Definition of optimal proportion of phased investment: A real options approach

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Abstract: Real options approaches can be employed to value the flexibility in the decision-making courses of phased investments. This paper focuses on the two-stage investment problems of venture firms. Firstly, based on the analysis of profit function and real options thinking, stochastic models are proposed to describe the uncertainties of such problems. Then the value function of decision-making flexibility is derived. Finally numerical techniques are employed to calculate the optimal proportions of a case and the influence of investment proportions upon the flexibility value is also analyzed.

Keywords: Phased investments, Optimal proportion, Real options.

Since venture capital is a phased investment, different venture firms use different amounts of capital diversely at their different stages of development. To yield high profits, such capital should be rationally allocated to avoid either hasty input at the beginning of the project or hesitation when more investment is called for urgently. Of course further investment should be terminated and initial capital be reclaimed as much as possible when the venture firms are faced with poor prospects, in which case flexibilities of phased investments should be considered in the decision-making courses instead of mechanically adopting the method of "investment now, or never". When solving these problems, the real options approach, with many of its successful applications to various fields, gains an evident advantage over the traditional DCF method. References are given to Brennan and Schwartz (1985), McDonald and Siegel (1986) as the renowned pioneers. For recent overviews, see Grenadier (1995), Kemna (1993), Sunnevag (1998) and Trigeogis (1993). And with the help of the real options approach, this paper attempts an evaluation model for the optimal proportion of phased investment.

1 Problem description

Consider a highly competitive firm whose hi-tech products have been legally licensed and ready to be launched to the market in a large scale. The investment for production and sales must be ascertained after the estimation of the market capacity. In view of various uncertainties on the market, the firm will usually put in the finances from venture capitalists phase by phase. Suppose there are two phases, namely the startup phase and the expansion phase. After doing business for a period of time, the firm could accumulate enough information to gain a clear insight into the possible returns. Now it's time for the firm to determine its further development. It could either super-add its investment to pursue more profits, or hold any further move because of the poor prospects. It might even give up the business trying to reclaim its initial investment as much as possible. That's the time when the firm has two options: One is a Europe call option, which means to put in the expansion investment so as to gain more profits later, and it can be carried out when the profits available exceed the investment; The other is a Europe put option, which means to reclaim the startup investment and give up its possible profits when the reclaimable investment exceeds the possible profits. And the time is the exact date when the two options are due. At this critical moment, the firm has to choose from the above-mentioned two choices. When the profits from the expansion investment exceed the expansion investment, the profits from the startup investment in the expansion phase should also exceed its reclaimable investment. Similarly, when the profits from the expansion investment lag behind the expansion investment, the profits from the startup investment in the expansion phase would also lag behind its reclaimable investment. So the probabilities that the two options are executed add up to 1. The values of the two options are therefore also the value that exists in the flexibilities of decision-making. This, together with the corresponding net profits from the startup investment, constitutes the value inherent in the decision-making of the firm's phased investment. When the cost of financing is deducted from this value, the value will become the net value. In this way, the optimal investment proportion that will maximize the net value can be obtained.

2 Modeling

Some stochastic models are proposed to describe uncertainties of the problems in section 2.

2.1 The critical moment of decision-making

If the startup investment proportion is x ($0 < x < 1$), then the expansion investment proportion must be $1 - x$. Generally speaking, a large proportion of startup investment will usually accelerate more information and more outstanding achievements—the perfection of its technology, the improvement of its products, the enhancement of its distribution, etc.—to a firm, and can therefore enable it to make further decisions against time. In this sense, the critical moment of decision-making is highly dependent on the investment proportion x in the first phase.

On the other hand, influenced by such factors as its achievements and competitors, the critical moment $\tau(x)$ of decision-making is uncertain. Usually it follows exponential distribution (Ottoo, 1998), the parameter is supposed to be a function $f(x)$, and the average time needed to make the decision is $(f(x))^{-1}$. From the above analysis we could see that as x increases, $f(x)$ increases too, while $\tau(x)$ decreases. Suppose $f(x) = bx$, b is the parameter that affects the average time, $b > 0$.

2.2 Stochastic processes of prices and sales

Thanks to a relatively smaller size and more flexible management, venture firms are able to realize prompt production according to the market demands. Free from the anxiety of overstock, we might as well consider that the production is equal to the sales.

Suppose R_t stands for the marginal profit of the products at time t, m_t the instantaneous sales,

p_t the unit price of the product at t, C the marginal cost of the product, and n the maximum capacity of the project, and then we may conclude that $R_t = p_t - C$. It's alright to say that C is only a function of n, which has nothing to do with m_t; that is $C = C(n)$ (Dangl, 1999).

Both p_t and m_t can be supposed to follow GBM (Pennings and Lint, 2000),

$$dp_t = \alpha_p p_t dt + \sigma_p p_t dW_1(t) \tag{1}$$

$$dm_t = \alpha_m m_t dt + \sigma_m m_t dW_2(t) \tag{2}$$

Usually firms will decide the amount of the startup investment according to the initial sales, so in a sense the amount is proportional to the sales. Suppose m_0 stands for the initial sales when the investment proportion is 1, xm_0 stands for the initial sales when the investment proportion is x.

Since C is a constant when the maximum capacity of a specific project n is fixed, R_t and p_t follow a same geometric Brownian motion. They're only different with initial values. Suppose R_0 stands for the initial value of R_t. Then we have

$$dR_t = \alpha_R R_t dt + \sigma_R R_t dW_1(t)$$

and $\alpha_R = \alpha_p, \sigma_R = \sigma_p$. Notice that Brownian motion $W_1(t)$ means the risky factors influencing the marginal profits, while Brownian motion $W_2(t)$ stands for those influencing the instantaneous sales. Usually these two factors are correlated with each other by the correlation coefficient ρ, so $EW_1(t)W_2(t) = t\rho$. For high-tech products, generally, prices and sales have a negative correlation, in which case $\rho < 0$.

If y_t stands for profit flow, then $y_t = R_t m_t$. Using Ito's Lemma, we can get that y_t follows the geometric Brownian motion:

$$dy_t = \alpha_y y_t dt + \sigma_y y_t dW(t) \tag{3}$$

in which

$$\alpha_y = \alpha_R + \alpha_m + \rho\sigma_R\sigma_m,$$

$$\sigma_y^2 = \sigma_R^2 + \sigma_m^2 + 2\rho\sigma_R\sigma_m$$

$$W(t) = \frac{\sigma_R W_1(t) + \sigma_m W_2(t)}{\sqrt{\sigma_R^2 + \sigma_m^2 + \rho\sigma_R\sigma_m}}$$

where α_y and σ_y denote the expected drift rate and volatility of y_t. $W(t)$ is a standard Wiener process. y_t synthesizes the developing trends and uncertainties of R_t and m_t.

2.3 The profit functions

Suppose the decision of expansion investment is made at $\tau(x)$, then the discounted profit value of the expansion investment, $V_2(t)$ is

$$V_2(t) = E\{\int_t^{\infty} e^{-\mu s} y_s ds \mid \mathcal{F}_t\}, \quad t \geq \tau(x) \tag{4}$$

in which μ means the individual discount rate of the firm. Due to the fact that the expected profits of the firm are finite, $V_2(t)$ is by no way an infinitude, thus $\mu > \alpha_y$. Through stochastic calculus, we now have

$$V_2(t) = \frac{y_t}{\mu - \alpha_y} e^{-\mu t}, \quad t \geq \tau(x) \tag{5}$$

Using Ito's Lemma, we'll also have:

$$dV_2(t) = \alpha_V V_2 dt + \sigma_V V_2 dW_t, \quad t \geq \tau(x) \tag{6}$$

in which

$$\alpha_V = \alpha_y - \mu,$$

$$\sigma_V^2 = \sigma_y^2 = \sigma_R^2 + \sigma_m^2 + 2\rho\sigma_R\sigma_m.$$

Eq.(6) states that $V_2(t)$ follows a geometric Brownian motion. α_V and σ_V denote the expected drift rate and volatility of $V_2(t)$. Since $\alpha_V = \alpha_y - \mu < 0$, the mean value of $V_2(t)$ is the decreasing function of t, that is to say, the firm's expected profit is becoming smaller and smaller as time goes on.

Similarly, we get the discounted profit value of the startup investment $V_1(t)$:

$$V_1(x) = E\int_0^{\infty} e^{-\mu t} y_t dt = \frac{x R_0 m_0}{\mu - \alpha_y}$$

while the discounted profit value of the startup investment before $\tau(x)$ is

$$V_1(x)\Big|_{t:0\to\tau(x)} = E\int_0^{\tau(x)} e^{-\mu t} y_t dt = \frac{xR_0 m_0}{\mu-\alpha_y}(1-e^{-(\mu-\alpha_y)\tau(x)})$$

and the discounted profit value of the startup investment after $\tau(x)$ is

$$V_1(x)\Big|_{t:\tau(x)\to\infty} = E\int_{\tau(x)}^{\infty} e^{-\mu t} y_t dt = \frac{xR_0 m_0}{\mu-\alpha_y} e^{-(\mu-\alpha_y)\tau(x)}$$

2.4 The executable probability of the two options

If the total investment is I, then the startup investment and the expansion investment must be xI and $(1-x)I$ respectively. Since the firm can carry out but one plan at the critical moment of decision-making, it could either superadd its investment to pursue more profits, or give up the business trying to reclaim its initial investment as much as possible. Thus the firm doesn't acquire two completely independent options. Instead, their executable probabilities add up to 1.

Let a denote the probability of executing the put option. According to the above-mentioned analysis, we have

$$a = P(V_2(\tau(x)) \le (1-x)I).$$

Correspondingly, the probability of executing the call option is $1-a$. From Eq.(5), we have

$$V_2(\tau(x)) = (1-x)V_0 \exp\{\sigma_V W(\tau(x)) + (\alpha_V - \frac{\sigma_V^2}{2})\tau(x)\} \tag{7}$$

Thus

$$P(V_2(\tau(x)) \le (1-x)I) = N(d) \tag{8}$$

In which

$$d = \frac{\ln\{\frac{I}{V_0}\} + (-\alpha_V + \frac{1}{2}\sigma_V^2)\tau(x)}{\sigma_V\sqrt{\tau(x)}}.$$

And $(1-x)V_0$ is the mean value of $V_2(x)$ at $\tau(x)$. From Eq.(5), we can have

$$V_0 = \frac{R_0 m_0}{\mu-\alpha_y} e^{-(\mu-\alpha_R-\alpha_m)\tau(x)} \tag{9}$$

2.5 The value inherent in the decision-making of phased investment

Since the discounted profit value of the expansion investment $V_2(t)$ follows a geometric Brownian motion, we now have the value of the call option using Black—Scholes pricing formula.

$$F_c = (1-x)V_0 \exp(-\delta\tau(x))N(d_1) - (1-x)I \exp(-r\tau(x))N(d_2) \quad (10)$$

Notice that $N(\cdot)$ is representing the cumulative normal distribution. Since the project is non-traded, δ denotes the dividend analogy of the below-equilibrium return shortfall for a company that does not hold the project. It is similar to the dividend yield of financial options and its value is $\mu - \alpha_V$. r denotes the risk-free interest rate.

$$d_1 = \frac{\ln\{\frac{V_0}{I}\} + (r - \delta + \frac{1}{2}\sigma_V^2)\tau(x)}{\sigma_V \sqrt{\tau(x)}},$$

$$d_2 = d_1 - \sigma_V \sqrt{\tau(x)}$$

The value of the put option is:

$$F_p = \gamma x I \exp(-r\tau(x))N(-d_2 + \frac{\ln(\gamma)}{\sigma_V \sqrt{\tau(x)}}) - xV_0 \exp(-\delta\tau(x))N(-d_1 + \frac{\ln(\gamma)}{\sigma_V \sqrt{\tau(x)}}) \quad (11)$$

in which γ stands for the reclaimable proportion of the startup investment.

$$\gamma = \frac{T' - \tau(x)}{T'}$$

T' stands for the equipment's longevity of service. The longer the service, the more γ approaches 1.

Since a and $1-a$ are the probability of executing the put option and the call option, the values of the two options add up to $aF_p + (1-a)F_c$. $V_1(x)$ represents the total profit from the startup investment, then $(1-a)V_1(x)$ denotes the corresponding profit under the probability of executing the call option, and $aV_1(x)\big|_{t:0\to\tau(x)}$ denotes the corresponding profit under the probability of executing the put option. The financing costs under the two probabilities are $(1-a)\beta I$ and $a\beta xI$ respectively.

Thus the net value inherent in the decision-making of phased investment $G(x,\tau(x))$ is:

$$G(x,\tau(x)) = (1-a)V_1(x)\big|_{t:0\to\infty} + aV_1(x)\big|_{t:0\to\tau(x)} - xI + aF_p$$
$$+ (1-a)F_c - (1-a)\beta I - a\beta xI \quad (12)$$

where β stands for the rate of the financing costs. Notice that $G(x,\tau(x))$ denotes the value at $\tau(x)$, that means $G(x,\tau(x))$ is the function of random variable $\tau(x)$.

Thus we get the average of the net value $F(x)$ by calculating the expectation of $G(x,\tau(x))$.

$$F(x) = EG(x,\tau(x)) \qquad (13)$$

Eq.(13) can be expressed specifically as follows:

$$F(x) = V_1(x) - (x+\beta)I - bxV_1(x)\int_0^{\infty} e^{-(\mu-\alpha_y+bx)s} N(d)ds + bx\beta I(1-x)\int_0^{\infty} e^{-bxs} N(d)ds$$

$$+ bx\int_0^{\infty} e^{-bxs} N(d)F_p ds + bx\int_0^{\infty} e^{-bxs}(1-N(d))F_c ds \qquad (14)$$

2.6 The optimal investment proportion

Obviously, the optimal startup proportion is the x that maximizes $F(x)$. To calculate x, numerical techniques can be employed in the complex Eq.(14). By providing x with many different values, we can get different values of $F(x)$. For the sake of calculation convenience, $F(x)$ is divided into 7 parts for stimulation. See the detailed consequences below.

3 A case for computation

Consider a biotechnology company that is highly competitive because both FDA and CCCD (China Certification Committee for Drug) have licensed its unique products. Now the company is ready to launch its products to the market in a large scale. The investment for production and sales has been ascertained after the estimation of the market demand. Suppose the overall finance will be injected into the two phases. Due to the diminishment of technical risk, the main risk the firm faces comes from markets and embodies in the volatilities of prices and sales.

The values of corresponding variables are given in Table 1.

Table 1 Values of corresponding variables

Variable	Notation	Value
Risk-free rate of interest	r	0.04
Parameter of waiting time	b	1
Correlation coefficient between prices and sales	ρ	-0.4
Individual discount rate of firms	μ	0.1
Total investment	I	1×10^6

Initial unit profit of products	R_0	20
Initial sales of products	m_0	2000
Reclaimable proportion of investment	γ	1
Price rate of return	α_p	-0.07
Sales rate of return	α_m	0.16
Volatility of prices	σ_p	0.27
Volatility of sales	σ_m	0.38
Rate of financing costs	β	0.08

Other computable variables see Table 2.

Table 2 Values of computable variables

Variable	Notation	Value
Profit flow rate of return	α_y	0.04896
Investment profit rate of return	α_V	-0.05104
Volatility of profit flow	σ_y	0.3677
Volatility of investment profit	σ_V	0.3677
Dividend yield	δ	0.15104

40 groups of $F(x)$ are calculated for analysis. See Table 3.

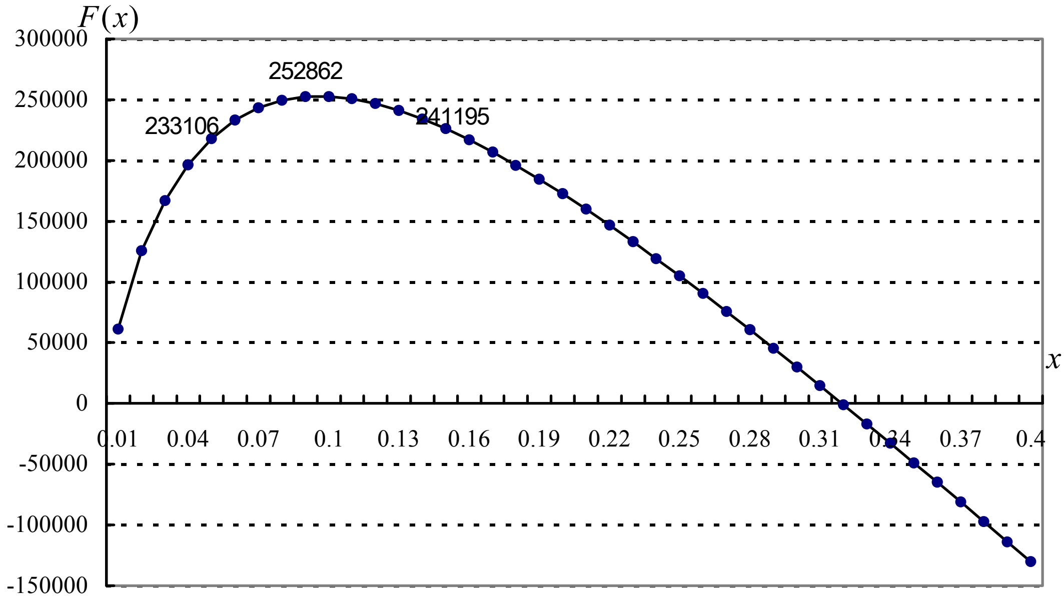
Table 3 The effect of the startup investment proportion x on the values of $F(x)$

x	$F(x)$	x	$F(x)$
0.01	61378	0.21	159871
0.02	125552	0.22	146752
0.03	166967	0.23	133218
0.04	196571	0.24	119315
0.05	217970	0.25	105085
0.06	233106	0.26	90563

0.07	243291	0.27	75783
0.08	249488	0.28	60774
0.09	252425	0.29	45559
0.10	252664	0.30	30162
0.11	250649	0.31	14604
0.12	246731	0.32	-1098
0.13	241195	0.33	-16927
0.14	234275	0.34	-32869
0.15	226162	0.35	-48910
0.16	217017	0.36	-65039
0.17	206975	0.37	-81244
0.18	196149	0.38	-97516
0.19	184636	0.39	-113845
0.20	172519	0.40	-130225

According to Table 3, Figure 1 is put forward.

Figure 1 The impact of the startup investment proportion x on the values of $F(x)$



$F(x)$ is approaching the maximum value 252862 when x is approximately equal to 0.096. Although the startup investment is only 9.6% of the total investment, the value inherent in the decision-making of phased investment reaches 25.3% of the total investment. So such flexibility values shouldn't be neglected. Once the proportion x is adopted improperly, the value changes drastically. In this case, when x exceeds 0.32, $F(x)$ becomes negative and such decision-making is terrible. When the proportion is too small or too big, the value $F(x)$ is unavoidably very small or even negative. Risk-hating people are likely to adopt smaller proportions and hesitate to develop firms when facing chances, while risk-preferring people are likely to adopt bigger proportions and passive to make up to the loses incurred by too much startup investment. So it's very important to selecting the optimal proportion.

Table 4 Summary of **t**he effects of changes in the relevant parameters on the values of $F(x)$

Variable	Notation	Impact	Comments
Risk-free rate of interest	r	+	Effects of the put option are bigger than the call option.
Parameter of waiting time	b	First+, then-	$F(x)$ is highest at b =3.
Correlation coefficient between prices and sales	ρ	+	Since ρ is negative, $F(x)$ increase with ρ.
Individual discount rate of firms	μ	-	
Total investment	I	First+, then-	$F(x)$ is highest at about I =0.8m.
Initial unit profit of products	R_0	+	
Initial sales of products	m_0	+	
Price rate of return	α_p	+	Since α_p is negative, $F(x)$ increase with α_p.
Sales rate of return	α_m	+	
Volatility of prices	σ_p	First-, then+	$F(x)$ is lowest at about σ_p =0.45.
Volatility of sales	σ_m	First-, then+	$F(x)$ is lowest at about σ_m =0.4.
Rate of financing costs	β	-	

Notes: As the level of a variable increase, other factors held constant at values as in Table 1 (Especially, x =0.096), $F(x)$ may rise (+), fall (-).

From Table 4, we can see $F(x)$ increases with risk-free rate of interest, correlation coefficient between prices and sales, initial unit profit and sales of products, price and sales rate of return, and decrease with individual discount rate of firms as well as rate of financing costs. $F(x)$ has maximum or minimum values with proper parameter of waiting time, total investment and volatility of prices and sales.

4 Conclusion

From the analysis of this paper, we can see the achievements contributed by the flexibility values in the decision-making courses as well as the importance of the optimal proportion. To realize the optimal objects, the relevant variables including the investment proportion should be ascertained properly. The real options approach proposed here focuses on the two-stage investment problems, estimating the above-mentioned values and proportions in case of market risks. It could be a useful tool for the decision-making of the phased investment if all relevant parameters can be exactly estimated. However, further considerations need to be taken into account as the practical problems might be multi-phased and much more complex. Further researches are needed to solve these problems.

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