REAL OPTIONS AND PREEMPTION UNDER INCOMPLETE INFORMATION

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Abstract

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Abstract

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1 Introduction

1.1 Real Options and Strategic Behaviour

Capital budgeting by corporations has been significantly influenced in recent years by the insights of the real options literature.[1] This literature stresses the fact that performing an irreversible investment when payoffs are stochastic involves sacrificing the option to invest in future. To maximize profits, one must therefore balance the profits foregone by delaying investment against the option value relinquished when the investment is made. If one follows this approach, the optimal investment time is often significantly later than the first date on which the present discounted value of future cash-flows exceeds zero.

In practice, investment timing decisions are often also affected by a firm's competitive environment. In some situations (which one may call cases of preemption), a firm fears that a competitor may seize an advantage by acting first. For example, a firm may preemptively occupy a market niche which is only large enough for one producer. In other situations (which one may label cases of attrition), firms do better

[1] Kester (1984) and Leslie and Michaels (1997) provide examples.

to delay until their competitors have acted. An example of attrition might be when an oil company delays exploratory drilling in order to see whether oil is discovered by a second firm with rights to an adjacent oil track.

Although corporate planners often recognize the practical importance of strategic considerations in investment timing decisions, it has until recently been unclear how strategic behaviour could be integrated with the contingent claims techniques employed in the real options literature.[2] Several recent studies, however, have begun the task of incorporating strategic elements into real options models. Smets (1993) and Grenadier (1996) analyze foreign direct investment and real estate development respectively using continuous-time, leader-follower games in which firms strategically choose trigger points for their investments. Trigeorgis (1996) and Smit and Trigeorgis (1997) look at strategic investment decisions by different firms using binomial models. Other relevant papers include Williams (1993), Leahy (1993) and Fries, Miller, and Perraudin (1997) which consider real investment decisions in a perfectly competitive industry equilibrium.

All these authors employ full-information, non-cooperative games.[3] Although clearly the right starting point for analysis, this approach has two limitations. First, the assumption of complete information is often unrealistic. A significant risk for many firms is that their conjectures about the behaviour of competitors will prove incorrect. Second, if information is complete and preemptive behaviour generates substantial losses in value (as is commonly true in these models), one may well ask

[2]Some practitioner discussions contain the intuition that preemption by competitors may affect the value of real options just as dividends on the underlying security affect the value of financial options. For example, Leslie and Michaels (1997) note that is important in real investment decisions "to reduce the value lost by waiting to exercise a real option. In financial options, this is the cost of waiting until the payment of a dividend. In a real business situation, the cost of waiting could be high if an early entrant were to seize the initiative. When first-mover advantages are significant, the dividends are correspondingly high, thus reducing the option value of waiting." Trigeorgis (1991) formalizes this intuition.

[3]Spatt and Sterbenz (1985) study investment in an industry equilibrium with incomplete information but their focus is not on pricing and the firms in their model are assumed to be symmetrically informed so the analysis is quite different from ours. Grenadier (1999) also looks at an investment problem with incomplete information but his payoff structure is very different from ours and hence the threat of preemption on which we (and most of the above papers) focus does not arise.

why firms cannot reach an agreement to split the surplus created by the real option. Simply assuming that bargaining is impossible without introducing frictions (such as incomplete information) into the model appears unsatisfactory.

In this paper, we show how one may incorporate incomplete information into an equilibrium model in which groups of firms invest strategically. Each firm faces a simple real option investment problem of the type investigated by McDonald and Siegel (1986) and Dixit (1989). We suppose, however, that the first firm to invest obtains the whole market and that other firms receive a zero payoff.[4] Furthermore, we assume that firms know their own cost of investment but only the distribution of their competitors' investment costs. As time goes by and competitors have not so far invested, each firm up-dates its conjecture about its competitors' investment cost distribution. Our framework yields intuitively reasonable expressions for investment triggers. These are simple generalizations of results from McDonald and Siegel (1986) and Dixit (1989) and are easy to employ in practical capital budgeting applications.

The structure of our paper is as follows. Section 2 analyses real option models for single firms which face either (i) a simple, non-strategic real option problem as in McDonald and Siegel (1986), and Dixit (1989), or (ii) face a threat of preemption. Section 3 constructs a perfect, Bayesian equilibrium with incomplete information in which two or more firms invest subject to threats of preemption from competitors. Section 4 examines empirical implications of the model and the impact of preemption on firm value. Section 5 concludes.

2 Optimal Investment by Single Firms

2.1 The Non-Strategic Firm

In this and the next subsection, we develop simple models of firm behaviour which serve as building blocks in our subsequent analysis of equilibrium with entry by multiple firms. We assume throughout that investors are risk neutral and can borrow and lend freely at a constant safe interest rate, $r > 0$. Introducing risk aversion hardly

[4] This assumption may be relaxed to allow partial reductions in payoffs.

changes the analysis since one may repeat the arguments using risk neutral rather than actual probabilities (see Cox and Ross (1976) and Harrison and Kreps (1979)).

We begin by supposing that there exists a single firm with the possibility of investing a sum k in an indivisible technology that yields a flow of income x_t, where x_t is a geometric Brownian motion:

$$dx_t = \mu x_t dt + \sigma x_t dB_t \quad , \tag{1}$$

for constants $\mu < r$ and $\sigma > 0$, and a standard Brownian motion, B_t.

Although simple, these assumptions allow us to capture many of the more important intuitions of the real options literature. Indeed, our setup represents a slight specialization of the much-cited studies by McDonald and Siegel (1986) and Dixit (1989). In our notation, the McDonald and Siegel (1986) model would amount to assuming that both x and k are both geometric Brownian motions. The only change this introduces in the analysis we perform below is that the state variable for the model becomes x_t/k_t rather than x_t. Dixit (1989), on the other hand, assumes that the post-investment firm cash flow is $x_t - w$ (in our notation) for a positive constant, w. Since this flow becomes negative when x_t falls below w, the firm may eventually wish to terminate production.[5] Valuing the firm requires that one simultaneously calculate option values associated with both entry and exit.

Since payoffs are independent of time, the firm's optimal investment strategy is to select a constant trigger, $\overline{x}$, and then to invest when x_t first crosses $\overline{x}$. If V_t is the value of the firm *before* investment has taken place, then under risk neutrality, V_t must satisfy the equilibrium condition:

$$rV(x|\overline{x}) = \mu x \frac{\partial V(x|\overline{x})}{\partial x} + \frac{\sigma^2}{2} x^2 \frac{\partial^2 V(x|\overline{x})}{\partial x^2} \quad . \tag{2}$$

Here, the left hand side is the return obtained from investing V in the safe bond while the right hand side is the expected capital gain from investing in the firm's equity.

After the investment has been made, the firm value is simply the expected discounted cash flow:

$$\mathrm{E}_t \left(\int_t^\infty x_s \exp[-r(s-t)] ds \right) = \frac{x_t}{r-\mu} \quad . \tag{3}$$

[5]In our case, with $w = 0$, after the investment has been made, the cash flow, x_t, is always positive and hence the firm will never wish to cease production.

Ruling out jumps in equity value at the moment of investment and imposing a no bubbles condition therefore yields the boundary conditions:

$$V(\overline{x}|\overline{x}) \quad = \quad \frac{\overline{x}}{r-\mu} - k \qquad \text{and} \qquad \lim_{x \downarrow 0} V(x|\overline{x}) \quad = \quad 0 \ . \tag{4}$$

The optimal trigger point for investment, $\overline{x}$, may be found by maximizing $V(x|\overline{x})$ with respect to $\overline{x}$. First and second order conditions for this maximization are:

$$\frac{\partial V(x|\overline{x})}{\partial \overline{x}} \quad = \quad 0 \qquad \text{and} \qquad \frac{\partial^2 V(x|\overline{x})}{\partial \overline{x}^2} \quad < \quad 0 \ . \tag{5}$$

Solving the differential equation in (2) subject to the boundary and maximization conditions, (4) and (5), yields:

Proposition 1 *Under the above assumptions, the value of the firm prior to investment is:*

$$V(x|\overline{x}_n) = \left(\frac{\overline{x}_n}{r-\mu} - k \right) \left(\frac{x}{\overline{x}_n} \right)^{\lambda} \ . \tag{6}$$

Here, λ is the positive root of the quadratic equation $\lambda(\lambda-1)\sigma^2/2 + \lambda\mu = r$. The firm's optimal strategy consists of investing when x_t first crosses $\overline{x}_n$, where

$$\overline{x}_n \quad \equiv \quad -\frac{\lambda}{1-\lambda} \ (r-\mu) \ k \ . \tag{7}$$

Proofs of this and subsequent results appear in the Appendix.

We shall refer to $\overline{x}_n$ as the *non-strategic trigger*, recognizing the fact that it is the optimal trigger when the firm's payoffs are independent of the actions of other firms. We follow Dixit (1989) in defining the *Marshallian trigger*, $\overline{x}_m$, as the point at which the total expected discounted income flow equals the cost of investment:

$$\overline{x}_m \quad \equiv \quad (r-\mu)\, k. \tag{8}$$

$\overline{x}_n$ exceeds $\overline{x}_m$ if there is an option value to waiting. Inspection of the formulae reveals that this is true if r are positive since then $\lambda > 1$.

2.2 Preemption

Now, consider the investment decision of a firm *threatened by preemption.* To motivate the model we develop, one may consider examples of investment decisions influenced by fear of preemption like those discussed by Lieberman and Montgomery (1988). They categorize preemptive behaviour into cases in which first-mover advantages arise from (i) technological leadership, (ii) preemption of assets, and (iii) buyer switching costs.

A good illustration of (ii) is the preemptive, spatial investment strategy followed by the US discount retailer Wal-Mart. This is described in a well-known Harvard Business School case study by Ghemawat (1986). In its early development through the 1960s and 70s, Wal-Mart's strategy consisted of establishing discount stores in small towns in the South Western United States which were clearly too small to contain more than one discount retailer. In the 1980s, Wal-Mart began opening warehouse clubs in densely populated areas. Ghemawat (1986) suggests that there were only about 100 metropolitan areas in the US with suitably large populations and first mover's advantages were considered important.[6]

The investment decisions faced by Wal-Mart illustrate rather well the combination of real investment decisions and preemptive pressure which we study in this paper. Both in its period of early expansion and in its subsequent development of warehouse clubs, Wal-Mart was exercising real options that were subject to potential preemption by competitors.

To model *a threat of preemption*, let us suppose that a firm, labeled i, can invest at a cost, k_i, in the income stream, x_t, already described; however, another firm, labeled j, may invest first, in which case i loses any further opportunity to invest. To introduce incomplete information, we assume that firm i conjectures that firm j invests when x_t first crosses some level $\overline{x}_j$, and that $\overline{x}_j$ is an independent draw from a distribution $F_j(\overline{x}_j)$. $F_j(\overline{x}_j)$ has a continuously differentiable density $F_j'(\overline{x})$ with positive support on an interval, $[\overline{x}_L, \overline{x}_U]$.

Our approach represents a quite extreme form of preemption in that i loses entirely its chance to invest if it is forestalled by j. This is fairly realistic in some of the investment problems faced by Wal-Mart above and, in any case, may be generalized

[6]One analyst quoted by Ghemawat (1986) put it: "Because warehouse clubs, as currently structured, depend on memberships - solicited directly to wholesale customers and to 'group' members through savings and loan clubs, credit unions and employee organizations - being the first warehouse club to solicit and introduce the concept in a market can be a major competitive advantage."

to situations in which firms lose part of the value of their real option when a competitor invests first.

The structure of learning implied by our assumptions is quite simple. Since j invests only when x_t first crosses a threshold, i learns about j when x_t hits a new high within the support of $F_j(\overline{x}_j)$. When this happens, if firm j invests, i learns that j's trigger level is the current x_t. Conversely, if j does *not* invest, firm i learns that j's trigger lies in a higher range of x values than it had previously believed (i.e., in $[\hat{x}_t, \overline{x}_U]$ where $\hat{x}_t \equiv \max_{0 \leq \tau \leq t}\{x_\tau\}$). Thus, i's *conditional* conjecture about the distribution of j's trigger, $F_j(\overline{x}_j|\hat{x}_t)$, is:

$$F_j(\overline{x}_j|\hat{x}_t) = \frac{F_j(\overline{x}_j) - F_j(\hat{x}_t)}{1 - F_j(\hat{x}_t)}. \tag{9}$$

To value the ith firm under the threat of preemption, denoted Z_{it}, we may derive the value for an arbitrary investment trigger, $\overline{x}_i$ and then maximize over this trigger. Z_{it} will of course depend not just on the publicly-observable profit variable x_t but also on $\hat{x}_t$. The latter is a sufficient statistic for all that firm i has learnt about firm j by time t.

As we show in the Appendix, a simple conditioning argument enables us to calculate firm value and the optimal investment trigger as:

Proposition 2 *The value of a firm i, which fears preemption by a firm j, prior to investment by either firm is*

$$Z_i(x, \hat{x}|\overline{x}_{is}) = \left(\frac{\overline{x}_{is}}{r - \mu} - k_i\right) \left(\frac{x}{\overline{x}_{is}}\right)^{\lambda} \frac{1 - F_j(\overline{x}_{is})}{1 - F_j(\hat{x})}. \tag{10}$$

Firm i's optimal investment strategy consists of investing when x_t first crosses $\overline{x}_{is}$, where $\overline{x}_{is}$ is given by:

$$\overline{x}_{is} \quad = \quad -\frac{\lambda + h_j(\overline{x}_{is})}{1 - \lambda - h_j(\overline{x}_{is})} \; (r - \mu) \; k_i. \tag{11}$$

Here, $h_j(x)$ denotes the hazard rate: $xF_j'(x)/(1 - F_j(x))$.

Note that the expression for the optimal investment trigger given in equation (11) is just a rearranged form of the first order condition $\partial Z_i(x, \hat{x}|\overline{x}_i)/\partial \overline{x}_i = 0$. A sufficient although not necessary condition for the second order condition $\partial^2 Z_i(x,\hat{x}|\overline{x}_i)/\partial\overline{x}_i^2 < 0$ to hold is that $xF'(x)/(1-F(x))$ is increasing in x. This is true for standard distributions such as uniform, negative exponential, Weibull and Pareto.

2.3 Interpreting the Solutions

The firm value which appears in Proposition 2 is very simple in that it equals the non-strategic firm value already encountered in equation (6) multiplied by *the probability that firm i 'wins the race' by being the first to invest*, namely $(1-F_j(\overline{x}_{is}))/(1-F_j(\hat{x}))$.[7]

The geometry of our solutions is illustrated by Figure 1[8] which shows the value of the non-strategic and of the strategic firm, $V(x|\overline{x}_n)$ and $Z_i(x,\hat{x}|\overline{x}_{is})$, plotted against x. Since $Z_i(x,\hat{x}|\overline{x}_{is})$ is a function of both x and $\hat{x}$, it is shown in the figure as a family of curves, each specific to a particular $\hat{x}$ value. If x rises enough, $\hat{x}$ will increase. When this happens, if firm j does not invest, this is good news for firm i and Z_i shifts onto a higher curve corresponding to the higher $\hat{x}$. If firm j invests, Z_i drops to zero.

The family of $Z_i(x,\hat{x}|\overline{x}_{is})$ curves is bounded to the right by the envelope defined as: $Z_i(x,\hat{x}|\overline{x}_{is})|_{\hat{x}=x}$ for different levels of x. The optimal trigger point for firm i, $\overline{x}_{is}$, is the point at which this envelope is tangent to the 'investment payoff line', $x/(r-\mu)-k$. From the expression in (11) and the fact that $h_j(x) \geq 0$, one may deduce that the optimal investment trigger, $\overline{x}_i$, lies between the Marshallian and the non-strategic triggers, i.e.,

$$\overline{x}_{im} \quad \leq \quad \overline{x}_{is} \quad \leq \quad \overline{x}_{in}. \tag{12}$$

Generalizing the two firm case, it is straightforward to solve for the value of firm i when it faces a threat of preemption by $n > 1$ firms. Firm i loses its option to invest if any one of the n other firms invests before it. But if each firm's investment trigger is independently distributed with a distribution function $F(\overline{x})$, then the distribution

[7]It is worth noting here that much of our analysis would still go through if x_t follows a more general diffusion than the geometric Brownian motion assumed above. In this case, one would replace the terms $(x_t/\overline{x})^\lambda$ that appear in equation (10) and what follows with the more general probability-weighted discount factor $\mathrm{E}_t(\exp[-r(T-t)])$ where T is the first time that x_t hits $\overline{x}$.

[8]This and subsequent figures assume $F_j(\overline{x}) = 1-(\overline{x}/x_L)^{-\alpha}$ and that parameters take the values: $\alpha = 5$, $\mu = 0$, $\sigma = 0.1$, $r = 0.025$ and $x_L = 0.094$.

of the *minimum* of the n competitor firms' triggers is simply:

$$F^{(n)}(\overline{x}) \equiv 1 - (1 - F(\overline{x}))^n. \tag{13}$$

Similar arguments to those employed above yield firm values and optimal triggers which are equal to the expressions in equations (10) and (11) except that $F^{(n)}$ replaces F_j in (10) and $n\, h_j$ replaces h_j in (11).

There are two limiting cases which are instructive to examine. First, consider what happens as the incomplete information case collapses to one with complete information. Suppose that there exist two symmetric firms each of which knows the other's characteristics and hence optimal strategy. Each will try to preempt the other by cutting its investment trigger until it is marginally below that of the other firm. The limit of this process is that both firms attempt to invest when x_t first hits $(r-\mu)k$, i.e., at the Marshallian trigger.

To see how this fits into our model, suppose two firms each conjecture that the other's trigger is a draw from a distribution F. Let this distribution be one of a sequence of continuously differentiable trigger distributions such that probability weight is increasingly concentrated on $\overline{x}_m$. As one moves along the sequence of distributions, $h_j(\overline{x})$ will explode in the neighborhood of $\overline{x}_m$. As one may see from equation (11), each firm's optimal trigger will converge to $\overline{x}_m$.

A second interesting limiting case occurs when the number of competing firms goes to infinity. From the discussion below equation (13), and the fact that $\lim_{n\to\infty} n\, h_j = \infty$, one may see that $\overline{x}_i$ converges to $k_i(r-\mu)$, the Marshallian trigger.[9] It is instructive to compare this result with Leahy (1993)'s finding that in a perfectly competitive industry equilibrium in which infinite numbers of firms exercise real options, the optimal trigger strategy consists of investing 'myopically' at the non-strategic trigger corresponding in our notation to $\overline{x}_n$.

The difference between our analysis and Leahy's is that in his model, the profit variable x_t is regulated at the trigger level due to the entry of multiple firms. Since there is consequently no up-side in the profit variable (x_t cannot take values above the trigger $\overline{x}$ after the date of entry), entry must take place at a point significantly

[9] So long as this latter exceeds x_L.

above the Marshallian trigger, if expected discounted profits prior to investment are to equal zero (as they must in the perfectly competitive case).

3 A Multi-Firm Equilibrium

3.1 Triggers and Characteristics

Drawing on the analysis in the last section of investment decisions by individual firms, we are now in a position to analyze a Bayesian Nash equilibrium in which firms invest strategically. Assume that there are two firms, labeled i=1,2, each of which can invest in the income flow, x_t, described above for a cost, k_i. When one firm invests, however, the opportunity to invest is lost to the other firm. If the two firms invest simultaneously, with probability 1/2, firm i (i=1,2) receives the income flow at a cost k_i while firm j ($j \neq i$) gets nothing.[10]

We introduce incomplete information by supposing that the ith firm observes its own cost, k_i, but knows only that k_j, $j \neq i$, is an independent draw from a distribution $G(k)$. $G(k)$ has a continuously differentiable density, $G'(k)$, with strictly positive support on an open interval (k_L, k_U).

The above assumptions mean that the multiple-firm equilibrium is related to those which arise in models of first price auctions under incomplete information with a continuum of types. To appreciate the similarity, consider an auctioneer who successively announces declining prices until one of two bidders agrees to buy the item for sale. Each bidder knows his own reservation value for the item being sold but only a continuous distribution for that of his competitor. As prices decline and the other bidder has not so far accepted a price, each bidder up-dates his prior distribution for the other's reservation value. Maskin and Riley (1986) examined such first price auctions. Their unpublished results are sketched in Fudenberg and Tirole (1993) while Fudenberg and Tirole (1986) provide a technically similar analysis of a model of industry

[10] The analysis would be the same under a more general 'tie-breaking rule', for example if each firm's probability of winning after an attempt to invest simultaneously was a constant in $(0, 1)$.

exit under certainty.[11]

Returning to the derivation of the model, we note that, in equilibrium, each firm's investment trigger $\overline{x}_i$ may be regarded as the level of a mapping $\overline{x}_i(k)$, i=1,2, from its cost parameter,[12] k_i, to its optimal strategic investment trigger, $\overline{x}_{is}$. Before solving for the equilibrium mappings, we must demonstrate some important properties that the mappings possess. In doing this, we need to impose a regularity condition on the distribution function, $G(k)$.

If the density $G'(k)$ falls too quickly in some range, the hazards $h_i(x)$ may decline so rapidly that the first order conditions cease to yield a one-to-one mapping between investment triggers and characteristics, k. A simple condition that rules out such problems and which is satisfied for most cases of interest is that,[13] for $k \in [k_L, k_U]$, $kG'(k)/(1-G(k))$ is increasing in k. As noted in the discussion following Proposition 2, most standard distributions possess this property.

Subject to the regularity condition on $G(k)$ just stated, one may obtain the following proposition.

Proposition 3 *Under the assumptions of this section, for each firm, i=1,2, the mapping, $\overline{x}_i(k)$, from the investment cost, k_i, to the optimal investment trigger, $\overline{x}_i$, is strictly increasing. If the investment cost distribution, $G(k)$, satisfies the regularity condition mentioned above, then $\overline{x}_i(k)$, is continuous. Finally, for $i = 1, 2$ the values of the $\overline{x}_i(k)$ functions coincide at the upper and lower ends of the support of the k distribution, i.e., $\overline{x}_1(k_L) = \overline{x}_2(k_L)$ and $\overline{x}_1(k_U) = \overline{x}_2(k_U)$.*

[11] The model we develop below differs from all the above in that the basic state variable, x_t, is a diffusion process rather than a deterministic variable, and agents learn about each other by observing, $\hat{x}_t$, the supremum of past levels of x_t.

[12] While it seems reasonable to suppose as we do that competing firms are ignorant of the level of each other's costs, our analysis could be developed under the assumption that firms have incomplete information about some other parameter of their competitor's optimal investment decision. The only essential requirement is that the parameter be monotonically related to the investment trigger. For example, if firms were operated by risk averse investors with different coefficients of relative risk aversion, each firm would decide an investment trigger which depended on a different risk-adjusted drift term, μ, for the state variable x_t.

[13] This condition is sufficient to rule out problems but not necessary as one may see from the proof of Proposition 3.

3.2 Deriving the Equilibrium Mappings

If the cost parameter and investment trigger are linked by a continuous, strictly increasing function $\overline{x}_i(k)$, the rational conjecture for firm i (i=1,2) to adopt is that the distribution of j's ($j \neq i$) investment trigger is:

$$F_j(\overline{x}) = G(k_j(\overline{x})). \tag{14}$$

where $k_j(x)$ is the inverse of $\overline{x}_j(k)$, and the support of the F_j distribution is $[x_L, x_U]$, where x_L and x_U equal $\overline{x}_i(k_L)$ and $\overline{x}_i(k_U)$ for i=1,2.

Rearranging the first order condition given in equation (11) and using the fact that:

$$h_j(x) \equiv \frac{xF_j'(x)}{1 - F(x)} = \frac{xG'(k)}{1 - G(k)} k_j'(x), \tag{15}$$

we obtain the following system of non-linear differential equations for the two functions, $k_i(x)$, i=1,2.

$$k_1'(x) = \frac{1 - G(k_1(x))}{G'(k_1(x))} \left(\frac{1}{x - (r - \mu)k_2(x)} - \frac{\lambda}{x} \right) \tag{16}$$

$$k_2'(x) = \frac{1 - G(k_2(x))}{G'(k_2(x))} \left(\frac{1}{x - (r - \mu)k_1(x)} - \frac{\lambda}{x} \right). \tag{17}$$

What boundary conditions must these equations satisfy? If $\hat{x}_t \to \overline{x}_U$ but neither firm has so far invested, then each firm knows that the other will act almost certainly in the next few instants. In consequence, the hazard of being preempted per unit of time explodes to infinity. As one may see from equation (11), this means that $k_i(\overline{x}_i) \to \overline{x}_i/(r - \mu)$, for i=1,2, i.e., if both firms are close to the upper end of the k support, the option value of waiting is eliminated. Thus, the relevant boundary conditions for (16) and (17) are: $k_i((r - \mu)k_U) = k_U$ for $i = 1, 2$.

Analysis of the system of differential equations in (16) and (17) permits us to deduce the following result:

Proposition 4 *Under the assumptions of this section, there is a unique, symmetric equilibrium in which each firm's optimal investment trigger is the solution to the differential equation:*

$$k'(\overline{x}) = \frac{1 - G(k)}{\overline{x}G'(k)} \frac{\overline{x} - \lambda(\overline{x} - (r - \mu)k)}{\overline{x} - (r - \mu)k} \qquad \text{subject to} \qquad k((r - \mu)k_U) = k_U. \tag{18}$$

3.3 The Isoelastic Case

In this subsection, we describe a simple distribution that yields convenient, closed-form solutions for investment triggers and firm values. This solution is very useful for practical applications of our model. Suppose that the investment cost distribution, $G(k)$, is isoelastic with a bounded support, i.e.,

$$G(k) \quad = \quad \frac{k_L^{-\alpha} - k^{-\alpha}}{k_L^{-\alpha} - k_U^{-\alpha}} \qquad \begin{array}{lll} \text{for} & k \in [k_L, k_U] & \\ \text{where} & 0 < k_L < k_U < \infty, \quad \text{and} & \alpha \neq 0. \end{array} \tag{19}$$

This distribution is commonly referred to as the Pareto distribution. One may note that it satisfies the regularity condition on $G(k)$ assumed above.

When k follows the Pareto distribution, our analysis above implies that the function $x(k)$ satisfies:

$$x'(k) = \frac{\alpha x k^{-\alpha-1}}{k^{-\alpha} - k_U^{-\alpha}} \, \frac{x - (r-\mu)k}{x - \lambda(x - (r-\mu)k)} \quad \text{subject to} \qquad x(k_U) = \frac{k_U}{r-\mu}. \tag{20}$$

This equation may be solved numerically with little difficulty. Alternatively, a simple closed form solution may be obtained by driving k_U, the upper bound of the support of $G(k)$, to infinity. An appealing feature of the solution one then obtains is that the ratio of the optimal investment trigger to the investment cost is constant.

Proposition 5 *If $G(k)$ satisfies equation (19) and $\alpha > 0$, as $k_U \to \infty$, the optimal investment trigger $\overline{x}(k)$ converges to a constant proportion of the investment cost, k, in that:*

$$\lim_{k_U \to \infty} \overline{x}(k|k_U) \quad = \quad - \frac{\lambda + \alpha}{1 - \lambda - \alpha} \, (r - \mu) \, k. \tag{21}$$

The practical benefits of this simple asymptotic solution should be stressed. As the limit of a sequence of triggers corresponding to models in which k has bounded support and the equilibrium is unique, the asymptotic solution has nice theoretical properties. On the other hand, its simplicity allows one to calculate analytically various interesting quantities including firm values and trigger distributions. We shall use this in our comparative statics and empirical implementation of the model below.

Figure 2 shows numerical solutions for $\overline{x}(k)$ for different upper truncation points, k_U, when $G(k)$ is the Pareto distribution. In all cases, the solutions start at the origin and increase monotonically. Furthermore, the solutions lie everywhere within the cone created by the lines corresponding to the Marshallian trigger, $(r-\mu)k$, and the non-strategic trigger, $(-\lambda/(1-\lambda))(r-\mu)k$. For finite k_U, the solutions terminate on the Marshallian trigger line, $(r-\mu)k$. When $k_U \to \infty$, $\overline{x}(k)$ converges to a straight line.

Figure 3 shows the Marshallian trigger, $\overline{x}_m = (r-\mu)k$, and the non-strategic trigger, $\overline{x}_n = (-\lambda/(1-\lambda))(r-\mu)k$ for different values of α but the same k_U. When $\alpha = -1$, $G(k)$ is the uniform distribution. Lowering α shifts probability weight towards k_U and hence moves $\overline{x}(k)$ towards the non-strategic trigger.

It is interesting to compare our real option solutions to those obtained by Merton (1973) for infinite maturity American options when there exists some constant mean arrival rate of ruin, ζ, per unit of time. In our notation, this would yield an investment trigger of $\lambda^*/(\lambda^*-1)(r-\mu)k$ where λ^* is the positive root to $\lambda^*(\lambda^*-1)\sigma^2/2+\lambda^*\mu = (r+\zeta)$. One may easily show that raising ζ unambiguously lowers the investment trigger. The resulting solution looks very similar to our asymptotic solution (see equation (21)) when k has a truncated Pareto distribution since the Pareto parameter, α, appears additively with λ. In the general case (see equation (11)), the hazard rate acts like a mean arrival rate of ruin which is increasing in the state variable.

4 Implications of Our Analysis

4.1 Investment Triggers and Firm Value

To assess the destruction of value implied by preemption under differing degrees of incomplete information, we calculated investment triggers and firm values using the parameters employed by Dixit (1989).[14] Let $\tilde{x}_s$, $\tilde{x}_n$, and $\tilde{x}_m$ denote the strategic, non-strategic and Marshallian triggers, integrated over the cost distribution, $G(k)$. We suppose that $G(k)$ is truncated Pareto and perform the calculations for different

[14] These are: $r = 0.025$, $\mu = 0$, $\sigma = 0.1$, and $\kappa = 4$.

Table 1: Investment Trigger Comparative Statics.

Dixit (1989) parameters: $r = 2.5\%$ $\mu = 0$ $\sigma = 10\%$ $\kappa = 4$.					
	$\alpha = 1.5$	$\alpha = 3$	$\alpha = 6$	$\alpha = 12$	$\alpha = 24$
$\tilde{x}_s$	0.13	0.12	0.11	0.11	0.10
$\tilde{x}_n$	0.16	0.16	0.16	0.16	0.16
$\tilde{x}_s/\tilde{x}_m$	1.30	1.21	1.13	1.07	1.04
$\tilde{x}_n/\tilde{x}_m$	1.56	1.56	1.56	1.56	1.56
$\tilde{Z}$	0.04	0.07	0.06	0.04	0.03
$\tilde{V}$	0.10	0.27	0.42	0.52	0.58
$\tilde{V}^*$	0.14	0.33	0.47	0.55	0.60
$\tilde{Z}/\tilde{V}$	0.41	0.26	0.15	0.09	0.04
$\tilde{Z}/\tilde{V}^*$	0.30	0.22	0.14	0.08	0.04

$\tilde{x}_s$, $\tilde{x}_n$, and $\tilde{x}_m$ denote the strategic, non-strategic and Marshallian triggers integrated over the investment cost distribution, $G(k)$. $\tilde{Z}$, $\tilde{V}$ and $\tilde{V}^*$ denote the strategic, non-strategic and cooperative firm values again integrated over $G(k)$.

values of the Pareto parameter, α.[15]

The ratio of the non-strategic to the Marshallian trigger implied by our model broadly resembles those found by Dixit (1989) in that $\tilde{x}_n/\tilde{x}_m = 1.56$ compared with Dixit's ratio (as reported in his Figure 3) of 1.36.[16] When we introduce strategic behaviour into our model, the ratio of the entry trigger to the Marshallian trigger is sharply lower than the ratio based on the non-strategic triggers, even when α is fairly low. For example, for $\alpha = 1.5$, the ratio of the strategic to the Marshallian trigger was around 1.3. As α increases, $\tilde{x}_s/\tilde{x}_m$ falls until by $\alpha > 20$, it is less than 1.05.

The lower part of Table 4 reports average firm value ratios for different α values

[15]Note that we hold the mean of $G(k)$ constant as we vary α (by altering k_L). This implies that $\tilde{x}_n$ and $\tilde{x}_m$ are independent of α.

[16]His entry trigger is slightly lower than ours because in his model firms have an exit option which reduces the degree to which entry is irreversible.

and under different assumptions about strategic behaviour. Again, by 'average', we mean that the relevant quantities are integrated over the cost distribution, $G(k)$. The ratio of the average strategic value ($\tilde{Z}$) to the average non-strategic value ($\tilde{V}$) is 0.41 for an α value of 1.5 but declines to 0.04 when α is 24.

If firms could cooperate, then they would agree to let the more efficient firm invest at the trigger level which fully exploits its option value of waiting. Integrating this value over the distribution of the minimum of two draws from the cost distribution, i.e., over $1 - (1 - G(k))^2$, we obtain the 'average cooperative firm value', which we denote $\tilde{V}^*$. The true cost of non-cooperation is better measured by the ratio $\tilde{Z}/\tilde{V}^*$ than $\tilde{Z}/\tilde{V}$. As one may see from the Table 4, $\tilde{Z}/\tilde{V}^*$ is 0.3 for the baseline volatility and value of $\alpha = 1.5$, but falls to 0.04 for $\alpha = 24$.

4.2 Complete versus Incomplete Information Cases

It is interesting to compare the value implications of complete and incomplete information cases directly. Figure 4 shows the value of two competing firms under complete information divided by the value of the same two firms when information is incomplete. The incomplete information values are calculated from the firms' point of view, i.e., conditional on knowing each firm's own cost but not that of its competitor.

As one may see in the figure, the value ratio equals zero when the two firms have identical costs since in that case the complete information option values are entirely destroyed. For a given cost level for firm 1, if one reduces the investment cost of firm 2, the ratio first rises and then falls. This reflects the fact that, initially, the fall in 2's cost translates under complete information into a substantial rise in value (exceeding that under incomplete information) as firm 2 is able to delay investment until just before x_t hits the Marshallian trigger of the less efficient firm 1.

Eventually, these value gains in the complete information cease because firm 2's non-strategic trigger falls below the Marshallian trigger of firm 1. In contrast, the value gains in the incomplete information case continue as k_2 is reduced and hence the ratio of the two values declines, ultimately falling below unity for very low k_2.[17]

[17]Indeed, in the limit as $k_2 \to k_L$, the probability that firm 2 wins goes to unity and firm 2's claim

4.3 Implications for Equity Price Behaviour

Our analysis clearly has implications for the timing of real investments by firms in environments with differing degrees of competition and imperfect information. One could therefore test our models through investigations of microeconomic data along the lines of the recent studies by Quigg (1993) and Moel and Tufano (1998). It is also interesting, however, to note the implications of our model for the equity returns of firms which hold real options.[18] The combination of preemptive behaviour and incomplete information means that these implications are quite rich.

First, equity return will exhibit discrete jumps associated with the resolution of incomplete information about competitors. The jumps will be positive or negative depending on whether the firm itself or a competitor invests. Thus, according to the model, equity return volatility comprises (i) value changes attributable to the evolution of publicly observable profit variables and (ii) changes attributable to "competitor risk". The signs of the jump sizes imply that while "profit risk" is positively correlated across firms within the same industry, competitor risk is negatively correlated.

Second, positive jumps will on average be larger than negative ones. This is true even if there are just two firms since, when an investment occurs, the market learns that at least one of the firms is more efficient than expected and thus the combined value of the two firms jumps up. With multiple competing firms, the asymmetry is even greater since the loss in the equity value of firms which do not invest is spread among several firms. Hence, there could be a tendency towards positive skewness in equity returns of firms holding real options subject to incomplete information and possible preemption.

Third, the fact that the market gradually learns about firms' types implies that there will be marked age effects in the higher moments of equity returns. In particular, volatility and kurtosis are likely to decline over time as investment announcements

value under incomplete information tends to the complete information value. Hence, as the claim value for the high cost agent under complete information is zero, and positive under incomplete information, the ratio will fall below unity.

[18] Berk, Green, and Naik (1997) and Pope and Stark (1997) discuss the implications of real option values for equity returns in an analogous fashion.

by the firm and its competitors progressively eliminate incomplete information.

5 Conclusion

Firms commonly engage in preemptive actions to gain advantages over competitors. In a survey reported by Bunch and Smiley (1992) and Smiley (1988), managers describe the strategies used by those in their industry to deter entry. Many of the strategies described involve substantial irreversible investments (for example, advertising, building up excess capacity, R&D and patenting, and filling market niches) and hence fit within the framework we develop in this paper.

In this article, we examine the implications of such preemptive behaviour for valuation and the timing of real investment. Our models provide a unifying theory in which the standard zero NPV investment rule and the real options investment rule (as in McDonald and Siegel (1986) and Dixit (1989)) appear as special cases.

The inclusion of incomplete information in our model yields quite rich implications for the equity return distributions of companies holding real options subject to possible preemption. In particular, the model predicts that returns on such equities will contain jumps. Volatility associated with these jumps will be negatively correlated across competing firms unlike more standard volatility attributable to news on the general prospects of the industry.

Appendix

Proof of Proposition 1

The derivation is standard. See for example, Dixit and Pindyck (1994). □

Proof of Proposition 2

Agents are risk neutral and the value of the firm may hence be written as a discounted expectation:

$$Z_{it} = \mathrm{E}_t \left[\left(\frac{\overline{x}_{is}}{r-\mu} - k_i \right) \exp[-r(T-t)] \middle| \text{firm i has the lowest trigger} \right]$$

$$\times \quad \text{Prob}\{\text{firm i has the lowest trigger}\}. \tag{22}$$

where T is a random stopping time. But using the results of Proposition 1, one may write this as:

$$Z_{it} = \left(\frac{\overline{x}_{is}}{r-\mu} - k_i\right)\left(\frac{x_t}{\overline{x}_{is}}\right)^{\lambda} \text{Prob}\{\text{firm i has the lowest trigger}\}. \tag{23}$$

Evaluating the probability gives the expression in equation (10). Taking a derivatives with respect to $\overline{x}_{is}$, yields a first order condition which, rearranged, gives equation (11). It is simple to confirm the statement in the text that the second order condition holds so long as the slope of the trigger density, $F''(x)$ is not too negative. We shall see that this requirement appears again in the proof of Proposition 3. □

Proof of Proposition 3

The $\overline{x}_i(k)$ Mappings are Non-Decreasing.

If k_i' and k_i'' are firm i types which prefer trigger strategies $\overline{x}_i'$ and $\overline{x}_i''$ respectively, then

$$\left[\frac{\overline{x}_i'}{r-\mu} - k'\right]\left(\frac{x}{\overline{x}_i'}\right)^{\lambda}\frac{1-F_j(\overline{x}_i')}{1-F_j(\hat{x})} \geq \left[\frac{\overline{x}_i''}{r-\mu} - k'\right]\left(\frac{x}{\overline{x}_i''}\right)^{\lambda}\frac{1-F_j(\overline{x}_i'')}{1-F_j(\hat{x})} \tag{24}$$

$$\left[\frac{\overline{x}_i''}{r-\mu} - k''\right]\left(\frac{x}{\overline{x}_i''}\right)^{\lambda}\frac{1-F_j(\overline{x}_i'')}{1-F_j(\hat{x})} \geq \left[\frac{\overline{x}_i'}{r-\mu} - k''\right]\left(\frac{x}{\overline{x}_i'}\right)^{\lambda}\frac{1-F_j(\overline{x}_i')}{1-F_j(\hat{x})}. \tag{25}$$

Subtracting the right (left) hand side of equation (25) from the left (right) hand side of (24), and using the fact that $(1-F_j(x))/x^{\lambda}$ is decreasing in x, implies that $\overline{x}_i(k)$ is non-decreasing in its argument for i=1,2.

$\overline{x}_1(k_L) = \overline{x}_2(k_L)$ and $\overline{x}_1(k_U) = \overline{x}_2(k_U)$

First, we show that $\overline{x}_i(k_L) = \overline{x}_L$ for i=1,2. To see this, suppose that $\overline{x}_i(k_L) < \overline{x}_j(k_L)$. Since the jth firm does not invest in an interval to the right of $\overline{x}_i(k_L)$, $\overline{x}_i(k_L) = -\lambda/(1-\lambda)(r-\mu)k_L$. But, $h_i(\overline{x}_j(k_L)) \geq 0$, and so $\overline{x}_j(k_L) = -(\lambda + h_i(\overline{x}_j(k_L)))/(1-\lambda-h_i(\overline{x}_j(k_L)))(r-\mu)k_L$ which is less than or equal to $\overline{x}_i(k_L)$. But this is a contradiction.

Second, we show that $\overline{x}_i(k_U)$ for i=1,2 have a common value, call it $\overline{x}_U$. Suppose not, and that, without loss of generality, $\overline{x}_i(k_U) < \overline{x}_j(k_U)$. As x approaches $\overline{x}_i(k_U)$, the jth firm knows that, with probability one, the ith firm will invest in the next few instants. Clearly, since the ith firm with cost k_U is willing to invest at this point, $\overline{x}_i(k_U) \geq k_U(r-\mu)$. Hence, the jth firm with investment cost arbitrarily close to k_U will always invest at or before $\overline{x}_i(k_U)$, contradicting our assumption that $\overline{x}_i(k_U) < \overline{x}_j(k_U)$.

There Are No Atoms in the $\overline{x}_i$ Distributions.

Next, we show that there are no atoms in the distribution of the $\overline{x}_i$, i=1,2, at points in $[x_L, x_U]$. Given that G is a continuous distribution by assumption, this is equivalent to saying that the inverse mapping, $k_i(\overline{x})$, can have no jumps in this interval. Suppose that there were an atom in the distribution of the $\overline{x}_i$, say at some point $x_0 \in [x_L, x_U]$. The fact that the mapping from k to $\overline{x}_i$ is non-decreasing implies that the firms in this atom must be those whose costs lie in some interval $[k^*, k^{**}]$. But the presence of the atom implies that no type j firm would want to invest in an interval $[x_0, x_0 + \epsilon]$ for small $\epsilon > 0$, since investing just before x_0 would be preferable. But, if no firm j invests in an interval to the right of x_0, then the type i firms with characteristics in $[k_i^*, k_i^{**}]$ must be investing non-strategically. But then if a type i firm with investment cost equaling k_i^* invests at x_0, one with cost $k_1 \in (k^*, k^{**}]$ will find it optimal to invest at a point to the right of x_0 since clearly $(-\lambda/(1-\lambda))(r-\mu)k_1 > x_0 = (-\lambda/(1-\lambda))(r-\mu)k^*$.

There Are No Gaps in the Support of the $\overline{x}_i$ Distributions.

The result amounts to showing that there are no jumps in the mappings $\overline{x}_i(k)$, i=1,2. Such a jump would imply a gap in the support of the $\overline{x}_i$ distribution at $[x', x''] \subset [x_L, x_U]$. The proof is quite long and we just sketch it here (see Lambrecht and Perraudin (1997) for full proofs). There are four cases to consider. Case A: Type j firms *also* have a gap starting at x'; Case B: Type j firms do not invest in an open interval that includes x'; Case C: Type j firms invest with positive probability throughout an open interval that includes x'; and Case D: Type j firms have a gap ending at x'. One may rule out cases A, B and C by showing that they imply contradictions. To rule out Case D, one may use the argument that for any hypothesized series of gaps in the i and j distributions, ordered by their left-most points, one may employ the above arguments for Cases A, B and C to rule out gaps starting from the left. □

Proof of Proposition 4

There Are No Asymmetric Equilibria.

Integrating the equations in (16) and (17) and subtracting the resulting expressions, one obtains:

$$\log\left(\frac{1-G(k_1(x))}{1-G(k_2(x))}\right) = \int_{\overline{x}_L}^{x} \left(\frac{1}{\nu-(r-\mu)k_1(\nu)} - \frac{1}{\nu-(r-\mu)k_2(\nu)}\right) d\nu. \qquad (26)$$

Equation (26) implies that if $k_1(x_0) = k_2(x_0)$ for some x_0, then $k_1(x) = k_2(x)$ for all x.

Suppose that $k_1(x) > k_2(x)$ for all x. Then, the left hand side is negative while the right hand side is positive. A similar contradiction is obtained if one assumes that $k_1(x) < k_2(x)$. Hence, $k_1(x) = k_2(x)$ for all x and there cannot be an asymmetric equilibrium.

To establish the existence and uniqueness of equilibrium, we may therefore study the solutions of the differential equation that appears in the proposition.

Uniqueness of an Equilibrium.

Initially, suppose an equilibrium exists. Though a Lipschitz condition does not apply to equation (18), one may nevertheless show uniqueness. Integrate the equation in the proposition to obtain:

$$\log\left[1 - G(k(x))\right] = \int_{\zeta}^{x} \frac{(\lambda-1)s - \lambda(r-\mu)k(s)}{s(s-(r-\mu)k(s))} ds. \tag{27}$$

Suppose that there exist two constants, x_L^*, and x_L^{**}, such that solutions to equation (27) with $\zeta = x_L^*$ and $\zeta = x_L^{**}$, respectively denoted $k^*(x)$ and $k^{**}(x)$, both satisfy the boundary conditions: $k^*(x_U) = k_U$ and $k^{**}(x_U) = k_U$, where recall that $x_U \equiv (r-\mu)k_U$. Without loss of generality, suppose that $x_L^* > x_L^{**}$. Since $k^*(x_L^*) = k^{**}(x_L^{**}) = k_L$ and $k^{**}(x)$ is strictly increasing, $k^*(x_L^*) < k^{**}(x_L^*)$. From (27),

$$\begin{aligned}\log\left[\frac{1-G\left(k^*(x)\right)}{1-G\left(k^{**}(x)\right)}\right] &= \int_{x_L^*}^{x}\left[\frac{s}{s-(r-\mu)k^{**}(s)} - \frac{s}{s-(r-\mu)k^*(s)}\right] ds \\ &- \log\left[1-G(k^{**}(x_L^*)\right]. \end{aligned} \tag{28}$$

Since $k^{**}(x_L^{**}) = k^*(x_L^*) = k_L$, $x_L^* > x_L^{**}$ and $k^{**}(x)$ is increasing, $k^{**}(x_L^*) > k^*(x_L^*)$. Hence, the right hand side of (28) is positive and increasing in x. Consider the behaviour of the expression in the log on the left hand side of (28) as $x \to k_U(r-\mu)$. Using l'Hopital's rule, one may show that, if $G'(k_U) > 0$, then $\lim_{x\to x_U} k'(x) = 2/(r-\mu)$. Similarly, if $G'(k_U) = 0$ but $G''(k_U) \neq 0$, one obtains: $\lim_{x\to x_U} k'(x) = 3/(2(r-\mu))$. (If $G^{(i)}(k) = 0$ for $i = 1, \ldots n$ and $G^{(n)}(k) \neq 0$, successive applications of l'Hopital's rule can be used to show that $k'(x)$ is finite at k_U.) If $G'(k_U) > 0$, it follows therefore from l'Hopital's rule that

$$\lim_{x\to k_U(r-\mu)}\left[\frac{1-G(k^*(x))}{1-G(k^{**}(x))}\right] = \frac{G'(k_U)}{G'(k_U)} \lim_{x\to k_U(r-\mu)}\left[\frac{k^{*\prime}(x)}{k^{**\prime}(x)}\right] = 1. \tag{29}$$

Hence, we have a contradiction. If $G^{(n)}(k_U) = 0$ for $1 \leq n \leq m$ and $G^{(m)}(k_U) \neq 0$ for some positive integer m, then the arguments in the last paragraph may be used to obtain a similar contradiction. Hence, if a solution to the differential equation exists, it is unique.

The Existence of an Equilibrium.

Defining $K(x,k) \equiv (1-G(k))/(xG'(k))(x-\lambda(x-(r-\mu)k))/(x-(r-\mu)k)$, we can rewrite the differential equation in (18) as

$$k'(x) = K(x,k) \qquad \text{subject to} \qquad k(x_U) = k_U, \tag{30}$$

where $x_U \equiv (r-\mu)k_U$. We are interested in solutions to (30 lying in the triangular region

$$\mathcal{A} \equiv \left\{(x,k) \ : \ k \leq \frac{x}{r-\mu} \ , \ k \geq \frac{\lambda-1}{\lambda}\frac{x}{r-\mu} \ , \ x \leq x_U\right\}. \tag{31}$$

Standard existence theorems are hard to apply directly since the boundary condition, $k(x_U) = x_U/(r-\mu)$, holds on the boundary of $\mathcal{A}$ and $K(x,k)$ is not continuous at (x_U, k_U). (For (x,k) such that $k = x/(r-\mu)$ and $x < x_U$, $K(x,k)$ is infinite, while for (x,k) such that $x = x_U$ and $k < k_U$, $K(x,k) = 0$.)

To show existence, we, therefore, proceed indirectly by proving existence and uniqueness for the differential equation in (30) subject to different boundary conditions, and then obtaining the solution to (30) subject to the actual boundary condition as the supremum over these other solutions.

Consider the set of functions, $k_\epsilon(x)$, indexed by $\epsilon \in (0, k_U - ((\lambda-1)/\lambda)x_U/(r-\mu))$, where for a particular ϵ, $k_\epsilon(x)$ is the solution to:

$$k'(x) = K(x,k) \qquad \text{subject to} \quad k(x_U) = k_U - \epsilon. \tag{32}$$

Define

$$\mathcal{A}_\delta \equiv \left\{(x,k) \ : \ k \leq \frac{x}{r-\mu} \ , \ k \geq \frac{\lambda-1}{\lambda}\frac{x}{r-\mu} \ , \ x \leq x_U + \delta\right\}, \tag{33}$$

for small $\delta > 0$. By the Extension Theorem for local solutions to differential equations (see Birkhoff and Rota (1989), Chapter 6, Theorem 11), there exists a solution to (32) in $\mathcal{A}_\delta$, such that, as x decreases from an initial level of x_U, either $k_\epsilon(x)$ is unbounded at some point in A or the solution crosses the boundary of $\mathcal{A}_\delta$.

It is clear that $k_\epsilon(x)$ can never cross the lower boundary of $\mathcal{A}_\delta$, since, for (x,k) such that $k = ((\lambda-1)/\lambda)x/(r-\mu)$, $k'(x) = 0$. Suppose that $k_\epsilon(x)$ were to cross the line defined by (x,k) such that $k = x/(r-\mu)$ at some point x_0 and that $k < x/(r-\mu)$ for all $x \in (x_0, x_U)$. As $x \downarrow x_0$, $k'_\epsilon(x) \to \infty$. Taking derivatives of both sides of the differential equation in (32), substituting for $k'_\epsilon(x)$ and taking terms that explode most rapidly, one may show that, as $x \downarrow x_0$,

$$k''_\epsilon(x) \to \left(\frac{1-G}{G'}\right)^2 \frac{r-\mu}{(x_0-(r-\mu)k_\epsilon(x_0))^3} \to \infty. \tag{34}$$

But, if as $x \downarrow x_0$, $k'(x) \to \infty$ and $k''(x) \to \infty$, then for some $x \in (x_0, x_U)$, $k_\epsilon(x) > x/(r-\mu)$, contradicting the assumption that x_0 is the highest point below x_U for which $k_\epsilon(x) = x/(r-\mu)$. Hence, for all $x < x_U$, $(x, k_\epsilon(x))$ is in the interior of $\mathcal{A}_\delta$.

The set of functions, $k_\epsilon(x)$ for $\epsilon \in (0, k_U - ((\lambda-1)/\lambda)x_U/(r-\mu)$ is therefore uniformly bounded and the uniqueness of the solutions implies that the solutions for two values, ϵ_1 and ϵ_2 do not intersect for $\epsilon_1 \neq \epsilon_2$. Hence, if $\epsilon_1 < \epsilon_2$, $k_{\epsilon_1}(x) > k_{\epsilon_2}(x)$ for all $x \leq x_U$. For any x, as $\epsilon \downarrow 0$, $k_\epsilon(x)$ is increasing and bounded. Define $k_0(x)$ as the supremum over $\epsilon > 0$ of the $k_\epsilon(x)$. Since the $k_\epsilon(x)$ are uniformly bounded above this supremum is finite. Since the $k_\epsilon(x)$ do not intersect, $k_0(x)$ satisfies the differential equation in (18). Clearly, $k_0(x_U) = x_U/(r-\mu)$ so $k_0(x)$ also satisfies the boundary condition in (18). Hence, there exists a solution to the differential equation $\square$

Proof of Proposition 5

The solution to the differential equation in (20) depends on k_U in two ways: (i) directly through the fraction preceding the bracketed term in the differential equation itself, and, (ii) through the boundary condition. The solution to the differential equation will be continuous in its dependence on k_U as it appears in the differential equation itself. Hence, to study the limiting solution, we can solve instead the simpler equation:

$$\overline{x}'(k) \quad = \quad \frac{\alpha \overline{x}}{k} \left[\frac{\overline{x} - (r-\mu)k}{\overline{x}(1-\lambda) + \lambda(r-\mu)k} \right] \quad \text{subject to} \quad \overline{x}(k_U) = (r-\mu)k_U. \tag{35}$$

and then consider what happens as $k_U \to \infty$. Adopting the change of variable: $v(k) \equiv \overline{x}(k)/k$, we obtain:

$$v'(k) \quad = \quad -\frac{v}{k} \left[\frac{(1-\lambda-\alpha)v + (\lambda+\alpha)(r-\mu)}{(1-\lambda)v + \lambda(r-\mu)} \right]. \tag{36}$$

Integrating, substituting back for $v = k/x$, and taking exponentials on both sides, we obtain:

$$C\ k \quad = \quad \left[\frac{\overline{x}}{k} + \frac{(\lambda+\alpha)(r-\mu)}{1-\lambda-\alpha} \right]^{\frac{-\alpha}{(\lambda+\alpha)(1-\lambda-\alpha)}} \left(\frac{\overline{x}}{k} \right)^{\frac{-\lambda}{\lambda+\alpha}} \tag{37}$$

where C is a constant to be determined from the boundary condition. We know that $\overline{x}/k$ lies in the cone defined by the Marshallian and non-strategic triggers and hence is bounded away from zero and infinity. But as $k \to \infty$, this is consistent with equation (37) only if $C \to 0$. This then implies that the square bracketed term in (37) converges to zero as $k \to \infty$. One may conclude that the optimal trigger for the ith firm as the $k_U \to \infty$ is equal to the particularly simple form that appears in the proposition. $\square$

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FIGURE 1: VALUE OF CLAIM

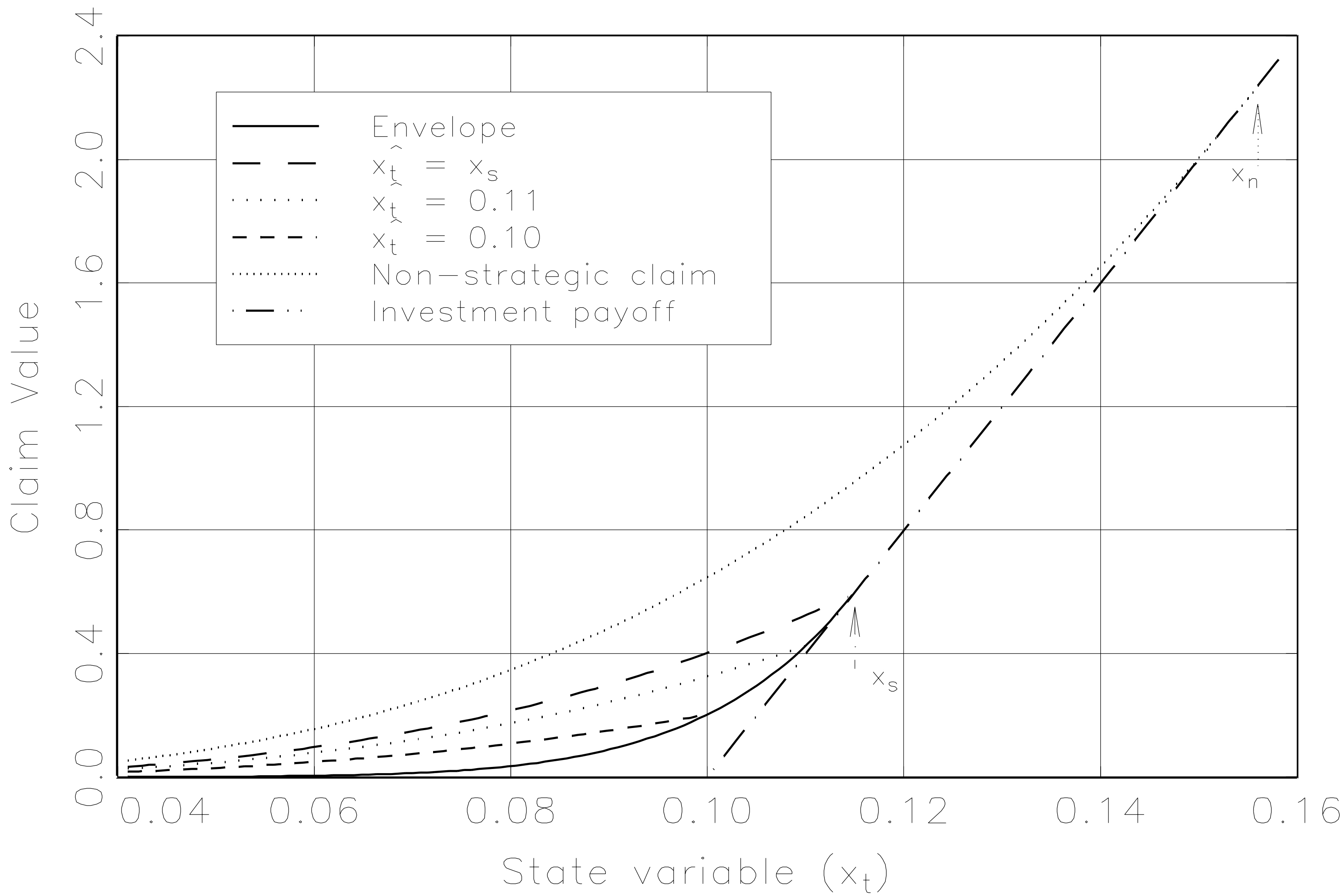

FIGURE 2: COST-TRIGGER MAPPINGS AND k_u.

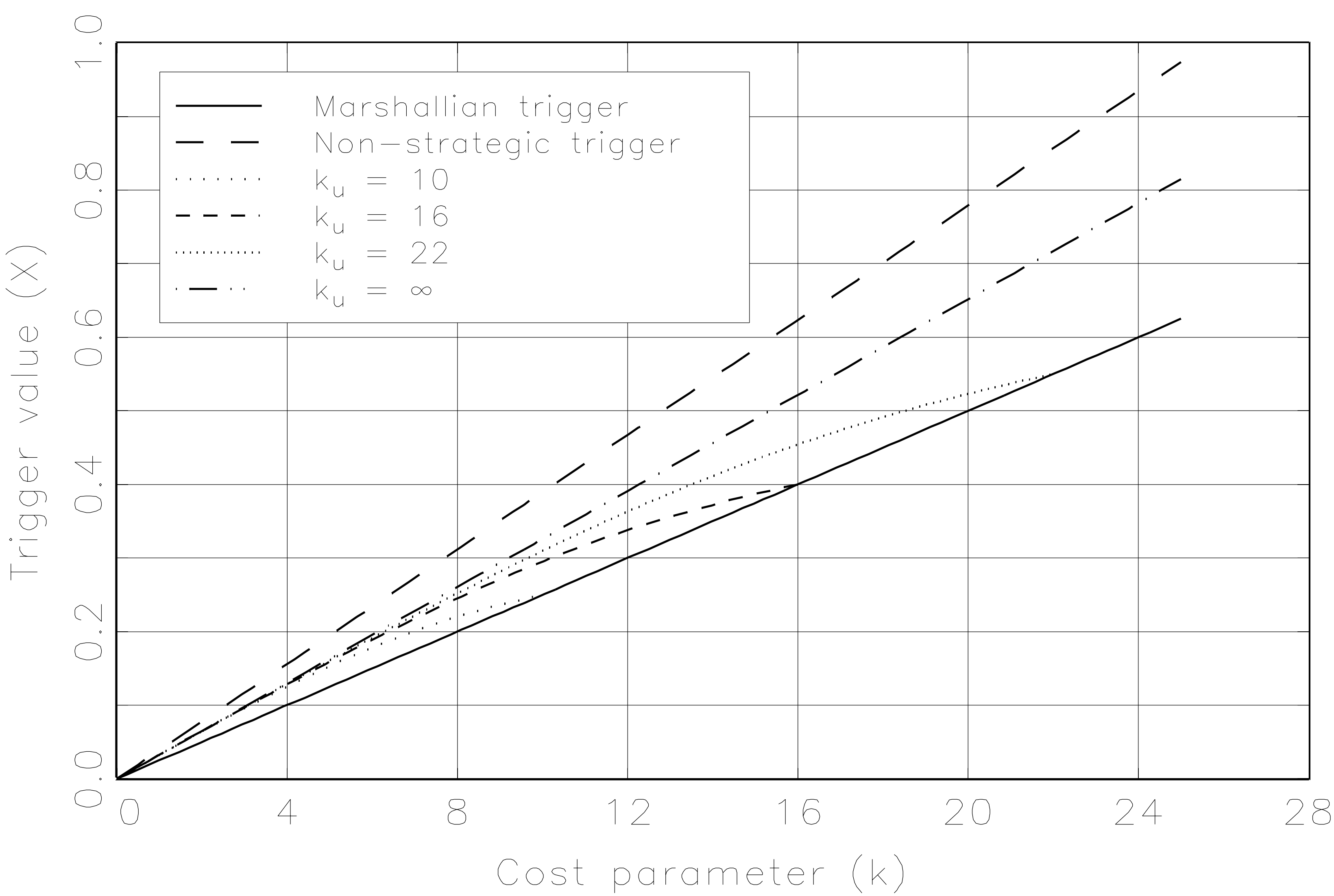

FIGURE 3: COST–TRIGGER MAPPINGS AND α

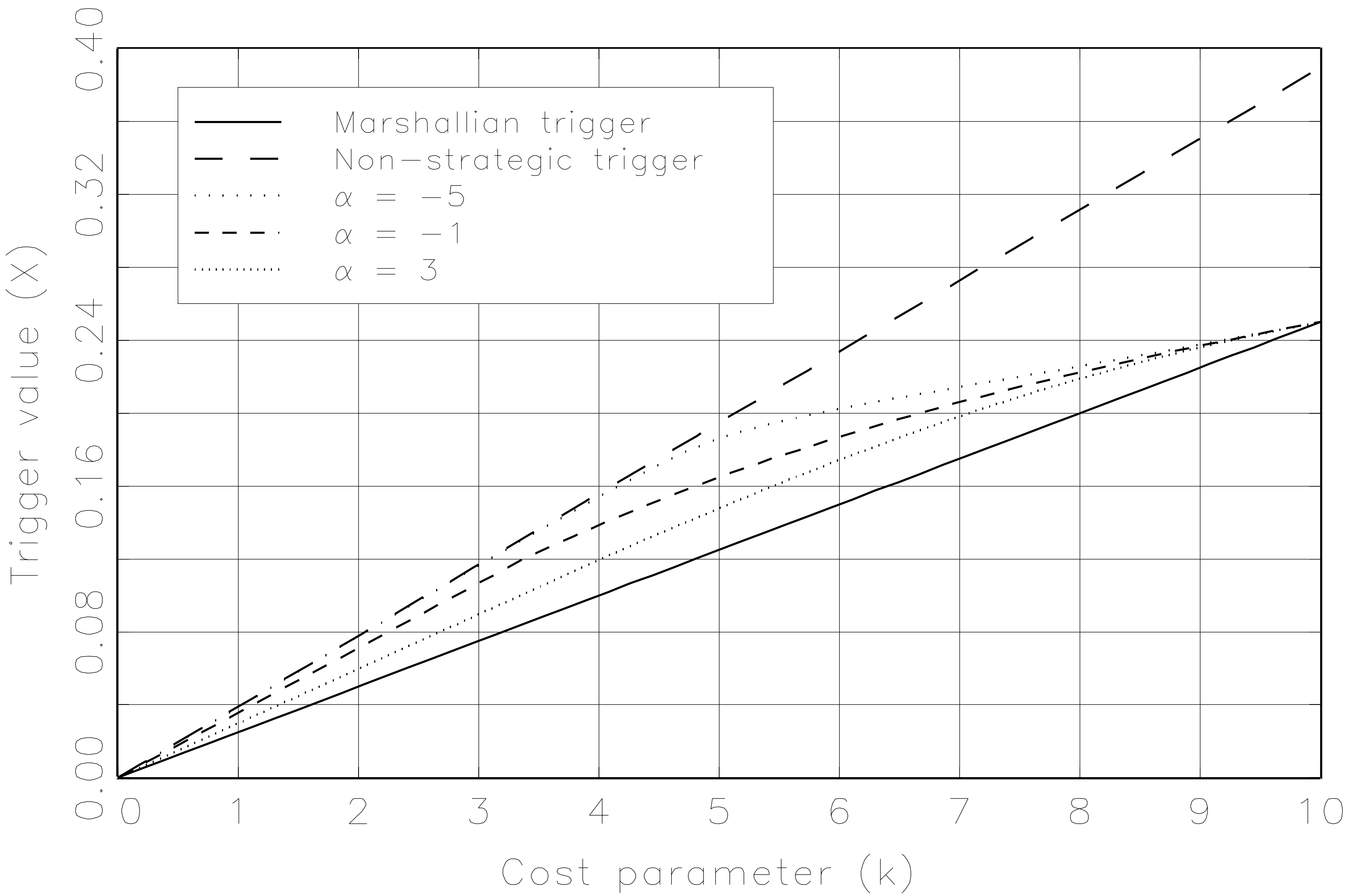

FIGURE 4: VALUE RATIOS FROM FIRM'S VIEWPOINT

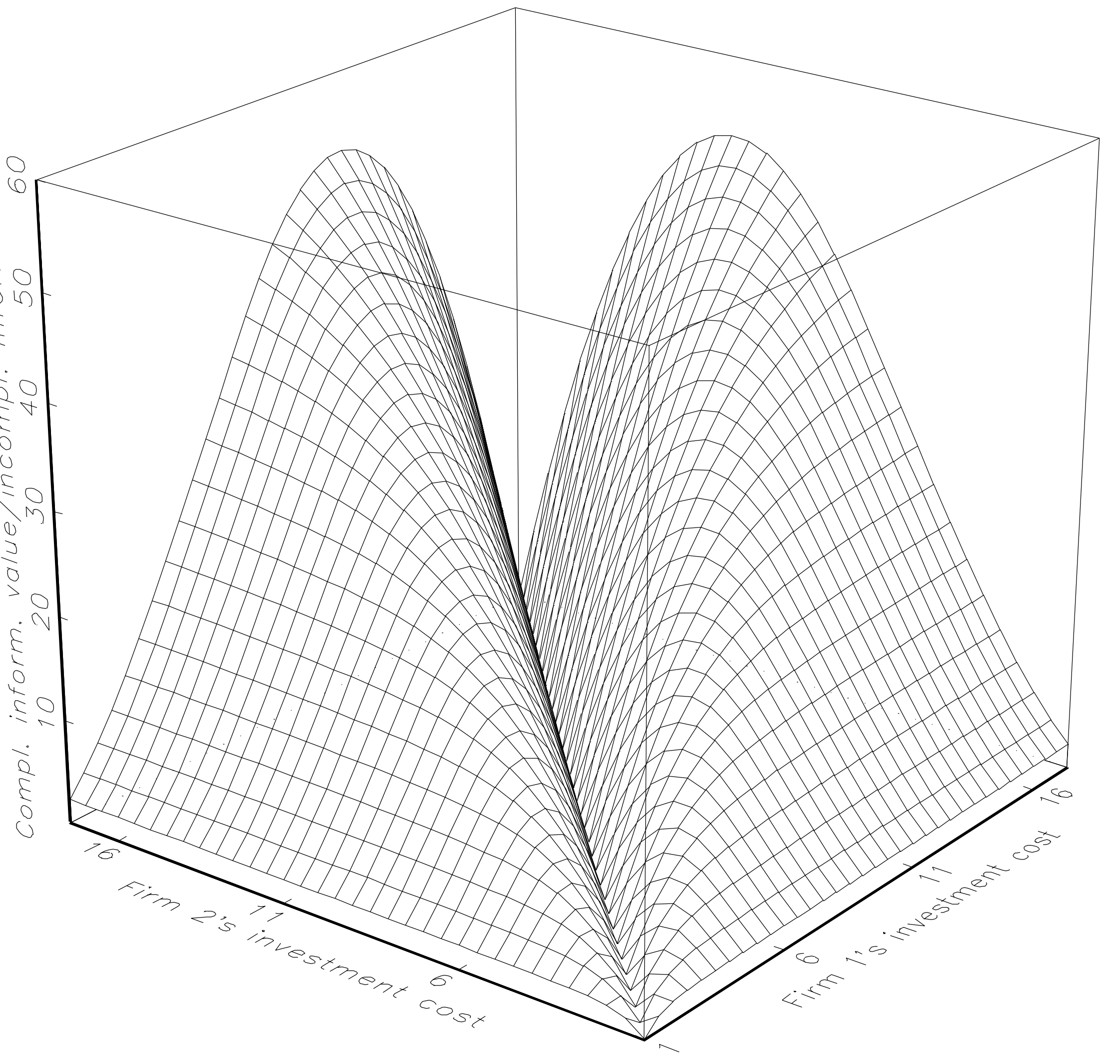

Figure 1 The claim value for firm 1, $Z_1(x_t, \hat{x}_t)$, is plotted against the potential earnings, x_t, for different levels of $\hat{x}$, the maximum level of x_t so far reached. When x_t hits a new high (i.e., $x_t = \hat{x}_t$), the firm's competitor may invest in which case Z_{1t} jumps to zero. If not, Z_{it}, shifts onto a higher sub-solution corresponding to a higher level of $\hat{x}_t$. The non-strategic firm value, $V(x_t)$, for a monopoly firm is also shown.

Figure 2 Trigger levels for real option exercise are plotted against the firm's cost of investment, k. The triggers shown include (i) the Marshallian trigger ($\overline{x}_m$) at which the investment would just break even, (ii) the monopoly firm trigger ($\overline{x}_n(k)$), and (iii) the optimal strategic trigger, $\overline{x}_s(k)$. $\overline{x}_s(k)$ is plotted for different values of k_U, the upper end of the support of the investment cost distribution of the firm's competitor.

Figure 3 The strategic trigger ($\overline{x}_s(k)$) is plotted against k for different values of the incomplete information parameter, α. When incomplete information is substantial (low α), $\overline{x}_s$ lies close to the non-strategic trigger, $\overline{x}_n$. When x_t exceeds 0.24, and neither firm has invested, each knows the other is likely to invest in the near future and fear of preemption is substantial. Thus, $\overline{x}_s(k)$ approaches $\overline{x}_m$ in this region.

Figure 4 The ratio of the total value of two firms under complete and incomplete information is plotted against the firms' investment costs, k_1 and k_2. Values are calculated from the firms' standpoints in that the incomplete information value for firm i ($i = 1, 2$) is conditional on i's investment cost, k_i. The full information value is the value of the firm with the lowest investment cost (call it firm 1), investing at the minimum of its unconstrained trigger, $\overline{x}_{1n}$, and of the Marshallian trigger of the less efficient firm, $\overline{x}_{2m}$.