Long-term profitability, mean-reversion in earnings and optimal capital structure

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Abstract

We develop a dynamic trade-off model with mean-reversion in earnings and multiple stages of lumpy investments with infrequent leverage adjustments at investment points. We provide insights on the impact of earnings dynamics with respect to long-term profitability, mean reversion speed and volatility of earnings on firm value, the dynamics of leverage and credit spreads. We also provide managerial implications regarding the optimal timing of investment and default related to model parameters and in particular the characteristics of the earnings process. Our model shows that the relation between current profitability and leverage generally follows a U-shape and thus the empirically observed negative relation between profitability and leverage is plausible for a certain range of earnings. Our analysis highlights the importance of identifying the data generating process of earnings since it has important implications on understanding firms' capital structure decisions such as financial conservatism and low leverage phenomena.

1. Introduction

Dynamic models which consider adjustments in leverage and real investments provide a vehicle for understanding the debt conservatism puzzle and why financial flexibility and earnings and cash flow volatility are important for firms (see Graham and Harvey, 2001).[1] Strebulaev (2007) concludes that a proper study of the evolution of capital structure requires a model that combines both dynamic capital structure decisions and real investment. While theoretical models combining capital structure and real investment decisions are well-developed (e.g., Hennessy and Whited (2005), Titman and Tsyplakov (2003), and Hackbarth and Mauer (2012)), a better link between the nature of shocks (i.e. temporary or permanent) on firm's cash flows and corporate financial decisions still appears a challenging task.

In this paper we develop a model where earnings follow an arithmetic mean-reverting (AMR) process. The firm has multiple stages of investment options and makes infrequent leverage adjustments at investment points. The literature studying the effect of mean reversion on the investment and disinvestment decisions of firms has used mostly the geometric mean-reverting (GMR) process (Sarkar, 2003; Tserkrekos, 2010; Metcalfe and Hassett, 1995). The same type of process is employed also in Sarkar and Zapatero (2003), where they reformulate Leland (1994) trade-off model incorporating mean reversion in the corporate earnings process and study debt financing without investment decisions. With GMR, however, cash flows can never be negative. Our model differs from these studies in that cash flows can be negative, consistent with the empirical evidence.[2] We also incorporate multiple investment and financing stages and alternative priority rules for debt thus casting Hackbarth and Mauer (2012) model within a mean reversion framework.

Allowing for negative earnings might increase the importance of financial flexibility. For example, for any given level of leverage, the expected advantage to debt is likely to be of lower value with more frequently lower cash flows. Also the difference between investment and default thresholds and the implied option values in equity and debt valuation with GMR and AMR are likely to have important implications especially for multi-staged projects, capital expansion and valuation of debt-financed projects. Our focus on earnings dynamics is also motivated by empirical studies that show that not properly taking into account earnings

[1] DeAngelo et al. (2018) shows that firms retain financial flexibility by adjusting leverage.

[2] Gorbenko and Strebulaev (2010) show that between 1987 and 2005, approximately 17% to 25% of all quarterly cash flows for the full COMPUSTAT sample are negative.

or cash flow dynamics may lead to misleading findings.[3] By focusing on this specific AMR process, we provide new results on the impact of the earnings dynamics and in particular long-term profitability, mean reversion speed and volatility on firm value, and the dynamics of leverage and credit spreads. We show also how to estimate the parameters of the process and show that this type of process characterizes a vast majority (about 60%) of the universe of non-financial, non-regulated firms in the COMPUSTAT database with 40 consecutive quarterly observations (that allows for reasonable accuracy in estimation of model parameters).

One of our theoretical contributions is that we challenge the traditional interpretation of the reasons between the negative relation between profitability and leverage which is often suggested to be against trade-off models and in favour of pecking order theory (see Shyam-Sunder and Myers (1999) and follow on work). We show that for firms characterized with mean reverting earnings, the relation between earnings and profitability could be negative. Our framework thus extends this insight first discussed in Sarkar and Zapatero (2003) in the presence of multiple investment stages and also allowing for negative earnings. However, compared to their results we show that more generally leverage has a U-shape relationship with profitability. The U-shape is driven by the alternative magnitudes of increase between equity and debt values where debt value incremental improvement (relative to equity) is higher when profitability is high. Strebulaev (2007) provides an alternative explanation for the negative relation between leverage and earnings, showing that this effect may also be well captured in a dynamic trade-off framework where firms make infrequent adjustments in leverage. Danis et al. (2014) show that this empirical regularity can be explained by a dynamic trade-off model where firms make infrequent capital structure adjustment (so called "inaction" models).[4] In addition, our analysis predicts that leverage is decreasing in earnings volatility and growth options (i.e., the expansion factor of future revenues) and positively related to the mean reversion speed and long-term profitability. During times of "lumpy"

[3] One such example is leverage mean reversion (e.g. found in Fama and French, 2002 and Flannery and Rangan 2006). Chen and Zhao (2007) and Chang and Dasgupta (2009) also discuss hazards of not properly employing earnings dynamics.

[4] While the leverage ratios is negative related to profitability in the inaction region, they show that at times when firms are at their optimal level of leverage, the cross-sectional correlation between profitability and leverage is positive. This implies that when firms actually take actions to adjust leverage then indeed higher profitability is positively related with leverage. Our analysis provides an alternative explanation that relates to the stochastic process dynamics of earnings within a multistage investment setting. This implies that further empirical investigation may be needed to distinguish among alternative explanations by considering earnings dynamics and firms' investment decisions.

investment and adjustments in leverage, we show that leverage ratios increase relative to earlier levels for firms with higher volatility in earnings, and higher growth option expansion. Leverage decreases relative to previous levels for firms with stronger mean reversion in earnings and larger long-term profitability levels and has a U-shape with respect to the level of initial earnings.

Allowing for transitory shocks has been traditionally part of the analysis of optimal cash management policies (and not for analysing optimal capital structure). For example, Décamps et al. (2016) focus on liquidity and risk management policies for firms facing financing frictions and are subject to permanent and temporary cash flow shocks. More relatedly to our framework, Gorbenko and Strebulaev (2010) provide a contingent claim trade-off model with both temporary and permanent shocks. The temporary component of the shocks is driven by Poisson jump shocks that arrive in discrete time and then fade in expectation over time so that earnings mean-revert to permanent cash low levels. These transitory shocks are characterized by their longevity (or speed of mean-reversion), their arrival intensity, and their magnitude. In their model equity holders benefit disproportionately from positive shocks, whereas debtholders bear higher costs due to negative shocks. Debtholders thus demand ex ante compensation, which reduces equity holders' desire to rely on debt, resulting in "debt conservatism" which brings theoretical predictions closer to empirical evidence. In line with this prediction, in our model a lower speed of mean reversion-which implies that earnings shocks can be at distance from long-term profitability for longer periods of time-also results in lower leverage levels. We show however that the mechanisms which drive the positive relation between leverage and the speed of mean reversion differ depending on the level of long-term profitability. When long-term profitability is low, higher leverage is driven by increases in debt value and decreases in equity value. However, when long-term profitability is high both equity and debt increase, albeit the latter increases at a higher rate. Furthermore, in our model "debt-conservatism" can be further exasperated when firms face moderate or low levels of long-term profitability.

Our analysis is also more broadly related with work focusing on investment dynamics. Hennessy and Whited (2005) develop a dynamic trade-off model with endogenous choice of leverage, dividends and real gradual investments. Instead, our model is focusing on lumpy investments. Dudley (2012) considers the interaction of investment and financing focusing on time-to-build and large investment outlays.

In addition to the empirical predictions, our framework provides a number of managerial implications regarding optimal investment timing and default decisions. In particular, we show that optimal investment is delayed for firms with earnings which are more volatile and have low levels of long-term profitability or firms with low levels of expansion (growth) options. The impact of mean reversion hinges upon the level of long-term profitability. When long-term profitability is high, an elevated degree of mean reversion accelerates investment. On the other hand, investment is postponed when profits mean-revert faster to low long-term profitability levels. Optimal default is delayed for firms with higher earnings volatility, higher levels of growth options and long-term profitability. The optimal default threshold exhibits a U-shape with respect to mean reversion speed when long-term profitability is high and is decreasing with mean reversion speed when long-term profitability is low.

Our paper is organized as follows. Section 2 describes the model, Section 3 presents the numerical sensitivity results and summarizes the model predictions. Section 4 shows the estimation of earnings process and applies it to US data, while Section 5 concludes. Appendix 1 shows the notation of various variables of the theoretical model, Appendix 2 the derivation of the homogeneous differential equation solution, Appendix 3 the derivation of solution for the basic and general claims involving two boundaries within a mean-reverting framework, and Appendix 4 the proofs for security and firm values presented in the main text. Appendix 5 shows the details of the estimation procedure for the parameters of the mean reverting process and the empirical test employed for stationarity.

2. The model

2.1. Model assumptions

We model a firm with existing assets generating net cash flow or earnings (EBIT+depreciation) x. The earnings stream x follows an arithmetic mean-reverting process as follows:

$$dx = q(\theta - x)dt + \sigma dz \tag{1}$$

where q defines the mean reversion speed, θ defines the long-term mean to which earnings revert, σ the project earnings volatility and dz is the increment to a standard Brownian Motion process. The firm has a growth opportunity to increase earnings to a level $e\,x$ at an optimal time. The firm selects an optimal level of perpetual debt $Db(x)$ at time zero (stage 1) with a promised (coupon) payment R_0.and pays corporate taxes at a constant rate τ with a full-loss offset scheme.

The bankruptcy trigger x_b is endogenously and optimally chosen by equity holders by maximizing equity value. When earnings x drop to the low threshold level x_b then the firm goes bankrupt and the original debt holders take over and obtain the firm's unlevered assets $Ub(x)$ net of proportional bankruptcy costs *b,* 0<*b*<1. On the other hand, if earnings rise to a high level x_I then the firm makes a capital (growth) investment *I* and expands earnings by *e* > 1, thus earnings after investment become $v = e\,x$. The optimal timing for investment is chosen to maximize equity holders market value of equity ("second-best investment"). Using Ito's lemma[5], post investment earnings also follow a mean-reverting process of the following form:

$$dv = q(e\theta - v)dt + e\sigma dz \tag{2}$$

Thus, after investment, earnings follows an AMR process with standard deviation $e\sigma$ and long term mean $e\theta$.

New investment can be financed by additional perpetual debt $Da(x)$ with coupon R_1. Post investment, equity holders select the earnings level v_L which triggers bankruptcy. In the event of bankruptcy priority rules define the amount of unlevered assets obtained by original and subsequent debt holders. Similarly to Hackbarth and Mauer (2012) we allow for commonly observed priority rules which include absolute priority of original debt, pari-passu (equal priority) and absolute priority for subsequent debt holders.

The optimization of capital structure is performed by selecting the initial coupon R_0 and subsequent coupon level R_1 jointly with optimally chosen investment and default levels. R_0 is chosen to maximize initial firm value (equity plus initial debt financing obtained) while R_1 is chosen to maximize equity plus the proceeds from the new debt issue. This amounts to "second-best financing" as suggested in Hackbarth and Mauer (2012). We do not focus on agency considerations in this paper and thus do not consider a comparison with a "first-best" optimization for either the selection of investment timing and/or financing. "First-best"

[5] One could assume that *e* is stochastic and follows a Wiener process, that is *de=sdz.* In this case, (2) becomes a more involved expression..

investment timing would be the one that caters for debt holders value by maximizing firm (instead of equity only) value and "first-best financing" allows that the choice of R_1 caters for the dilution effects on initial debt (see Hackbarth and Mauer, 2012 for further details).

2.2. Security and firm valuation after investment

2.2.1. Equity and unlevered assets after investment

Equity value after investment is equal to:

$$Ea(v) = Ea_p(v) - Ea_p(v_L)\left(\frac{P_1(v)}{P_1(v_L)}\right) \tag{3}$$

where $v = ex$ are expanded cash flows following investment and

$$Ea_p(v) = \left(\frac{1}{q+r}v + \frac{q\theta^*}{r(q+r)} - \frac{R_0+R_1}{r}\right)(1-\tau) \tag{4}$$

with $\theta^* = e\theta$.

In equation (3) the term $P_1(\cdot)$ is defined in equation (5a) below. Equation (5b) also defines $P_2(\cdot)$ that will be used in subsequent equations for the value of securities.

$$P_1(x) = e^{\frac{1}{4}\left(\frac{(x-\theta)\sqrt{2q}}{\sigma}\right)^2} D_\nu\left(\frac{(x-\theta)\sqrt{2q}}{\sigma}\right) \tag{5a}$$

$$P_2(x) = e^{\frac{1}{4}\left(\frac{(x-\theta)\sqrt{2q}}{\sigma}\right)^2} D_\nu\left(-\frac{(x-\theta)\sqrt{2q}}{\sigma}\right). \tag{5b}$$

where $D_\nu(z) = \frac{1}{2^\xi\sqrt{\pi}}\left[\cos(\xi\pi)\,\Gamma\left(\frac{1}{2}-\xi\right)\,y_1(a,z) - \sqrt{2}\sin(\xi\pi)\,\Gamma(1-\xi)\,y_2(a,z)\right]$ (6)

$$z = \frac{x-\theta}{\bar{\sigma}}, \bar{\sigma} = \sigma/\sqrt{2q}$$

$$a = -\nu - \frac{1}{2}, \nu = -\frac{r}{q} < 0$$

$$\xi = \frac{1}{2}a + \frac{1}{4}$$

$\Gamma(\cdot)$= is the Gamma function

$$y_1(a,z) = e^{-\frac{z^2}{4}}\ {}_1F_1\left(\frac{1}{2}a + \frac{1}{4}; \frac{1}{2}; \frac{z^2}{2}\right)$$

$$y_2(a,z) = z\,e^{-\frac{z^2}{4}}\ {}_1F_1\left(\frac{1}{2}a + \frac{3}{4}; \frac{3}{2}; \frac{z^2}{2}\right)$$

In the above ${}_1F_1\ (\alpha;\beta;z) = M(\alpha;\beta;z)$ is the confluent hypergeometric function (see Abramowitz and Stegun, 1972). The Gamma function is defined as follows:

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$$

where the integral converges for n > 0. Note that $\Gamma(n+1) = n\Gamma(n)$, so for integer *n* this function coincides with the factorial function, that is, $\Gamma(n+1) = n!$.

Note that in equation (3) the term $Q(v) = \frac{P_1(v)}{P_1(v_L)}$ can be interpreted as the value of a basic claim which pays one dollar when v_L is reached from above from v.

The value of unlevered assets after investment is:

$$Ua(v) = \left[\frac{1}{q+r} v + \frac{q\theta^*}{r(q+r)}\right](1-\tau) \tag{7}$$

In expression (7) the term $\frac{1}{q+r} v$ represents the transitory component and the constant $\frac{q\theta^*}{r(q+r)}$ is a permanent component. Note that when q = 0, then expression (7) simplifies to $v(1\text{-}\tau)/r$, which is the value for an arithmetic process with zero drift. When the earnings level changes, the value $Ua(v)$ is affected only by the transitory part. Since the transitory part is a decreasing function of the speed of reversion q, if mean reversion becomes stronger (q increases), the transitory part becomes less important and if q goes to infinity, it disappears. To avoid negative liquidation values for initial debt holders at bankruptcy we ensure that $Ua(v)$, as well as $Ub(x)$, do not drop below zero at the bankruptcy thresholds (see appendix equations A29 and A44).

2.2.2. Debt and firm value after investment

Debt value after investment for the initial debt issued at time zero $Da_0(v)$ and the second debt issued at the investment trigger $Da_1(v)$ are given by:

$$Da_i(v) = \frac{R_i}{r} + \left(Da_i(v_L) - \frac{R_i}{r}\right)\left(\frac{P_1(v)}{P_1(v_L)}\right) \tag{8}$$

where $Da_i(v_L)$ depends on the priority structure. In the case of equal priority of the two debt issuers, liquidation proceeds are shared depending on the scale of payments:

$$\beta_0 = \frac{R_0}{R_0+R_1}, \qquad \beta_1 = 1-\beta_0 = \frac{R_1}{R_0+R_1}$$

Thus, with equal priority the boundary condition for debt becomes:

$$Da_i(v_L) = \beta_i\ (1-b)\ Ua(v_L) \tag{9}$$

In the case the first lender has secured priority to other creditors ("me-first" for initial debt) then the boundary conditions become:

$$Da_0(v_L) = min\ \left[(1-b)\ Ua(v_L), \frac{R_0}{r}\right] \tag{10a}$$

$$Da_1(v_L) = (1-b)\ Ua(v_L) - Da_0(v_L) \tag{10b}$$

In the case that second lender have secured priority to other creditors then the boundary conditions become:

$$Da_1(v_L) = min\ \left[(1-b)\ Ua(v_L), \frac{R_1}{r}\right] \tag{11a}$$

$$Da_0(v_L) = (1-b)\ Ua(v_L) - Da_1(v_L) \tag{11b}$$

Firm value after investment is then given by the sum of equity plus debt values after investment:

$$Fa(v) = Ea(v) + Da_0(v) + Da_1(v) \tag{12a}$$

Replacing equation (3) and equations (8) for $Ea(v), Da_0(v)$ and $Da_1(v)$ in equation (12a) above we obtain an alternative characterization of firm value as follows:

$$Fa(v) = Ua(v) + TBa(v) - BCa(v) \tag{12b}$$

where $Ua(v)$ is given in equation (7) and $TBa(v), BCa(v)$ are defined as follows:

$TBa(v) = \left(\frac{R_0+R_1}{r}\right)\tau - \left(\frac{R_0+R_1}{r}\right)\tau\left(\frac{P_1(v)}{P_1(v_L)}\right)$, $BCa(v) = bUa(v_L)\left(\frac{P_1(v)}{P_1(v_L)}\right)$. We also define $NBa(x) = TBa(v) - BCa(v)$ as a summary measure of the net benefits of debt after investment.

2.3. Valuation before investment

2.3.1. Equity and unlevered value before investment

Equity value before investment $Eb(x)$ is given by:

$$Eb(x) = \left(Ea(e\ x_I) - I + Da_1(e\ x_I) - Eb_p(x_I)\right)J(x) - Eb_p(x_b)\ L(x) + Eb_p(x) \tag{13}$$

where $Eb_p(x) = \left(\frac{1}{q+r}x + \frac{q\theta}{r(q+r)} - \frac{R_0}{r}\right)(1-\tau)$.

$J(x)$ in equation (13) defines the value of a basic claim that pays one dollar if x hits trigger x_I and zero when it hits x_b. Similarly, we define a basic claim $L(x)$ that pays one dollar if x hits trigger x_b and zero when it hits x_I. The solutions to these basic claims are as follows (see Appendix 2):

$$J(x) = \frac{P_2(x_b)}{D(x_I,x_b)} P_1(x) - \frac{P_1(x_b)}{D(x_I,x_b)} P_2(x) \tag{14}$$

$$L(x) = -\frac{P_2(x_I)}{D(x_I,x_b)} P_1(x) + \frac{P_1(x_I)}{D(x_I,x_b)} P_2(x)$$

where $D(x_I, x_b) = P_1(x_I)P_2(x_b) - P_1(x_b)P_2(x_I)$.

The value of unlevered assets before investment is given by:

$$Ub(x) = \left[\frac{1}{q+r}x + \frac{q\theta}{r(q+r)}\right](1-\tau) \tag{15}$$

2.3.2. Debt and firm value before investment

Initial (t = 0) debt value is given by:

$$Db(x) = \frac{R_0}{r} + \left(Da_0(x_I) - \frac{R_0}{r}\right)J(x) + \left((1-b)\,Ub(x_b) - \frac{R_0}{r}\right)L(x) \tag{16}$$

where equation for $Da_0(x)$ is given in equation (8) and $Ub(x)$ in equation (15).

Thus, firm value before investment is the sum of equity plus debt before investment:

$$Fb(x) = Eb(x) + Db(x) \tag{17a}$$

Replacing equation (13) for $Eb(x)$ and equation (16) for $Db(x)$ we obtain the following breakdown of firm value at t = 0:

$$Fb(x) = UB(x) + Ua(v_I)\,J(x) + TBb(x) + TBa(v_I)\,J(x) - BCb(x) - BCa(v_I)J(x) - I\,J(x) \tag{17b}$$

where $UB(x) = Ub(x) - Ub(x_I)J(x)$ with $Ub(\cdot)$ given in equation (15), $TBb(x) = \frac{\tau R_0}{r} - \frac{\tau R_0}{r}J(x)$-$\frac{\tau R_0}{r}L(x)$ and $BCb(x) = bUb(x_b)L(x)$. We also define the net benefits of debt at t = 0 as $NBb(x) = TBb(x) - BCb(x)$.

2.4. Optimal investment, default and capital structure

In this section we describe smooth pasting (optimality) conditions. First, we demand that the derivative of equity after investment at v_L should be zero to ensure that equity holders choose the bankruptcy trigger optimally following investment. This implies the condition:

$$Ea'(v_L) = 0 \,. \tag{18}$$

Note that the optimality condition in equation (18) can be stated in terms of the underlying x with the condition $\tilde{E}a'(x_L) = 0$ where $\tilde{E}a(\cdot)$ is equation defined in equation (3) evaluated at $v = e\,x$. Similarly, we demand that the derivative of equity value before investment should be zero at bankruptcy trigger x_b:

$$Eb'(x_b) = 0. \tag{19}$$

We use "second-best investment" optimization for the investment trigger x_I which accounts for raising the optimal new level of debt financing, however, it does not account for the effect of investment on existing debt holders. This translates to:

$$Eb'(x_I) = \tilde{E}a'(x_I) + \tilde{D}a_1{}'(x_I) \tag{20}$$

Note that under "first-best investment" optimization (not used in our subsequent analysis) equity holders would take into account the best interest of debt issuers by optimizing firm value. This would imply the following condition $Fb'(x_I) = \tilde{F}a'(x_I)$ where $\tilde{F}a'(x_I)$ is equation (12) replacing $v = e\,x$. "First-best investment" would be useful for analysing agency issues which is not the goal of this paper.

The optimal capital structure is selected by performing a dense grid search for both the initial and subsequent coupon levels such that R_0 and R_1 maximize firm value at t = 0 (see equation, 17a) by applying optimally chosen investment and default levels (see equations 18, 19 and 20). This optimization identifies the initial and subsequent debt levels in the firm's capital structure. In our solution R_0 is chosen to maximize initial firm value (equity plus initial debt financing) while R_1 (see equation 20) is chosen to maximize equity plus the new debt proceeds ("second-best financing"). We do not focus on "first-best" optimization for either the selection of investment timing and/or financing.

3. Model predictions

In this section we provide numerical sensitivity results with respect to the relevant model parameters. We provide implications relating the effect of the volatility of earnings (σ), mean reversion speed (q), long-term profitability (θ), earnings level (x), growth option expansion factor (e) and investment cost (I). We explore the impact of these variables on firm value, leverage ratio levels and the change in leverage ratios, and the credit spreads. We also explore the effect of alternative priority rules for debt. In the last part of this section we focus on the development of testable empirical predictions relating to leverage ratios along the time and cross section dimensions.

Our base case parameters are as follows. We use a normalized level of current earnings at the level x = 1. Following Hackbarth and Mauer (2012) we use a risk-free rate of r = 0.06, a tax rate τ = 0.15. For the growth option we also follow the same study and use $e = 2$ and an investment cost I = 10 (i.e., the cost of investment is ten times the current earnings level as used in Hackbarth and Mauer, 2012). We use proportional bankruptcy costs $b = 0.5$ as in Leland (1994). For the mean-reverting stochastic process parameters we follow Sarkar and Zapatero (2003) and use σ = 0.4, mean reversion speed q = 0.1 and long-term mean θ = 1.

For all subsequently reported results we report sensitivity until $x < x_I$ remains valid and that the value of unlevered assets at default thresholds is never negative. For the latter we check that in reported simulations the conditions $v_L < v_A$ and $x_B < x_A$ are satisfied (see discussion following equations A29 and A44 in the Appendix). Where necessary to show a particular direction we provide more densely applied sensitivity results to a particular parameter.

Table 1 provides sensitivity results with respect to the volatility of earnings σ. In Panel A, consistent with a real options explanation we observe that an increase in volatility results in a delay of the option to invest (x_I increases) and a delay in default decisions (v_L and x_b thresholds decrease). In Panel B we observe that a higher earnings volatility has a (minor) U-shape effect on firm value (Fb(x)), decreases the value of unlevered assets (Ub(x)), results in a lower leverage ratio at t = 0 (Lev_b), lower net benefits of debt (NBb(x)) and decreases the expected present value of investment costs (Inv_b). The latter effect implies that investment becomes less likely to occur. As expected (see also Sarkar and Zapatero, 2003), leverage decreases with volatility. Despite the decrease in leverage, credit spreads increase with σ. Higher volatility also reduces leverage ratios and has a positive impact on credit spreads at the investment trigger. With respect to leverage dynamics, we find that there is an increase in leverage relative to prior levels. At higher volatility, investment is triggered at a higher revenue level which enables the firm to move to higher levels of leverage. In short, firms with higher volatility of earnings would delay investment but would exhibit larger increases in leverage when expansion and leverage adjustments take place.

We have also explored whether the above sensitivity results change when long-term profitability is different (results not shown for brevity). We find that when long-term profitability is high an increase in volatility reduces firm value. This is intuitive since higher volatility increases the likelihood of moving away from highly valuable future prospects. The opposite result is obtained when long-term profitability is low in which case firm value becomes strictly increases with volatility. Despite these differences with our base case, all other results remain: the investment and default is delayed, leverage decreases, credit spreads increase and changes in leverage at the investment trigger relative to previous levels increase..

[Insert Table 1 here]

Table 2 provides sensitivity results with respect to mean reversion speed q. We observe that for the base case levels of long-term profitability an increase in mean reversion speed q decreases x_I which accelerates investment (see Panel A) and increases firm value (see Panel B). The acceleration of investment (as also indicated by the higher $Inv_b(x)$) makes intuitive sense: using our base case the growth option is quite attractive[6] implying that a higher speed makes it even more likely to remain profitable once investment is triggered--this further increases the moneyness of the option and leads to an acceleration of investment. This intuition is confirmed since when long-term profitability is low we observe the opposite result (see case θ = 0.5). Indeed, when long-term profitability is low (θ = 0.5) investment is delayed when mean reversion speed q increases and results in lower firm values. The results of the lower long-term profitability case also show that an increase in the speed of mean reversion results in more conservative optimal coupon levels. In contrast, for the base case (higher long term profitability) coupons increase with the speed of mean reversion. However, interestingly the leverage ratio is increasing in the persistence of shocks around long-term profitability (i.e., with higher q), irrespective of the long-term profitability level. Gorbenko and Strebulaev (2010) have also shown that more persistence shocks results in higher

[6] In order to get a sense of the growth option attractiveness (moneyness) for our the base case parameters one should note from equation (25) that if earnings at the investment trigger were to remain around their t = 0 value of x =1 then Ua(v)=28.33>I=10. This implies that even looking at the value of unlevered assets alone (excluding the net benefits of debt) the growth option at the current level of earnings is attractive. We also explore a case where long-term profitability implied by growth options is less attractive and discuss some differences in the generated predictions. Importantly, our main cross-sectional and time-series predictions regarding leverage ratios remain unaffected by growth option moneyness, albeit for different reasons as explained in the main text.

leverage ratios, however our model highlights the different mechanisms driving this result between high vs low long-term profitability. In both the case of high and low long-term profitability debt values increase with the persistence of shocks (q) since a higher persistence of earnings results in lower overall uncertainty which benefits debt holders. However, it should be noted that at high long-term profitability equity values also increase with the speed of mean reversion, albeit at a lower rate compared to debt while with low long-term profitability equity values decrease. Also, with respect to leverage dynamics, for both cases of θ, we observe that an increase in the speed of mean reversion results in a decrease in leverage at investment relative to previous levels.

Credit spreads decrease as the speed of mean reversion increases for both the base case of long-term profitability θ and the case where θ is low, albeit for different reasons. For the case of low θ, this is driven by the more conservative coupon levels and delayed default, whereas in the case of high θ it is driven by the reduction in risk since a higher speed to high long-term profitability provides assurance that the firm will more likely remain away from default.

The observed slight U-shape effect of firm value for higher speed of mean reversion of the base case can be more clearly interpreted when contrasted with the case of low θ where firm values are strictly decreasing in the speed of mean reversion. For θ values higher than the base case (i.e., very positive long-term prospects), an increase in the speed of mean reversion would result in a strictly increasing effect on firm value with respect to the speed of mean reversion. Thus, our base case parameters which correspond to an "average" profitability result in two opposing effects creating a U-shape. Similarly, in panel A, we also find that v_L and x_b have a U-shape relationship with mean reversion speed q for the base case. It is thus expected that with high θ the default threshold would be increasing in q. Our results regarding the default trigger can also be contrasted with Gorbenko and Strebulaev (2010). They find that an increase in the persistence of shocks results in an acceleration of default (see, p. 2604) explaining that adverse shocks of longer duration imply that the firm continues to be in financial distress for longer. Similarly, in our model when long-term profitability is low and the speed of mean reversion is high, this implies that shocks stay persistently at low levels for longer periods of time; for this case we also find that default is accelerated. Surprisingly, however, we find that initial default thresholds may be accelerated in our model even when speed of mean reversion is high towards a high long-term profitability, i.e., even when profitability is expected to stay at high levels. This occurs because the firm utilizes this potential by taking much more debt initially (see high initial coupon levels in panel A for the base case θ and high speed of mean reversion q).

[Insert Table 2 here]

Table 3 shows that an increase in long-term profitability (θ) accelerates investment as indicated by the lower x_I (see panel A) and the higher expected investment costs ($Inv_b(x)$) (see panel B). A higher long-term profitability creates a U-shape with respect to default thresholds v_L and x_b. As expected, higher long-term profitability increases firm value, increases the initial leverage ratio and reduces credit spreads. The positive relationship between θ and the leverage ratio and the negative between θ and credit spreads holds also at the investment threshold. Interestingly, the results show that leverage decreases relative to previous levels when long-term profitability is higher. This is driven by the earlier threshold where investment takes place when long-term profitability is high thus not allowing for high leverage levels at the investment trigger compared to earlier levels.

[Insert Table 3 here]

As we have seen earlier (Table 3), long-term profitability had a positive effect on leverage ratios. This is not necessarily the case with respect to current profitability levels as we show below. Table 4 shows a negative relation between leverage and profitability levels x (see panel B) for the base case parameters exists for a wide range of x values. For high value of x the relation becomes positive at high x, i.e., there is an overall U-shape relation of leverage with x[7].

[Insert Table 4 here]

The negative relation between current earnings levels and leverage observed for low x values is driven by the higher positive impact of an improvement in profit levels on equity compared to debt value when earnings are low.[8] Thus, while debt values and the net benefits of debt (NBb(x)) improve with x (see panel B), the positive effect on equity values outweighs the effect on debt values and leads to a reduction in leverage ratios. This latter result was discussed in Sarkar and Zapatero (2003) within a GMR setting and is shown to also hold in our dynamic multistage framework with an AMR process. However, we show that for high current profitability levels the result may be reversed and debt values may increase more

[7] We have conducted additional sensitivity analysis and the U-shape is robust to alternative parameterizations such as different long-term profitability or different mean-reversion speeds.
[8] Equity value is not reported but can be calculated as the difference between firm value and debt value.

rapidly compared to equity (hence resulting in an increase in the leverage ratio). This is intuitive since at high profitability levels x the significant reduction in risk benefits debt holders more compared with equity holders who have incrementally more to gain under a riskier environment due to more valuable options both on the upside due to investment options and on the downside protection due to limited liability.

Panel A also shows that the investment trigger increases. However since the increase is not as significant compared to incremental increase in x investment is actually accelerated. The acceleration of investment can also be seen by the increase in the expected value of investment costs (Inv_b(x)) (see Panel B). In Panel A, we also find that x_b increases with x while v_L remains rather flat (only slightly increases at higher x levels).

In Panel B we observe (as expected) that higher profitability increases firm and debt values, the net benefits of debt and the expected costs of the investment option (Inv_b). Interestingly, a higher revenue level x creates a similar to leverage slight U-shape for credit spreads at t = 0. At the investment trigger both leverage and credit spreads decrease, albeit slightly. We thus observe that the change in leverage at the investment trigger relative to the initial level follows a U-shape (similarly to the leverage ratio at t = 0).

Table 5 shows the impact of the growth option expansion factor. As expected, a higher growth expansion factor accelerates investment, delays default and improves firm value. Despite the increase in the net benefits of debt, the leverage ratio at t = 0 decreases which is in agreement with the well-documented debt conservatism for firms with growth options (see Graham and Harvey, 2001).[9] On the other hand, the leverage ratio increases at the investment trigger and there is a more notable increase in credit spreads. Finally, relating to the dynamics of leverage we find that leverage exhibits an increase relative to earlier levels when the expansion factor is higher.

[Insert Table 5 here]

Opposite directional effects to the one discussed above for the expansion factor are observed with respect to capital investment cost level (I) and are thus not shown for brevity.

Table 6 shows sensitivity results with respect to the priority rules of debt at default. We focus only on the case of "me-first" priority for initial debt (see equations 10a and 10b) and contrast

[9] It should be noted that coupon levels increase at t = 0 but since the improvement in equity value is more significant than that of debt the leverage ratio decreases. Credit spreads at t = 0 do not show any notable increase which also appears to support a debt "conservatism" argument.

it with the results of equal priority used in our earlier analysis. We show sensitivity with respect to volatility using "me-first" priority so these results can be contrasted with those of Table 1 where a similar sensitivity was conducted with equal priority for debt holders.[10]

[Insert Table 6 here]

Compared to the case of equal priority (see Table 1, panel B) we observe a slight increase in firm value under a "me-first" priority rule for initial debt. This is similar to the result of Hackbarth and Mauer (2012) where they find that differences in firm value under different priority rules is relatively small.[11] When the initial debt holders have priority in default we observe a more significant conservatism of debt raised at t = 0. This can be seen both by the initial coupon R_0 and initial debt level raised at t = 0 which are both smaller under the "me-first" for initial debt compared to the case of equal priority. Instead, the firm under "me-first" for initial debt priority rule preserves more financial flexibility to issue more debt when the investment is exercised. Indeed, R_1 at the investment trigger is higher under "me-first" compared to the case of equal priority. Despite the increase in the coupon of new debt, at the investment the total leverage and overall credit spreads remain rather similar between the two cases due to the counterbalancing effect caused by lower initial leverage under "me-first". On the other hand, note that the initial conservatism in debt levels combined with higher protection for initial debt and more delayed default (see Panel A compared to the case of Table 1) substantially decreases initial (t = 0) credit spreads relative to the equal priority rule. With respect to other firm policies (see Panel A), we observe that under "me-first" for initial debt the firm delays investment more (see also the lower expected investment cost incurred in Panel B). We finally observe that the directional effects with respect to the sensitivity with respect to volatility remain the same as in the case of equal priority. We however note on the more pronounced increase in leverage at investment relative to initial leverage that exists under "me-first" compared to the equal priority case.

Table 7 summarizes the predictions of the model concerning the parameters providing guidance for future empirical work in the area. In the next section we further facilitate this step by providing the framework for estimating the parameters of the continuous time process.

[10] We have conducted extensive sensitivity across all other parameters using "me-first" for initial debt. Similarly to Hackbarth and Mauer (2012) we find no significant differences in firm values or leverage decisions. Importantly, the predictions highlighted in the rest of the paper remain the same under "me-first" priority.
[11] They also point out that me-first is closer among all rules to the (ideal) optimal priority rule.

[Insert Table 7 here]

4. Mean reversion process estimation

We close this section by showing how to estimate the parameters of the continuous time process and investigate the prevalence of earnings mean reversion in the data. Appendix 5 describes the process of estimating the mean reversion speed q, the long-term mean and the volatility of the earnings process, as well as, the test used to identify whether a firm's earnings follows mean reverting process or a non-stationary process.

Our initial sample is from the quarterly COMPUSTAT database between 1961 and 2019. We exclude financial firms (Standard Industrial Classification (SIC) codes 6000 to 6999) and regulated firms (SIC codes 4900 to 4999). We require that a firm is included in the analysis for testing for mean reversion if it has at least 40 consecutive quarterly observations (10 year of data).

The total number of non-financial and non-regulated firms in the sample before and after the requirement of at least 40 consecutive observations is given shown in Table 8. The table also shows the number of firms classified as mean reverting using the full sample available for each firm.

[Insert Table 8 here]

Figure 1 illustratively shows a selection of two firms from our sample, one found to be mean-reverting and one which is non-stationary. Clearly, firm 1 earnings revert to a long-term mean whereas firm 2 appears not to revert to a long-term mean level.

[Insert Figure 1 here]

Our analysis shows that 60% of the firms with available data are classified as mean reverting showing the importance and need for further empirical work to distinguish the impact of earnings dynamics on firm policies.

5. Conclusions

We have developed a dynamic trade-off model with mean-reversion in earnings and multiple investment stages dynamic leverage adjustments at investment points. Our results challenge the traditional interpretation relating the empirical observation of a negative relation between profitability and leverage since for firms characterized with mean reverting earnings, the relation between earnings and profitability can be negative or even U-shape. We provide further insights on the impact of the earnings dynamics and in particular long-term profitability, mean reversion speed and volatility on firm value, leverage levels and dynamic changes in leverage and credit spreads. We also provide managerial implications regarding the optimal timing of investment and default and optimal capital structure related to earnings dynamics.

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Table 1. Sensitivity analysis with respect to earnings volatility (σ)

Panel A: Optimal coupon and thresholds

σ	R_0	R_1	x_I	v_L	x_b
0.24	0.72	0.46	1.015	-0.658	-0.415
0.3	0.61	0.5	1.135	-1.028	-0.772
0.4	0.52	0.55	1.341	-1.501	-1.191
0.5	0.46	0.64	1.541	-1.856	-1.563

Panel B: Values at t = 0 and the investment trigger T

	Values at t = 0							Values at investment trigger T		
σ	Fb(x)	Db(x)	Ub(x)	NBb(x)	Lev_b	Inv_b	Cr_b	Lev_T	ΔLev	Cr_T
0.24	20.822	11.644	28.187	2.475	0.559	9.840	0.0018	0.616	0.057	0.0018
0.3	20.580	9.711	27.373	2.080	0.472	8.873	0.0028	0.553	0.081	0.0028
0.4	20.556	7.997	26.720	1.692	0.389	7.856	0.0050	0.485	0.096	0.0052
0.5	20.815	6.778	26.551	1.532	0.326	7.268	0.0079	0.450	0.124	0.0082

Notes: In the above sensitivity the following was used: current earnings at the level x = 1, risk-free rate of r = 0.06, tax rate τ = 0.15 and proportional bankruptcy costs b = 0.5. For modelling the growth option we use e = 2 , investment cost I = 10. For the mean-reverting stochastic model parameters we vary σ and use a mean reversion speed q = 0.1 and long-term mean of earnings θ = 1. ΔLev stands for change in leverage and is calculated as Lev_T - Lev_b. Base case parameters is highlighted in red.

Table 2. Sensitivity analysis with respect to mean reversion speed (q)

Panel A: Optimal coupon and thresholds

Base case (θ = 1)

q	R_0	R_1	x_I	v_L	x_b
0.1	0.52	0.55	1.341	-1.501	-1.191
0.15	0.65	0.45	1.238	-1.795	-1.209
0.2	0.78	0.45	1.149	-1.810	-1.134
0.25	0.88	0.47	1.077	-1.738	-1.053
0.3	0.96	0.49	1.021	-1.623	-0.958

Case with lower long term profitability (θ = 0.5)

q	R_0	R_1	x_I	v_L	x_b
0.1	0.28	0.53	1.644	-1.120	-0.903
0.15	0.24	0.35	1.673	-1.531	-0.984
0.2	0.23	0.28	1.703	-1.847	-1.068
0.25	0.25	0.27	1.730	-2.003	-1.087
0.3	0.27	0.29	1.755	-2.052	-1.105

Panel B: Values at t = 0

Base case (θ = 1)

	Values at t = 0							Values at investment trigger T		
q	Fb(x)	Db(x)	Ub(x)	NBb(x)	Lev_b	Inv_b	Cr_b	Lev_T	ΔLev	Cr_T
0.1	20.556	7.997	26.720	1.692	0.389	7.856	0.0050	0.485	0.096	0.0052
0.15	20.771	10.486	27.234	2.174	0.505	8.637	0.0020	0.545	0.040	0.0020
0.2	21.166	12.787	27.683	2.707	0.604	9.224	0.0010	0.628	0.024	0.0010
0.25	21.524	14.526	28.019	3.139	0.675	9.634	0.0006	0.698	0.023	0.0006
0.3	21.814	15.898	28.253	3.469	0.729	9.908	0.0004	0.752	0.023	0.0004

Case with lower long term profitability (θ = 0.5)

	Values at t = 0							Values at investment trigger T		
q	Fb(x)	Db(x)	Ub(x)	NBb(x)	Lev_b	Inv_b	Cr_b	Lev_T	ΔLev	Cr_T
0.1	12.154	3.633	16.091	0.890	0.299	4.827	0.0171	0.388	0.089	0.0158
0.15	10.502	3.453	14.124	0.618	0.329	4.240	0.0095	0.345	0.016	0.0097
0.2	9.679	3.562	12.678	0.596	0.368	3.595	0.0046	0.342	-0.026	0.0051
0.25	9.251	4.005	11.486	0.664	0.433	2.899	0.0024	0.378	-0.055	0.0027
0.3	9.011	4.405	10.490	0.737	0.489	2.216	0.0013	0.427	-0.062	0.0016

Notes: In the above sensitivity the following was used: current earnings at the level x = 1, risk-free rate of r = 0.06, tax rate τ = 0.15 and proportional bankruptcy costs b = 0.5. For modelling the growth option we use e = 2 , investment cost I = 10. For the mean-reverting stochastic model parameters we use σ = 0.4, long-term mean θ = 1 and vary mean reversion speed q. ΔLev stands for change in leverage and is calculated as Lev_T - Lev_b. Base case parameters highlighted in red.

Table 3. Sensitivity analysis with respect to long-term profitability (θ)

Panel A: Optimal coupon and thresholds

θ	R_0	R_1	x_I	v_L	x_b
0.6	0.31	0.53	1.571	-1.217	-0.974
0.75	0.36	0.54	1.473	-1.356	-1.094
1	0.52	0.55	1.341	-1.501	-1.191
1.25	0.73	0.58	1.237	-1.544	-1.234
1.5	0.99	0.61	1.156	-1.504	-1.210
1.75	1.29	0.63	1.093	-1.411	-1.132
2	1.64	0.62	1.047	-1.282	-0.976
2.25	2.03	0.57	1.017	-1.161	-0.764

Panel B: Values at t = 0

	Values at t = 0							Values at investment trigger T		
θ	Fb(x)	Db(x)	Ub(x)	NBb(x)	Lev_b	Inv_b	Cr_b	Lev_T	ΔLev	Cr_T
0.6	13.635	4.246	18.243	0.970	0.311	5.578	0.0130	0.405	0.094	0.0126
0.75	16.063	5.238	21.488	1.167	0.326	6.591	0.0087	0.430	0.104	0.0090
1	20.556	7.997	26.720	1.692	0.389	7.856	0.0050	0.485	0.096	0.0052
1.25	25.429	11.561	31.741	2.396	0.455	8.708	0.0031	0.548	0.093	0.0032
1.5	30.532	15.923	36.579	3.214	0.522	9.261	0.0022	0.611	0.089	0.0022
1.75	35.763	20.952	41.283	4.093	0.586	9.613	0.0016	0.667	0.081	0.0016
2	41.063	26.787	45.883	5.004	0.652	9.824	0.0012	0.716	0.064	0.0012
2.25	46.398	33.325	50.414	5.928	0.718	9.944	0.0009	0.754	0.036	0.0009

Notes: In the above sensitivity the following was used: current earnings at the level $x = 1$, risk-free rate of r = 0.06, tax rate τ = 0.15 and proportional bankruptcy costs $b = 0.5$. For modelling the growth option we use $e = 2$, investment cost $I = 10$. For the mean-reverting stochastic model parameters we use σ = 0.4, mean reversion speed q = 0.1 and vary long-term mean of earnings θ. ΔLev stands for change in leverage and is calculated as Lev_T - Lev_b. Base case parameters highlighted in red.

Table 4. Sensitivity analysis with respect to current profitability level (x)

Panel A: Optimal coupon and thresholds

x	R_0	R_1	x_I	v_L	x_b
-0.4	0.30	0.8	1.323	-1.481	-1.660
-0.25	0.32	0.8	1.324	-1.481	-1.616
0	0.37	0.7	1.327	-1.481	-1.508
0.25	0.41	0.7	1.330	-1.481	-1.422
0.5	0.44	0.6	1.332	-1.481	-1.358
0.75	0.48	0.6	1.336	-1.501	-1.274
1	0.52	0.6	1.341	-1.501	-1.191
1.25	0.59	0.5	1.352	-1.501	-1.049
1.3	0.62	0.45	1.357	-1.501	-0.989
1.35	0.68	0.4	1.369	-1.481	-0.873

Panel B: Values at t = 0

	Values at t = 0							Values at investment trigger T		
x	Fb(x)	Db(x)	Ub(x)	NBb(x)	Lev_b	Inv_b	Cr_b	Lev_T	ΔLev	Cr_T
-0.4	10.391	4.583	13.440	1.176	0.441	4.225	0.0055	0.492	0.051	0.0054
-0.25	11.372	4.912	14.596	1.225	0.432	4.449	0.0051	0.491	0.059	0.0054
0	13.039	5.688	16.576	1.318	0.436	4.855	0.0051	0.491	0.055	0.0053
0.25	14.762	6.310	18.716	1.407	0.427	5.361	0.0050	0.491	0.064	0.0054
0.5	16.563	6.780	21.069	1.493	0.409	6.000	0.0049	0.490	0.081	0.0053
0.75	18.476	7.393	23.688	1.589	0.400	6.801	0.0049	0.486	0.086	0.0052
1	20.556	7.997	26.720	1.692	0.389	7.856	0.0050	0.485	0.096	0.0052
1.25	22.882	9.050	30.322	1.807	0.396	9.247	0.0052	0.484	0.088	0.0052
1.3	23.389	9.509	31.118	1.832	0.407	9.561	0.0052	0.483	0.077	0.0052
1.35	23.915	10.418	31.908	1.856	0.436	9.848	0.0053	0.485	0.050	0.0052

Notes: In the above sensitivity the following was used we vary the current earnings at the level x . Other parameters used are risk-free rate of r = 0.06, tax rate $\tau = 0.15$ and proportional bankruptcy costs $b = 0.5$. For modeling the growth option we use $e = 2$, investment cost $I = 10$. For the mean-reverting stochastic model parameters we use $\sigma = 0.4$, mean reversion speed q = 0.1 and long-term mean of earnings $\theta = 1$. ΔLev stands for change in leverage and is calculated as Lev_T - Lev_b. Base case parameters highlighted in red.

Table 5. Sensitivity analysis with respect to growth expansion factor (e)

Panel A: Optimal coupon and thresholds

e	R_0	R_1	x_I	v_L	x_b
1.35	0.49	0.43	3.569	-0.625	-0.844
1.5	0.48	0.48	2.399	-0.814	-0.897
1.75	0.48	0.52	1.667	-1.185	-1.060
2	0.52	0.55	1.341	-1.501	-1.191
2.25	0.59	0.56	1.155	-1.799	-1.275
2.5	0.74	0.5	1.038	-2.076	-1.212

Panel B: Values at t = 0

	Values at t = 0							Values at investment trigger T		
e	Fb(x)	Db(x)	Ub(x)	NBb(x)	Lev_b	Inv_b	Cr_b	Lev_T	ΔLev	Cr_T
1.35	15.026	7.510	14.327	0.863	0.500	0.164	0.0052	0.365	-0.135	0.0042
1.5	15.390	7.350	16.694	1.035	0.478	2.339	0.0053	0.433	-0.045	0.0050
1.75	17.506	7.377	21.995	1.405	0.421	5.894	0.0051	0.469	0.048	0.0051
2	20.556	7.997	26.720	1.692	0.389	7.856	0.0050	0.485	0.096	0.0052
2.25	23.977	9.058	31.066	1.928	0.378	9.018	0.0051	0.493	0.115	0.0053
2.5	27.578	11.319	35.196	2.139	0.410	9.756	0.0054	0.498	0.088	0.0054

Notes: In the above sensitivity the following was used: current earnings at the level x = 1, risk-free rate of r = 0.06, tax rate $\tau = 0.15$ and proportional bankruptcy costs b = 0.5. For modelling the growth option we vary e and use investment cost I = 10. For the mean-reverting stochastic model parameters we use $\sigma = 0.4$, use a mean reversion speed q = 0.1 and long-term mean of earnings $\theta = 1$. ΔLev stands for change in leverage and is calculated as Lev_T - Lev_b. Base case parameters highlighted in red.

Table 6. Sensitivity with respect to priority rule: "me-first" priority for initial debt with sensitivity with respect to volatility

Panel A: Optimal coupon and thresholds

σ	R_0	R_1	x_I	v_L	x_b
0.23	0.68	0.51	1.006	-0.601	-0.471
0.3	0.58	0.54	1.163	-1.007	-0.825
0.4	0.5	0.59	1.386	-1.461	-1.218
0.5	0.45	0.67	1.599	-1.818	-1.565

Panel B: Values at t = 0 and the investment trigger T

	Values at t = 0							Values at investment trigger T		
σ	Fb(x)	Db(x)	Ub(x)	NBb(x)	Lev_b	Inv_b	Cr_b	Lev_T	ΔLev	Cr_T
0.23	20.888	11.166	28.273	2.551	0.535	9.935	0.0009	0.624	0.090	0.0016
0.3	20.585	9.417	27.161	2.068	0.457	8.644	0.0016	0.552	0.095	0.0029
0.4	20.562	7.924	26.469	1.679	0.385	7.586	0.0031	0.487	0.102	0.0052
0.5	20.823	6.899	26.290	1.521	0.331	6.989	0.0052	0.451	0.119	0.0082

Notes: In the above sensitivity the following was used: current earnings at the level $x = 1$, risk-free rate of $r = 0.06$, tax rate $\tau = 0.15$ and proportional bankruptcy costs $b = 0.5$. For modelling the growth option we use $e = 2$, investment cost $I = 10$. For the mean-reverting stochastic model parameters we vary σ, use a mean reversion speed $q = 0.1$ and long-term mean of earnings $\theta = 1$. ΔLev stands for change in leverage and is calculated as Lev_T - Lev_b. Base case parameters highlighted in red. In this sensitivity results we use "me-first" priority for first debt (see equations 10a and 10b).

Table 7. Summary of directional effects on firm value, firm investment and default policies and leverage dynamics

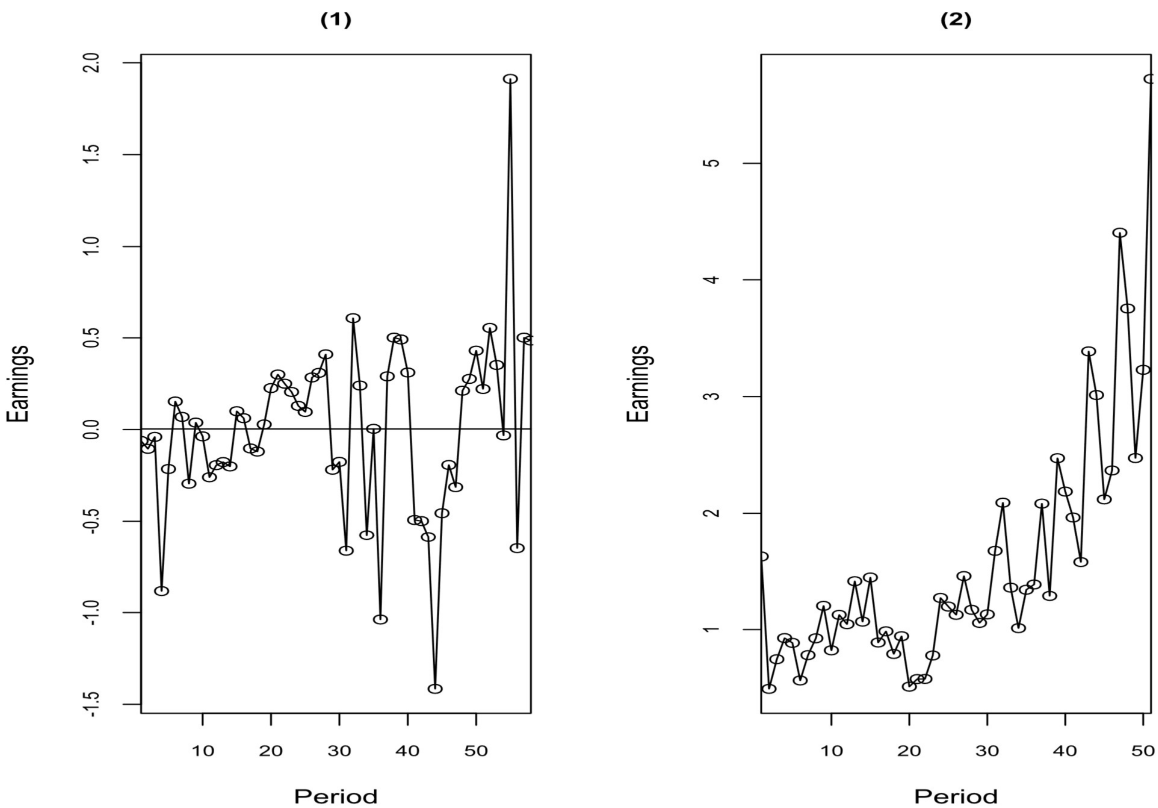
Parameter	Fb(x)	x_I	v_L	x_b	Lev_b	Lev_T	ΔLev
Volatility (σ) [1]	U	+	-	-	-	-	+
Speed of mean reversion (q)[2]	+	-	-	U	+	+	-
Long-term profitability (θ)	+	-	U	U	+	+	-
Current earnings level (*x*)	+	+	-	+	U	-	U
Growth expansion factor (e)	+	-	-	-	-	+	+
Capital investment cost (I)	-	+	+	+	+	-	-
Me-first priority rule (compared with equal priority)	+	-	-	-	-	No significant change	+

Notes: The above summary sensitivity results are based on the base case parameters. The following notes concern changes in the observed results depending on long-term profitability levels: 1. Firm value is increasing when long-term profitability is low and decreasing when long-term profitability is high. All other results remain unchanged 2. When long-term profitability is low then firm value is decreasing in q, the investment trigger is increasing and default triggers are decreasing.

Table 8. Number of non-financial and non-regulated firms that can be classified as mean reverting

All firms including financial and regulated	38,205
Financial firms	11,053
Regulated	929
Total non-financial and non-regulated firms	26,223
Number of firms with N = 40 consecutive earnings (oibdpq)	5,325
N. of firms classified as mean reverting	3,200

Figure 1. Examples of classifications of earnings processes



Notes: Plot 1 shows an example of a mean reverting process. It refers to firm "AM COMMUNICATIONS INC" with CUSIP number 001674100 with estimated mean reversion parameters $\theta = 0$, $q = 1.97$ and $\sigma = 0.69$. Plot 2 shows an example of a firm found to be non-stationary ("ABS INDUSTRIES INC" with CUSIP = 000781104). Both plots shows their earnings (oibdpq) unadjusted for seasonality for the whole periods of consecutive available data for each firm.

Appendix 1: Notation

$Eb(x)$ = Equity before investment (equity in stage 1).

$Fb(x)$ = Firm value before investment.

$Ub(x)$ = Value of unlevered assets before investment.

$TBb(x)$ = Tax benefits before investment.

$BCb(x)$ = Bankruptcy costs before investment.

$Db(x)$ = Debt before investment.

R_0= Coupon for $Db(x)$.

x_b = Bankruptcy threshold before investment.

x_I = Investment trigger

$Ea(x)$ = Equity after investment (equity in stage 2).

$Fa(x)$ = Firm value after investment.

$Ua(x)$ = Value of unlevered assets after Investment.

$TBa(x)$ = Tax Benefits after Investment.

$BCa(x)$ = Bankruptcy costs after Investment.

$Da_0(x)$ = Debt value of debt obtained at time zero after investment.

$Da_1(x)$ = Debt value of debt obtained at the investment trigger after investment.

R_1= Coupon for $Da_1(x)$.

x_L = Bankruptcy threshold following investment (in stage 2).

τ = Corporate tax rate

b = Proportional to unlevered assets bankruptcy costs

β_0 = share of initial debt holders at bankruptcy in stage 2 under equal priority.

$\beta_1 =$ share of second debt holders at bankruptcy in stage 2 under equal priority.

$I =$ Investment cost

$R_T = R_0 + R_1$

$Lev_b = Db(x) / Fb(x)$: Leverage ratio at t = 0

$Cr_b = R_0 / Db(x) - r$: Credit spread of initial debt at t = 0

$Inv_b = I \cdot J(x)$ = Expected present value of investment costs

$NBb(x) = TBb(x) - BCb(x)$: Net benefits of debt

At the investment trigger:

$Lev_T = (Da_0(x) + Da_1(x)) / Fa(x)$: Total leverage ratio at the investment trigger

$\Delta Lev = Lev_T - Lev_b$: Change in leverage relative to initial stage

$Cr_b = R_0 / Da_0(x) - r$: Credit spread of initial debt at the investment trigger

$Cr_T = (R_0 + R_1)/(Da_0(x) + Da_1(x)) - r$: Credit spread of total debt at the investment trigger

Appendix 2: Derivation of the homogeneous differential equation solution

Following standard replication arguments (example, Dixit and Pindyck, 1994, p.180) any contingent claim $P(x)$ on underlying asset x that follows the mean reversion process defined in equation (1) should satisfy[12]:

$$T(P(x)) = \frac{1}{2}\sigma^2 P''(x) - q(x-\theta)P'(x) - rP(x) = 0, \quad x \in \Re \tag{A1}$$

To find the general solution of this homogeneous differential equation first set $\bar{\sigma} = \sigma/\sqrt{2q}$ and make the following change of variables:

$$z = \frac{x-\theta}{\bar{\sigma}}.$$

Then $P(x) = u(z)$, $P'(x) = \frac{1}{\bar{\sigma}} u'(z)$ and $P''(x) = \frac{1}{\bar{\sigma}^2} u''(z)$. Thus equation (A1) is transformed to:

[12] To derive this general contingent claim differential equation we assume risk-neutral investors and hence that the total required return on holding an asset in equilibrium is r = a(x)+δ where a(x)=q(θ-x) is the capital (gains) of asset x and δ the convenience yield. Thus, the implied convenience yield of holding the underlying asset *x* is δ=r-a(x). A similar approach is followed in Sarkar and Zapatero (2003). A detailed proof is available upon request.

$$q\, u''(z) - qz\, u'(z) - ru(z) = 0, \quad z \in \Re. \tag{A2}$$

Setting also $u(z) = w(z)e^{\frac{z^2}{4}}$, with $\nu = -\frac{r}{q} < 0$, deduce that $u'(z) = e^{\frac{z^2}{4}}\left(w'(z) + w(z)\frac{z}{2}\right)$ and $u''(z) = e^{\frac{z^2}{4}}\left(w''(z) + zw'(z) + w(z)\frac{1}{2}(1 + \frac{z^2}{2})\right)$. A simple calculation then shows that equation (A2) can be rewritten into:

$$w''(z) - \left[\frac{1}{4}z^2 - \left(\nu + \frac{1}{2}\right)\right] w(z) = 0, \quad z \in \Re. \tag{A3}$$

Equation (A3) is the real version of Weber's equation (Abramowitz and Stegun, 1972), that is:

$$w''(z) - \left[\frac{1}{4}z^2 + a\right] w(z) = 0, \quad z \in \mathbb{C}, \tag{A4}$$

where $a = -\nu - \frac{1}{2}$. The general solution of equation (A3) is given by:

$$w_g(z) = C_1 U(a, z) + C_2 U(a, -z). \tag{A5}$$

With C_1 and C_2 general constants and where:

$$U(a, z) = \frac{1}{2^{\xi}\sqrt{\pi}}\left[\cos(\xi\pi)\,\Gamma\left(\frac{1}{2} - \xi\right)\, y_1(a, z) - \sqrt{2}\sin(\xi\pi)\,\Gamma(1 - \xi)\, y_2(a, z)\right] \tag{A6}$$

with

$$\xi = \frac{1}{2}a + \frac{1}{4},$$

$$y_1(a, z) = e^{-\frac{z^2}{4}}\ {}_1F_1\ \left(\frac{1}{2}a + \frac{1}{4};\, \frac{1}{2};\, \frac{z^2}{2}\right) \tag{A7}$$

and

$$y_2(a, z) = z\, e^{-\frac{z^2}{4}}\ {}_1F_1\ \left(\frac{1}{2}a + \frac{3}{4};\, \frac{3}{2};\, \frac{z^2}{2}\right) \tag{A8}$$

where ${}_1F_1\ (\alpha; \beta; z) = M(\alpha; \beta; z)$ is the confluent hypergeometric function (see Buchholz, 1969, Borodin and Salminen, 2002).

As a result, the general solution is now written as:

$$w_g(z) = C_1 D_\nu(z) + C_2 D_\nu(-z) \tag{A9}$$

Since in our case the variable z in equation (A3) is real, then the general solution of (A3) is expressed by:

$$w_g(z) = C_1 D_\nu(z) + C_2 D_\nu(-z)\,, \qquad z \in \Re.$$

Two useful asymptotic properties of the two linear independent solutions of equation (A3) (Buchholz, 1969) are the following:

$$\lim_{z\to\infty} e^{\frac{z^2}{4}} D_\nu(z) \;=\; \lim_{z\to\infty} \; z^\nu(1 + O(z^{-2}) = 0, \quad for \;\; \nu < 0 \qquad \text{(A10)}$$

and

$$\lim_{z\to\infty} e^{\frac{z^2}{4}} \; D_\nu(-z) \sim \frac{\sqrt{2\pi}}{\Gamma(-\nu)} \; \lim_{z\to\infty} \; e^{\frac{z^2}{2}} \; z^{-\nu-1} = \infty. \qquad \text{(A11)}$$

We can now get the general solution of equation (A2) to be:

$$u_h(z) = C_1\, e^{\frac{z^2}{4}} D_\nu(z) + C_2\, e^{\frac{z^2}{4}} D_\nu(-z), \qquad z \in \Re.$$

In addition, deduce that the solution of equation (A1) is given by:

$$P(x) = C_1\, e^{\frac{1}{4}\left(\frac{(x-\theta)\sqrt{2q}}{\sigma}\right)^2} D_\nu\left(\frac{(x-\theta)\sqrt{2q}}{\sigma}\right) + C_2 e^{\frac{1}{4}\left(\frac{(x-\theta)\sqrt{2q}}{\sigma}\right)^2} D_\nu\left(-\frac{(x-\theta)\sqrt{2q}}{\sigma}\right),\; x \in \Re. \qquad \text{(A12)}$$

For simplicity of presentation denote the general solution of (A1) as

$$P(x) = C_1 P_1(x) + C_2 P_2(x), \qquad \text{(A13a)}$$

with

$$P_1(x) = \; e^{\frac{1}{4}\left(\frac{(x-\theta)\sqrt{2q}}{\sigma}\right)^2} D_\nu\left(\frac{(x-\theta)\sqrt{2q}}{\sigma}\right),$$

and

$$P_2(x) = e^{\frac{1}{4}\left(\frac{(x-\theta)\sqrt{2q}}{\sigma}\right)^2} D_\nu\left(-\frac{(x-\theta)\sqrt{2q}}{\sigma}\right),$$

with equations (A10) and (A11) giving that:

$$\lim_{x\to\infty} P_1(x) = 0 \qquad \text{(A13b)}$$

$$\lim_{x\to-\infty} P_1(x) = \infty \qquad \text{(A13c)}$$

$$\lim_{x\to\infty} P_2(x) = \infty \tag{A13d}$$

$$\lim_{x\to-\infty} P_2(x) = 0 \tag{A13e}$$

Appendix 3: Derivation of solution for basic and general claims involving two boundaries

A3.1.Basic claim paying one dollar at v_L after investment

Consider the following differential equation problem:

$$T^*\big(Q(v)\big) = 0, \quad v \in \Re \tag{A14}$$
$$\lim_{v\to\infty} Q(v) = 0$$
$$Q(v_L) = 1$$

where $T^*(\theta^*, \sigma^*) \equiv T(e\theta, e\sigma)$. The solution for $Q(v)$ is given by applying (A13a):

$$Q(v) = C_1 P_1(v) + C_2 P_2(v)$$

Applying the first boundary condition in (A14) combined with equation (A13d) gives $C_2 = 0$. Then the second boundary condition gives $C_1 = \frac{1}{P_1(v_L)}$. Thus, the solution for this basic claim paying one dollar at v_L after investment is:

$$Q(v) = \frac{P_1(v)}{P_1(v_L)} \tag{A15}$$

A3.2. Basic claims for homogeneous equations before investment

$J(x)$ and $L(x)$ are basic claims where $J(x)$ pays one dollar at x_I and zero when x_b is reached and $L(x)$ pays one dollar at x_b and zero when x_I is reached.

A. Derivation of $J(x)$

Consider the following differential equation problem:

$$T\big(J(x)\big) = 0, \quad x \in \Re \tag{A16}$$
$$J(x_I) = 1$$
$$J(x_b) = 0$$

The solution $J(x)$ satisfies (A13) hence:

$$J(x) = C_1 P_1(x) + C_2 P_2(x)$$

Applying the boundary conditions in (A16) results in:

$$C_1 = \frac{P_2(x_b)}{D(x_I, x_b)}, \qquad C_2 = -\frac{P_1(x_b)}{D(x_I, x_b)},$$

where

$$D(x_I, x_b) = P_1(x_I)P_2(x_b) - P_1(x_b)P_2(x_I).$$

Thus, the solution for $J(x)$ is:

$$J(x) = \frac{P_2(x_b)}{D(x_I,x_b)} P_1(x) - \frac{P_1(x_b)}{D(x_I,x_b)} P_2(x). \tag{A17}$$

B. Derivation of $L(x)$

Consider now the corresponding problem for $L(x)$ which is given by:

$$T\big(L(x)\big) = 0, \quad x \in \Re \tag{A18}$$
$$L(x_I) = 0$$
$$L(x_b) = 1$$

Applying the boundary conditions results in the following solutions for the constants:

$$C_1 = -\frac{P_2(x_I)}{D(x_I, x_b)}, \qquad C_2 = \frac{P_1(x_I)}{D(x_I, x_b)}.$$

Thus the solution for $L(x)$ is:

$$L(x) = -\frac{P_2(x_I)}{D(x_I,x_b)} P_1(x) + \frac{P_1(x_I)}{D(x_I,x_b)} P_2(x) \tag{A19}$$

A3.3. Basic claims for linear homogeneous equations

Consider now the following problem regarding a contingent claim $N(x)$:

$$T\big(N(x)\big) = 0, \quad x \in \Re \tag{A20}$$
$$N(x_I) = A$$
$$N(x_b) = B$$

It can be easily shown that the solution of problem (A20) can be written in terms of the basic claims $J(x)$ and $L(x)$ in the following way:

$$N(x) = A\ J(x) + B\ \ L(x). \tag{A21}$$

A3.4. Basic claims for non-homogeneous equations

Consider now a more general contingent claim $M(x)$ which may pay $g(x)$ expressed by:

$$T\big(M(x)\big) + g(x) = 0, \quad x \in \Re \tag{A22}$$
$$M(x_I) = A$$
$$M(x_b) = B$$

Since $T(\cdot)$ is a linear differential operator then the general solution is given by the expression:

$$M(x) = M_h(x) + M_p(x), \tag{A23}$$

where $M_h(x)$ is a solution of a corresponding homogeneous problem

$$T\big(M_h(x)\big) = 0$$

(that is $g(x) = 0$) and $M_p(x)$ is one solution of problem (A20). To find which boundary conditions $M_h(x)$ should satisfy notice that:

$$M_h(x_I) = M(x_I) - M_p(x_I) = A - M_p(x_I)$$

$$M_h(x_b) = M(x_b) - M_p(x_b) = B - M_p(x_b)$$

The problem for $M_h(x)$ is in the form of problem (A20) and its solution is given by equation (A21) . Thus, we obtain the solution:

$$M_h(x) = \Big(A - M_p(x_I)\Big)J(x) + \Big(B - M_p(x_b)\Big)L(x).$$

As a result the solution for the value of $M(x)$ is:

$$M(x) = \Big(A - M_p(x_I)\Big)J(x) + \Big(B - M_p(x_b)\Big)L(x) + M_p(x). \tag{A24}$$

Equation (A24) is general enough to value securities (equity, debt) and firm value prior to investment depending on the payment $g(x)$ (which define $M_p(x)$ for the particular claim) and the boundary values *A* and *B*. Note that for debt holders $g(\cdot)$ is not a function of *x*.

Appendix 4: Detailed proofs of security and firm valuation solutions

A4.1. General solution of the problem

Consider the differential equation of the form:

$$T\big(y(x)\big) + ax + b = 0, \quad x \in \Re. \tag{A25}$$

The general solution of this problem is given by $y_g(x) = y_h(x) + y_p(x)$, where $y_h(x)$ is a solution of $T\big(y(x)\big) = 0$ and $y_p(x)$ (particular solution) is one solution of equation (A25). From equation (A13a) we have that $y_h(x) = C_1P_1(x) + C_2P_2(x)$. For the particular solution consider that $y_p(x) = k_1x + k_2$. Then $y_p'(x) = k_1$ and $y_p''(x) = 0$. Plugging in equation (A25) where $T(\cdot)$ is given by equation (A1) we get:

$$-q(x-\theta)k_1 - r(k_1x + k_2) + ax + b = 0.$$

Rearranging the terms one gets:

$$(-(q+r)k_1 + a)x + q\theta k_1 - rk_2 + b = 0.$$

This gives that:

$$k_1 = \frac{a}{q+r}$$

and

$$k_2 = \frac{1}{r}\left(\frac{q\vartheta a}{q+r} + b\right)$$

Thus, the general solution of equation (A25) is given by:

$$y_g(x) = C_1P_1(x) + C_2P_2(x) + \frac{a}{q+r}x + \frac{1}{r}\left(\frac{q\vartheta a}{q+r} + b\right) \tag{A26}$$

A4.2. Values after investment

A4.2.1. Value of unlevered assets

The value of unlevered assets after investment satisfies the following differential equation:

$$T^*\big(Ua(v)\big) + v(1-\tau) = 0, \quad v \in \Re. \tag{A27}$$

The general solution of equation A27 is given by equation (A26) with $a = 1 - \tau$ and $= 0$:

$$Ua(v) = C_1P_1(v) + C_2P_2(v) + \left[\frac{1}{q+r}v + \frac{q\theta^*}{r(q+r)}\right](1-\tau)$$

(A28)

The value of unlevered assets must also satisfy the following boundary conditions:

$$\lim_{v\to\pm\infty} Ua(v) = Ua_p(v) \qquad \text{(A28)}$$

Equation (A13b) then suggests that $C_2 = 0$ and equation (A13c) suggests that $C_1 = 0$. Thus

$$Ua(v) = \left[\frac{1}{q+r}v + \frac{q\theta^*}{r(q+r)}\right](1-\tau) \qquad \text{(A29)}$$

$Ua(v)$ can turn negative for sufficiently negative v. The value of v_A at which the value of unlevered assets is zero is the solution of $Ua(v_A) = 0$ which suggests that $v_A = -\frac{q\theta^*}{r}$. Since the value of unlevered assets is obtained (net of bankruptcy costs) by debt holders when the firm goes bankrupt at optimally determined v_L we need to ensure that if $v_L < v_A$ debt holders do not obtain a negative value and thus if $v_L < v_A$, $Ua(v_L)$ is set to zero.

A4.2.2. Equity value

Equity value after investment satisfies the following differential equation:

$$T^*\big(Ea(v)\big) + (v - R_0 - R_1)(1-\tau) = 0, \quad v \in \Re \qquad \text{(A30)}$$

The general solution of equation A30 is given by equation (A26) with $a = 1 - \tau$ and $b = -(1-\tau)(R_0+R_1)$:

$$Ea(v) = C_1P_1(v) + C_2P_2(v) + \left(\frac{1}{q+r}v + \frac{q\theta^*}{r(q+r)} - \frac{R_0+R_1}{r}\right)(1-\tau).$$

(A31)

Equity must also satisfy

$$\lim_{v\to\infty} Ea(v) = Ea_p(v) \qquad \text{(A32)}$$

and

$$Ea(v_L) = 0 \qquad \text{(A33)}$$

Equation (A13d) suggests that $C_2 = 0$ and by (A33) we obtain that

$$C_1 = -\frac{Ea_p(v_L)}{P_1(v_L)}$$

Thus, we obtain that:

$$Ea(v) = Ea_p(v) - Ea_p(v_L)\frac{P_1(v)}{P_1(v_L)} \qquad \text{(A34)}$$

Setting $v = ex$ define

$$\tilde{E}a(x) = Ea(ex) = Ea_p(ex) - Ea_p(v_L)\frac{P_1(ex)}{P_1(v_L)} \qquad \text{(A35)}$$

A4.2.3. Debt values

Debt value after investment for the initial debt issued at time zero $Da_0(v)$ and the second debt issued at the investment trigger $Da_1(v)$ satisfy the following:

$$T^*\big(Da_i(v)\big) + R_i = 0, \qquad i = 0{,}1 \qquad v \in \Re \qquad \text{(A36)}$$

The general solution of equation (A36) is given by equation (A26) with $a = 0$ and $b = R_i$:

$$Da_i(v) = C_1P_1(v) + C_2P_2(v) + \frac{R_i}{r}$$

(A37)

Debt must also satisfy two boundary conditions. The first one is given by:

$$\lim_{v\to\infty} Da_i(v) = \frac{R_i}{r} \qquad \text{(A38)}$$

The second boundary depends on the priority structure. Under equal priority:

$$Da_i(v_L) = \beta_i\ (1-b)\ Ua(v_L) \qquad \text{(A39)}$$

In the case the first creditors have secured priority to other creditors then the boundary conditions become:

$$Da_0(v_L) = min\ \left[(1-b)\ Ua(v_L), \frac{R_0}{r}\right] \qquad \text{(A40)}$$

$$Da_1(v_L) = (1-b)\ Ua(v_L) - Da_0(v_L))$$

In the case that second debt holders have secured priority to other creditors then the boundary conditions become:

$$Da_1(v_L) = min\ \left[(1-b)\ Ua(v_L), \frac{R_1}{r}\right] \quad \text{(A41)}$$

$$Da_0(v_L) = (1-b)\ Ua(v_L) - Da_1(v_L)$$

Equation (A36) combined with (A13d) suggests that $C_2 = 0$. Thus $Da_i(v) = C_1 P_1(v) + \frac{R_i}{r}$. Depending on priority structure, applying boundary conditions (A39), (A40) or (A41) deduce that:

$$C_1 = \frac{Da_i(v_L) - \frac{R_i}{r}}{P_1(v_L)}$$

and thus

$$Da_i(v) = \frac{R_i}{r} + \left(Da_i(v_L) - \frac{R_i}{r}\right)\left(\frac{P_1(v)}{P_1(v_L)}\right) \quad \text{(A42)}$$

Setting $v = ex$ define

$$\widetilde{D}a_i(x) = Da_i(ex) = \frac{R_i}{r} + \left(Da_i(ex_L) - \frac{R_i}{r}\right)\left(\frac{P_1(ex)}{P_1(ex_L)}\right) \quad \text{(A43)}$$

A4.3. Values before investment

A4.3.1 Value Unlevered before investment

Following similar arguments as the ones used to derive the value of unlevered assets after investment one can show that the value of unlevered assets before investment $Ub(x)$ is given by:

$$Ub(x) = \left[\frac{1}{q+r}x + \frac{q\theta}{r(q+r)}\right](1-\tau) \quad \text{(A44)}$$

To avoid negative liquidation values for initial debt holders at bankruptcy if $x_B < x_A$ then $Ub(x_B) = 0$ where $x_A = -\frac{q\theta}{r}$ is the threshold where $Ub(x)$ becomes zero.

A4.3.2. Debt value before investment

Debt $Db(x)$ satisfies the following differential equation:

$$T\big(Db(x)\big) + R_0 = 0, \qquad x \in \Re \qquad \text{(A45)}$$

The general solution of equation (A43) is given by equation (A26) with $a = 0$ and $b = R_0$:

$$Db(x) = Db_h(x) + \tfrac{R_0}{r} \qquad \text{(A46)}$$

Debt before investment must also satisfy the following boundary conditions:

$$Db(x_I) = Da_0(x_I)$$

$$Db(x_b) = (1-b)\, Ub(x_b).$$

Equation (A24) then suggests that the solution of the problem is given by:

$$Db(x) = \left(Da_0(x_I) - \tfrac{R_0}{r}\right) J(x) + \left((1-b)\, Ub(x_b) - \tfrac{R_0}{r}\right) L(x) + \tfrac{R_0}{r} \qquad \text{(A47)}$$

A4.3.3 Equity and firm value before investment

The equity function before investment satisfies the following differential equation:

$$T\big(Eb(x)\big) + (x - R_0)(1-\tau) = 0, \quad x \in \Re \qquad \text{(A48)}$$

The general solution is given by equation (A26) with $a = 1 - \tau$ and $b = -(1-\tau)R_0$:

$$Eb(x) = Eb_h(x) \;+\; Eb_p(x) = C_1 P_1(x) + C_2 P_2(x) + \left(\frac{1}{q+r} x + \frac{q\theta}{r(q+r)} - \frac{R_0}{r}\right)(1-\tau)$$

Equity should also satisfy the following boundary conditions:

$$Eb(x_I) = Ea(v_I) - I + Da_1(v_I)$$

$$Eb(x_b) = 0$$

Equation (A24) then implies that solution of the problem is:

$$Eb(x) = \big(Ea(v_I) - I + Da_1(v_I) - Eb_p(x_I)\big) J(x) - Eb_p(x_b)\, L(x) + Eb_p(x) \qquad \text{(A49)}$$

Note that with $v_I = e x_I$ this becomes:

$$Eb(x) = \big(Ea(e\, x_I) - I + Da_1(e\, x_I) - Eb_p(x_I)\big) J(x) - Eb_p(x_b)\, L(x) + Eb_p(x)$$

Firm value before investment is then given by the sum of equity plus debt after investment:

$$Fb(x) = Eb(x) + Db(x) \qquad \text{(A50)}$$

Appendix 5: Estimation procedure for the parameters of the continuous time process

The continuous time model dynamics for the earnings in equation (1) need to be translated in a suitable discrete time approximation for estimation. To do that we first note that the solution of the stochastic differential equation (SDE) in equation (1) is of the following form (see Lo and Wang, 1995):

$$x_t = x_0 e^{-qt} + \theta(1 - e^{-qt}) + \sigma \int_0^t e^{q(s-t)} dz_s \qquad \text{(A50)}$$

The (conditional) expected value and variance of x_t can be obtained by solving the Kolmogorov forward equation (as in Dixit and Pindyck, 1994, p.90). These moments have also been derived in Lo and Wang, (1995). The conditional expected value of earnings is:

$$E(x_t) = x_0 e^{-qt} + \theta(1 - e^{-qt}) = \theta + (x_0 - \theta)e^{-qt} \qquad \text{(A51)}$$

Note that for $q > 0$ as $t \to \infty$, $E(x_t) \to \theta$ which confirms the mean-reverting nature of the process. The variance of the variable x is also obtained as follows:

$$Var(x_t) = \frac{\sigma^2}{2q}(1 - e^{-2qt}) \qquad \text{(A52)}$$

Note that as the mean reversion speed q increases the variance of x decreases.

The solution of the SDE implies the following discrete version of solution can be used to generate the dynamics of x (see Dixit and Pindyck, 1994, p.76, eq.19):

$$\Delta x_t = \theta(1 - e^{-q}) + (e^{-q} - 1)x_{t-1} + \sigma\sqrt{\frac{1-e^{-2q}}{2q}} Z_t \qquad \text{(A53)}$$

where $Z_t \sim N(0,1)$. The above specification implies that the error volatility per unit of interval is:

$$\sigma_\varepsilon = \sigma\sqrt{\frac{1-e^{-2q}}{2q}} \qquad \text{(A54)}$$

Equation (A53) can also be used to simulate the stochastic dynamics of the continuous process (see below). To estimate the mean reversion speed (q), long-term mean (θ) and volatility (σ) in equation (A53) we estimate the following AR(1) model in discrete time (see Dixit and Pindyck, 1994, p.76):

$$\Delta x_t = a + b x_{t-1} + \varepsilon_t \tag{A55}$$

We then associate the estimated constant, slope and error term volatility of (A55) with the continuous time model approximation analogue in equation (A53) which results in the following solution for the parameters:

$$q = -\ln\left(1 + \hat{b}\right) \tag{A56}$$

$$\theta = -\frac{\hat{a}}{\hat{b}} \tag{A57}$$

$$\sigma = \sigma_\varepsilon \sqrt{\frac{-2\ln\left(1+\hat{b}\right)}{\left(1+\hat{b}\right)^2 - 1}} \tag{A59}$$

For the dynamics above to be meaningful we need that $-1 < \hat{b} < 0$ so that we obtain a positive mean reversion speed. Note that the smaller the coefficient $\hat{b}$ the larger the speed of mean reversion q while as $\hat{b} \rightarrow 0$ we have $q \rightarrow 0$. To avoid $\hat{b} = 0$ and ensure that earnings dynamics in (A55) remains stationary we employ an Augmented Dickey-Fuller and thus in the final estimated version we include a time-trend and lags of the change of earnings as follows:

$$\Delta x_t = a_0 + a_1 t + b x_{t-1} + \sum_{i=1}^{h} \beta_i \Delta x_{t-i} + \varepsilon_t \tag{A60}$$

For the Augmented Dickey-Fuller test we estimate (A60) for the de-seasonalized series and test the null of non-stationary series $\hat{b} = 0$ versus the alternative of stationary series $\hat{b} < 0$. We have used $h = 4$ lags in all specifications.