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A CVP Analysis with Irreversible and Recurrent Real Options.*

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20th March 2002

Abstract

In this short paper, (Kulatilaka, 1988) model of FMS management is reinterpreted as a real options dynamic programming (DP) version of traditional Cost Volume Profit (CVP) analysis. Numerical examples replicate results reported in chapter 4 example 1.H. and chapter 7 of (Dixit and Pindyck, 1994). Moreover, a different version of (Kulatilaka, 1988) numerical example is analyzed. Results include not only the value of the flexible plant, decomposed into its base value and the value of the flexibility options (namely abandonment option, production mode switching option, mothballing option and waiting to invest option), but also mode bounds (threshold curves) are derived not only for the beginning time but also for the whole life of the project. In conclusion, this paper shows how much powerful is Kulatilaka's General Real Option Pricing Model (GROPM) in reaching through simple numerical methods results that others, e.g. (Dixit and Pindyck, 1994), get through very difficult symbolic stochastic algebra.

JEL classification code:G13, G31.

*A previous version of this paper was circulated under the title *Kulatilaka '88 as a CVP Analysis in a Real Options Framework: A Review, GAUSS^{tm} Codes and Numerical Examples* and it is posted on http://papers.ssrn.com. The author wishes to thank participants to AFFI 2001, Namur, BE; Bocconi Corporate Finance Autumn 2001 Meeting, Milan, Ita; EWGFM Autumn 2001 Haarlem, NL; Giardino Giusti Seminars Autumn 2001, Verona, Ita for helpful comments and suggestions. The usual disclaimer applies.

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Introduction

In this short paper, the (Kulatilaka, 1988) is reinterpreted as a dynamic generalization of the traditional cost volume profit analysis (CVP) under uncertainty, see (Hilliard and Leitch, 1975), (Chan and Lauhland, 1976) and (Lee, 1993) in (Aggarwal, 1993). In the latter articles the CVP analysis is used in a comparative statics framework without taking account of the flexibility options of any industrial plant investment project. As a matter of fact, before investing in an industrial plant, it is possible to wait until the uncertainty conditioning firm operations has been resolved, option to wait. Moreover, it is possible to shut down temporarily the plant when it becomes sufficiently[1] unprofitable, mothballing option. To continue, it is possible to switch between production modes, switching option. Last, but not least, it is possible to abandon the project, abandonment option.

All these flexibility options are quite common in any industrial, mining and transportation investment project. Because of this, the model of (Kulatilaka, 1988) can be extended to a variety of industrial plants that is much wider than the flexible manufacturing systems in a narrow sense.[2]

This paper is organized as follows. In section 1, the (Kulatilaka, 1988) model is revisited in a CVP budgeting framework. In this section, I have tried to bridge the gap between traditional CVP analysis and real options interpretation of investment project flexibilities. This has been done using a notation which is homogeneous with later Kulatilaka papers. Finally, the computational issues in applying the Bellman Dynamic Programming algorithm in a Walrasian general equilibrium framework[3] are explained using a general exposition and a couple of Mickey Mouse examples which draw a parallelism between a lattice and a grid specifications of the continuous stochastic process discretizations.

In section 2, some numerical examples have been derived. Examples inputs are intended to draw the reader's attention on the capabilities of the dynamic programming algorithm and its specification given in this paper[4].

In particular, the value of a plant is decomposed in its naked value, without flexibility options, and in the value of various combinations of options.[5] Finally, the mode bounds have been derived for each option

[1] I will show that this kind of judgment is rather more difficult that at first sight.

[2] Moreover, after (Kulatilaka, 1988) the analysis of FMS in the real options literature has had further developments which have delved better into problems which are very much specific to numerical control manufacturing machines, see for instance (Chen et al., 1998).

[3] Namely, the measure of probability to use in computing expected values and the appropriate interest rate to compute present values.

[4] In another paper I will apply these codes to real data from the shipping industry, the cement industry and the oil industry. Some of the data required are of quite easy availability, e.g. the state variable represented by some commodity prices, some others, instead, require a case study for each different industry and plant. Therefore, they deserve a dedicated paper in order to examine thoroughly the CVP analysis of the cost structure of the average plant in these industries and the interaction among the real options.

[5] It should be stressed the fact that real options seldom respect the value additivity principle. More often their values are

with respect to any other production mode.[6] Section 3 reports summary and conclusions and it sets the blueprint for further extensions of the General Real Options Pricing model of Kulatilaka in a CVP budgeting framework.

1 A CVP Analysis with Irreversible and Recurrent Real Options

Decisions concerning an investment project differ depending on the horizon on which they are taken, their irreversibility and the kind of management (passive or active) of the project when coping with a changing environment. In traditional CVP analysis decisions can be taken according to both a myopic single period comparative statics and a multi period static expectations, passive management framework. The shortcomings of this approach is that it cannot take account of decisions irreversibility, e.g. for investment or abandonment, and, for the same reason, of project active management flexibility, e.g. switching to the best production mode or mothballing the project. Moreover, CVP analysis is not capable of taking into account the fact that reversible decisions too, having transition costs, can have different value depending on the different sequence in which they are taken.

Although that is true, CVP analysis helps to bridge the gap that widened lately between accounting and budgetary control literature on one side and financial economics models evaluating real investments, in particular real options literature, on the other. As a matter of fact, CVP analysis can be considered a useful tool to specify correctly the interactions among the various profitability drivers and design accordingly the real option valuation framework. On the other hand, budgetary control procedures should take the necessary feedback from financial modeling of investment projects, focusing their attention on the variables considered most important in determining profitability.[7] In this section, CVP analysis is very briefly introduced as a comparative statics benchmark to decisions taken in a dynamic programming framework using a notation similar to (Kulatilaka, 1995).

Suppose to have an industrial plant which can be operated in at least two production modes. In the first one, production Mode A, net operating income is given by equation 1 while for production mode B the same variable is given in equation 2. To help intuition of the reader, one may think of a capital intensive plant, mode A, compared to a work intensive one, mode B. In the former fixed costs are high while variable are low, thus producing a very high contribution (gross) margin per unit produced and sold. In the latter the reverse is true.

super or sub additive.

[6] This result is not given in (Kulatilaka, 1988) although it shows how much powerful is this simple DP algorithm when compared to similar results derived through difficult stochastic algebra, see for instance (Dixit and Pindyck, 1994) page. 111 for a difficult derivation of the abandonment threshold curve and chapter 7 for the other mode bounds.

[7] For an attempt to recast budgetary control targets in terms of real options see chapter 8 of (Trigeorgis, 1996).

$$NOI_a = \pi^m \left(\theta_t^i\right)_a = Q \cdot (P - vc) - F = \theta \cdot 2 - .6 \tag{1}$$

$$NOI_b = \pi^m \left(\theta_t^i\right)_b = Q \cdot (P - vc) - F = \theta \cdot 0.6 - .1 \tag{2}$$

where:[8]

Q	:	quantity produced and sold, measured in units of production;
P	:	price for each unit produced;
vc	:	variable costs for each unit produced, those are costs that are linked to the unit of production, such as raw materials, energy, hours of direct work, freight and other logistics costs;
F	:	total fixed costs, those are cost that do not vary with the amount of production, such as depreciation, overhead cost for administration and skilled labor force;

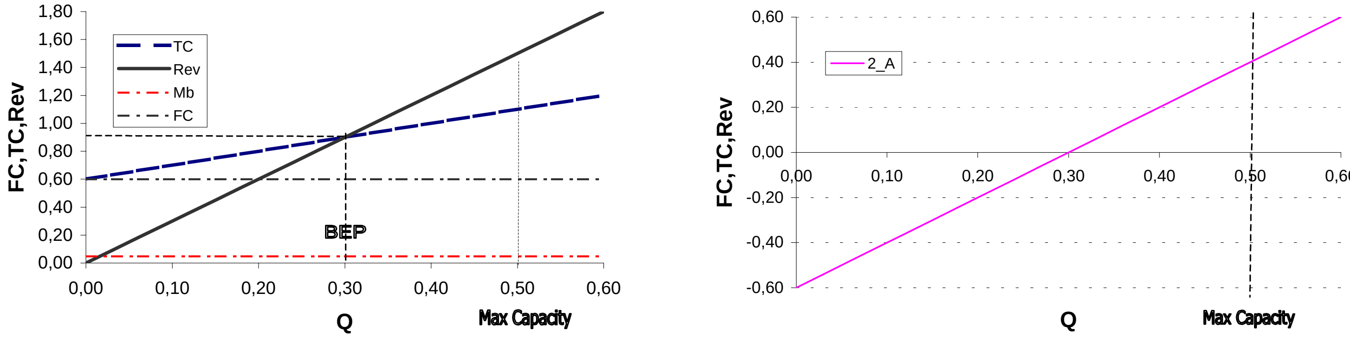
In the traditional CVP analysis Q, the quantity produced and sold, is chosen as state variable. In (Kulatilaka, 1988) any exogenous variable which is not under control and cannot be forecasted with a high degree of confidence can be chosen as state variable. The choice will depend on how much crucial is that variable in determining the investment profitability. In conclusion, the most important profitability driver should be chosen as state variable.

The choice of Q in CVP analysis is due to the fact that this kind of study of a firm profitability is aimed at separating the profits due to individual production lots as programmed in specific production budgets covering sub periods of the year. Quantity produced and sold is a pivotal variable in production budgets. Because of this, accounting students use to give a comparative statics overview of how the firms breaks even using a diagram like the one reported in figure 1 panel A, which comes usually along with the representation of profit functions on different levels of Q, see panel B. In these graphs the decision to operate or not is taken on a single period horizon, i.e. it is a myopic comparative statics framework in which one period profitability is compared across different and alternative production modes.

This way of representing firm profitability as dependent on an exogenous variable can be considered as a useful tool to specify correctly the interactions among the various profitability drivers. In order to get profits as represented in equations 1 and 2 a lump sum is invested at time $t = 0$, $I = 1$. Then, in each period of the life of the plant, set equal to 120 budget periods, a sum is spent on fixed costs together with another, fluctuating with the state variable Q. Depending on the assumptions made on the management of the plant and on the behavior of the state variable, several kind of values of the plant can be derived.

Under the hypothesis of no change throughout the whole life of the project of the initial level of the

[8] Those variables are derived under an accrual basis of accounting. With some adjustment those same variables can be transformed into the cash flows used for capital budgeting purposes. Although the basis of accounting changes, the classification between fixed and variable with respect to θ stays the same and it is quite crucial for real option derivation



Panel A: Break even analysis of production mode A. Panel B: Profit function for mode A against θ.

Figure 1: CVP Analysis of an industrial plant.

Legend: Assuming quantity Q produced and sold as state variable,
Graph in Panel A reports the revenue function $S = Q \cdot P$, the total fixed cost function $FC = F\ \forall Q$, the total variable cost function $VC = vc \cdot Q$ and the total cost function $TC = F + vc \cdot Q$.
Graph in Panel B reports the net operating income $NOI = Q(P - vc) - F$ for production mode A whose cost and revenue functions are reported in panel A.

state variable[9] and of a passive management of the plant, the BEP diagram can be translated into an NPV diagram, showing the level of the state variable for which $NPV > 0$, see figure 2, left hand graph, (Rao, 1992) page. 227. Present values have been computed using a risk free interest rate $r_f = 5\%$.

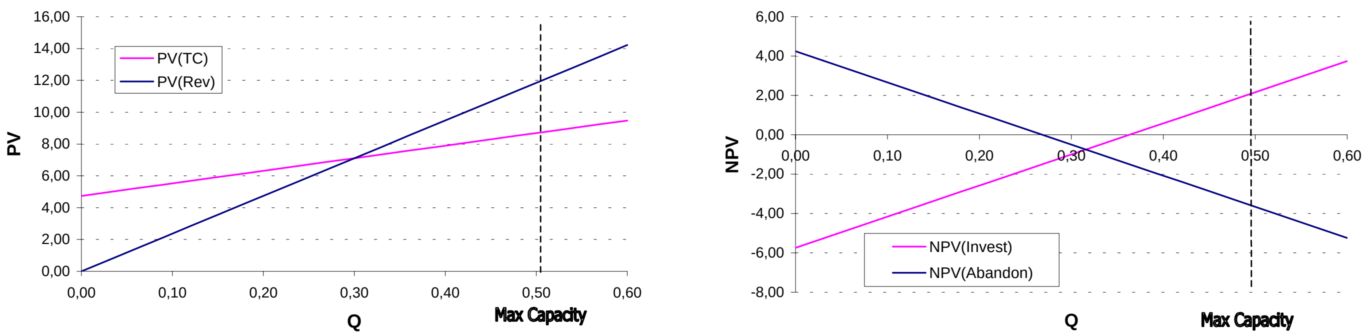


Figure 2: Mirrored NPV for CVP Analysis of an industrial plant.

Legend: Assuming quantity Q produced and sold as state variable, and a life of the investment project of 10 years divided into 12 budget period for each year, interest rates are $r_f = 5\%$. Present values are computed as monthly annuities, $a_{\overline{120|},5\%}$.
PVO represents the present value of outflows, $TC = F + vc \cdot Q$; PVI represents the present value of inflows, $S = Q \cdot P$. $NPV_{invest} = -I - PVO + PVI$, $NPV_{abandon} = +E + PVO - PVI$

The hypothesis under which values of the project are derived in figure 2 is very restrictive and useless in practice. As a matter of fact, it is quite unusual that a state variable like Q does not change during the whole life of the project. Moreover, it is definitely irrational that when the state variable pulls profitability in the losses region the plant will be kept running in the ordinary production mode. Under the hypothesis of

[9] These are the so called "static expectations", see page 219 of (Dixit and Pindyck, 1994).

static expectations it is possible to compute threshold levels for both investing and abandoning the project, see right hand graph in figure 2, (Dixit and Pindyck, 1994) page 225. As a matter of fact, while in a pure comparative statics framework the plant starts producing profits for $\theta > .30$, in order to have a positive net present value I need to have a $\theta > 0.3633$. Instead, considering a negative scrapping value $E = -.5$, a net present value for abandoning the project can be computed for each level of θ. In this case, at time $t = 0$ I have $NPV_{abandon} > 0$ for $\theta < 0.2683$. Even under the hypothesis of static expectations there is a range of values of θ for which even if the plant is profitable in the short run, the investment is not implemented, to be specific, $.30 < \theta < .3633$. Moreover, there is a range of values of the state variable for which, even if the plant generates short run losses, it is not abandoned, namely, $.2683 < \theta < .30$. Overall, the interval $.2683 < \theta < .3633$ resembles to an hysteresis band in which the plant keeps being operated in the same mode in which it entered the interval. Static expectations and passive management of the plant do not take into account options that could be taken temporarily. As a matter of fact, both investing and abandoning modes are irreversible states of the investment project.

Instead, if the state variable reaches very low levels, the plant will be closed temporarily, mothballed, waiting to be reopened when the state variable is back at convenient levels. Therefore, for both production modes A and B, I need to have the minimum costs that are incurred when the plant temporarily stops production, aka minimal maintenance costs.[10]

Mode	Description	Profit flow per period
1	waiting to invest	0
2	production mode A	$\pi^m \left(\theta_t^i\right)_a$
3	production mode B	$\pi^m \left(\theta_t^i\right)_b$
4	shut down	-MB
5	abandon	0

Table 1: Modes of production, their descriptions and payoffs

Moreover, when the level of the state variable will be particularly high, it would be convenient to switch to the more aggressive production mode, reducing variable costs and increasing fixed costs, production mode A. On the other hand if the level of the state variable is exceptionally low for many production budgets, then it would be wise to abandon the project. This last remark leads naturally to conclude that the very initial investment could be implemented waiting for uncertainty resolution. All these different payoffs are summarized in table 1 and depicted in figure 3. It is important to notice the indifference levels of θ between

[10] In (Kulatilaka, 1988) these costs are the same as fixed costs F. Actually, considering that a switching cost is paid to mothball the plants, mothballing mode costs and fixed cost in production mode can differ. Moreover, it is quite straightforward to show in a CVP graphical analysis that if mothballing costs are equal to fixed costs, then it is never convenient to mothball the plant because losses are at most equal to fixed costs for $Q = 0$. See also page 229 of (Dixit and Pindyck, 1994).

the production modes, see from left to right $\theta_{4\to2}$, $\theta_{1\to2}$, $\theta_{2\to3}$. These levels are derived without taking into account any transition cost. Hence they can be defined as Marshallian thresholds. They are a good benchmark for comparing optimal management in a dynamic and in a comparative statics framework. As a matter of fact they would represent the levels for which the plant mode is changed to the most profitable one in a myopic single period comparative statics framework. Blatantly enough, figure 3 reports threshold levels for all modes but the abandonment mode. The reason is that while the others are all alternative recurrent modes, abandonment mode is an absorbing, i.e. irreversible, mode. In other words, when the plant is abandoned, it would not be any longer possible to switch back to any production mode.

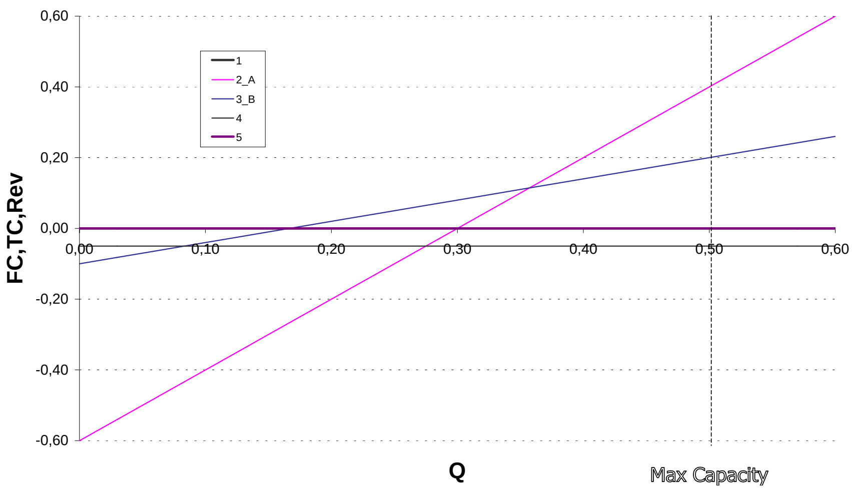


Figure 3: Profit Functions for the Fixed Modes against θ.

Legend: Assuming quantity Q produced and sold as state variable. Mode 1: waiting to invest; Mode 2: production mode A, capital intensive; Mode 3: production mode B, work intensive; Mode 4: mothballed state; Mode 5: abandoned project.

This last observation shows why comparative statics analysis is not appropriate for the study of even the simplest of the industrial plants. As a matter of fact production modes listed in table 1 should be considered as alternative choices that are made in an optimal *sequence*, depending on the time series behavior of the state variable. These sequential choices would take into account the levels of θ for which payoff curves cross in figure 3 for all production modes but the abandonment mode. Being this an absorbing state, choices with respect to abandonment should be taken choosing the production mode which is most profitable not in the individual period[11] but the one that maximizes investment value within an optimal sequence of production mode choices. Moreover, this kind of path dependency in mode production choice extends to the other production modes too when taking into account the costs of transition from one mode of production to the other. In this case, it is not possible to represent modes payoffs in one graph like the one reported in

[11] This would be a simple comparative statics rationale in choosing among production modes.

figure 3. Instead, I need a graph for each beginning mode of production drawing the value of the plant for any subsequently chosen mode, see figure 4.

$$\delta = \begin{pmatrix} \delta_{1,1} & \delta_{1,2} & \delta_{1,3} & \delta_{1,4} & \delta_{1,5} \\ \delta_{2,1} & \delta_{2,2} & \delta_{2,3} & \delta_{2,4} & \delta_{2,5} \\ \delta_{3,1} & \delta_{3,2} & \delta_{3,3} & \delta_{3,4} & \delta_{3,5} \\ \delta_{4,1} & \delta_{4,2} & \delta_{4,3} & \delta_{4,4} & \delta_{4,5} \\ \delta_{5,1} & \delta_{5,2} & \delta_{5,3} & \delta_{5,4} & \delta_{5,5} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & +\infty & +\infty \\ +\infty & 0 & .005 & .01 & +\infty \\ +\infty & .005 & 0 & .01 & +\infty \\ +\infty & .01 & .01 & 0 & +\infty \\ +\infty & +\infty & +\infty & .5 & 0 \end{pmatrix} \tag{3}$$

Sunk costs of transition from one production mode to the other have very direct intuitive meaning, see expression 3 for the matrix of transition costs. Costs of staying in one mode are set to zero and are reported in the main diagonal of the matrix. Costs of passing from the waiting to invest mode 1 to production mode 2 or 3 are $\delta_{1,2} = \delta_{1,3} = 1$. Costs of passing from one production mode to the other are $\delta_{3,2} = .005$ and $\delta_{2,3} = .005$.[12]. Costs of mothballing the plant from either production mode are $\delta_{2,4} = .01$ and $\delta_{3,4} = .01$. Costs of starting up again in any production mode from the mothballed state are $\delta_{4,2} = .01$ and $\delta_{4,3} = .01$.

Plant production modes 2,3,4 are recurrent states, see page 140 of (Ross, 1993), meaning that the dynamic system goes from one state to the other an infinite number of times. Instead, plant abandonment is an absorbing state, meaning that it is impossible to come back from that state. Moreover, following (Kulatilaka, 1988) model and other authors, e.g. (Dixit and Pindyck, 1994) page 230, this state is made accessible only through the mothballed state. Because of this, transition costs are set equal to $+\infty$ for all the transitions from and to mode 5 except $\delta_{4,5} = .5$. When the mothballed plant is abandoned it is supposed to be taken by an environmental services company for a sum that is one half of the initial investment.[13] Hence there is a cost in abandoning the project.[14] Moreover, state 1, waiting to invest, is an unaccessible state from any of the other production modes, see page 143 of (Ross, 1993). As a matter of fact the decision to invest is irreversible. Because of irreversibility, transition costs from any other state to state 1 are set to $+\infty$. To the same token, other two non rational transitions have infinite costs, namely from the waiting to invest to the mothballed and the abandoned state.[15]

[12] In this case they are set as symmetric but they do not need to be necessarily so.

[13] This scrap value may be positive if wreckage materials can be sold at a profit. Frequently, these materials require cleanup and environmental restoration. In those cases transition costs to the abandonment state can be very high and comparable to the initial investment.

[14] A kind of barrier to exit in industrial organization literature.

[15] In the actual implementation of the transition cost matrix, $+\infty$ has been set to a very high number when compared to other variable levels.

It is worth noting that transition costs that have finite levels should be in certain proportions that make economic sense. For instance, the costs of restarting $\delta_{4,2}, \delta_{4,3}$ should be always smaller than the costs of starting operations from scratch. Moreover, this latter cost should be always larger than the payoff from scrapping the plant in order to rule out a "money machine" of rapid cycles of investment and abandonment. There are not particular constraints on the other transition costs.

Taking into account transition costs, see figure 4, it is not correct to compare plant operating modes using individual period profit flows. Instead, the plant values operated in subsequently chosen mode should be compared. Indifference levels of θ among the production modes differ from those reported in figure 3. For instance, in graph 1 in panel A, levels of the state variable θ can be partitioned in three regions, according to the choice that is taken after investing the lump sum in the industrial plant, from mode 1 to modes 1,2 and 3.

$$\begin{array}{rcccll} & & & & & \underline{\text{Mode Choice}} \\ & & \theta & \leq & \theta^*_{1,3} & \text{do not invest;} \\ \theta^*_{1,3} & \leq & \theta & \leq & \theta^*_{3,2} & \text{invest and operate in mode 3;} \\ \theta^*_{3,2} & \leq & \theta & & & \text{invest and operate in mode 2;} \end{array}$$

While, in graph 2 in panel A, payoffs from reaching state 2,3,4 being in state 2 are partitioned according to the following levels of θ.

$$\begin{array}{rcccll} & & & & & \underline{\text{Mode Choice}} \\ & & \theta & \leq & \theta^*_{3,4} & \text{mothball the plant;} \\ \theta^*_{3,4} & \leq & \theta & \leq & \theta^*_{3,2} & \text{switch to mode 3;} \\ \theta^*_{3,2} & \leq & \theta & & & \text{keep operating in mode 2;} \end{array}$$

Moreover, see graph 3 in panel B, payoffs from reaching states 2,3,4 being in state 3 are partitioned in the same way by the following thresholds.

$$\begin{array}{rcccll} & & & & & \underline{\text{Mode Choice}} \\ & & \theta & \leq & \theta^*_{3,4} & \text{mothball the plant;} \\ \theta^*_{3,4} & \leq & \theta & \leq & \theta^*_{3,2} & \text{keep operating in mode 3;} \\ \theta^*_{3,2} & \leq & \theta & & & \text{switch to mode 2;} \end{array}$$

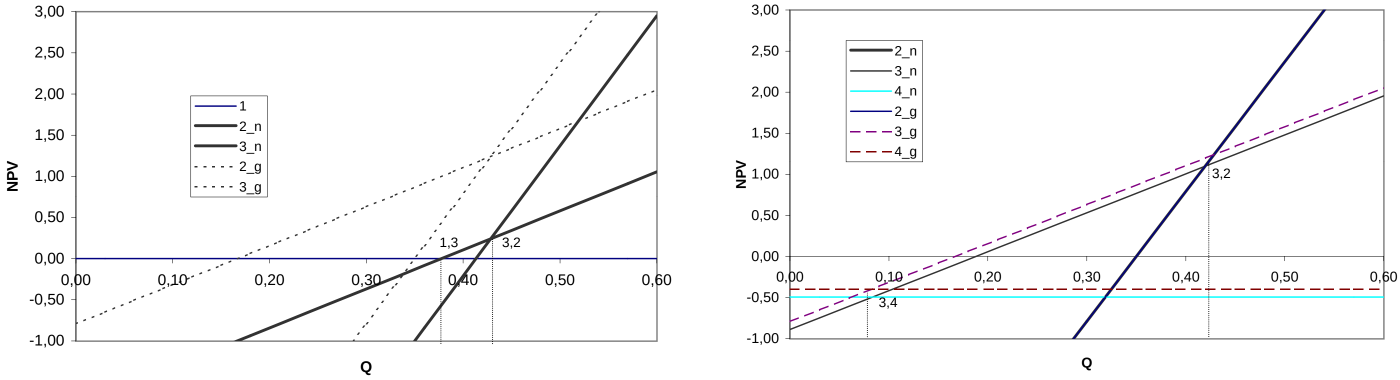
Finally, being in the mothballed state, see graph 4 in panel B, payoffs from reaching states 2,3 or 5 are given by the following partition of the state variable

$$\begin{array}{rcccll} & & & & & \underline{\text{Mode Choice}} \\ & & \theta & \leq & \theta^*_{3,4} & \text{keep mothballing the project;} \\ \theta^*_{3,4} & \leq & \theta & \leq & \theta^*_{3,2} & \text{start again operating in mode 2;} \\ \theta^*_{3,2} & \leq & \theta & & & \text{start again operating in mode 3;} \end{array}$$

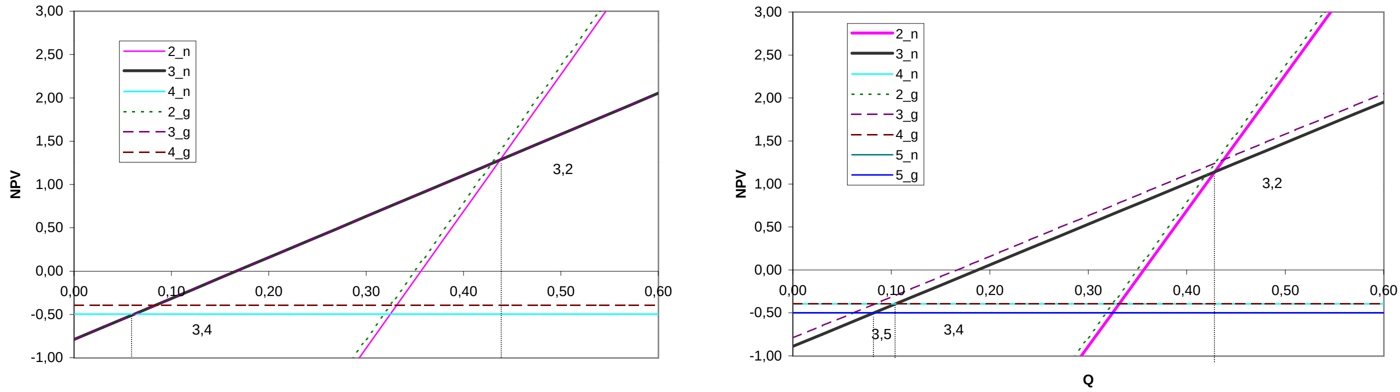
It is worth noting that in this last partition the mothballing state is a recurrent state.[16] As a matter of fact

[16] Numbers could be "massaged" to have the mothballing state as a transient state. In that case the NPV from mothballing the project could have been set at a lower level than the NPV from abandoning the project. As a consequence, abandonment would have always been preferred to mothballing.

when comparing its payoff to the one of the abandonment state, the system is reflected back into the choice of mothballing the project.



Panel A: Payoffs for the passage from Mode 1 to 1,2,3 and from Mode 2 to 1,2,3.



Panel B: Payoffs for the passage from Mode 3 to 1,2,4 and from Mode 4 to 1,2,4.

Figure 4: Profit Functions for the Fixed modes against θ net of transition costs.

Legend: Assuming quantity Q produced and sold as state variable. Mode 1: waiting to invest; Mode 2: production mode A, work intensive; Mode 3: production mode B, capital intensive; Mode 4: mothballed state; Mode 5: abandoned project. Deponent $_{g/n}$ indicates gross or net of transition costs.

Having shown the compound, path dependent, nature of the mode choices, it is necessary to derive the optimal sequence of choices using the appropriate dynamic planning algorithm. Following (Kulatilaka, 1988) I have chosen Richard Bellman Dynamic Programming (DP), (Bellman, 1957).

There are at least two ways to tackle this problem. The first is to set up an Hamilton-Jacoby-Bellman (H-J-B) Equation, see page 408 (Judd, 1998) and solve it, see for instance (Knudsen et al., 1999). This method uses continuous variables and difficult stochastic algebra to derive solutions that are often not so much straightforward. Certainly this method is unsuitable for pedagogical purposes. The second method consists in discretizing the problem and deriving results numerically, see page 409 (Judd, 1998). There are several methods to derive results of a dynamic programming problem, e.g. the value function iteration or the policy function iteration. All these methods rely on the (Denardo, 1967) contraction mapping theorem for dynamic programming.[17] Although these methods are certainly easier than the symbolic pde solutions of the H-J-B equation, they would require a quite long deviation from the main topics within a corporate finance course.

Therefore, I have chosen to apply the same backward induction method suggested by (Trigeorgis, 1996) in a numerical example on page 177. There the stochastic variable is discretized in a standard binomial lattice, here, following (Kulatilaka, 1993), it is discretized in a grid allowing a specification of the stochastic process that is richer than the standard Brownian motion random walk (RW), i.e. an Ornstein Uhlenbeck process which covers the RW process as a particular case. The advantage of the grid over the lattice specification stays in the fact that it is much straightforward to specify the range of variation of the state variable according to economic intuition, e.g. Q, quantity produced and sold, cannot exceed the maximum plant production capacity.

This review is organized around this parallelism between a lattice and a grid discretization of the stochastic process generating the state variable. In section 1.1 notation is reviewed in an effort to use the same symbols for all the problems that can be solved in the Kulatilaka general model of real options, see (Kulatilaka, 1988), (Kulatilaka, 1993) and (Kulatilaka, 1995).

In section 1.2 it is shown what is the right probability measure to compute expected values when the stochastic state variable is discretized in a grid. In section 1.3 it is shown under which conditions the drift rate of the Ornstein Uhlenbeck process can be considered a certainty equivalent drift rate, (Cox et al., 1985) lemma 4.

In section 1.4 the implementation of the Bellman dynamic programming recursion is analyzed using not only a general notation but also a couple a *Mickey Mouse* examples which are intended to help intuition in grasping this passage, usually the most unfriendly in all the classes in corporate finance I have taught.

[17] For an application of this theorem to a stationary discounted dynamic programming problem in a numerical example see (Harris, 1987) page 28.

1.1 Notation

Following (Kulatilaka, 1995) I define:

θ_t^j : state variable at time $t = 1, \ldots, N_t = (T/\Delta\, t)$ and level $j = 1, \ldots, N_s = (S/\Delta\, S)$, a scalar in a univariate model;

$\pi^m\left(\theta_t^i\right)$: profit function in the interval $(t,\ t + \Delta\, t)$ for the state variable level θ_t^i and the plant operating in mode m $\forall\, m = 1, \ldots, M$;

$p_{i,j}$: $Pr\left(\theta_t^j | \theta_{t-1}^i\right)$ probability of passing from state i to state j after $\Delta\, t$;

$c_{m,l}$: cost for passing from production mode m to mode l;

$F\left(\theta_t^i, m, t\right)$: Value of the plant for a level of the state variable θ^i, conditional on an entering mode m, valued at time t;

$S\left(\theta_T^i, T\right)$: Salvage value for each level of the state variable θ^i at time $t = T$, for any entering mode, hence unconditional on the mode;

ρ : discount factor, when using an EMM $\rho = \left(1 + r_{f,1/m}\right)^{-1}$. where $r_{f,1/m}$ is the expected return rate for the fraction $1/m$ of the year;

1.2 Specifications of $p_{i,j}$

The exogenous uncertainty faced by the firm is summarized by a state variable θ_t^j which follows a diffusion process, see equation 4.

$$d\,\theta = \alpha\left(\theta, t\right)\ d\,t + \sigma\left(\theta, t\right)\ d\,Z \tag{4}$$

where:

$\alpha\left(\theta, t\right)$: instantaneous drift rate;
$\sigma^2\left(\theta, t\right)$: instantaneous variance rate;
$d\,t$: time differential;
$d\,Z$: standard Wiener process, normally distributed with $E\left(d\,Z\right) = 0$ and $Var\left(d\,Z\right) = E\left(\left(d\,Z\right)^2\right) = d\,t$.

Since most of the variables not controlled by a firm tend to revert on some long run level, [18] i.e. they are mean reverting processes, the diffusion in equation (4) is specified as an arithmetic Ornstein-Uhlenbeck process[19] of the form given in expression (5), see (Dixit and Pindyck, 1994) page 74.

$$d\,\theta = \eta \cdot \left(\overline{\theta} - \theta\right)\ d\,t + \sigma_\theta\ d\,Z \tag{5}$$

[18] For instance, a commodity price tends to revert towards its long run marginal price.

[19] The arithmetic version of the O-U process does not guarantee that the state variable does not reach negative levels. An adjustment in the estimation of the process parameters to get a geometric O-U is contained in (Sick, 1995) in (Jarrow et al., 1995).

where, in addition to the previous notation:

η : the speed of reversion, e.g. for $\eta = 0$ the process becomes a Brownian motion while for $\eta > 0$ the process tends to be mean reverting;

$\overline{\theta}$: the normal level of θ, i.e. the level at which θ tends to revert;

In order to discretize this process over the range of possible values that θ can take, following (Kulatilaka, 1993) page. 279 Appendix, I assume that, over the interval Δt with which the continuous variable is discretized,

$$\Delta\,\theta_t \sim N\left(\eta\left(\overline{\theta} - \theta_t\right) \cdot \Delta\,t, \sigma_\theta^2 \cdot \Delta\,t\right)$$

The stochastic process governing θ_t^i is discretized through a one step transition probability matrix, (Kulatilaka, 1988), (Kulatilaka, 1993) and (Kulatilaka, 1995). This means that the levels of θ_t^i can be easily described by a grid for states $i = 1, \ldots, S$, see page 369 (Hull, 1997), see figure 5, where the investment horizon is discretized in Δt intervals while the difference between the maximum, $i = S$, and the minimum, $i = 1$, that the state variable can reach, is discretized in $\Delta\theta$ steps.

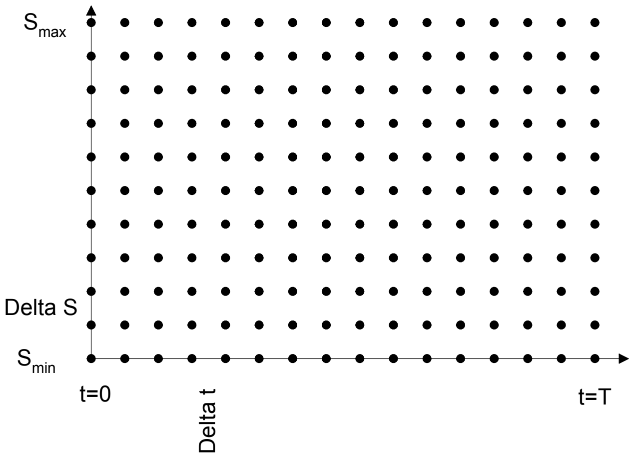


Figure 5: A Grid for the DP Approach

The probability of reaching any of the inside nodes $j = 2, \ldots, S-1$ being in any node $i = 1, \ldots, S$ is defined as

$$Prob\left(\theta_{j,t}|\theta_{i,(t-1)}\right) = \Phi\left[\frac{-\eta\left(\overline{\theta} - \theta_t^i\right) \cdot \Delta\,t + (j-i+1/2) \cdot \Delta\,\theta}{\sigma_\theta \cdot \theta_t^i \cdot \sqrt{\Delta\,t}}\right] - \Phi\left[\frac{-\eta\left(\overline{\theta} - \theta_t^i\right) \cdot \Delta\,t + (j-i-1/2) \cdot \Delta\,\theta}{\sigma_\theta \cdot \theta_t^i \cdot \sqrt{\Delta\,t}}\right] \tag{6}$$

Instead for the boundary states, e.g. state $j = 1, B$, being in any node $i = 1, \ldots, S$, one step transition probabilities are

$$Prob\left(\theta_{1,t}|\theta_{i,(t-1)}\right) = \Phi\left[\frac{-\eta\left(-\overline{\theta}-\theta_t^i\right)\cdot\Delta\,t+(1-i+1/2)\cdot\Delta\,\theta}{\sigma_\theta\cdot\theta_t^i\cdot\sqrt{\Delta\,t}}\right] \quad (7)$$

$$Prob\left(\theta_{B,t}|\theta_{i,(t-1)}\right) = 1-\Phi\left[\frac{-\eta\left(\overline{\theta}-\theta_t^i\right)\cdot\Delta\,t+(B-1-i+1/2)\cdot\Delta\,\theta}{\sigma_\theta\cdot\theta_t^i\cdot\sqrt{\Delta\,t}}\right] \quad (8)$$

Notice that probabilities in equation (7) are just those in (8) in reverse order. [20]

In an alternative approach, see (Trigeorgis, 1996), the stochastic process governing θ_t^i is discretized through a binomial lattice, see figure 6.

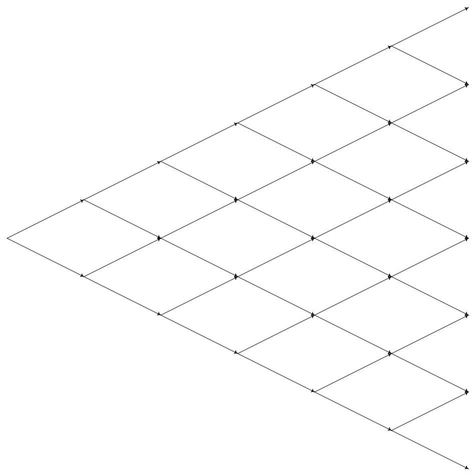


Figure 6: A Lattice for the DP Approach

In this case the probability of the two following nodes is the usual equivalent martingale measure used in the binomial option pricing model of Cox Ross Rubinstein (1979) (Cox et al., 1979).

$$q_u = \frac{(1+r_f)-d}{u-d}$$
$$q_d = 1-q_u$$

where:

r_f : risk free rate over the interval $\Delta\,t$;
u : $e^{\sigma\cdot\sqrt{\Delta\,t}}$
d : $e^{-\sigma\cdot\sqrt{\Delta\,t}}$

[20] Equation A2 in (Kulatilaka, 1993) is slightly different. As a matter of fact there is a $(B-i+1/2)$ instead of $(B-1-i+1/2)$. In other words it seems that the number of nodes in the grid in which the $S_{max}-S_{min}$ has been missed for the number of intervals. No references are given nor a derivation is reported in the reference just cited.

1.3 The Discount Rate

Equations (6)-(8) provide a natural measure of probability being impossible to construct an EMM for a grid. It is shown that this natural measure of probability can actually be considered as a EMM being the drift rate of the Ornstein-Uhlenbeck process adjusted for the risk premium. Because of this, it is possible to use the risk free rate to compute present values.

Following (Cox et al., 1985) (Lemma 4) as quoted in (Kulatilaka, 1993), I can use these equations to compute a measure of probability that replaces the actual drift rate of the underlying stochastic process with a *certainty equivalent drift rate* (CEDR). This CEDR can be obtained subtracting to the actual drift rate on θ the risk premium that would be expected on that asset in a market equilibrium model, e.g. (Lintner, 1965), (Sharpe, 1964), (Mossin, 1966) CAPM.

In general, I have that on a non dividend paying asset,

$$
\begin{aligned}
E\left(R_j\right) &= r_f + \left[E\left(R_m\right) - r_f\right] \cdot \beta_j \\
& \quad \text{in a Sharpe Lintner Mossin CAPM framework} \\
\mu &= r_f + RP \\
& \quad \text{the drift rate expected in equilibrium} \\
CEDR = \mu - RP &= r_f \\
& \quad \text{the CE drift rate, (Cox et al., 1985)}
\end{aligned}
$$

Instead, on a dividend paying asset expected drift, capital gain, and expected return differ. Then I have,

$$
\begin{aligned}
g = E\left(R_j\right) - \delta &= r_f + \left[E\left(R_m\right) - r_f\right] \cdot \beta_j - \delta \\
& \quad \text{expected drift, capital gain, in a Sharpe Lintner Mossin CAPM framework} \\
g &= r_f + RP - \delta \\
& \quad \text{the drift rate expected in equilibrium} \\
CEDR = g - RP &= r_f - \delta \\
& \quad \text{the CE drift rate, (Cox et al., 1985)}
\end{aligned}
$$

In this case,

$$
g - RP = r_f - \delta
$$

$$
\begin{aligned}
& \text{define } \delta = (r_f + RP) - \mu \text{ and substitute} \\
= \;& r_f - [(r_f + RP) - \mu] \\
& \text{let } RP = 0 \text{ for the specific asset considered} \\
= \;& r_f - (r_f - \mu) \\
& \text{let } \mu = \eta\left(\overline{\theta} - \theta_t\right)\text{, the Ornstein-Uhlenbeck drift rate.} \\
= \;& r_f - \left[r_f - \eta\left(\overline{\theta} - \theta_t\right)\right] \\
CEDR = g - RP \;=\;& \eta\left(\overline{\theta} - \theta_t\right)
\end{aligned}
$$

It is important to notice that CEDR changes from time to time being function of θ_t. This means that the CE measure of probability in equations (6)-(8) should be recomputed each time given the current level of θ reached by the stochastic process within the grid. The intuition of this time varying probability measure is that the stochastic process has a state dependent dividend. Having shown that the O-U drift rate is equivalent to the CEDR in a world of risk neutral agents, the probability measure in equations 6-8 is an EMM in a world of risk neutral agents . Therefore it is possible to discount the expected value computed on this measure of probability with a risk free interest rate, see (Kulatilaka, 1993) page 274.[21]

1.4 DP Algorithm Implementation

The implementation of the (Bellman, 1957) dynamic programming recursion is explained in two steps. To begin with, the general formulation is given. Then, two *Mickey Mouse* examples are given one for a lattice the other for a grid specification of the discretization of the state variable. In general, the backward Bellman recursion process is set up as follows

$$
\begin{aligned}
t &= T \\
F\left(\theta_t^i, m, t\right) &= F\left(\theta_t^i, T\right) = S\left(\theta_T^i\right)
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
t &= T-1 \\
F\left(\theta_{T-1}^i, m, T-1\right) &= \max_{l} \left\{\pi\left(\theta_{T-1}^i, l, T-1\right) - c_{m,l} + \frac{E_{T-1}\left[S\left(\theta_T\right)\right]}{1+r_f}\right\}
\end{aligned}
\tag{10}
$$

[21] In the same way, see (Cox et al., 1979), in the case of a lattice specification of the stochastic process, the expected value computed on the risk neutral EMM is discounted with the rate paid on a risk free asset, zero coupon, of equal life.

$$\begin{aligned} t &= T-2 \\ F\left(\theta^i_{T-2}, m, T-2\right) &= \underset{l}{max} \left\{ \pi\left(\theta^i_{T-2}, l, T-2\right) - c_{m,l} + \frac{E_{T-2}\left[F\left(\theta_{T-1}, l, T-1\right)\right]}{1+r_f} \right\} \end{aligned} \tag{11}$$

$$\begin{aligned} &\forall\, t \\ F\left(\theta^i_{t}, m, t\right) &= \underset{l}{max} \left\{ \pi\left(\theta^i_{t}, l, t\right) - c_{m,l} + \frac{E_{t}\left[F\left(\theta_{t+1}, l, t+1\right)\right]}{1+r_f} \right\} \end{aligned} \tag{12}$$

Depending on how the stochastic process governing θ^i_t is specified, the expected value at each step is computed in one of the two following ways:

1. stochastic process discretized in a binomial **lattice**, (Trigeorgis, 1996) page 180:

$$E_t\left[F\left(\theta_{t+1}, l, t+1\right)\right] = q^* \cdot F\left(\theta^+_{t+1}, l, t+1\right) + (1-q^*) \cdot F\left(\theta^-_{t+1}, l, t+1\right)$$

2. stochastic process discretized in a $N_t = (T/\Delta\, t)$ by $N_s = (S/\Delta\, S)$ **grid** on (time)-(level of θ), Kulatilaka (1988), (1993), (1995), where Δt and $\Delta\, S$ are the steps chosen to discretize time and the state variable respectively:

$$E_{t,\theta^i_t}\left[F\left(\theta_{t+1}, l, t+1\right)\right] = \sum_{k=1}^{N} F\left(\theta^k_{t+1}, l, t+1\right) \cdot p_{i,k}$$

In general, the backward dynamic programming solution works as follows:

- at time $t = T$ for *any* mode of operation and any state θ, the value of the plant is equal to its salvage value, (Kulatilaka, 1995), if this is set to zero the process simply starts at $T-1$, see equation (1) in (Kulatilaka, 1988). When the salvage value is different from zero, see equation (9) above, the value of the plant in not conditioned by the operating mode in which the plant is closed. Hence, I have N_s expressions like (9) one for each level in which it is discretized the state variable.[22]

- at time $t = T-1$ for *each* entering mode of operation and any state θ I have to maximize over the production modes choosing the one which maximizes the current profit,[23] at time $t = T-1$. This choice has no influence on the salvage value. At the end of this step I have M equations [24] like (10) for each of the levels of state θ.

- at time $t = T-2$ for *each* m-th entering mode of operation and any state θ I have to maximize over the production mode choosing the l-th one which maximizes together:

[22] Instead, in the case of a binomial lattice discretization, I have a salvage value for each of the final nodes.
[23] Net of switching costs.
[24] One for each entering mode.

– the current profit net of switching costs:

$$\pi\left(\theta^i_{T-2}, l, T-2\right) - c_{m,l};$$

– the continuation value:

$$\frac{E_{T-2}\left[F\left(\theta_{T-1}, l, T-1\right)\right]}{1+r_f};$$

the expected value at the numerator of the last expression is computed choosing among the expected values computed in equ. (10) of all the $l = 1, \ldots, M$ modes for the appropriate levels of θ_{T-1}. In the case of the lattice specification those are the adjacent nodes, in the case of a grid specification they are simply all the levels of θ_{T-1}.

Hereby two *Mickey Mouse* applications are reported: the first on a binomial lattice, the second on a grid. In a three period binomial lattice discretization,$t = 0, 1, 2$, with a two mode plant $m = A, B$, see (Trigeorgis, 1996) page. 184, I have at time $t = 2$

$$\begin{aligned}
F\left(\theta_2^{++}, A, 2\right) &= max\left[\pi\left(\theta_2^{++}, A, t\right) - 0, \pi\left(\theta_2^{++}, B, t\right) - c_{A,B}\right] && (13)\\
F\left(\theta_2^{+-}, A, 2\right) &= max\left[\pi\left(\theta_2^{+-}, A, t\right) - 0, \pi\left(\theta_2^{+-}, B, t\right) - c_{A,B}\right] && (14)\\
F\left(\theta_2^{--}, A, 2\right) &= max\left[\pi\left(\theta_2^{--}, A, t\right) - 0, \pi\left(\theta_2^{--}, B, t\right) - c_{A,B}\right] && (15)
\end{aligned}$$

$$\begin{aligned}
F\left(\theta_2^{++}, B, 2\right) &= max\left[\pi\left(\theta_2^{++}, A, t\right) - c_{B,A}, \pi\left(\theta_2^{++}, B, t\right) - 0\right] && (16)\\
F\left(\theta_2^{+-}, B, 2\right) &= max\left[\pi\left(\theta_2^{+-}, A, t\right) - c_{B,A}, \pi\left(\theta_2^{+-}, B, t\right) - 0\right] && (17)\\
F\left(\theta_2^{--}, B, 2\right) &= max\left[\pi\left(\theta_2^{--}, A, t\right) - c_{B,A}, \pi\left(\theta_2^{--}, B, t\right) - 0\right] && (18)
\end{aligned}$$

and at time $t = 1$ I have

$$F\left(\theta_1^{+}, A, 1\right) = max\left\{\begin{array}{c} \pi\left(\theta_1^{+}, A, 1\right) - 0 + PV\left[E\left(F\left(\theta_2, A, 2\right)\right)\right], \\ \pi\left(\theta_1^{+}, B, 1\right) - c_{A,B} + PV\left[E\left(F\left(\theta_2, B, 2\right)\right)\right] \end{array}\right\} \qquad (19)$$

where $E\left(F\left(\theta_2, A, 2\right)\right)$ is computed on equations 13 and 14;

and $E\left(F\left(\theta_2, B, 2\right)\right)$ is computed on equations 16 and 17;

$$F\left(\theta_1^-, A, 1\right) = max\left\{ \begin{array}{c} \pi\left(\theta_1^-, A, 1\right) - 0 + PV\left[E\left(F\left(\theta_2, A, 2\right)\right)\right], \\ \pi\left(\theta_1^-, B, 1\right) - c_{A,B} + PV\left[E\left(F\left(\theta_2, B, 2\right)\right)\right] \end{array} \right\} \tag{20}$$

where $E\left(F\left(\theta_2, A, 2\right)\right)$ is computed on equations 14 and 15;

and $E\left(F\left(\theta_2, B, 2\right)\right)$ is computed on equations 17 and 18;

$$F\left(\theta_1^+, B, 1\right) = max\left\{ \begin{array}{c} \pi\left(\theta_1^+, A, 1\right) - c_{B,A} + PV\left[E\left(F\left(\theta_2, A, 2\right)\right)\right], \\ \pi\left(\theta_1^+, B, 1\right) - 0 + PV\left[E\left(F\left(\theta_2, B, 2\right)\right)\right] \end{array} \right\} \tag{21}$$

where $E\left(F\left(\theta_2, A, 2\right)\right)$ is computed on equations 13 and 14;

and $E\left(F\left(\theta_2, B, 2\right)\right)$ is computed on equations 16 and 17;

$$F\left(\theta_1^-, B, 1\right) = max\left\{ \begin{array}{c} \pi\left(\theta_1^-, A, 1\right) - c_{B,A} + PV\left[E\left(F\left(\theta_2, A, 2\right)\right)\right], \\ \pi\left(\theta_1^-, B, 1\right) - 0 + PV\left[E\left(F\left(\theta_2, B, 2\right)\right)\right] \end{array} \right\} \tag{22}$$

where $E\left(F\left(\theta_2, A, 2\right)\right)$ is computed on equations 14 and 15;

and $E\left(F\left(\theta_2, B, 2\right)\right)$ is computed on equations 17 and 18;

and for $t = 0$

$$F\left(\theta_0, A, 0\right) = max\left\{ \begin{array}{c} \pi\left(\theta_0, A, 1\right) - 0 + PV\left[E\left(F\left(\theta_2, A, 2\right)\right)\right], \\ \pi\left(\theta_0, B, 1\right) - c_{A,B} + PV\left[E\left(F\left(\theta_2, B, 2\right)\right)\right] \end{array} \right\} \tag{23}$$

where $E\left(F\left(\theta_1, A, 1\right)\right)$ is computed on equations 19 and 20;

and $E\left(F\left(\theta_1, B, 1\right)\right)$ is computed on equations 21 and 22;

$$F\left(\theta_0, B, 0\right) = max\left\{ \begin{array}{c} \pi\left(\theta_0, A, 1\right) - c_{B,A} + PV\left[E\left(F\left(\theta_1, A, 2\right)\right)\right], \\ \pi\left(\theta_0, B, 1\right) - 0 + PV\left[E\left(F\left(\theta_1, B, 2\right)\right)\right] \end{array} \right\} \tag{24}$$

where $E\left(F\left(\theta_1, A, 1\right)\right)$ is computed on equations 19 and 20;

and $E\left(F\left(\theta_1, B, 1\right)\right)$ is computed on equations 21 and 22;

Instead, in a grid discretization, the DP algorithm works as follows. In this *Mickey Mouse* model the horizon is discretized in three times $t = 0, 1, 2$, and the state variable in four levels $\theta_t^i\ \forall i = 1, \ldots, 4$. The plant has two production modes as before $m = A, B$.

At time $t = 2$ for every level θ_2^i, the value function is equal to the salvage value only

$$F\left(\theta_2^i, A, 2\right) = S\left(\theta_2^i\right) \tag{25}$$
$$F\left(\theta_2^i, B, 2\right) = S\left(\theta_2^i\right) \tag{26}$$

I have $S = 4$ times $M = 2$ values for these equations: each mode for the last possible level of the state variable. In this case I have chosen to specify the salvage value as unconditional on entering mode.

At time $t = 1$ for every level θ_1^i, the value function is equal to

$$F\left(\theta_1^i, A, 1\right) = max\left\{\begin{array}{c} \pi\left(\theta_1^i, A, 1\right) - 0 + \rho \cdot E_1\left[S\left(\theta_2\right)\right], \\ \pi\left(\theta_1^i, B, 1\right) - c_{a,b} + \rho \cdot E_1\left[S\left(\theta_2\right)\right] \end{array}\right\} \tag{27}$$

$$F\left(\theta_1^i, B, 1\right) = max\left\{\begin{array}{c} \pi\left(\theta_1^i, A, 1\right) - c_{b,a} + \rho \cdot E_1\left[S\left(\theta_2\right)\right], \\ \pi\left(\theta_1^i, B, 1\right) - 0 + \rho \cdot E_1\left[S\left(\theta_2\right)\right] \end{array}\right\} \tag{28}$$

At time $t = 0$ for every level θ_0^i, the value function is equal to

$$F\left(\theta_0^i, A, 0\right) = max\left\{\begin{array}{c} \pi\left(\theta_1^i, A, 1\right) - 0 + \rho \cdot E_0\left[F\left(\theta_1, A, 1\right)\right], \\ \pi\left(\theta_1^i, B, 1\right) - c_{a,b} + \rho \cdot E_0\left[F\left(\theta_1, B, 1\right)\right] \end{array}\right\} \tag{29}$$

$$F\left(\theta_0^i, B, 0\right) = max\left\{\begin{array}{c} \pi\left(\theta_1^i, A, 1\right) - c_{b,a} + \rho \cdot E_0\left[F\left(\theta_1, A, 1\right)\right], \\ \pi\left(\theta_1^i, B, 1\right) - 0 + \rho \cdot E_0\left[F\left(\theta_1, B, 1\right)\right] \end{array}\right\} \tag{30}$$

Expectations at time $t = 0$ of the value function at time $t = 1$ in equations 29 and 30 are computed on the value function for the same mode in each of the θ_1^i.

2 Numerical Examples

In this section I have evaluated two stylized industrial plants:

1. a simple one production mode plant, very much like those used in basic CVP analysis. Its operating options, namely to wait, to mothball, to abandon, are analyzed choosing as state variable the quantity produced and sold. In this example I focus on downside risk options;

2. a two production modes plants, very much like the one described in section 1. In addition to operating options listed for the previous one, in this case the plant can be exercised in two production modes

which differ on the relative level of fixed costs and relative costs chosen in each budget period. Hence, in addition to downside risk options there is one upside risk switching option.

Both models have been developed over an annual period meaning that production mode decisions can be revised year after year. This should be taken as an example only. As a matter of fact, the period in which to divide the useful life of a plant should be chosen considering technical features such as minimum resetting time for plant machinery and organization and the length of the period covered by budgets within the year. Finally, in this paper I have focused my attention on cost structures. Therefore, I do not give further details about the estimation of the Ornstein Uhlenbeck parameters, see (Alesii, 2000) for a review of the econometric methods about this.

2.1 A Simple Production Plant

This example slightly resembles the one on page 238 (Dixit and Pindyck, 1994). Here I differentiate the example using a different cost structure that has both fixed and variable costs with the quantity produced and sold Q. Having chosen this state variable I need an upper bound because quantities produced and sold which exceed the maximum production capacity do not have any economic meaning. Because of this for the examples that follow, Q is not the absolute value of units produced and sold but the percentage of plant capacity which is actually exploited, therefore $0 < Q < 1$. The stochastic process generating Q is an Ornstein Uhlenbeck process with the following parameters $\eta = .05$; $\overline{\theta} = .5$; $\sigma_\theta = .5$. Risk free rate is supposed to be $r_f = 5\%$ and the risk adjusted rate $\mu = r_f$ being the systematic risk of the investment project $\beta = 0$.

The plant data can be summarized as follows:

I	=	40	initial investment;
E_m	=	2	one time costs of mothballing;
E_s	=	-5	one time costs of scrapping (if negative wreckage is sold at a profit);
R	=	4	one time cost of reactivating;
M	=	1.5	cost of being in the mothballed state;
C	=	7	fixed cost of operating;
P	=	23	gross revenue from operating per unit sold;
Vc	=	3	Unit variable cost;
Cm	=	20	contribution margin;
T	=	50	life of the investment project;

The investment project has then the following real options:

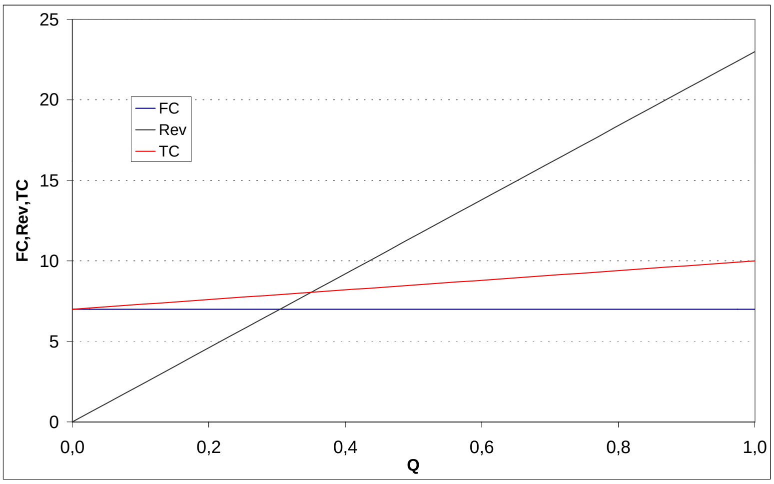
1. :waiting;
2. :operating;
3. :mothballing;
4. :abandoned.

Data about transition costs can be summarized as in expression 31.

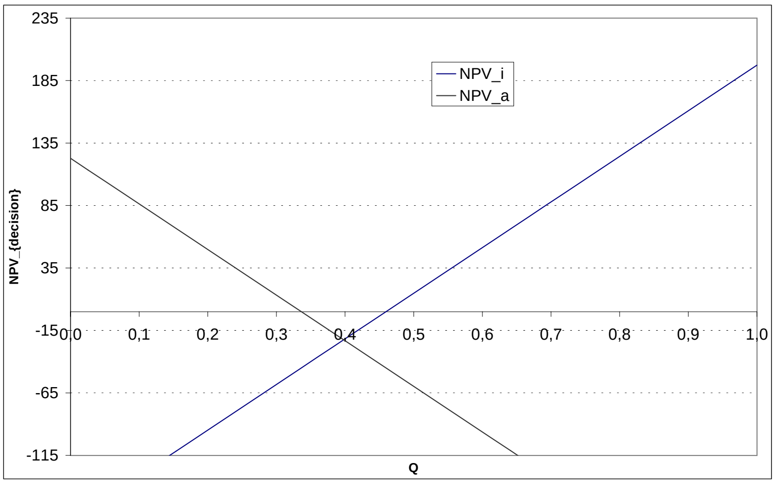
$$\delta = \begin{pmatrix} \delta_{1,1} & \delta_{1,2} & \delta_{1,3} & \delta_{1,4} \\ \delta_{2,1} & \delta_{2,2} & \delta_{2,3} & \delta_{2,4} \\ \delta_{3,1} & \delta_{3,2} & \delta_{3,3} & \delta_{3,4} \\ \delta_{4,1} & \delta_{4,2} & \delta_{4,3} & \delta_{4,4} \end{pmatrix} = \begin{pmatrix} 0 & -40 & +\infty & +\infty \\ +\infty & 0 & -2 & +\infty \\ +\infty & -4 & 0 & 5 \\ +\infty & +\infty & +\infty & 0 \end{pmatrix} \tag{31}$$

In a static expectations framework the plant could be operated at a profit for $Q > Q_{Bep} = .3500$, see panel A in figure 7. Although that is true, the 40 M\$ would be invested, only if $Q > Q_{Invest} = .4596$. Instead, it would be wrecked only if $Q < Q_{abandon} = .3363$, see panel B in figure 7. Finally, on a pure comparative statics framework, the plant would be mothballed if $Q < Q_{mothball} = .2750$. In conclusion, in a static expectations framework too there is a kind of hysteresis region in which decisions taken with respect to the long run differ from those taken in the short run. As a matter of fact, in the interval $.3500 < Q < .4596$ although the plant is profitable it would not be built. Instead, in the interval $.3363 < Q < .3500$ although the plant is not profitable, it is not abandoned.[25] Overall, there is a kind of hysteresis band, $.3363 < Q < .4596$ in which the plant is kept in the same state in which it entered this region disregarding short run profitability and taking into account only long run net present value of the invest and abandon decisions.

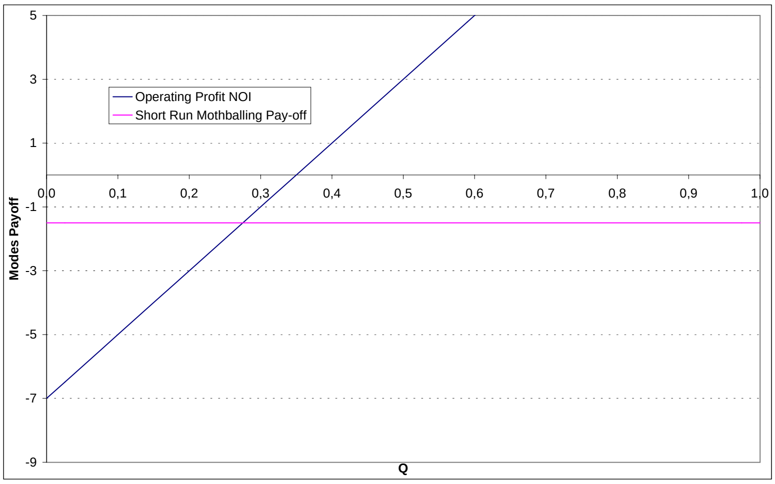
[25] As a matter of fact, once the investment is implemented, if the state variable moves in the interval $.3363 < P < .3500$, under the hypothesis of static expectations, the value of the plant is smaller if it is abandoned than if it is kept working.



Panel A: BEP diagram for the one production mode plant.



Panel B: Mirrored NPV for CVP Analysis in panel A.



Panel C: Short Run Payoffs from Operating and Mothballing Modes.

Figure 7: Profit Functions for the Fixed Modes against θ.

Legend: Assuming quantity Q produced and sold as state variable. FC: Fixed Costs, Rev: Revenues, TC: Total Costs; NPV_i: net present value from the decision to invest; NPV_a: net present value from the decision to abandon; NOI: net operating income set equal to operating cash flows;

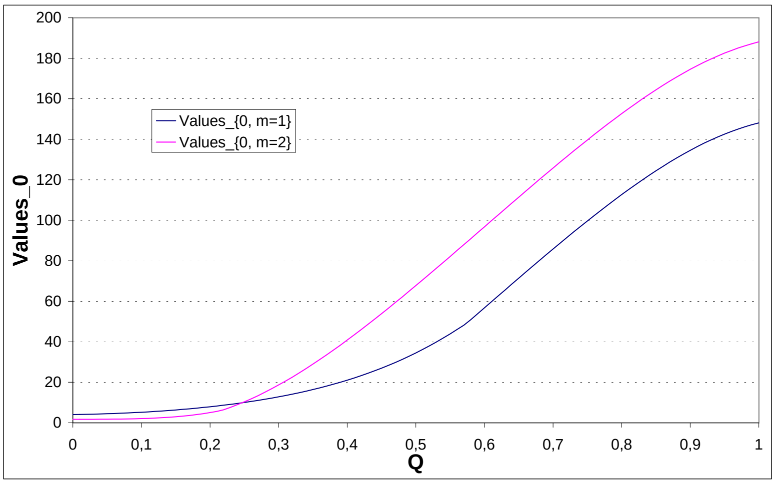
The value of the plant was computed including the flexibility options which in this case are expecially downside risk options. Following (Dixit and Pindyck, 1994) cap. 7, the values of the plant are reported for both waiting to invest and operating modes at time $t = 0$, see figure 8. In panel A the values of the plant at time $t = 0$ are reported for both mode 1: (Waiting to invest) and mode 2: (Operating). It is interesting to examine the difference between these two curves, see panel B of figure 8. The intuitive meaning of this variable is the incremental value of becoming active or of implementing the investment project, see equation 32.

$$Gain(Q) = Values_{0,m=2} - Values_{0,m=1} \tag{32}$$

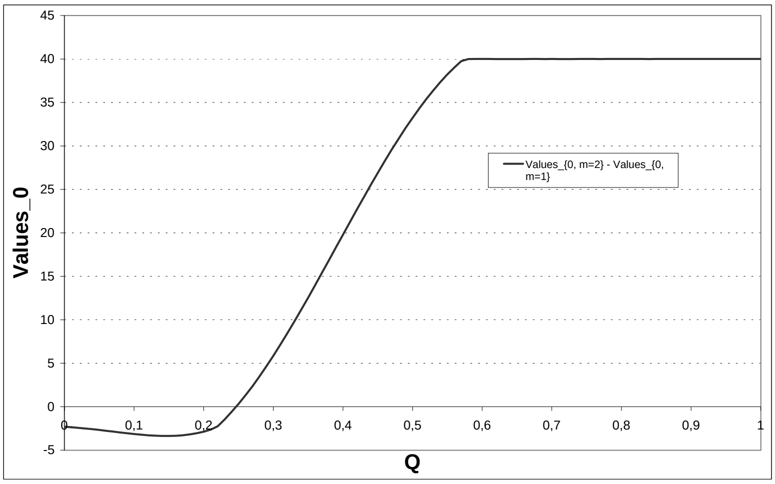
The plot of the incremental value of becoming active is a cross check of mode bounds levels at time $t = 0$. As a matter of fact it is actually used by Dixit and Pindyck to derive threshold levels to invest and abandon the project, see figure 7.6 and 7.5 on page 229 of (Dixit and Pindyck, 1994). The "s" shape of the curve in panel B is influenced by the amounts that the investment project requires to be implemented $I = 40$, upper bound, and the one that it yields in case it is abandoned, lower bound, $E = 5$. As the state variable increases, the advantage of becoming active levels off to an ordinate that is equal to the lump sum to invest at time $t = 0$, in this case $I = 40$. The abscissa for which $Gain(Q)$ levels off is the threshold level of the state variable for the optimal exercise of the waiting option. For all the levels for which the incremental value of becoming active is less than $I = 40$ it is convenient to wait. For levels of Q that exceed the threshold, it is convenient to invest $I = 40$ and become active.

As the state variable decreases, the incremental value of becoming active decreases and it becomes negative for some levels. Had it reached the level $E = -5$, the sum reaped in case of abandonment, the plant would have been worth more if abandoned than if operated and the abscissa for which this happened would have been the threshold for which to abandon the plant. This does not happen for any level of Q because in this case in addition to the abandonment option like in the Dixit Pindyck example I have waiting and mothballing options too. Because of the presence of these options there is no level of Q for which it would be optimal to abandon the plant at time $t = 0$. This is also due to the fact that the abandoned state is not reachable from the wait to invest or the operative mode A state.[26]

[26] As it is shown below, there is a threshold for abandonment even for $t = 0$, see figure 11. In figure 8 it cannot be observed simply because the abandoned state is not accessible through the states represented, to wait and to operate, but only through the mothballed state.



Panel A: Values of the Plant at time $t = 0$: $Values_{0,m=1}$: all inclusive options, beginning mode m=1 waiting to invest; $Values_{0,m=2}$: all inclusive options, beginning mode m=2, operating mode.



Panel B: Incremental Value of Becoming Active: $G(P) = Values_{0,m=2} - Values_{0,m=1}$.

Figure 8: Values of the Plant Under Ornstein Uhlenbeck Evolution of θ and Flexibility Options.
Legend: Assuming quantity Q produced and sold as state variable. Mode 1: waiting to invest; Mode 2: production mode; Mode 3: mothballed state; Mode 4: abandoned project. $Values_{0,m=1}$: Value of the plant with *all* the options at time $t = 0$; $Values_{0,m=2}$: Value of the plant with *all* the options but already operating at time $t = 0$. The difference between the two values is the incremental value of becoming active, $Values_{0,m=2} - Values_{0,m=1}$ resembles to figures 7.1 page. 220 and 7.6 page.229 in (Dixit and Pindyck, 1994)

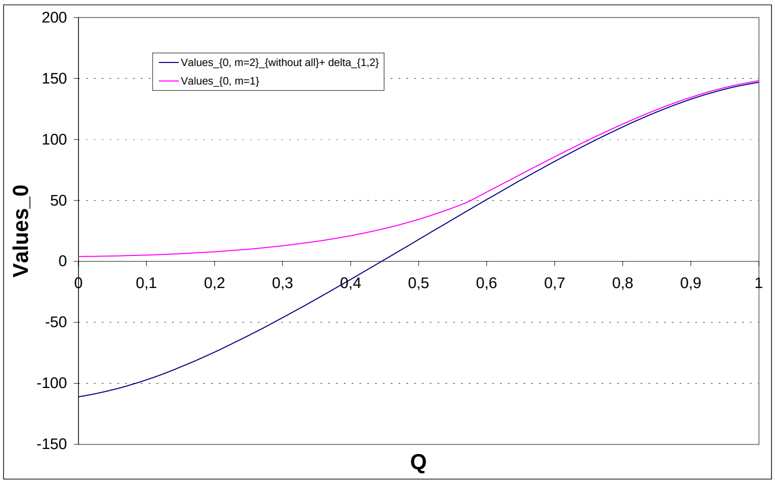
It is interesting to compare the value of the plant with all the options with its "naked value" or the value of the plant without any of the downside risk options, that is a plant working in any contingency in the operating mode, see figure 9. The value of the plant with all options is the one already shown in panel A of figure 8, value of the plant in the waiting mode 1. Instead, the value of the plant without all options has been computed substituting for any payoff previously used the CVP equation so that the profit generated in any of the four modes would be always the same of the operating mode. Moreover, since in this way the initial lump sum invested $\delta_{1\rightarrow 2}$ is not taken into account, this is subtracted from the expected value of the plant as previously computed.

The combined value of the three downside options is higher for low levels of the state variables, see panel B of figure 9. This result is intuitively immediate and does not deserve further explanations. It is worth noting that in this case the value of the options is a relevant percentage of the investment project level for $Q_{BEP} = .35$. As a matter of fact, at this level of the state variable the expanded NPV of the plant is $NPV_{exp} = 16.43$. Of this value, options accounted for $O.V. = 47.18$ being the "naked value" of the plant $NPV_{nak} = -30.76$. This can be summarized as in equation 33. Blatantly enough, real options are a relevant part of the initial investment.

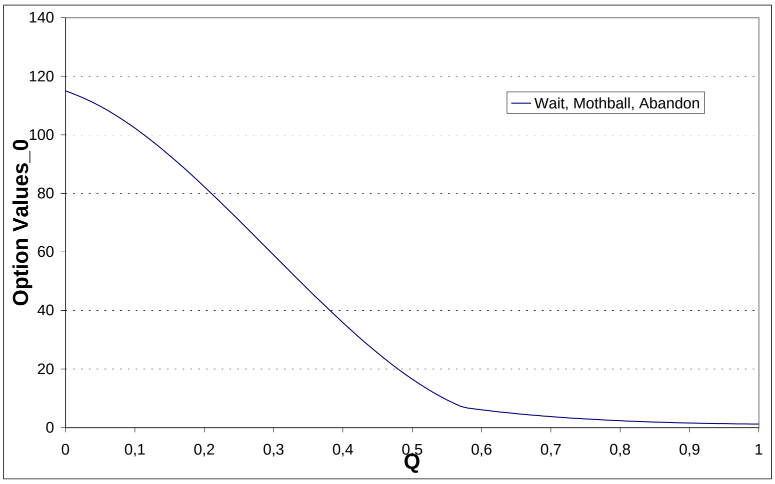
$$NPV_{exp} = NPV_{nak} + O.V = -30.76 + 47.18 = 16.43 \tag{33}$$

From a practical point of view it is important to disentangle the value of the individual options from the combined options value. This can be done computing the value of the plant excluding some real options. For instance, in order to compute the value of the option to abandon *within* the combined value depicted in panel B of figure 8 I have computed the value of the plant without the options to mothball and to wait, see panel A of figure 10. Then I have computed the value of the option to wait and to mothball just subtracting the value of the all option inclusive plant from the one previously computed, see panel B of figure 10. Finally, I have subtracted the combined value of the options to wait and to mothball from the all options combined value and I have obtained the value of the options to abandon by difference, see panel C in figure 10.

This method to obtain option values can be criticized on the grounds that option values are not perfectly additive. Because of this, the value of the option to abandon may be different in the presence of an option to wait or to mothball. As a consequence, intuition suggest that the value of the abandonment option that I have computed is overvalued with respect to the one included in the all option plant. As a cross check of the non perfect additivity of real options, I have computed the value of the option to wait and compared it to the combined value of the option to wait and to mothball, see panel B of figure 9. As it would have been expected the values of these two options are subadditive. As a matter of fact, having the option to wait reduces the value of the opportunity to mothball and the other way around.



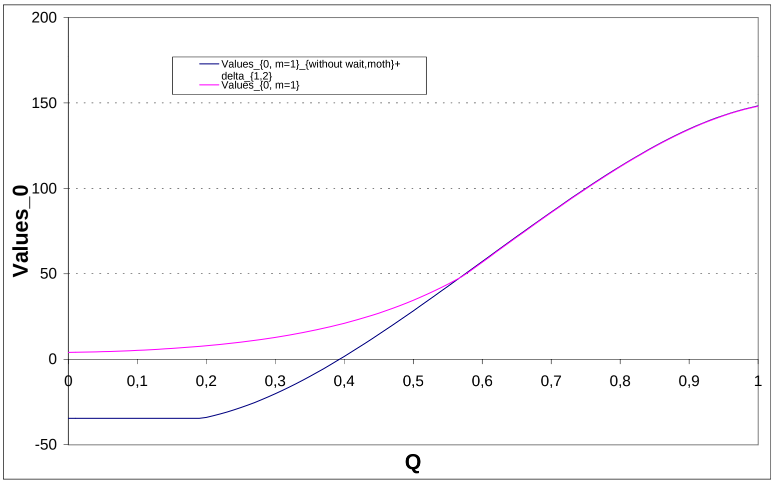
Panel A: Values of the plant at time $t = 0$ for the plant with and without all downside risk options.



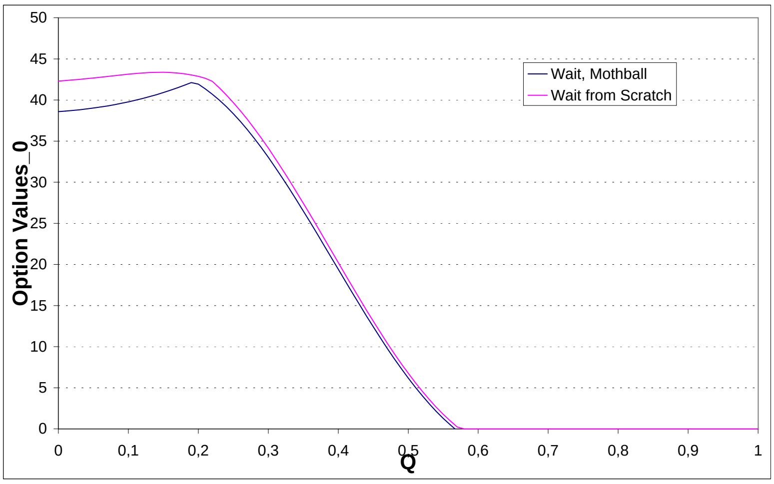
Panel B: Value of the combined options to wait, to mothball and to abandon.

Figure 9: Plant and Options Values against θ: All Options.

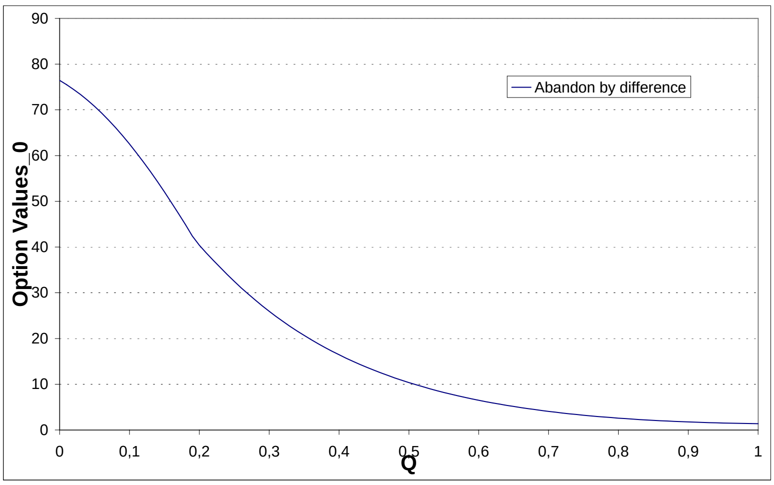
Legend: Assuming quantity Q produced and sold as state variable. The value of the plant with all options is the value of the plant at time $t = 0$ in the waiting mode, the value of the plant without all options, naked value, is the expected value of the plant working in any contingency in the operating mode. The difference in ordinates between the two is the combined value of the options to wait, to mothball and to abandon.



Panel A: Values of the plant at time $t = 0$ for the plant with and without options to wait and to mothball.



Panel B: Value of the combined options to wait, to mothball and of the option to wait computed from scratch.



Panel C: Value of the option to abandon computed by difference between the value of the all three combined options and the value of the two options shown in panel B.

Figure 10: Plant and Options Values against θ: To Wait and to Mothball Options.

Legend: Assuming quantity Q produced and sold as state variable. The value of the plant with all options is the value of the plant at time $t = 0$ in the waiting mode, the value of the plant without options to wait and to mothball is the expected value of the plant working in any contingency in the operating mode but abandoned for levels below the abandonment threshold. The difference in ordinates between the two is the combined value of the options to wait and to mothball.

Together with the value of the plant, it is very much important from a practical point of view to derive mode bounds, i.e. the threshold levels for which it is optimal to change production mode within a dynamic program. While both Kulatilaka and Dixit and Pindyck, cap.7 (Dixit and Pindyck, 1994), derived these threshold levels for time $t = 0$ only, it is easy to derive them for the whole life of the project. This is particularly important for those thresholds that change a lot during the life of the project.

For instance, the abandonment threshold[27] is the boundary of the region of the state variable for which the plant would be optimally abandoned. This level is very low at the beginning of the life of the project and it grows towards the end of the useful life of the plant. The shape of the investment implementation threshold is slightly similar. Around the end of the project it would not be convenient to invest but for very high levels of the state variable.

Between those two levels the mothballing and restarting thresholds are reported. They form an hysteresis region being the restarting level much higher than the mothballing one. It is important to notice that after time $t = 27$ the option to mothball is completely offset by the option to abandon being the latter bound stricter than the former. The option to reactivate the project is instead exercised for levels much lower than those for which the investment is implemented.

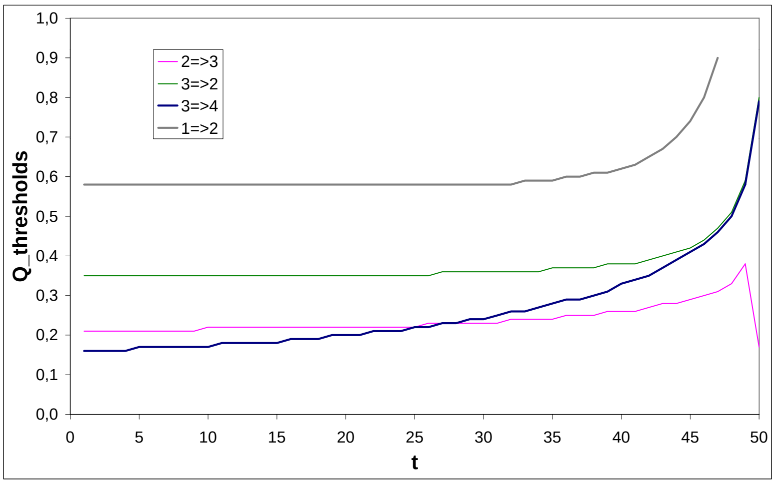


Figure 11: Mode Boundaries during the Project Life.

Legend: Assuming quantity Q produced and sold as state variable. Four threshold levels for a dynamical program are represented: $Q_{1\to2}$: levels for Q for which it is optimal to pass from the waiting to invest mode 1 to the operating mode 2; $Q_{2\to3}$: levels for which it is optimal to pass from the operating mode to the mothballed plant mode 3; $Q_{3\to2}$: levels for which it is optimal to restart the plant; $Q_{3\to4}$: levels for which it is optimal to abandon the plant, mode 4, being in mode 3 mothballed plant.

[27]Whose shape resembles the one reported in (Dixit and Pindyck, 1994) page 111 where it was derived through difficult stochastic algebra.

2.2 A Two Production Modes Plant

This example slightly resembles to (Kulatilaka, 1988). There it was conceived as an FMS system, here, more generally it is intended as a representation of a simple manufacturing plant in which it is possible to trade off between per budget fixed and variable costs. For instance, it is possible to buy some machinery in order to perform the same task that in the alternative mode was performed by another company for a cost variable with the quantity produced and sold. In this way the contribution margin is increased together with the break even point. This makes this production mode more risky with respect to the other with less fixed costs and more variable costs. Therefore, I define this production mode Aggressive while the other Conservative.

To give the reader a benchmark of comparison and for not to laden this paper with another data set, I simply add a more aggressive production mode to the one of the previous example. The cost volume profit equations 34 and 35 summarize the differences between the two production modes.

$$NOI_A = \pi^m \left(\theta_t^i\right)_A = Q \cdot (P - vc) - F = \theta \cdot 20 - 7 \tag{34}$$

$$NOI_B = \pi^m \left(\theta_t^i\right)_B = Q \cdot (P - vc) - F = \theta \cdot 35 - 14 \tag{35}$$

In addition to plant data already given in section 2.1, the following switching costs should be considered in computing the value of the flexible plant:

$$\begin{array}{lcll} E_{A \to B} & = & 1 & \text{one time costs of aggressive switch } A \to B; \\ E_{B \to A} & = & 1 & \text{one time costs of conservative switch } B \to A; \end{array}$$

The investment project has then the following different modes of operations:

1. :waiting;
2. :operating A (conservative);
3. :operating B (aggressive);
4. :mothballing;
5. :abandoned;

Before being actually implemented the investment project is "dressed" with all the real options, namely option to wait; to operate in A or B, to mothball and to abandon. Although that is true, while waiting to invest, the direct real option available is to start operations in the conservative mode. Operating in mode A, the plant offers the following real options: to switch to mode B and to mothball. Being in the mothballed state, instead, the plant can be restarted in the conservative production mode A or it can be abandoned. In conclusion, not all the modes of operation are accessible from any other. This corresponds to the transition cost matrix reported in expression 36.

	Threshold Levels	
	Mode B	Mode A
Q_{invest}	0.4626	0.4596
$Q_{abandon}$	0.3922	0.3363
Q_{bep}	0.4000	0.3500
$Q_{mothballing}$	0.3571	0.2750

Table 2: Threshold Levels for Operating Modes

Legend: thresholds are derived under the hypothesis of static expectations, $\theta_{invest} = \frac{FC}{CM} + \frac{I}{CM \cdot a_{\overline{n}|i}}$, $\theta_{abandon} = \frac{FC}{CM} + \frac{E}{CM \cdot a_{\overline{n}|i}}$, $\theta_{moth} = \frac{FC - E_m}{CM}$.

$$\delta = \begin{pmatrix} \delta_{1,1} & \delta_{1,2} & \delta_{1,3} & \delta_{1,4} & \delta_{1,5} \\ \delta_{2,1} & \delta_{2,2} & \delta_{2,3} & \delta_{2,4} & \delta_{2,5} \\ \delta_{3,1} & \delta_{3,2} & \delta_{3,3} & \delta_{3,4} & \delta_{3,5} \\ \delta_{4,1} & \delta_{4,2} & \delta_{4,3} & \delta_{4,4} & \delta_{4,5} \\ \delta_{5,1} & \delta_{5,2} & \delta_{5,3} & \delta_{5,4} & \delta_{5,5} \end{pmatrix} = \begin{pmatrix} 0 & -40 & -\infty & -\infty & -\infty \\ -\infty & 0 & -1 & -2 & -\infty \\ -\infty & -1 & 0 & -\infty & -\infty \\ -\infty & -4 & -\infty & 0 & 5 \\ -\infty & -\infty & -\infty & -\infty & 0 \end{pmatrix} \tag{36}$$

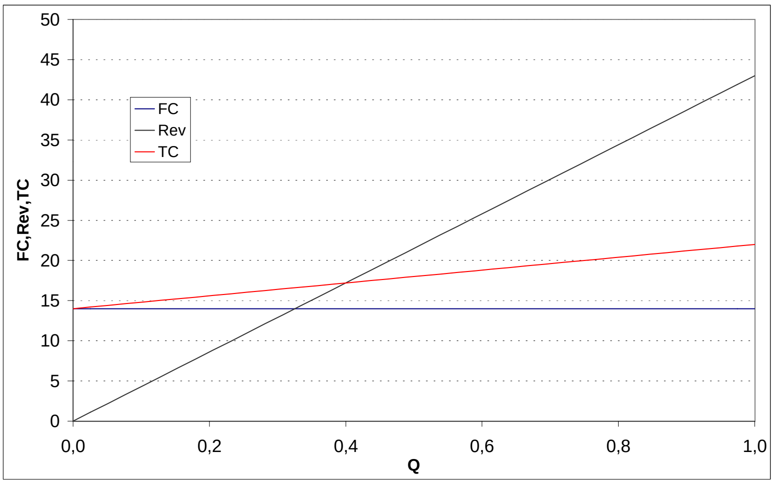
In a static expectation framework, for operating mode B the breakeven chart is reported in panel A in figure 12. BEP, investment, mothballing and abandonment thresholds are higher than those found for operating mode A, see table 2.

In a dynamic management and Ornstein Uhlenbeck expectations framework, as expected, the value of the plant is increased by the presence of the option to switch to a more aggressive production mode, see panel A in figure 13 expecially for higher levels of the state variable, instead the option to switch is not worth much for very low levels of the state variable, see panel B in figure 13. Computing the value of the plant managed in a dynamic program optimally switching between production modes only, it is possible to get, by difference with the value of the all options "dressed" plant, the combined value of the downside risk options, i.e. option to wait, to mothball and to abandon. These options value has just the opposite behavior of the upside risk option to switch to the aggressive mode. Computing the value of a one production mode A plant, "naked" of all options, it is possible to get by difference the aggregate value of the downside and upside options. These have a "V" shape having lower value for levels in which an hysteresis effect prevails and it is optimal to keep the plant in the mode in which it entered the hysteresis region. It is worth noting that upside and downside risk options have a super additive value that enhances their value in the trough represented by the hysteresis region.

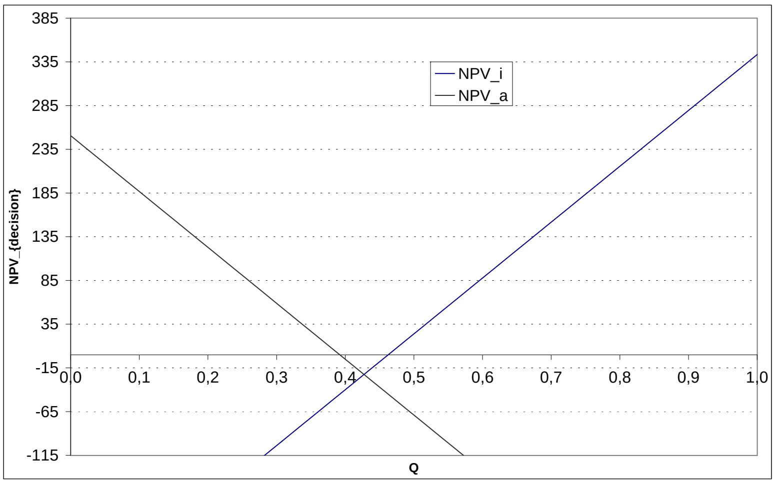
It is possible to cross-check what just stated observing the mode bounds derived for the six possible passages between modes, see figure 14. In panel A activation and abandonment thresholds are derived. In this case too, the abandonment threshold has the same shape as in (Dixit and Pindyck, 1994) page 111. These two thresholds have been put in one graph since they are respectively non recurrent and absorbing. Their shape has a quite clear intuitive explanation. It would be not worth abandoning the project in the early stages of the industrial plant life because, in the presence of an "aggressive" production mode, the expected value of the plant would be much higher if mothballed for very low levels. Moreover, it would not be convenient to start the plant for a quite high, higher than under static expectations, level of Q. This threshold increases taking into account that the project should have enough time to repay the investor and this could be done only for very high levels of Q in a short period.

Between the activation and abandonment thresholds the plant is managed choosing the optimal recurrent mode according to the aggressive and conservative switch thresholds and the mothballing and reactivating ones. Two almost non overlapping hysteresis regions are created by the recurrent modes thresholds. As a matter of fact, the reactivation threshold is lower than the conservative switch bound. Because of this the plant will always be in the "conservative" mode when crossing the mothballing bound. Moreover, it will be restarted in the same "conservative" mode. Because of this between the restarting and the conservative switch thresholds, the plant will always be exercised in the "conservative" production mode A.

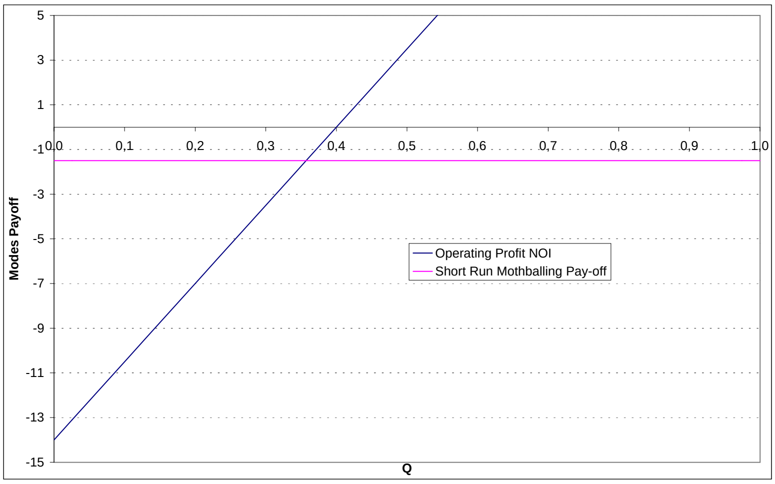
In the last periods of the life of the project the reactivation and the mothballing bounds widen a lot making it unprofitable to restart (mothball) operations but for very high (low) levels of Q. Instead, switches between the two production modes are governed by much more stable thresholds. Although that is true, in this latter case too the hysteresis region is widened in the last periods of the project life. The presence and the shape of these hysteresis regions confirms the explanation given above for the "V" shape of aggregate value of the upside and downside risk real options.



Panel A: BEP diagram for the *aggressive* production mode plant.



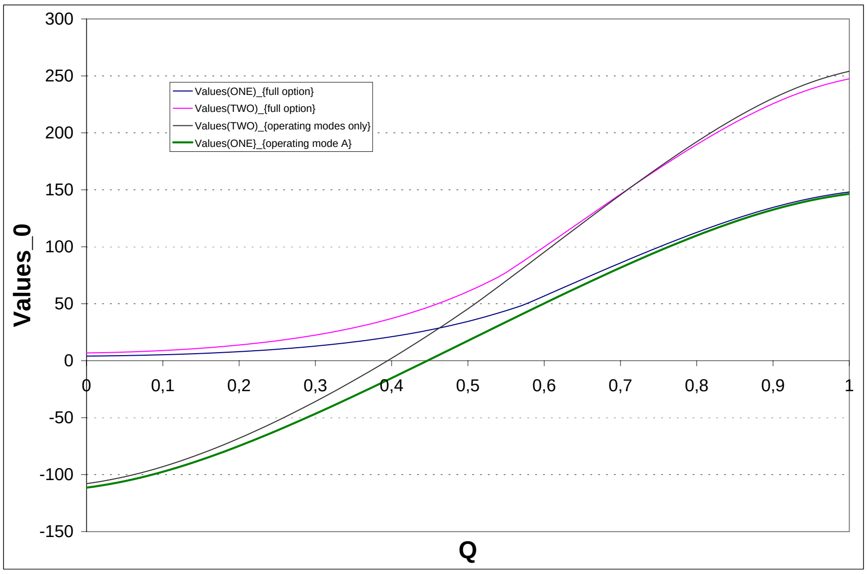
Panel B: Mirrored NPV for CVP Analysis in panel A.



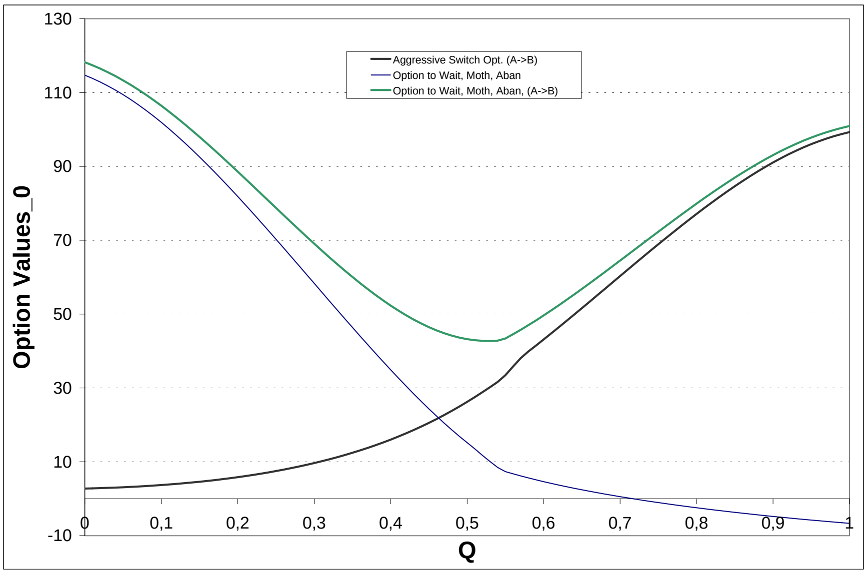
Panel C: Short Run Payoffs from Operating Mode B and Mothballing Modes.

Figure 12: Profit Functions for the Fixed Modes against θ.

Legend: Assuming quantity Q produced and sold as state variable. FC: Fixed Costs, Rev: Revenues, TC: Total Costs; NPV_i: net present value from the decision to invest; NPV_a: net present value from the decision to abandon; NOI: net operating income set equal to operating cash flows;



Panel A: Values of the plant at time $t = 0$ for the plant with and without options.



Panel B: Value of the combined options to wait, to mothball and to abandon and to switch.

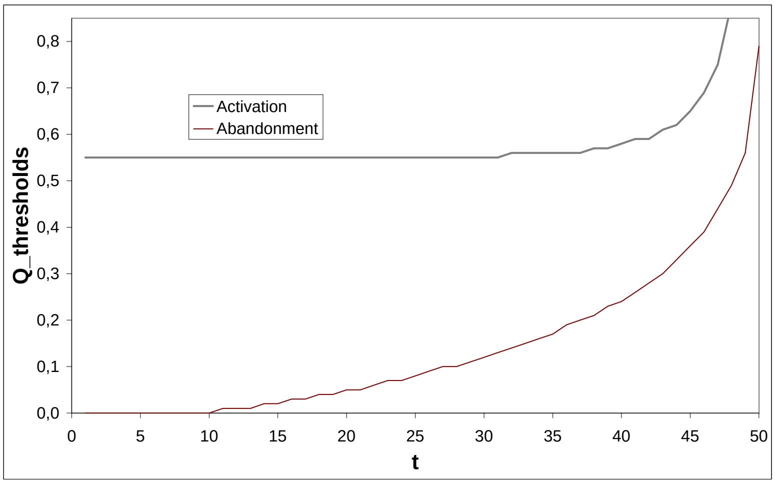
Figure 13: Plant and Options Values against θ: All Options.

Legend: Assuming quantity Q produced and sold as state variable. The value of the plant with all options is the value of the plant at time $t = 0$ in the waiting mode, the value of the plant without all options, naked value, is the expected value of the plant working in any contingency in the best operating mode or in the only mode available, Mode A. The difference in ordinates between the

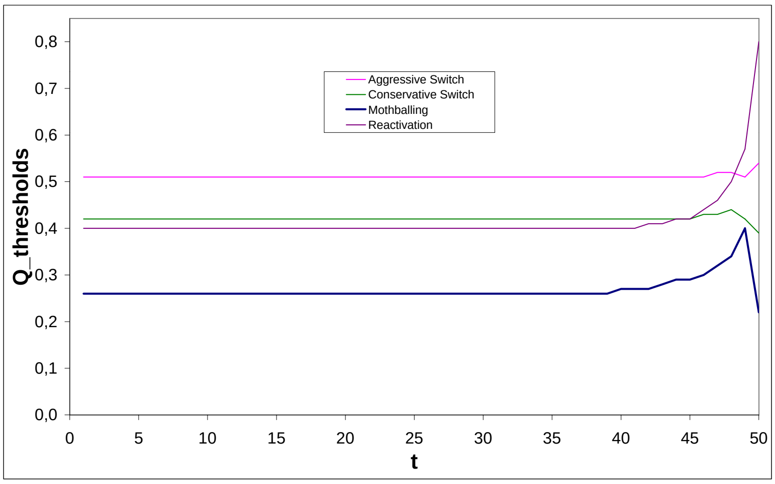
$$\left\{\begin{array}{c}\text{Aggressive } A \to B\\ \text{Switch}\\ \text{Option}\end{array}\right\} = \left\{\begin{array}{c}\text{All Options}\\ \text{Two Operating Modes (A,B)}\\ \text{Plant Value}\end{array}\right\} - \left\{\begin{array}{c}\text{All Options}\\ \text{One Operating Mode (A)}\\ \text{Plant Value}\end{array}\right\}$$

$$\left\{\begin{array}{c}\text{Waiting, Mothballing}\\ \text{and Abandonment}\\ \text{Combined Option Value}\end{array}\right\} = \left\{\begin{array}{c}\text{All Options}\\ \text{Two Operating Modes (A,B)}\\ \text{Plant Value}\end{array}\right\} - \left\{\begin{array}{c}\text{Two Operating Modes Only}\\ \text{(A,B)}\\ \text{Plant Value}\end{array}\right\}$$

$$\left\{\begin{array}{c}\text{Waiting, Mothballing}\\ \text{Abandonment and } A \to B \text{ switch}\\ \text{Combined Option Value}\end{array}\right\} = \left\{\begin{array}{c}\text{All Options}\\ \text{Two Operating Modes (A,B)}\\ \text{Plant Value}\end{array}\right\} - \left\{\begin{array}{c}\text{One Operating Mode Only}\\ \text{(A)}\\ \text{Plant Value}\end{array}\right\}$$



Panel A: Investment Implementation and Abandonment Thresholds.



Panel B: Other Thresholds.

Figure 14: Mode Boundaries during the Project Life.

Legend: Assuming quantity Q produced and sold as state variable. Six threshold levels for a dynamical program are represented, levels of Q for which it is optimal to:

$Q_{1\to2}$	:	pass from the waiting to invest mode 1 to the operating mode 2 (investment threshold);
$Q_{2\to3}$	:	pass from the operating mode A "conservative" to operating mode B "aggressive" (aggressive move threshold);
$Q_{3\to2}$	:	reverse the previous move, (conservative move threshold);
$Q_{2\to4}$	:	mothball the plant, (mothballing threshold);
$Q_{4\to2}$	:	restart the plant, (restarting threshold);
$Q_{4\to5}$	:	abandon the plant being in mode 3 mothballed plant (abandonment threshold).

3 Conclusions

This paper is aimed to entail in traditional CVP analysis a dynamic perspective in decision making, taking into account not only irreversible decisions, such as investing in or abandoning a project, but also recurrent decisions, e.g. mothballing plant, restarting operations, switching to a different production mode. This is performed drawing a parallelism between traditional CVP analysis and the Kulatilaka real options approach to investment evaluation. The traditional concept of *break even point* is reinterpreted in a dynamic framework and substituted for time varying thresholds between operating modes of the investment project.

How to solve the dynamic programming problem involved by Kulatilaka GROPM is explained drawing a parallelism between the DP implementation on a lattice reported in (Trigeorgis, 1996) on page 177 and the one actually implemented in this paper on a grid. A couple of *Mickey Mouse* examples explain plainly the mechanics of backward induction.

As a completely new result, it is worth noting the derivation of mode bounds that neither (Kulatilaka, 1988) nor (Dixit and Pindyck, 1994) chapter 7 derive for the whole useful life of the project. The analysis of the latter provides a kind of cross check for threshold levels at time $t = 0$ and for the shape of the abandonment threshold, see page 111 in (Dixit and Pindyck, 1994). The amazing thing is that I reach the same results that Dixit and Pindyck get through very difficult stochastic algebra[28]. While their method is certainly more elegant and based on solid theoretical grounds, the one used in this paper is coarser and sometimes controversial[29] from a theoretical point of view. Although that is true, the former is very difficult to teach in an MBA class requiring long and tiresome diversions to build up ad hoc stochastic algebra tools. Moreover, it is very difficult to apply requiring numerical methods that even Dixit and Pindyck do not suggest[30]. Instead, Kulatilaka approach is intuitively direct even in its numerical solution method and it is easy to teach. Because of this, I expect the latter method to become widely used in real business situations.

[28] Although Professor Pindyck claims in an e-mail to the author that chapter 7 model can be easily extended to any time of the life of the project, it seems to me that it is not easy, if ever possible, to "tame" the non linear system, eq.(9)-(12) on page 218, to give us all the thresholds for the whole life of the project.

[29] There are a number of issues that are still to be understood:

- why should the plant have a $\beta = 0$? See section 1.3;
- the choice of the θ boundary levels heavily conditions results and it is not so clear how should the interval be selected expecially when the variable is completely out of control;
- Cumulating probabilities for the boundary levels of the state variable θ has not a clear meaning. If it means that the transition probability is with respect to that level and not to higher levels, then I can use as θ even variables than have an upper or lower bound, e.g. quantity produced and sold bounded upward by the maximum production capacity of the plant. If this is not the case, then these kind of models cannot be applied to state variable that have an economic meaning only below a certain level. From this point of view a lattice discretization accommodates better state variables with barriers, see page 476 (Hull, 1997);

[30] How should be solved the non linear system of equations on page 218 of (Dixit and Pindyck, 1994)?

There are many extensions to the General Real Option Pricing Model (GROPM) of Kulatilaka in a CVP framework analysis. These are possible thanks to the simple numerical approach that avoids difficult stochastic algebra which is unmanageable beyond some levels of sophistication.

The first extension is a multivariate CVP analysis with irreversible and irreversible real options where all the primitive variables in the CVP analysis are specified as stochastic. This in turn can be implemented on a discretization like the multivariate binomial lattice of (Boyle et al., 1989) or in a generalization of the univariate grid to accommodate both Brownian motions and O-U simultaneously. The second extension is to derive the value at risk of plants which have real options in order to compare them with those that have rigid technologies. This could be done levering on the Ornstein Uhlenbeck specification of the data generating process in order to construct Monte Carlo experiments. Finally, the Cartesian product of these extensions could really make GROPM a model general enough to tackle almost any kind of real options problem. Although that is true, many theoretical issues wait in ambush the student of real options when it comes to fill all the cells of these extensions Cartesian products.

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