck, Robert (1991). “Irreversibility, Uncertainty, and Investment,” *Journal of Economic Literature* 29: 3, 1110-1148.

Ross, Stephen (1995). “Uses, Abuses, and Alternatives to the Net-Present-Value Rule,” *Financial Management* 24: 3, 96 – 102.

Siegel, Daniel R; James L Smith; and James L Paddock (1987). “Valuing Offshore Oil Properties With Option Pricing Models,” *Midland Corporate Finance Journal* 5, Spring, 22-30.

Trigeorgis, Lenos (1991). “Anticipated Competitive Entry and Early Preemptive Investment in Deferrable Projects,” *Journal of Economics and Business* 43, 143-156.

Trigeorgis, Lenos (1993). “Real Options and Interactions With Financial Flexibility,” *Financial Management* 22: 3, 202-224.

Trigeorgis, Lenos (1996). “Real Options: Managerial Flexibility and Strategy in Resource Allocation,” *The MIT Press*, Cambridge (MA).

Table 1. Portfolio that replicates the payoffs of a PAEO with ΔT and T maturities.

At the Evaluation Date	**At moment** DT	
	If P<= P*	If P> P*
Long Position s(V,D,T)	s(V,D,T)	s(V,D,T)
Long Position s(V,P*.D,DT)	0	V-P*.D
Short Position c(s(V,D,T),(P*-1).D,DT)	0	P*.D-D-s(V,D,T)
	s(V,D,T)	V-D

Table 2. Input values for the valuation of each of the three investment projects

	Project A	Project B	Project C
V	1,844,575	2,419,106	1,785,776
D	1,662,000	1,662,000	2,622,000
(T-t)	4 years	4 years	4 years
δ_v	0.0648	0.0667	0.0676
δ_d	0	0	0
σ_v	0.3058	0.3058	0.3058
σ_d	0.2202	0.2202	0.2202
ρ(v,d)	0.2532	0.2532	0.2532

Table 3. The NPV, AEO, DO and IO values (in Euros) and the respective contributions of the NPV and DO to the IO value and the impact of the DO for each of the three investment projects

Measure	Project		
	A	B	C
Net Present Value	182,575	757,106	- 836,224
Value of the European Exchange Option (EEO)	288,459	553,673	100,713
Value of the pseudo-American Exchange Option	332,298	658,851	108,668
Value of the Finite-Lived American Exchange Option (AEO)	346,911	693,910	111,320
Value of the Deferment Option (DO)	164,336	0	947,544
Total Value of the Investment Opportunity	346.911	757,106	111.320
% NPV	52.6%	100,0%	(751,1%)
% Value of the Deferment Option	47.4%	0%	851.1%
Increase in Value when Considering the DO	90.0%	0%	------

Table 4. The value and timing of the IO based on the traditional NPV and AEO methodologies

	Value		Timing	
Project	AEO methodology	Traditional NPV	AEO methodology	Traditional NPV
A	364,911	182,575	Defer the Project	Invest Now
B	757,106	757,106	Invest Now	Invest Now
C	111,320	- 836,224	Defer the Project	Abandon the Project

Table 5. Value of a Finite-Lived American Exchange Option for Project A when S = 1,844,575; D = 1,662,000; δd = 0; T = 4 years; and NPV = 182,575. All values in Euros.

		ρ = 0.20			ρ = 0.25			ρ = 0.30		
σV	**σD**	**δv = 5%**	**δv = 7%**	**δv = 9%**	**δv = 5%**	**δv = 7%**	**δv = 9%**	**δv = 5%**	**δv = 7%**	**δv = 9%**
25%	**20%**	333.338	289.293	248.819	322.702	278.784	239.199	311.781	268.065	229.285
	25%	367.946	321.236	280.581	355.948	309.762	269.536	343.535	299.629	258.174
	30%	408.198	359.802	318.508	395.246	347.382	306.107	381.844	334.540	293.491
30%	**20%**	379.106	331.918	290.935	367.946	321.236	280.581	356.436	310.228	269.938
	25%	408.198	359.802	318.508	395.246	347.382	306.107	381.844	334.540	293.491
	30%	443.143	393.366	349.550	428.922	379.659	336.390	414.098	365.464	324.242
35%	**20%**	428.073	378.882	335.646	416.600	367.866	326.631	404.786	356.529	315.218
	25%	452.385	402.254	358.089	438.838	389.227	345.575	424.829	375.767	332.655
	30%	482.517	431.265	385.992	467.349	416.655	371.934	451.622	401.520	357.384

Table 6. Value of a Finite-Lived American Exchange Option for Project B when S = 2,419,106; D = 1,662,000; δd = 0; T = 4 years; and NPV = 757,106. All values in Euros.

		ρ = 0.20			ρ = 0.25			ρ = 0.30		
σV	**σD**	**δv = 5%**	**δv = 7%**	**δv = 9%**	**δv = 5%**	**δv = 7%**	**δv = 9%**	**δv = 5%**	**δv = 7%**	**δv = 9%**
25%	**20%**	706.767	638.348	576.690	696.446	629.010	567.558	686.269	619.641	558.296
	25%	736.008	670.830	607.792	724.136	659.272	596.812	713.916	647.662	585.699
	30%	776.183	707.673	646.355	763.157	695.206	633.678	749.785	684.592	620.817
30%	**20%**	747.064	681.851	618.225	736.008	670.830	607.792	724.619	659.736	597.254
	25%	776.183	707.673	646.355	763.157	695.206	633.678	749.785	684.592	620.817
	30%	811.874	742.006	678.440	797.214	727.881	664.668	782.154	713.402	652.200
35%	**20%**	796.388	727.086	663.893	784.693	715.840	654.690	772.741	704.374	642.994
	25%	821.438	751.233	687.433	807.435	737.727	674.269	793.072	723.895	662.925
	30%	852.942	781.680	717.089	837.025	766.289	702.100	820.647	750.470	686.689

Table 7. Value of a Finite-Lived American Exchange Option for Project C when S = 1,785,776; D = 2,662,000; δd = 0; T = 4 years; and NPV = -836,224. All values in Euros.

		ρ = 0.20			ρ = 0.25			ρ = 0.30		
σV	**σD**	**δv = 5%**	**δv = 7%**	**δv = 9%**	**δv = 5%**	**δv = 7%**	**δv = 9%**	**δv = 5%**	**δv = 7%**	**δv = 9%**
25%	**20%**	93.490	72.139	55.423	85.251	65.054	49.398	77.013	58.033	43.487
	25%	122.106	97.151	77.079	111.944	88.205	69.273	101.712	79.266	61.537
	30%	157.912	129.110	105.396	146.125	118.524	95.951	134.186	107.863	86.502
30%	**20%**	131.781	105.724	84.615	122.106	97.151	77.079	112.352	88.563	69.584
	25%	157.912	129.110	105.396	146.125	118.524	95.951	134.186	107.863	86.502
	30%	190.800	158.926	132.275	177.198	146.550	121.073	163.357	134.021	109.797
35%	**20%**	176.435	145.858	120.449	165.681	136.119	111.680	154.784	126.295	103.879
	25%	199.735	167.087	139.692	186.669	155.162	128.862	173.377	143.085	117.948
	30%	228.480	194.395	164.669	214.394	180.520	151.951	198.994	166.409	139.076

Table C1. The Value of the Finite-Lived American Exchange Option using the Uncorrected and Corrected Carr (1988) Approximations

Project	Corrected Formula	The Carr Formula	Pseudo-AEO
A	346,911	303,072	332,298
B	663,910	588,733	658,851
C	111,320	103,385	108,668

Table D1. The Financial Characteristics of the Projects. Cost of Capital = 10%. All values in €.

Project	Investment Cost	PV of the Cash-Flows	NPV
A	1,662,000	1,844,575	182,575
B	1,662,000	2,419,106	757,106
C	2,622,000	1,785,776	(836,224)