Predatory Pricing and the Value of Corporate Cash Holdings

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Abstract

We analyze the interaction between firms' payout policies and their decisions in product markets in a continuous-time stochastic game between two firms. One of these is financially constrained, whereas the other is not. Contrary to the standard literature we allow firms to choose production and payout strategies, and focus on the effect of predation incentives on both. We find that predation induces fewer dividend payouts. Furthermore, the liquidity position of the constrained firm has an economically significant effect on the production choices of both firms and, thus, on the evolution of profits, cash holdings and stock returns.

"In this race the horse with the poorest record, or no record, must carry the greatest weight."

— J. K. Galbraith, *American Capitalism (1952)*

1 Introduction

The elaboration of the reasons behind exceptionally large cash balances in U.S. firms has received considerable attention in the corporate finance literature (Cole, 2014; Pinkowitz *et al.*, 2015). The classical precautionary motive, introduced by Keynes (1936), has been broadly acknowledged as an important factor in explaining this phenomenon. According to this view, cash holdings are more valuable for financially weak firms since they can be used as a safety cushion against bad times in the future. This is supported by empirical evidence, which shows that financially constrained firms have a propensity to hoard more cash than their unconstrained rivals (Almeida *et al.*, 2004; Campello *et al.*, 2010; Denis and Sibilkov, 2010; Faulkender and Wang, 2006).

Recent significant increases in corporate cash holdings can be attributed, among other things, to increased riskiness of cash flows (Bates *et al.*, 2018, 2009; Hugonnier *et al.*, 2014). The probability of default increases with cash flow volatility, thereby strengthening the precautionary motive. Empirical evidence also suggests that the intensity of competition in product markets creates a strategic motive for cash hoarding. This suggests that firms hold cash as a "war chest" against competitors (Fresard, 2010; Hoberg *et al.*, 2014; Lyandres and Palazzo, 2016). This effect is typically more pronounced in the presence of predatory threats (Alimov, 2014; Chi and Su, 2016; Haushalter *et al.*, 2007). Intuitively, there exists an important economic feedback between cash flow volatility and the aggressiveness of competition in product markets. Larger cash flow volatility of financially constrained firms increases their bankruptcy probabilities. This implies that these firms are more likely to become prey in the product market. In turn, this increases their riskiness. Therefore, the effect of predation can be amplified by increased volatility. However, the literature looks at these effects in isolation.

In this paper, we investigate corporate cash management decisions in a dynamic continuous-time game between two firms who maximize the present value of expected dividend flows. We focus on the incentives of firms to engage in anti-competitive practices in the product markets and the consequences of these decisions for their optimal dividend payout policies. In our model, these incentives are driven by an asymmetry in firms' financial strengths. This is relevant for markets that are dominated by large and financially strong firms who are in competition with smaller firms with weaker financial positions. In order isolate the effect of predation, we consider the simplest case possible where this asymmetry is extreme, so that the roles of predator and prey are predetermined. More specifically, in our model, one firm is assumed to operate under financial constraints (the prey), whereas its rival (the predator) does not face liquidity concerns and, thus, is not subject to default risk.

Our model combines aspects of both industrial organization and finance. Firms dynamically choose both their production quantities (*à la* Cournot) and dividend payouts in order to maximize the expected value of discounted future dividends. The constrained firm's cash pile is assumed to be subject to stochastic shocks, which are observed by both firms, so that our model exhibits symmetric but imperfect information. The randomness of the shocks adds an element of unpredictability that allows the unconstrained firm to try to load the dice in its favor by producing more than the (static) Cournot quantity. In our model, such an action reduces the drift of the constrained firm's cash process. Negative stochastic shocks may then reduce the constrained firm's stock of cash and, eventually, force this firm out of business. At that point, the unconstrained firm becomes a monopolist. In this way, strategic motives for cash hoarding of the constrained firm add *market share considerations* to the classical tradeoff between survival (accumulating cash) and maximization of the shareholder value (paying out dividends). We find that the liquidity position of the financially constrained firm has an important effect on the production choices of *both* firms and, thus, on the evolution of, e.g., consumer surplus and profits.

The conclusions derived from our model offer several contributions to both the financial and industrial organization literature. First, we characterize the optimal payout policy of a financially constrained firm in the presence of predation in the product market. In our model's equilibrium, the constrained firm's dividend policy takes the form of a reflecting barrier. For low levels of cash, the firm should not pay out any dividends, whereas the firm should pay out all its earnings as soon as its cash pile reaches a certain cash target.

In our model, a necessary condition for the possibility of predation arising as an equilibrium phenomenon is that at least one firm is financially constrained. In addition, predation can only take place while one such firm does not pay out dividends. If neither firm has any problem in accessing external financing, or if the cash constrained firm chooses to pay out all its profits always, then the equilibrium in our model is for both firms to produce the (static) Cournot quantity in every period.

Second, we find that predation in product markets has an effect on the optimal dividend policy, and vice versa. The desire to pay out dividends limits the extent to which firms will engage in predation. On the contrary, the imperative to keep the firm liquid makes the constrained firm more cautious in paying out dividends. More specifically, our model suggests that as the intensity of predation in the product market and/or the cash flow volatility increase, the constrained firm raises its cash target and postpones dividends payouts. These results are pertinent in view of recent empirical literature, which suggests that cash flow volatility has significant explanatory power for the observed increase in firms' cash holdings (Bates *et al.*, 2009) and that firms in more competitive industries with larger predatory threats hoard more cash and pay lower dividends (Hoberg *et al.*, 2014). The new insight provided by our model is that there is an inherent connection between these effects. The precautionary effect, resulting from increased volatility, is amplified by the presence of predation. Larger cash flow volatility leads to more aggressive competition, which in turn affects the expected cash flow growth of the constrained firm, thereby inducing a higher cash target. This emphasizes an important role in the setting of optimal dividend policies for the decisions of firms in their product markets.

Third, our model suggests that under certain conditions, the marginal value of cash for the constrained firm is non-monotonic in its cash position. Previous studies have found significant differences between financially constrained and unconstrained firms, with the marginal value of cash being higher for the latter (Almeida *et al.*, 2004; Faulkender and Wang, 2006). Faulkender and Wang (2006) also consider the implications for within-firm variation in cash. They find that, on average, the marginal value of cash declines with firms' cash holdings. This is consistent with predictions of theoretical models with structures similar to ours, such as Décamps *et al.* (2011); Jeanblanc-Picqué and Shiryaev (1995) that disregard product market competition. However, we find a qualitatively different result, that is, the marginal value of cash of a financially constrained firm increases for intermediate levels of cash reserves due to aggressive competition. This occurs particularly when the risk-adjusted discount rate is relatively low, that is, when the unconstrained firm has stronger incentives to fight for the possibility of a future monopoly position. This result leads to the testable hypothesis that the marginal value of cash for firms in different financial positions may respond differently to an increase in cash in the presence of predation threat. In addition, this suggests that OLS estimates might not capture the potential non-monotonicity and that the use of quantile regression may be more suitable to verify if there are significant differences in marginal values of cash for different quantiles.

Fourth, our model provides several testable implications for the effect of cash hoarding on return dynamics in the presence of aggressive competition. The result that the financial policy of competitors influences the cash position of its rivals is not new. Our model, however, allows us to establish the effect of the liquidity position of the constrained firm on the volatility of stock returns for *both* firms. Consistent with Décamps *et al.* (2011), in our model the financial position of the constrained firm negatively affects the volatility of its own stock returns. The novel implication is that the liquidity position of the financially constrained firm negatively affects the return volatility of its competitor. This effect arises due to aggressive competition in the product market, implying that predation creates a channel through which the competitive interactions of firms in the product market feed through into financial markets. In particular, the unconstrained firm is able to manipulate the rate of cash accumulation by its rival. As a result, the idiosyncratic cash flow volatility of the constrained firm becomes a factor in the unconstrained firm's return dynamics. The presence of predation also implies that there is a jump in the volatility of the financially unconstrained firm's returns at the moment its financially constrained rival leaves the market. This effect can only occur in the presence of predatory behavior, because otherwise the volatility of the unconstrained firm's returns is not affected by its competitor's liquidity position. Thus, our model allows for the identification of aggressive competition as a potential source for volatility jumps.

These findings are related to empirical observations of a substantial increase in idiosyncratic return volatility in recent years presented by Irvine and Pontiff (2008), who attribute this increase to intensified competition. Our model, however, provides a cautionary tale about the dynamic relationship between idiosyncratic volatility and the competitive environment. A large idiosyncratic volatility implies more competition in the short-run in our model. This competition, however, is of a predatory kind, which in the long-run may very well lead to less competition as the predator is able to achieve a monopoly position. Our main insight here is that high levels of competition may not be sustainable in the long-run. This would be consistent with a stream of literature, recently excellently contextualized by Philippon (2019), which argues that the intensity of (product-market) competition has been reducing in the US in the 21st century, mainly due to increased market concentration. This phenomenon was already observed by Galbraith (1952), who, as the quote at the top of the paper shows, was concerned about the future of US capitalism in light of the competitive advantages of financial strength bestowed upon (large and powerful) conglomerates.

Methodologically, our paper is closely related to the stream of literature on dynamic liquidity management and optimal payout policy under financial constraints. The seminal theoretical contributions in this area, such as Jeanblanc-Picqué and Shiryaev (1995) and Radner and Shepp (1996), support the precautionary motive as a main driver of cash hoarding. In these models, the accumulated cash process is modeled as a Brownian motion with a constant drift that triggers the bankruptcy of the firm once depleted to zero. They find that the firm pays out dividends once the firm's cash reserves reach a certain target. A number of extensions of these early models have been developed to account for various issues such as random interest rates, issuance costs, and risk management (Akyildirim *et al.*, 2014; Décamps *et al.*, 2011; Højgaard and Taksar, 1999). Our study is another step in this direction, which focuses on the effect of aggressive competition in product markets. The unique feature of our model is that the cash-flow growth of a liquidity constrained firm is determined endogenously though the feedback effect of competition in the product markets on its liquidity position. Among the studies that also relax the assumption of constant drift is Décamps and Villeneuve (2007), who consider a setting in which a firm can undertake an irreversible investment to boost the growth prospects of its cash process. Gryglewicz (2011) adds an element of randomness to the drift of the Brownian motion by assuming that the cash flows are subject to long-term uncertainty. Jiang and Pistorius (2012) incorporate the regime switch in the drift of the cash process. In our model, the drift of the cash process is continuously controlled by firms because of competitive interaction in the product markets. In this setting, we obtain qualitatively different results from the existing literature.

Our paper is also related to the stream of literature that considers the impact of product market competition on the nexus between firms' investments and stock returns. Grenadier (2002) derives the equilibrium investment strategies in an oligopolistic industry where each firm holds a sequence of capacity expansion options. Aguerrevere (2009) extends Grenadier (2002) by allowing firms to change their capacity utilization and concludes that the impact of the intensity of product market competition on return dynamics varies depending on current demand. Recently, Morellec and Zhdanov (2019) have built on these contributions and discovered that product market competition leads to a negative volatility skew in option prices. Gu (2016) studies the interaction effect between R&D investment and product market competition. They find that this effect is positively related to expected stock returns. All these studies, however, disregard the effect of financial constraints and, thus, the possibility of market exit. In our model, the possibility of bankruptcy of the financially constrained firm creates incentives for its financially strong rival to engage in aggressive competition in the product market to drive the constrained firm out of the market. This results in novel implications for the return dynamics where the ability of the financially strong firm to manipulate the volatility of the competitor affects its own return volatility. In this way, an exogenous cash flow shock of a financially constrained firm is endogenized in the return volatility of its rival.

Several studies explicitly consider exit decisions in a framework that combines capital structure decisions and industry dynamics. Kovenock and Phillips (1997) empirically show that firms with high leverage in highly concentrated industries are more likely to close plants and less likely to undertake new investments. In an empirical investigation of trucking companies, Zingales (1998) emphasizes the negative effect of leverage and, thus, insufficient financial strength (fatness) on the likelihood of firm survival. Lambrecht (2001) provides a theoretical contribution by studying the impact of capital structure on the entry and exit decisions in a duopoly. Unlike our paper, they focus on the relationship between firms' financing decisions on industry dynamics, rather than their production policies, and thus, ignore the effect of predation. The interaction between financing and production decisions is investigated by Miao (2005), who focuses on the costs and benefits of the debt issue. However, that model focuses on a perfectly competitive market in which firms cannot exert influence over their competitors' cash flows. In our paper, we take the opposite view and ignore the effect of external financing to study imperfect competition.

Among the studies that specifically consider cash hoarding of financially constrained firms under different market structures are Della Seta (2011); Morellec *et al.* (2014); Povel and Raith (2004). Apart from Povel and Raith (2004), however, these contributions typically overlook the context of predation. Unlike Povel and Raith (2004), we are able to separate predation incentives and purely competitive effects. This is because in our model, in the absence of financial constraints, firms would always choose the static Cournot production strategy. On the contrary, financial constraints do not affect the optimal production strategy in the monopoly case, because the monopoly profits maximize the drift of the firm's cash stock process. Therefore, the monopoly quantity also maximizes the survival probability. Thus, the only reason for a firm to depart from its usual static strategy is to predate its rival.

The idea that firms use their financial strength to engage in anti-competitive behavior is not new and dates back to the early literature on predatory pricing and, in particular, the *deep pocket argument*, introduced by McGee (1958), and later studied by Telser (1966) and Benoit (1983). In their models a more resourceful incumbent could potentially drive a financially constrained entrant out of the market by means of aggressive pricing. The main conclusion of this stream of literature is, however, that under perfect information no price war will be observed in equilibrium due to the temporary nature of price cuts. The modern view on predatory pricing does not support these early conclusions. However, more recent models strongly rely on asymmetric information in explaining anti-competitive behavior (Bonatti *et al.*, 2016). We show that predation can also be an equilibrium phenomenon also in environments without information asymmetries. In particular, we go back to a complete information setting and the traditional "deep pockets" hypothesis. We investigate whether the results from the early literature that suggest the irrationality of predation still hold in a dynamic setting where at least one firm is capital constrained and firms' profits are subject to stochastic shocks. Unlike the Chicago school, we show that predation is not a rare event but results from rational behavior. Quite remarkably, we can demonstrate this by solving an optimal control problem with a relatively simple structure.

At the same time, our model differs from the literature, such as Bolton and Scharfstein (1990), which explains predation by agency conflicts in environments where firms are cash constrained. Such models essentially add noise to the profits that are observed by shareholders so that they cannot establish their actual values. Managers, on the contrary, have prefect foresight. In contrast, our results do not hinge on the presence of private information. Instead, in our model information is symmetric in the sense that all agents are exposed to uncertainty in the same way and, as thus, none of them have the prefect foresight about the future. In fact, even in the presence of perfect monitoring by shareholders of managers one cannot avoid predation in our model. The primary drivers of predatory behavior are cash flow volatility and financial constraints. This point of view is supported by Bates *et al.* (2009) who empirically study increases in US firms' cash balances and conclude that these increases are primarily attributed to cash flow risk rather than to agency conflicts.

The remainder of the paper is organized as follows. Section 2 presents the model setup. Section 3 introduces the benchmark model in which the firms behave myopically in the product market. In Section 4 we derive optimal production profiles and payout strategies and characterize the equilibrium policies. Section 5 analyzes model implications for financial and product markets. Section 6 provides some conclusions.

2 The Model

In this section we introduce a continuously-repeated dynamic stochastic game between two firms that choose their quantity and dividend policies. In our model the firms are symmetric in the product market, while asymmetric in terms of their financial strength. One firm (Firm 2) operates under a liquidity constraint, and goes bankrupt if its cash reserves fall below zero. Its competitor (Firm 1) is an unconstrained firm in the sense that it is not subject to liquidity default.

In the product market, firms compete *à la* Cournot, i.e. they choose quantities at every point in time. Unconstrained Firm 1 pays out these profits as dividends. The financially constrained Firm 2 adds profits to its cash reserve, which is subject to a (white noise) liquidity shock. It then decides, in addition, how much of its cash reserve to pay out to shareholders as dividends.

The inverse demand in the product market is given by the function $P : \mathbb{R}_+ \to \mathbb{R}_+$, with $P' \leq 0$. The (identical) production technology is represented by a cost function $C : \mathbb{R}_+ \to \mathbb{R}_+$, which is continuous on $(0, \infty)$, with $C' > 0$, $C'' \geq 0$, and $C(0) = 0$. The profit of firm i, $i = 1, 2$, is $\pi(q_i, q_j) = P(q_i + q_j)q_i - C(q_i)$. We assume that there is a unique quantity $q_M > 0$ that maximizes $P(q)q - C(q)$ and that $\pi_M \triangleq P(q_M)q_M - C(q_M) \in (0, \infty)$. The values q_M and π_M are the monopoly quantity and profit, respectively. Furthermore, we assume that the Cournot quantity is unique and well-defined. That is, we assume that there is a unique maximizer, q_D, of the mapping $q \mapsto P(q + q_D)q - C(q)$, which is the Cournot quantity. We further denote the Cournot profit by $\pi_D \triangleq P(2q_D)q_D - C(q_D)$. Note that per-period profits are bounded above by π_M.[1]

[1] Sufficient conditions for q_M and q_D to be well-defined are $\frac{\partial^2 \pi(q_i, q_j)}{(\partial q_i)^2} + \left| \frac{\partial^2 \pi(q_i, q_j)}{\partial q_i \partial q_j} \right| < 0$ or equivalently

Uncertainty over the liquidity position of the constrained firm is driven by a standard Brownian motion $B = (B(t))_{t\geq 0}$, which is defined on a canonical probability space $(\Omega, \mathscr{F}, \mathsf{P})$. Let $\mathbf{F} = (\mathscr{F}(t))_{t\geq 0}$ be the filtration generated by B. Let $\mathscr{T}$ be the set of $\mathbf{F}$-adapted stopping times. A firm's operational activities, i.e. production, lead to a stream of profits.

DEFINITION 1. *A* production policy *is an* $\mathbf{F}$*-adapted and non-negative process* $q = (q(t))_{t\geq 0}$.

The set of production policies is denoted by $\mathscr{Q}$. Any part of profits that is not paid out as dividends by Firm 2 is assumed to be added to its cash reserve.

DEFINITION 2. *A* dividend policy *is an* $\mathbf{F}$*-adapted, non-negative, and non-decreasing process* $Z = (Z(t))_{t\geq 0}$.

The set of dividend policies is denoted by $\mathscr{D}$. A *strategy* $\xi = (q, Z) \in \mathscr{Q} \times \mathscr{D}$ for a firm consists of a production policy $q \in \mathscr{Q}$ and a dividend policy $Z \in \mathscr{D}$.

Since the constrained firm can use its instantaneous profits either to pay out dividends, or to accumulate cash reserves, it can exert control over its free cash process through its production policy and through its choice of dividend policy. Through the Cournot assumption in the product market, the unconstrained firm's production policy also influences the constrained firm's free cash process.

The crucial ingredient of our model is that Firm 2's cash reserves are subject to liquidity shocks, driven by the Brownian motion B. For a pair of strategies $\xi := (\xi_1, \xi_2)$, the *controlled cash process* (CCP) of Firm 2 is the process X_ξ, which, for all $t \geq 0$, is defined by

$$X_\xi(t) \triangleq X(0) + \int_{0+}^{t} \pi(q_2(s), q_1(s))ds + \sigma B(t) - Z_2(t), \quad X(0) > 0. \tag{1}$$

Since Firm 2 is financially constrained, a series of negative shocks to profits can drive the firm into bankruptcy when its cash reserves are depleted. Its *bankruptcy time* is given by the stopping time

$$\tau \triangleq \inf\{\, t \geq 0 \mid X_\xi(t) \leq 0 \,\}.$$

We need to ensure that Firm 2 no longer produces anything nor pays out dividends after it has gone bankrupt and that Firm 1 acts as a monopolist after Firm 2's bankruptcy.

DEFINITION 3. *The pair of strategies* $\xi = (\xi_1, \xi_2)$ *is* admissible *if on* $\{\, t \geq \tau \,\}$ *it holds that*

1. $q_1(t) = q_M$,
2. $q_2(t) = 0$, *and*
3. Z_2 *is a constant process.*

The sets of admissible production and dividend policies for Firm i are denoted by $\mathscr{Q}_i$ and $\mathscr{D}_i$, respectively. Finally, let $\Xi \triangleq \Xi_1 \times \Xi_2 \triangleq (\mathscr{Q}_1 \times \mathscr{D}_1) \times (\mathscr{Q}_2 \times \mathscr{D}_2)$ denote the set of all pairs of admissible strategies.

$2P' + q_i P'' + |P' + q_i P''| < C''$ (**?**). In the case of affine demand, $P(q) = a - bq$ and linear production costs, cq this condition reduces to $b > 0$.

The unconstrained firm in this setting chooses its optimal quantity policy by maximizing the present value of its expected future profits. This is because in the absence of a liquidity constraint, its value does not depend on the the ability to build reserves. Thus, its cash policy is irrelevant. This is in line with the empirical results that cash flow sensitivity of cash is not statistically different from zero for financially unconstrained firms Almeida *et al.* (2004).

For a pair of strategies $\xi \in \Xi$, we define the value of Firm i as:

$$V_i(x;\xi) \triangleq \mathsf{E}_x\Big[\int_0^\infty e^{-\rho s} dZ_i(s)\Big]. \tag{2}$$

Here $\rho > 0$ denotes the risk-adjusted discount rate and E_x denotes the expectation operator associated with P under the condition that $X(0) = x$, P-a.s.

Note that firms face the following tradeoff. When earning positive profits, it may have incentives to transfer them to cash reserves to avoid bankruptcy in the future. This, in turn, comes at a cost of paying out less dividends to shareholders. Since we assumed that Firm 1 is not financially constrained it has no incentive to build up cash reserves, but may have an incentive to use its financial strength to drive the competitor out of the market.

3 The Benchmark

Before considering how interactions in the product market affect the payout policy of Firm 2, we present a benchmark in which we ignore Firm 1's dividend policy and assume that the quantity choice of each firm is fixed. Under this assumption, the drift of process X is constant and the solution to Firm 2's optimal dividend problem is well known from the literature, see, e.g., Jeanblanc-Picqué and Shiryaev (1995) and Radner and Shepp (1996). In this case the optimal dividend policy is given by the local time of process X_ξ at an endogenously determined boundary; see Proposition 1. After bankruptcy of For $(q_1, q_2) \in \mathbb{R}^2_+$ with $P(q_1 + q_2) > c$, and some admissible dividend policy $(Z_2(t))_{t\geq 0}$, we define the *constant production policies* $\tilde{q}_i \in \mathscr{Q}_i$, $i = 1, 2$, to be the (admissible) production policies for all $t \geq 0$,

$$\tilde{q}_1(t) = \begin{cases} q_1 & \text{if } t < \tau \\ q_M & \text{if } t \geq \tau \end{cases}, \quad \text{and} \quad \tilde{q}_2(t) = \begin{cases} q_2 & \text{if } t < \tau \\ 0 & \text{if } t \geq \tau \end{cases}. \tag{3}$$

For these policies, the per-period profits (on $\{t < \tau\}$) are

$$\pi_1 \triangleq \pi(q_1, q_2), \quad \text{and} \quad \pi_2 \triangleq \pi(q_2, q_1). \tag{4}$$

For any admissible dividend policy $\tilde{Z}_2 \in \mathscr{D}_2$, the controlled cash process then is

$$X_{\tilde{\xi}(t)} = X(0) + \pi_2 t + \sigma B(t) - \tilde{Z}_2(t), \quad t \geq 0,$$

where $\tilde{\xi} = (\tilde{q}_1, (\tilde{q}_2, \tilde{Z}_2))$.

PROPOSITION 1. *For the production policies in (3), with per-period profits as in (4) the optimal cash target of Firm 2, denoted by $\tilde{x}_q$, is equal to*

$$\tilde{x}_q = \frac{1}{2\Delta_2} \ln\left(\frac{\sigma^2\Delta_2^2 + \Delta_2\pi_2 - \rho}{\sigma^2\Delta_2^2 - \Delta_2\pi_2 - \rho}\right), \tag{5}$$

where $\Delta_2 = \sqrt{\left(\frac{\pi_2}{\sigma^2}\right)^2 + \frac{2\rho}{\sigma^2}}$. Furthermore, the value functions of Firms 1 and 2 are

$$v_1(x) = \begin{cases} \frac{\pi_M}{\rho} & \text{if } x \leq 0, \\ \frac{\pi_1}{\rho} + \mathrm{e}^{-\frac{\pi_2}{\sigma^2}x} \frac{(\pi_M-\pi_1)\left(\Delta_2 \cosh(\Delta_2(x-\tilde{x}_q)) + \frac{\pi_2}{\sigma^2}\sinh(\Delta_2(x-\tilde{x}_q))\right)}{\rho\left(\Delta_2\cosh(\Delta_2\tilde{x}_q) - \frac{\pi_2}{\sigma^2}\sinh(\Delta_2\tilde{x}_q)\right)} & \text{if } 0 < x < \tilde{x}_q, \\ \frac{\pi_1}{\rho} + \mathrm{e}^{-\frac{\pi_2}{\sigma^2}\tilde{x}_q} \frac{\Delta_2(\pi_M-\pi_1)}{\rho\left(\Delta_2\cosh(\Delta_2\tilde{x}_q) - \frac{\pi_2}{\sigma^2}\sinh(\Delta_2\tilde{x}_q)\right)} & \text{if } x \geq \tilde{x}_q, \end{cases} \tag{6}$$

and

$$v_2(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ \mathrm{e}^{-\frac{\pi_2}{\sigma^2}(x-\tilde{x}_q)} \frac{\sinh(\Delta_2 x)}{\Delta_2\cosh(\Delta_2\tilde{x}_q) - \frac{\pi_2}{\sigma^2}\sinh(\Delta_2\tilde{x}_q)} & \text{if } 0 < x < \tilde{x}_q, \\ (x-\tilde{x}_q) + \frac{\pi_2}{\rho} & \text{if } x \geq \tilde{x}_q, \end{cases} \tag{7}$$

respectively, where $v_1' \leq 0$, $v_1'' \geq 0$, $v_2' \geq 0$, and $v_2'' \leq 0$ on $(0, \tilde{x}_q)$.

Proposition 1 implies that the financially constrained firm's dividend policy takes the form of a reflected Brownian motion. For low levels of cash the firm should not pay out any dividends, while for large levels, the firm should pay out all its earnings. In this setting, the motives for building a cash reserve are purely precautionary and are reflected by the fact that the firm only pays out dividends once its cash reserve reaches the cash target, $\tilde{x}_q$. This cash target, in turn, reflects a balancing of current dividends against future dividends, the latter being conditional on the survival of the firm and, thus, on the firm's cash reserve.

The value function of Firm 2 is increasing and concave in x. For $x \leq 0$, the firm goes bankrupt and its value is equal to zero. For $0 < x < \tilde{x}_q$, the value function is equal to the discounted value of future dividends after reaching the payout region. For $x > \tilde{x}_q$, the firm pays out the instantaneous dividend $x - \tilde{x}_q$ and then follows the reflecting barrier policy from the cash level $\tilde{x}_q$. Thus, its value in this case is equal to the sum of an instantaneous dividend at $t = 0$ and the present value of future duopoly profits. The structure of our solution in (7) is similar to Jeanblanc-Picqué and Shiryaev (1995). The difference is that the boundary between the dividend-paying and non-dividend-paying regions, $\tilde{x}_q$, is now linked to the product market through the firms' production policies. In the special case where both firms choose to produce the static Cournot quantity, q_D, in each period, we denote Firm 2's cash target by $\tilde{x}_D$.

The value of Firm 1 is decreasing and convex in x. As soon as $X_\xi(t)$ hits zero, Firm 1 becomes a monopolist in the market and its value is equal to the discounted monopoly profits. Thus, the large cash reserves of the financially constrained competitor adversely affect the value of Firm 1 due to the decreased bankruptcy probability of Firm 2. For $x > 0$, its value function consists of the perpetual stream of duopoly profits and a correction term. For $0 < x < \tilde{x}_q$, this correction term accounts for the possibility of Firm 2's bankruptcy. For $x > \tilde{x}_q$, the correction term reflects the fact that Firm 2 may eventually leave its payout region and go bankrupt in the future.

A novel and interesting result here is that the value of the unconstrained firm is affected by the financial position of its rival, emphasizing the important interaction between financial and product markets. The channel for this interaction is corporate cash hoarding of the financially constrained firm. This result occurs even in this myopic model where firms choose a constant production policy.

Notably, if Firm 2 is a monopolist, it is in fact optimal to choose the static monopoly quantity, q_M, in each period, leading to a cash target denoted by $\tilde{x}_M$. Intuitively, in this case, by choosing the quantity that maximizes profits for each period the firm can accomplish two goals at once: to maximize the expected present value of future dividends and to increase the probability of survival. This result, however, does not hold when we introduce competition, as we demonstrate in Section 4.

4 The production–dividend game

We extend the analysis in Section 3 by allowing firms to compete in the product market. To summarize the discussion in Section 2, our production–dividend game is a tuple

$$\Gamma = \langle \{1, 2\}, (\mathscr{Q}_i \times \mathscr{D}_i, V_i)_{i=1,2} \rangle \,.$$

In our equilibrium construction we will focus on Markovian production policies, that is, we will look (with some abuse of notation) at production policies of the form $q(t) = q(X(t))$, $t \geq 0$, for some continuous function q, with the property that $q(x) \geq 0$, all $x \in \mathbb{R}$. We call this function q a *quantity function*. Continuity of the quantity functions ensures that the instantaneous profit function $x \mapsto \pi(q_2(x), q_1(x))$ is continuous, which, in turn, means that process X is well-defined. The set of continuous quantity functions q with the property that $q(0) = 0$ is denoted by $\mathscr{M}_2$, whereas the set of continuous quantity functions with the property that $q(0) = q_M$ is denoted by $\mathscr{M}_1$. The set of strategies is denoted by $\mathscr{Z} \triangleq \mathscr{Z}_1 \times \mathscr{Z}_2$, where $\mathscr{Z}_i \triangleq \mathscr{M}_i \times \mathscr{D}_i$. With every $\zeta_i \in \mathscr{Z}_i$ we associate the strategy $\xi_i^{\zeta_i} = (q_i^{\zeta_i}, Z_i) \in \mathscr{Q} \times \mathscr{D}$, where for all $t \geq 0$ we define

$$q_i^{\zeta_i}(t) \triangleq \begin{cases} q_i(X(t)) & \text{if } t < \tau, \\ q_M & \text{if } t \geq \tau \text{ and } i = 1, \\ 0 & \text{if } t \geq \tau \text{ and } i = 2. \end{cases}$$

DEFINITION 4. *A collection of strategies $\bar{\zeta} = (\bar{\zeta}_1, \bar{\zeta}_2) \in \mathscr{Z}$ is a* Nash equilibrium in Markovian quantity strategies *(NMS) in the production–dividend game Γ if for every $i \in \{1, 2\}$, $\zeta_i \in \mathscr{Z}_i$, and*

$x \geq 0$, *it holds that*

$$V_1(x; \xi^{\bar{\zeta}_i}, \xi^{\bar{\zeta}_j}) \geq V_1\left(x; \xi^{\zeta_i}, \xi^{\bar{\zeta}_j}\right).$$

It is obvious that for Firm 1 it is a dominant strategy to pay out any positive profits as dividends, because this firm never has an incentive to hoard cash. So, in what follows, we will focus on the equilibrium dividend policy of Firm 2.

Consider an interval $E = (0, \bar{x})$ and a pair of admissible quantity functions $q = (q_1, q_2) \in \mathscr{M}_1 \times \mathscr{M}_2$, such that $P(q_1(x) + q_2(x)) > 0$ for all $x \in E$.[2] The *characteristic operator* on the set of twice differentiable functions on E, $C^2(E)$, is given by

$$\mathscr{L}^q \varphi(x) \triangleq \frac{1}{2}\sigma^2 \varphi''(x) + \pi(q_2(x), q_1(x))\varphi'(x) - \rho\varphi(x), \quad x \in E. \tag{8}$$

The characteristic operator captures several drivers of the firms' value dynamics. The first term represents the noise related to market uncertainty. The second term captures the effect of product market competition in the presence of a liquidity constraint. If the constrained firm had unlimited access to outside capital this term would be zero, because there would be no incentives to hoard cash. The second term can also be interpreted as the value of predation for the unconstrained firm, since in the absence of predatory incentives, the value function would not be affected by the change in its rival's cash stock.

The *Hamilton-Jacobi-Bellman* (HJB) equations for the firms are defined on $C^2(E)$ as

$$\mathscr{H}_1^{q_2}\varphi(x) \triangleq \sup_{q_1 \geq 0} \left\{ \mathscr{L}^{q_1,q_2}\varphi(x) + \pi(q_1, q_2(x)) \right\} = 0, \quad x \in E, \quad \text{and} \tag{9}$$

$$\mathscr{H}_2^{q_1}\varphi(x) \triangleq \max\left\{ \sup_{q_2 \geq 0} \mathscr{L}^{q_1,q_2}\varphi(x), 1 - \varphi'(x) \right\} = 0, \tag{10}$$

respectively.

With every $\varphi \in C^2$ that satisfies (10), we associate the *continuation region*,i.e. the region where Firm 2 does not pay out dividends,

$$\mathscr{C}_2^{q_1,\varphi} = \left\{ x > 0 \,\middle|\, \sup_{q_2 \geq 0} \mathscr{L}^{q_1,q_2}\varphi(x) = 0 \text{ and } \varphi'(x) > 1 \right\}, \tag{11}$$

and its complement, where Firm 2 pays out dividends,

$$\mathscr{S}_2^{q_1,\varphi} = \left\{ x > 0 \,\middle|\, \sup_{q_2 \geq 0} \mathscr{L}^{q_1,q_2}\varphi(x) < 0 \text{ and } \varphi'(x) = 1 \right\}. \tag{12}$$

Note that

$$\mathscr{C}_2^{q,\varphi} \cup \mathscr{S}_2^{q,\varphi} = (0, \infty) \text{ and } \mathscr{C}_2^{q,\varphi} \cap \mathscr{S}_2^{q,\varphi} = \emptyset.$$

[2] Here $\bar{x}$ is going to play the role of the constrained firm's cash target.

Furthermore, for any $\zeta \in \mathscr{Z}$, we define the process X^ζ, by

$$
\begin{aligned}
X^\zeta(t) \triangleq & X(0) + \int_0^t \pi\left(q_2(X^\zeta(s)), q_1(X^\zeta(s))\right) ds \\
& + \sigma B(t) - Z(t), \quad t \geq 0,
\end{aligned}
\tag{13}
$$

with bankruptcy time

$$
\tau^\zeta \triangleq \inf\left\{ t \geq 0 \,\middle|\, X^\zeta(t) \leq 0 \right\}.
$$

DEFINITION 5. *Let $q = (q_1, q_2) \in \mathscr{M}_1 \times \mathscr{M}_2$ be a pair of admissible quantity functions and assume that $\varphi \in C^2$ is such that $\mathscr{H}_2^{q_1}\varphi = 0$. The* dividend process associated with q and φ *is an* $\mathbf{F}$*-adapted, non-negative, and non-decreasing process $Z \in \mathscr{D}$ such that,*

1. *$X^{q,Z}(t) \in \bar{\mathscr{C}}_2^{q,\varphi}$, on $\{ t < \tau^{q,Z} \}$, and*
2. *$\int_0^{\tau^{q,Z}} \mathbb{1}_{X^{q,Z}(t) \in \mathscr{C}_2^{q,\varphi}} dZ(t) = 0$.*

We denote the dividend process associated with q and φ by $Z^{q,\varphi}$. The first condition ensures that $Z^{q,\varphi}$ is such that the closure of the continuation region is never exited. The second condition ensures that $Z^{q,\varphi}$ only has local time on the boundary of $\mathscr{C}_2^{q,\varphi}$, that is, that dividends are only paid on $\partial\mathscr{C}_2^{q,\varphi}$. That is, $Z^{q,\varphi}$ makes $X^{q,Z}$ a *reflected Brownian motion.*

In the remainder of the paper we will focus on a market with a linear inverse demand function, $P(q) = a - bq$ and linear cost function $C(q) = cq$, for $0 < c < a$ and $b > 0$. For a function $\varphi_1 \in C^2$, we define, on $\{x > 0 | \varphi_1'(x) < 3\}$, mappings $x \mapsto \bar{q}_1(x)$ and $x \mapsto \bar{q}_2(x)$ by

$$
\begin{aligned}
\bar{q}_1(x) &\triangleq \frac{a-c}{b}\frac{1-\varphi_1'(x)}{3-\varphi_1'(x)}, \quad \text{and} \\
\bar{q}_2(x) &\triangleq \frac{a-c}{b}\frac{1}{3-\varphi_1'(x)},
\end{aligned}
\tag{14}
$$

respectively.

These functions describe the instantaneous quantity choices in (Cournot) equilibrium. They follow from the best-response equations

$$
\begin{aligned}
q_1 &= \frac{a-c}{2b} - \frac{1}{2}(1 + \varphi_1'(x))q_2, \quad \text{and} \\
q_2 &= \frac{a-c}{2b} - \frac{1}{2}q_1,
\end{aligned}
$$

respectively. Note here that the reaction curve of Firm 2 is of the same form as in the static Cournot model. This is the case because this firm has no strategic power in this game. Thus, Firm 2 acts as a *prey* and simply best-responds to the actions of Firm 1, who acts as the *predator*. The predator's reaction curve is now affected by the sensitivity of its value to changes in the prey's cash reserves. Thus, $\varphi_1'(x)$ acts as a measure of the aggressiveness of competition: the larger it is in absolute value, the greater the incentive to the predator of deviating from the static Cournot best response.

The following proposition gives a verification result for the construction of an NMS.

PROPOSITION 2. *Suppose that there exist*

1. *a constant* $\bar{x} > 0$, *and*
2. *functions* $\varphi_1, \varphi_2 \in C^2$ *on* $(0, \bar{x})$, *with* $\varphi_2' > 1$ *on* $(0, \bar{x})$,

such that

1a) $\varphi_1(0) = \pi_M/\rho$	*1b)* $\varphi_2(0) = 0$,
2a) $\mathscr{L}^{\bar{q}_1,\bar{q}_2}\varphi_1 + \pi(\bar{q}_1, \bar{q}_2) = 0$ *on* $(0, \bar{x})$	*2b)* $\mathscr{L}^{\bar{q}_1,\bar{q}_2}\varphi_2 = 0$ *on* $(0, \bar{x})$,
3a) $\varphi_1'(\bar{x}-) = 0$	*3b)* $\varphi_2'(\bar{x}-) = 1$,
	4b) $\varphi_2''(\bar{x}-) = 0$,

Then the pair of strategies $\bar{\zeta} = (\bar{q}_i, \bar{Z}_i)_{i=1,2}$, *with* $(\bar{q}_1, \bar{q}_2)$ *as in* (14),

$$
\begin{aligned}
d\bar{Z}_1(t) &= 1_{X(t)>0}\pi(\bar{q}_1(x), \bar{q}_2(x))dt + 1_{X(t)\leq 0}\pi_M dt, \quad \textit{and} \\
\bar{Z}_2(t) &= 1_{t<\tau^{\bar{q},\bar{Z}_2}} Z^{\bar{q},\varphi_2}(t) + 1_{t\geq\tau^{\bar{q},\bar{Z}_2}} Z^{\bar{q},\varphi_2}(\tau^{\bar{q},\bar{Z}_2}),
\end{aligned}
$$

is an NMS of the production–dividend game Γ. *The value functions of the firms in this equilibrium are*

$$
V_1(x) = \begin{cases} \frac{\pi_M}{\rho} & \textit{if } x \leq 0 \\ \varphi_1(x) & \textit{if } 0 < x < \bar{x} \\ \varphi(\bar{x}) & \textit{if } x \geq \bar{x}, \end{cases}
$$

and

$$
V_2(x) = \begin{cases} 0 & \textit{if } x \leq 0 \\ \varphi_2(x) & \textit{if } 0 < x < \bar{x} \\ (x - \bar{x}) + \varphi_2(\bar{x}) & \textit{if } x \geq \bar{x}, \end{cases}
$$

respectively.

An equilibrium constructed in this way, assuming it exists, implies that, similar to the monopoly case, Firm 2's dividend policy is defined by a reflecting a barrier, $\bar{x}$, on its cash process. The new result now is that in the region where the constrained firm does not pay out dividends, the unconstrained firm has an incentive to engage in aggressive competition in the product market.

In equilibrium, the dividend-paying region and the region of aggressive competition are separated by the cash target, $\bar{x}$. This implies that predation only occurs when the financially constrained firm (the prey) does not pay out dividends. This result is intuitively clear because the only channel through which the unconstrained firm (the predator) can accelerate the bankruptcy time of the prey is the rate at which the latter accumulates cash. By lowering the profits of the prey, the predator lowers the drift of the prey's cash process, thereby increasing the probability of bankruptcy. However, once the prey starts paying out dividends it stops accumulating cash reserves. Although the drift of its cash process at $\bar{x}$ is equal to the (static) Cournot profit, its effective drift is zero because any positive profit is paid out in dividends. In our setting, firms can guarantee non-negative expected profits each period (because there are no fixed production costs). Therefore, in expectation, the predator is no longer able to affect the bankruptcy probability of the prey. Hence, the game ends up in the (static) Cournot equilibrium at $\bar{x}$.

An important question is whether the existence of an equilibrium in the production–dividend game is guaranteed. The following proposition answers in the affirmative and, in fact, proves the uniqueness of equilibria that can be constructed using Proposition 2.

PROPOSITION 3. *There exist unique functions φ_1 and φ_2 and a unique threshold $\bar{x} > 0$ that satisfy the conditions of Proposition 2.*

Throughout the remainder of this paper, we will refer to the equilibrium constructed in this way as the *predation equilibrium*. In this equilibrium, the financially unconstrained firm acts as the predator, whereas the financially constrained firm acts as the prey.

5 Results

The distinctive feature of our model is that the expected cash flow growth of the constrained firm is not exogenous. Rather, it is controlled by both firms. In particular, free cash flows depend on their production decisions in the product market. Notably, in a duopoly setting without profitability shocks the growth rate of the free cash flow is constant in equilibrium. In this case, firms can perfectly predict their future operational profits, leading to the result that it is always optimal for firms to choose (static) Cournot quantities. Our model accommodates a more general situation where the presence of profitability shocks causes deviations from this equilibrium. We can generate such deviations without imposing a complex dynamics on the cash process by simply assuming that disturbances in realized profits are white noise.

In this section, we present the implications of this feature for the firms' value functions, stock returns, production strategies and payout policies.

5.1 Value functions and product market effects

In our production–dividend game, continuous usage of quantity as a strategic instrument by Firm 1 in manipulating the competitor's liquidity position affects the value functions of both firms. Proposition 4 summarizes the main properties of the equilibrium value functions.

PROPOSITION 4. *The equilibrium value function of the unconstrained firm, V_1, is a decreasing and convex function of the level of cash reserves of the constrained firm, x.*

The equilibrium value function of the constrained firm, V_2, is increasing in its cash reserves and is concave for values of x close to zero as well as for values of x close to the cash target, $\bar{x}$.

Figure 1 presents a numerical illustration of the equilibrium value functions. In the following examples, we use a linear inverse demand curve $P(q) = a - bq$ and constant marginal production costs, $C(q) = cq$, for $0 < c < a$ and $b > 0$. Because in an arithmetic model, the drift and volatility terms measure absolute rather than relative quantities, the volatility parameter is not scale-free. We, therefore, introduce the scale-free parameter ϕ to determine appropriate volatility levels. For a given value of $\phi \in (0, 1)$, we find the volatility level σ which gives $(1 - \phi)\pi_D$ as the lower bound in a 95% confidence interval of one-period profit deviation from the static Cournot profit, π_D.

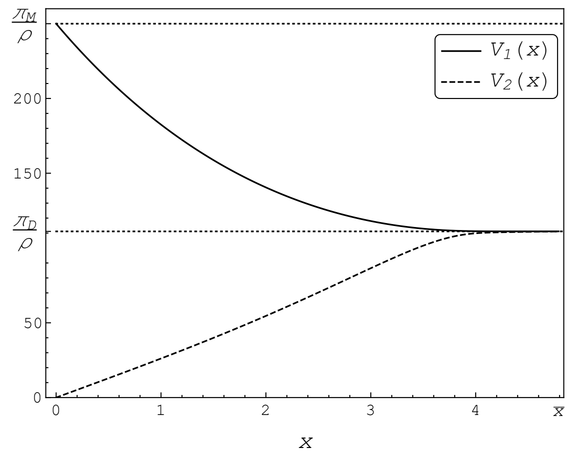


Figure 1: The value functions for different level of the cash reserves for the following set of parameter values: $(a - c) = 5$, $b = 0.5$, $\rho = 0.05$, and $\phi = 0.3$ (which gives $\sigma = 0.8503$).

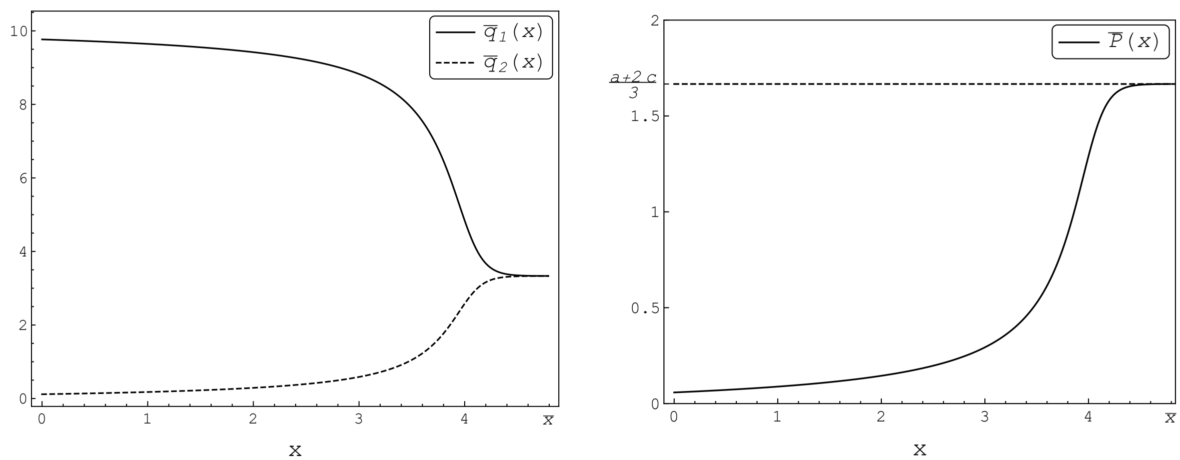
Note that once the cash target is reached, the equilibrium value function of Firm 2 is equal to the discounted value of an infinite stream of (deterministic) Cournot profits. However, Firm 1's equilibrium value function is somewhat higher: $V_1(\bar{x}) > V_2(\bar{x}) = \pi_D/\rho$. To interpret this result, recall that the optimal strategy of the constrained firm is to pay out all cash above $\bar{x}$, and not to distribute anything to the shareholders otherwise. Thus, from the moment its cash reserves hit $\bar{x}$ for the first time, the constrained firm expects to earn the discounted value of its deterministic profits. In addition, this means that the cash reserves of Firm 2 remain constant in expectation (the cash process is now a martingale). For the unconstrained firm, this implies that its value does not change with x, and the optimal production strategy is to maximize its per-period profits, which leads to static Cournot quantities in equilibrium for both firms. However, similar to our benchmark case, there is still a positive probability that Firm 2 goes bankrupt leading to the monopoly position for Firm 1. Hence, the equilibrium value function of Firm 1 is strictly larger than that of Firm 2 at $\bar{x}$.

Another important observation here is that in comparison to the myopic strategy of producing static Cournot quantities, the shareholders of the predator gain value, whereas the shareholders of the prey lose value due to aggressive competition in the product markets. Proposition 5 compares the values of the firms in the predation equilibrium and under the myopic strategy identified in our benchmark model.

PROPOSITION 5. *In the predation equilibrium, the value of Firm 1 is at least as large as under the myopic strategy, that is, $V_1 \geq v_1$. The equilibrium value of Firm 2 does not exceed its value* *under the myopic strategy, that is,* $V_2 \leq v_2$.

Thus, predation in the product markets results in value transfer from the shareholders of the prey and to the shareholders of the predator. In other words, there is a positive strategic externality for the unconstrained firm and a negative strategic externality for the constrained firm.

We now illustrate how the financial market effects feed through into the product market in our game's equilibrium. Figure 2 shows the firms' (instantaneous) equilibrium quantities, $\bar{q}_1$ and $\bar{q}_2$, as well as the (instantaneous) market price, in equilibrium, $\bar{P}$, as functions of the constrained firm's cash reserves.



(a) The equilibrium quantities

(b) The equilibrium price

Figure 2: The equilibrium quantities and price for different level of the cash reserves for the following set of parameter values: $(a - c) = 5$, $b = 0.5$, $\rho = 0.05$, and $\phi = 0.3$, (so that $\sigma = 0.8503$).

The dividend policy of the constrained firm creates a reflecting barrier on its cash process at $\bar{x}$, so that it pays out all cash in excess of this cash target. As seen in Figure 2(a), the unconstrained firm behaves as a predator in the product market, by increasing its quantity and thus indirectly lowering the price, in the region $(0, \bar{x})$ where the constrained firm does not pay out dividends. This affects the quantity choice of the constrained firm, which is forced to downscale its production to avoid bankruptcy because output choices are strategic substitutes. This results in a lower equilibrium price than one would obtain if firms would choose static Cournot quantities in every period, as is evident from Figure 2(b). However, if the constrained firm starts out with large enough cash reserves, in particular if they exceed the cash target $\bar{x}$, it immediately pays out dividends and the quantities equal the static Cournot equilibrium quantities.

This leads to the important result that the drift of the cash process of the constrained firm is not always constant when we allow for stochastic shocks, as illustrated in Figure 3. The first observation here is that the drift of the cash process of the cash-constrained firm is always weakly lower than in the situation without predation, where it always grows at a rate equal to the static Cournot duopoly profit. Another interesting effect illustrated in Figure 3 is that the instantaneous

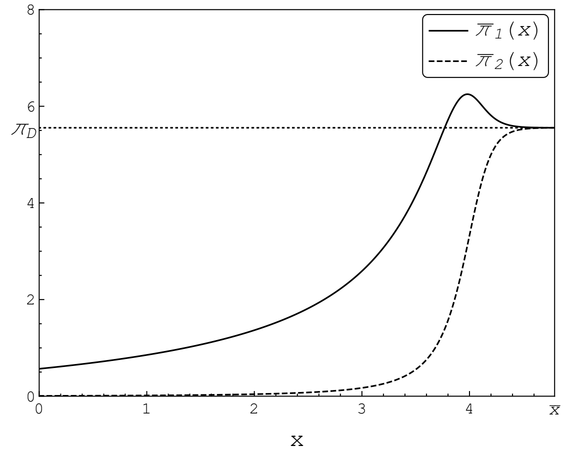


Figure 3: The instantaneous equilibrium profits for different level of the cash reserves for the following set of parameter values: $(a-c)=5$, $b=0.5$, $\rho=0.05$, and $\phi=0.3$ (so that $\sigma=0.8503$).

profit of Firm 1 is a non-monotonic function of x. Notably, the aggressive behavior in the product market may even lead to a higher per-period profit than in a static Cournot equilibrium.

The non-monotonic behavior of the instantaneous profit function of Firm 1 is driven by the fact that when it is making its quantity choice, it balances the future and instantaneous benefits and costs of overproduction (compared to the static Cournot quantity). For low values of x, the cash-constrained firm is likely to go bankrupt. Then the primary driver for the aggressive pricing policy of Firm 1 is to ensure a future monopoly position. In particular, it sacrifices immediate profits in order to increase the bankruptcy probability of the constrained firm and drive the competitor out of the market. The main implication here is that our model generates predation according to the classical definition (Joskow and Klevorick, 1979). Namely, the unconstrained firm resorts to aggressive pricing to increase the likelihood of bankruptcy for the competitor. The instantaneous profit then increases with x, since Firm 1 has less incentive to predate and, thus, to sacrifice immediate profits, when facing a financially stronger competitor.

In addition, we can identify another type of predatory behavior that does not lead to a decline in the instantaneous profits. This result is summarized in the following proposition.

PROPOSITION 6. *In our setting with affine demand and linear marginal costs, it holds that if* $V_1'(x) > -3$*, it holds that the instantaneous equilibrium profit of the unconstrained firm is larger than the static Cournot profit.*

For larger values of x, Firm 1 still overproduces, but bankruptcy of the competitor becomes less likely. The primary driver of overproduction is the ability to capture larger instantaneous profits by inducing the other firm to downscale its production. The constrained firm will do so, because it still cares about building up cash rather than paying out dividends.

5.2 Cash target

Our model also allows for an analysis in the other direction, that is, of how predation in product markets affects the dividend payout policy of the constrained firm. The key insight here is that, in the presence of predatory behavior, the financially constrained firm postpones its dividend payouts and builds up cash instead. This result is summarized in the following proposition, which compares the cash target in our predatory equilibrium to the (myopic) scenario where both firms always produce the static Cournot quantities that we analyzed in Section 3.

PROPOSITION 7. *The cash target in equilibrium in the production–dividend game is always higher than under the (myopic) strategy of always producing static Cournot quantities, that is,* $\bar{x} > \tilde{x}_D$.

Thus, we identify strategic interactions as an additional channel through which profit shocks and product market characteristics affect the optimal dividend policy. Figures 4 and 5 illustrate this effect by comparing two different cases: our predation equilibrium and the (myopic) strategy of always producing (static) Cournot quantities, as established in Proposition 1.

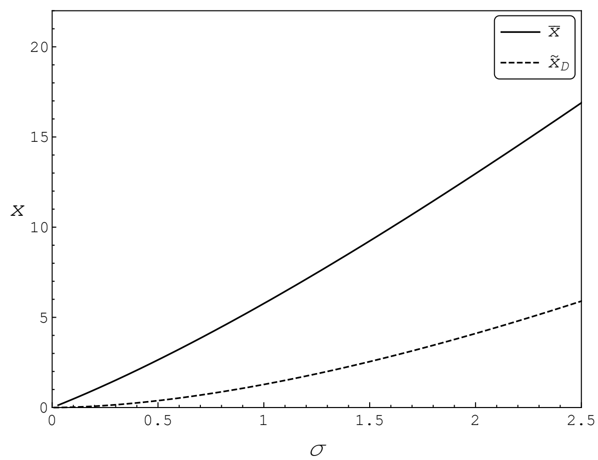


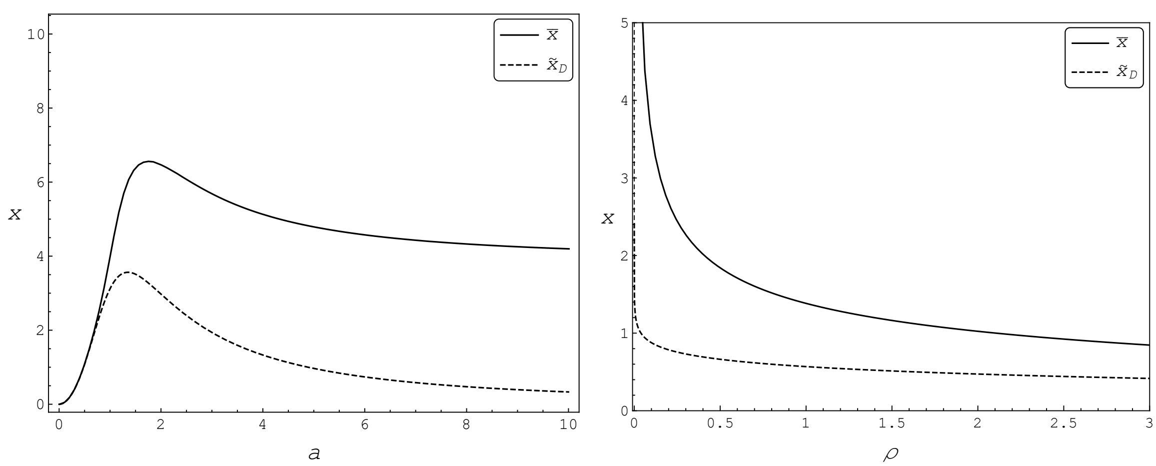
Figure 4: The cash targets in equilibrium duopoly, $\bar{x}$, and myopic duopoly, $\tilde{x}_D$, for the following set of parameter values: $b = 0.5$, $(a - c) = 5$, $\rho = 0.05$ and different values of σ.

Figure 4 depicts the effect of changes in the cash target resulting from an increase in volatility. If the constrained firm faces higher cash flow volatility, then the cash target, $\bar{x}$, increases. This is due to a classical precautionary effect that is also present if we do not take product market competition into account. In our model, this precautionary effect is amplified by a strategic effect. The closer the constrained firm is to bankruptcy, the further the equilibrium quantities will deviate from the static Cournot equilibrium quantities, causing the constrained firm to downscale its production. This results in a lower drift of its cash process, which, in turn, implies a larger cash target.

As a result, although dividend-paying and predation are mutually exclusive events, there exists an important feedback between them through the cash target. This target, $\bar{x}$, acts as the boundary between the dividend-paying and non-dividend-paying regions and is determined endogenously. It is affected by the presence of cash flow shocks both directly and indirectly. The direct (precautionary motive) and indirect (strategic effect, aggressiveness of competition) effects work in the same direction and increase the bankruptcy probability of the constrained firm. In turn, this leads to a higher cash target for this firm's dividend payouts. Thus, for larger values of cash flow volatility the difference between accommodating and predatory cash targets increases.

Note that in this model, for the limiting case where $\sigma = 0$, the optimal strategy of the firms would be to choose static Cournot quantities at every time $t \geq 0$ if the initial cash reserves of the constrained firm are positive. This is because, in the absence of shocks, the constrained firm can guarantee non-negative profits for each period, which can only increase its cash reserves over time. This, in turn, ensures that the constrained firm never goes bankrupt, so that the unconstrained firm has no incentive to engage in predatory pricing. Therefore, the optimal dynamic strategy for both firms is to maximize their instantaneous profits in every period, leading to the static Cournot equilibrium in every period.

Figure 5 illustrates the changes in the cash target as a result of an increase in the intercept of the inverse demand function, a, and the discount rate, ρ.



(a) For $\rho = 0.05$ and different values of $(a - c)$.

(b) For $(a - c) = 5$ and different values of ρ.

Figure 5: The equilibrium and myopic cash targets, $\bar{x}$ and $\tilde{x}_D$, respectively, for the parameter values: $b = 0.5$, and $\phi = 0.3$ (and, thus, $\sigma = 0.8503$).

First, consider the effect of a change in $(a - c)$ in Figure 5(a). Here, a can be interpreted as the maximum willingness to pay (by consumers) in the product market and c is the marginal production cost. The first interesting feature here is that the cash target is non-monotonic in $(a - c)$ in both the predatory and accommodating scenarios. This is because a higher value of $(a - c)$ increases the drift of the cash flow process. On the one hand, if the firm's cash flow grows faster (in expectation), it can afford to be "safer" once it starts paying out dividends by postponing the payout decision. On the other hand, a larger drift implies a smaller bankruptcy probability, which means that the constrained firm can afford an earlier payout. The value of the additional safety decreases with $(a - c)$, implying that the latter effect dominates for large values of $(a - c)$.

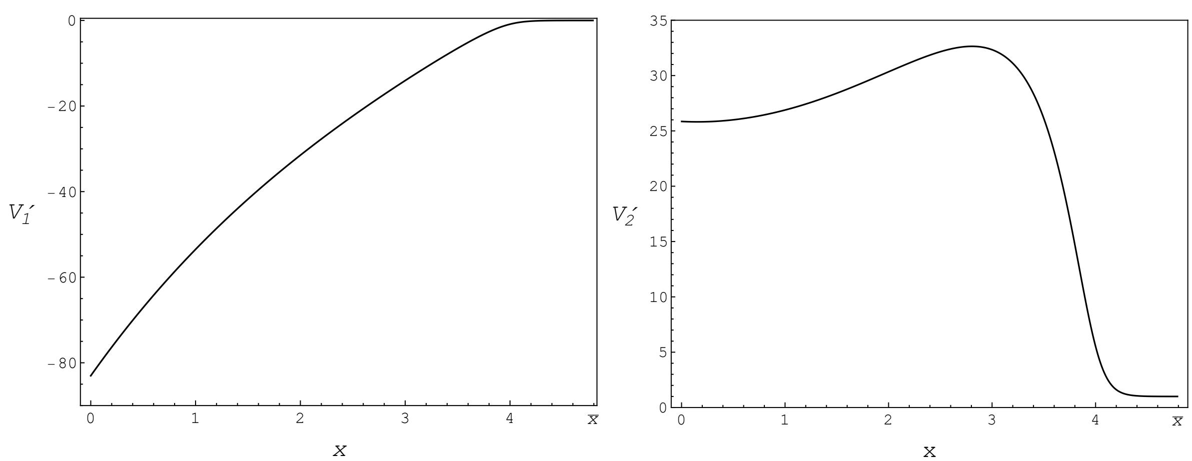
Another observation is that the largest difference between the predatory and accommodating scenarios is observed for intermediate values of $(a - c)$. This is because, on the one hand, for small values of $(a - c)$, the drift of the cash process of the constrained firm is small and the probability of bankruptcy is, consequently, large. Hence, predatory pricing does not contribute much to the resulting increase in the cash target. On the other hand, if $(a - c)$ becomes very large the predation incentive decreases, because the prey's cash reserves accumulate faster. This implies a larger impact of predation on the cash target for intermediate values of $(a - c)$.

From Figure 5(b) it is evident that an increase in the discount rate results in earlier dividend payouts as the firm becomes more impatient. The difference between the predatory and accommodating scenarios becomes smaller for large values of ρ as the incentive for aggressive competition is weakened by both sooner dividend payouts and more heavily discounted future monopoly gains.

5.3 Marginal value of cash

We now investigate the effect of product market competition on the marginal value of cash. Previous studies generally show that the value of every additional unit of cash for a constrained firm declines with an increase in its cash reserves (Décamps *et al.*, 2011; Faulkender and Wang, 2006). Here we show that this result does not always hold in the presence of predation.

Figure 6(a) depicts the marginal (equilibrium) value of the rival's cash for the unconstrained firm.



(a) For Firm 1.

(b) For Firm 2.

Figure 6: The marginal value of Firm 2's cash for different level of the cash reserves for the following set of parameter values: $(a - c) = 5$, $b = 0.5$, $\rho = 0.05$, and $\phi = 0.3$ (which gives $\sigma = 0.8503$).

From (14), V_1' represents the deviation from the static Cournot model and, thus, can be interpreted as a measure of the aggressiveness of competition in the product market. A larger value for V_1' implies a larger effect of predation on the equilibrium quantity choices at cash level x. First, as expected, V_1' is negative and approaches zero as the liquidity position of the constrained firm approaches its cash target. This reflects that the financial strength of the constrained firm has a negative impact on the value of the unconstrained firm. From Proposition 4, $V_1'' > 0$, which implies that the biggest incentive to predate occurs for small values of x, that is, when the cash-constrained firm is in a weak financial position. This is because acting aggressively increases the probability of bankruptcy of the constrained firm, which leads to a swift monopoly position for the unconstrained firm. Once the constrained firm starts paying out dividends, the unconstrained firm's incentives to predate disappear, as manipulating the drift of the constrained firm will not lead to an economically significant increase in the latter's bankruptcy probability.

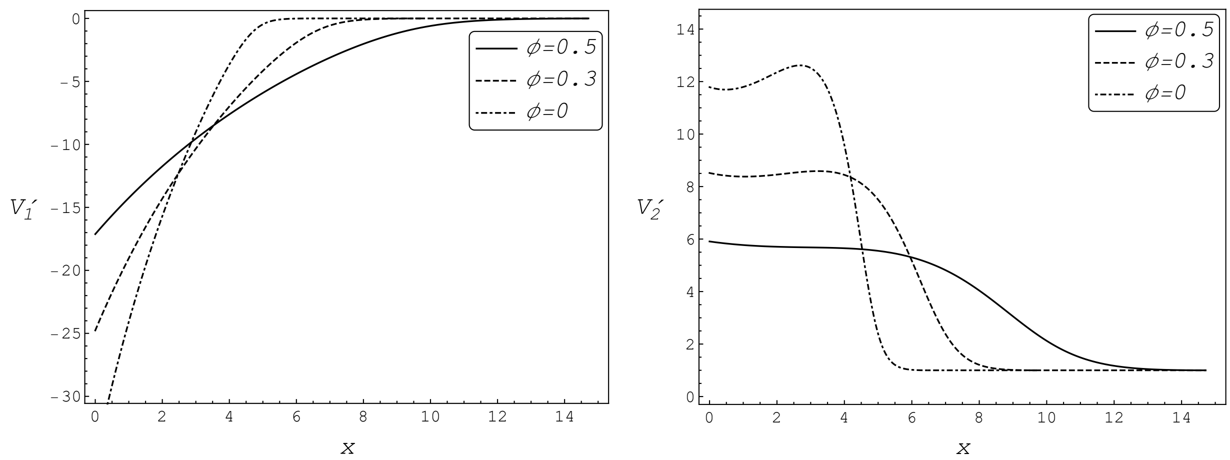
The results of Proposition 4 also imply that the marginal value of cash for the constrained firm may exhibit non-monotonic behavior. This effect arises due to the aggressive behavior of Firm 1 in the product market, because if the firms do not deviate from the static Cournot quantities, the marginal value of cash of Firm 2 declines, as follows from Proposition 1. If there exist cash levels for which the impact of product market competition is large relative to the overall (equilibrium) value, then the marginal value of cash for Firm 2 may increase in our model. We illustrate this situation in Figure 6(b).

Thus, whereas the literature predicts that the marginal value of cash is decreasing, our model predicts that the marginal value of cash can be increasing for intermediate values of the constrained firm's cash reserves. This is because in our model there are two opposing effects on firm value as the result of an additional unit of cash. On the one hand, there are the usual decreasing returns to investors' decreasing marginal utility of consumption. On the other hand, the marginal value of cash increases because it lowers the predatory threat and, thus, increases the constrained firm's chances of survival. The main empirical implication of our result is that the marginal value of cash is time varying and is affected by the firm's financial strength.

In what follows, we numerically investigate how these results depend on the underlying parameters. In particular, we find that the effect of predation leading to a non-monotonic marginal value of cash is strongest in environments with lower volatility, more profitable product markets and a smaller discount rate.

Figure 7 illustrates the marginal values of Firm 2's cash for different level of ϕ and, thus, σ.

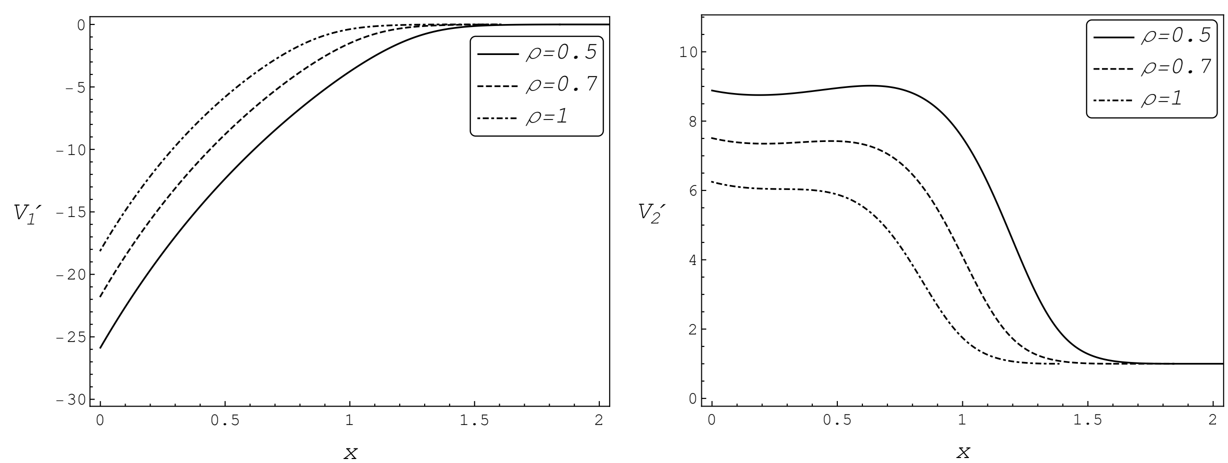
An increase in ϕ (and, thus, in σ) has an ambiguous effect on the aggressiveness of product market competition represented by V_1'. Notably, the highest cash flow volatility does not always directly translate into the fiercest predation for all cash levels. Instead, the contribution of the cash flow volatility of Firm 2 to Firm 1's predatory incentives is the largest for intermediate levels of cash reserves of the constrained firm. Intuitively, an increase in the cash flow volatility of the constrained firm has two effects on the incentives for aggressive competition. On the one hand, it increases the bankruptcy probability of Firm 2, thereby reducing the incentive to predate. On the other hand, the constrained firm postpones its dividend payouts, and thus, is exposed to predation even for larger cash balances. The former effect dominates for low values of x, when Firm 2 is the most vulnerable. Furthermore, we observe that in a more volatile environment the marginal value of cash for Firm 2 is strictly declining. This is because for high volatility, the contribution of an additional unit of cash to the probability of survival is relatively small.



(a) For Firm 1. (b) For Firm 2.

Figure 7: Marginal firm values as functions of Firm 2's cash reserves for the following set of parameter values: $(a - c) = 5$, $b = 0.5$, $\rho = 0.1$, and different values of ϕ (and, thus, σ).

Figure 8 illustrates the sensitivity of the marginal values to changes in the discount rate, ρ.

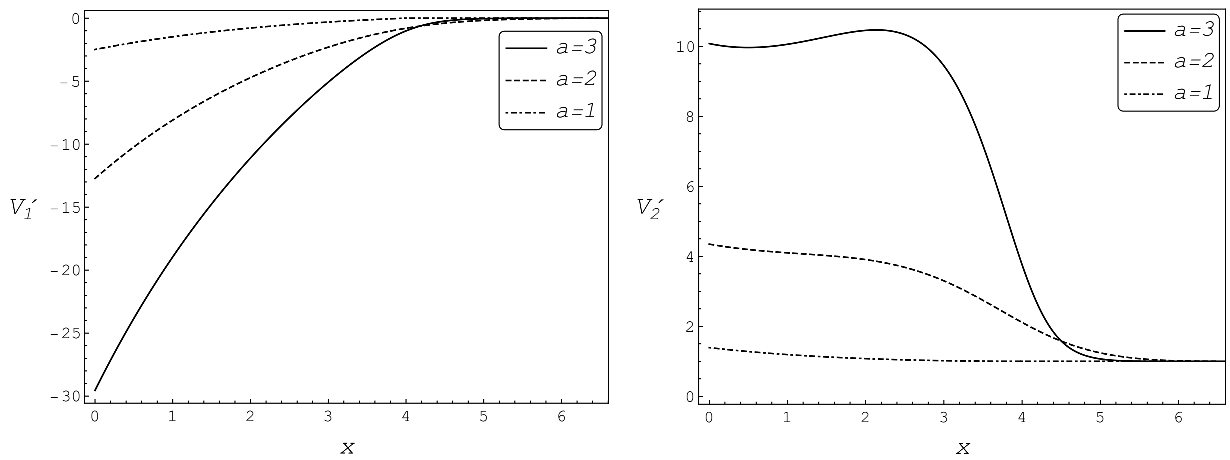


(a) For Firm 1. (b) For Firm 2.

Figure 8: The marginal value of Firm 2's cash for different level of the cash reserves for the following set of parameter values: $(a - c) = 5$, $b = 0.5$, and $\phi = 0.3$ (which gives $\sigma = 0.8503$), and different values of ρ.

As Figure 8 shows, the incentives to predate are lowest for environments with larger discount rates. This is because in such cases Firm 1 discounts its future payoffs more heavily, and thus, is less willing to sacrifice its immediate profits for a future monopoly position by engaging in predation.

Figure 9 depicts how the marginal value of the constrained firm's cash to both firms is affected by the demand intercept, a. We observe that for larger demand intercepts the predator behaves more aggressively. This is driven by the fact that, for larger a, the prospect of a future monopoly becomes more attractive. However, at the same time the competitor accumulates cash reserves



(a) For Firm 1. (b) For Firm 2.

Figure 9: The marginal value of Firm 2's cash for different level of the cash reserves for the following set of parameter values: $c = 0$, $b = 0.5$, $\rho = 0.05$, and $\phi = 0.3$ (which gives $\sigma = 0.8503$), and different values of a.

faster (in expectation) so that more effort is required to increase the probability of bankruptcy. As x increases and the bankruptcy of Firm 2 becomes less likely, the unconstrained firm has fewer incentives to predate because the duopoly profits also increase for larger values of a. The results for x close to the value of $\bar{x}$ are also driven by the non-monotonic effect of a change in a on the cash target of the constrained firm.

5.4 Stock return dynamics

We now derive pricing formulas for the stocks of both firms. In the absence of arbitrage opportunities, the stock price must be equal to the expected present value of future dividends. Given the equilibrium strategies from Proposition 2, the equilibrium stock price dynamics for Firms 1 and 2 are given by the equations

$$S_1(t) = \mathsf{E}\Big[\int_t^\infty e^{-\rho(s-t)}\pi(\bar{q}_1(s), \bar{q}_2(s))ds\Big|\mathscr{F}_t\Big], \quad t \geq 0, \tag{15}$$

$$S_2(t) = \mathsf{E}\Big[\int_t^\infty e^{-\rho(s-t)}d\bar{Z}_2(s)\Big|\mathscr{F}_t\Big], \quad t \geq 0, \tag{16}$$

respectively.

This implies that the stock prices at any time $t \geq 0$ are functions of the current cash position of the constrained firm. Using the fact that the value functions satisfy the HJB equations, a straightforward application of Itō's lemma shows that

$$\frac{dS_1(t)}{S_1(t)} = \rho dt - \sigma_1(S(t))dB(t), \tag{17}$$

$$\frac{dS_2(t)}{S_2(t)} = \rho dt + \sigma_2(S(t))dB(t) \tag{18}$$

where the stock return volatility of Firm i is denoted by $\sigma_i(v) = \left|\sigma \frac{V_i'(V_i^{-1}(v))}{v}\right|$ for $i = 1, 2$.

Evidently, the volatility of the stock returns of both firms is heteroskedastic. For the prey, this result is in line with Décamps *et al.* (2011), although here it is due to the presence of predation. An interesting and novel result in our model is that the predator's returns are also heteroskedastic even though the predator is not *directly* affected by the stochastic shocks to the prey's cash hoard. This is because the stochastic process that drives liquidity in our model is not exogenously given, and thus the liquidity of the constrained firm becomes a risk factor affecting the value of the unconstrained firm. A financially strong firm is in a position to manipulate the idiosyncratic volatility of its competitor, and it is in the interest of the shareholders to accept this higher volatility due to the upside for Firm 1 to become a monopolist. Thus, the private risk of a cash constrained firm has an effect on the other firm's stock return dynamics through the actions of the latter in the product markets. While we analyze the issue at the industry level, a recent paper by Babenko *et al.* (2016), which makes a similar point about a link between idiosyncratic risk to systematic risk.

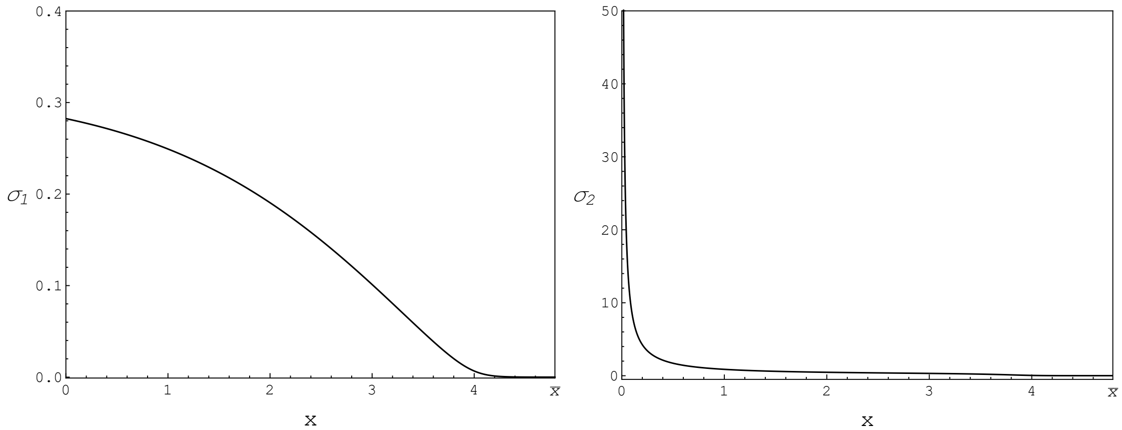
Proposition 8 identifies the effect of an increase in the cash hoard of Firm 2 on the stock return volatility for both firms. It states that stock return volatility is higher when the constrained firm is in a financially weaker position. Thus, in the context of our model, high stock return volatility is a symptom of high levels of predation and, thus, of a high likelihood of the emergence of a monopoly in the product market.

PROPOSITION 8. *The stock return volatility of both firms declines with the cash position of the constrained firm,* $\sigma_2'(x) < 0$, *and* $\sigma_1'(x) < 0$.

Figure 10 illustrates the volatility of returns for each firm as a function of the level of cash reserves of the constrained firm.

When the cash reserves of Firm 2 are close to zero, Firm 1 engages in the most aggressive competition, increasing the probability of Firm 2's bankruptcy and, thus, making a monopoly scenario highly likely. If, however, the constrained firm is strong enough, the unconstrained firm does not have incentives to behave aggressively, thereby increasing the likelihood of a duopoly scenario. The largest volatility of stock returns is attained under fiercest predation.

The stock return volatility of the constrained firm is not only time varying but also discontinuous at 0. This discontinuity occurs because of the interaction between the product and financial markets. In the absence of competition, the return volatility of the unconstrained firm is not affected by changes in its rival's cash position. In our model, however, the unconstrained firm becomes a monopolist and will, consequently, give up its aggressive strategy in the product market at the moment its opponent leaves the market. This causes a jump in the unconstrained firm's return



(a) Firm 1 (b) Firm 2

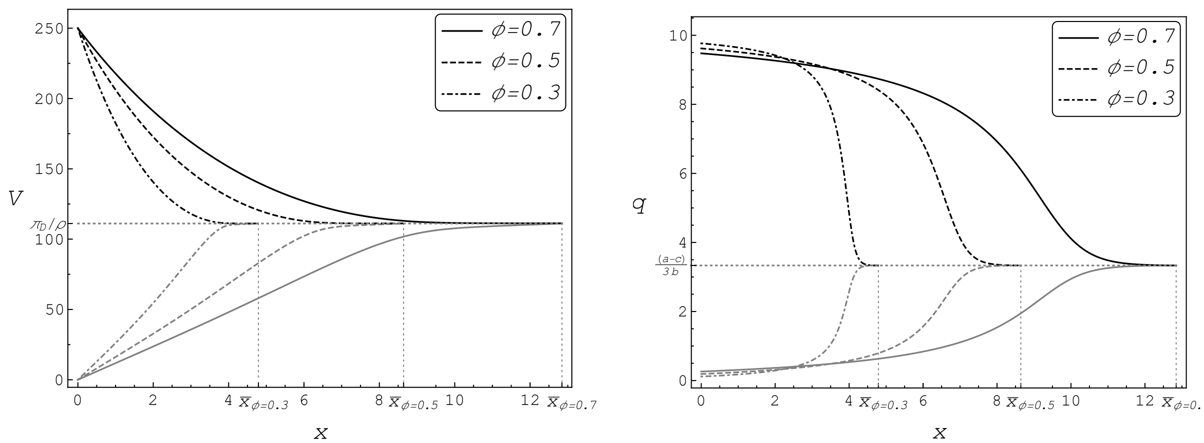
Figure 10: The volatility of stock returns for different level of the cash reserves of Firm 2 for the following set of parameter values: $(a - c) = 5$, $b = 0.5$, $\rho = 0.05$, and $\phi = 0.3$ (so that $\sigma = 0.8503$).

volatility. The model, therefore, allows us to identify an additional factor that may cause jumps in return volatility: industry structure dynamics.

Although we do not explicitly incorporate the possibility of entry, our model has direct implications for entry decisions in concentrated markets dominated by a financially strong incumbent. Our results suggest that to withstand aggressive competition an entrant must secure a cash buffer well in excess of the entry cost. Aggressive competition in product markets increases the bankruptcy probability for the cash constrained firm, creating entry barriers and, thus, increasing the risk of a firm's cash flows leading to an increased cost of financing. This is consistent with the empirical findings of Hou and Robinson (2006) that firms in highly concentrated industries earn lower returns. Our model predicts that firms in such markets command lower stock returns due to aggressive competition in product markets.

5.5 Sensitivity of the value functions and quantities

To illustrate some additional features of our model, we consider the changes in quantities and equilibrium value functions as a result of an increase in financial and product market parameters. In Figures 11, 12 and 13 the black curves correspond to the values and quantities of the unconstrained firm, whereas the gray curves represent the ones for the constrained firm. Consider the effect of the increased volatility shown in Figure 11(a).



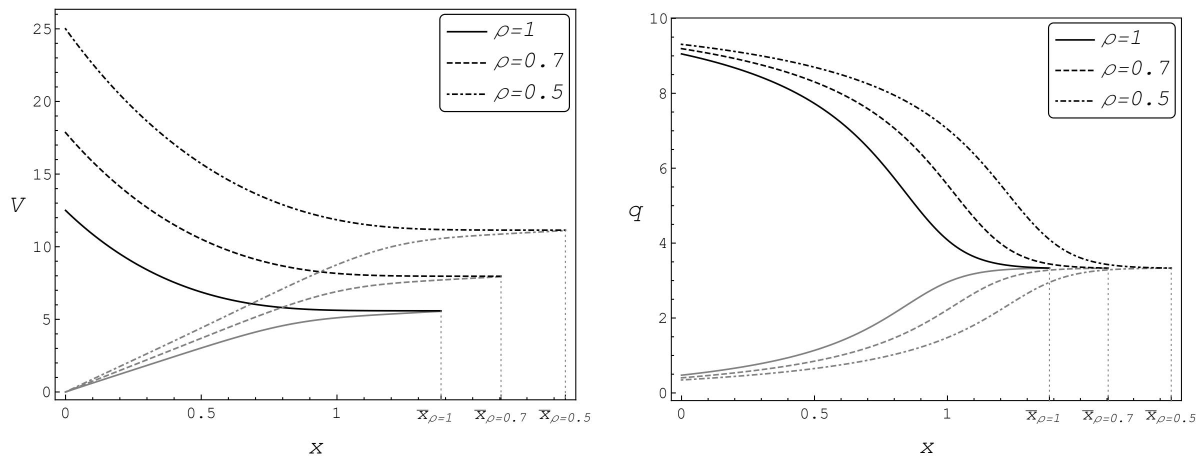
(a) The value functions.

(b) The quantities.

Figure 11: The equilibrium value functions $V = (V_1, V_2)$ and quantities $q = (\bar{q}_1(x), \bar{q}_2(x))$ for the following set of parameter values: $a = 5$, $b = 0.5$, $c = 0$, $\rho = 0.05$ and different values of ϕ (and, thus, σ).

When cash flow volatility of the constrained firm increases, we observe an increase in the cash target, and thus, a longer period of aggressive competition. This implies that both equilibrium value functions react monotonically to an increase in σ with V_1 increasing, and V_2 decreasing. However, the effect of volatility on equilibrium quantities is, however, non-monotonic. This directly follows from our results on V_1'. For low x, the unconstrained firm overproduces less for higher levels of volatility because negative cash flow shocks in this case already ensure a sufficiently large bankruptcy probability. As the cash reserve of Firm 2 increases, much effort is required by the unconstrained firm to exploit the constrained firm's financial weakness, which leads to a decline in predation intensity.

Figure 12 depicts the value functions and quantities for different values of the discount rate ρ.

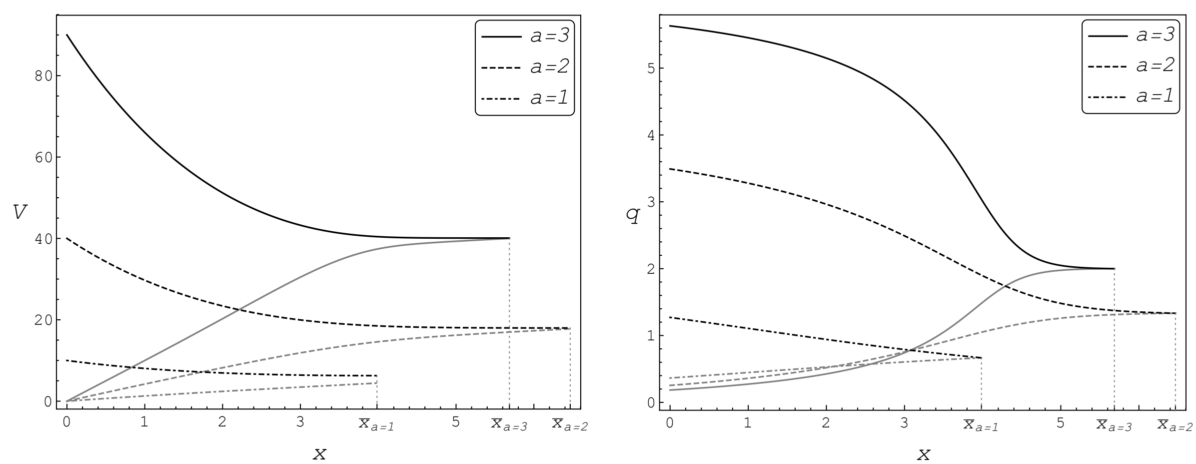


(a) The value functions. (b) The quantities.

Figure 12: The equilibrium value functions $V = (V_1, V_2)$ and quantities $q = (\bar{q}_1(x), \bar{q}_2(x))$ for the following set of parameter values: $\sigma = 0.8503$, $b = 0.5$, $(a - c) = 5$, and different values of ρ.

A higher discount rate decreases the values of both firms as well as their equilibrium quantities. This is because both the cash target and incentives to predate decrease with ρ.

Figure 13 illustrates the effect of an increase in the intercept of the inverse demand function.



(a) The value functions. (b) The quantities.

Figure 13: The equilibrium value functions $V = (V_1, V_2)$ and quantities $q = (\bar{q}_1(x), \bar{q}_2(x))$ for the following set of parameter values: $\sigma = 0.8503$, $b = 0.5$, $c = 0$, $\rho = 0.05$ and different values of a.

A large intercept increases the value of the constrained firm as it implies that its cash reserves accumulate at a faster rate. It also increases future profits in both monopoly and duopoly. As discussed earlier, in this case the attractiveness of the monopoly position for the unconstrained firm increases, thereby inducing more intensive predation for low values of x. In addition, for a large a, it becomes harder for the unconstrained firm to induce the bankruptcy of the constrained firm for a fixed level of financial strength. For larger x, the predation intensity declines, implying that the unconstrained firm has less incentive to overproduce for large values of a. This leads to a non-monotonic effect on the quantity policy of Firm 2.

Our model also allows for an investigation of the effects of financial constraints and market uncertainty on the (expected) consumer surplus. The expected present value of the consumer surplus in our model is equal to

$$CS(x;\xi) \triangleq \mathsf{E}_x\Big[\int_0^\infty e^{-\rho s}\frac{b}{2}(q_1(s)+q_2(s))^2 ds\Big], \quad t \geq 0. \tag{19}$$

For $x \in (0,\bar{x})$ the expected consumer surplus satisfies the following HJB equation

$$\frac{1}{2}\sigma^2 CS''(x) + \bar{\pi}_2(x)CS'(x) - \rho CS(x) + \frac{b}{2}(\bar{q}_1(x)+\bar{q}_2(x))^2 = 0. \tag{20}$$

The required boundary conditions are $CS(0) = \frac{(a-c)^2}{8br}$ and $CS(\bar{x}-) = \frac{2(a-c)^2}{9br}$, so that when Firm 2 goes bankrupt the consumer surplus is equal to that of a monopoly, whereas as soon as it starts paying out dividends the consumer surplus is equal to that of the Cournot duopoly (i.e., myopic benchmark). Figure 14 illustrates the consumer surplus as a function of the constrained firm's cash reserves for different values of volatility.

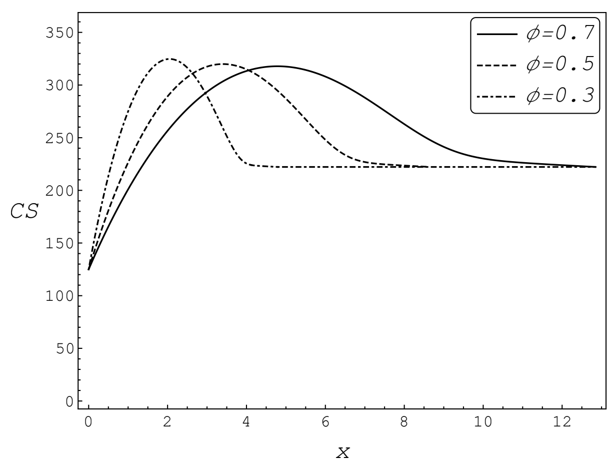


Figure 14: Expected consumer surplus for the following set of parameter values: $b = 0.5$, $(a-c) = 5$, $\rho = 0.05$ and different values of ϕ (and, thus, σ).

There are two opposing effects of predation on consumer welfare. On the one hand, predation leads to lower prices in the short run, which benefits consumers. On the other hand, predation makes it more likely that a monopoly emerges in the longer run, which lowers consumer welfare. We find that either of these effects can be dominant, depending on the firms' cash positions. In particular, there exists a "sweet spot" where cash hoards are low enough to lead to price reductions due to predation, but high enough to ensure the survival of firms for some time.

5.6 Implications for bankruptcy probabilities

To illustrate how market uncertainty affects industry dynamics and dividend policy, we simulate the probabilities of the constrained firm ever paying out dividends, before going bankrupt, for different levels of cash reserves. These are reported in Tables 1 and 2, respectively. We compare these with the corresponding probabilities in the accommodating scenario, where the firms always produce Cournot equilibrium quantities. In each table, the values on the diagonal represent the probabilities for the initial cash reserve, which is equal to 50% of the equilibrium cash target.

Table 1: Payout probability.[a]

	Initial cash level, x						
Volatility, σ	0.0659	0.3428	0.7129	1.5180	2.3956	3.3331	Cash target, $\bar{x}$
0.0283	57.123%	100 %	100 %	100 %	100 %	100 %	0.0318
0.1417	20.746%	89.362%	100 %	100 %	100 %	100 %	0.6855
0.2834	15.718%	67.973%	98.207%	100 %	100 %	100 %	1.4257
0.5669	11.856%	49.058%	84.305%	99.960%	100 %	100 %	3.0360
0.8503	10.356%	40.110%	72.369%	98.707%	99.997%	100 %	4.7911
1.1338	9.448%	34.800%	63.964%	95.683%	99.952%	100 %	6.6662

[a] Probabilities of the constrained firm to reach the pay out region before going bankrupt based on 100,000 simulation runs for the following set of the parameter values: $a = 5$, $b = 0.5$, $c = 0$, $\rho = 0.05$.

Table 1 illustrates the simulated probabilities for the cash constrained firm to reach the payout region. The main implication here is that for a given value of initial cash reserves, larger cash flow uncertainty decreases the payout probability. This is due to both an increased cash target, and a larger likelihood that negative shocks will drive the firm into bankruptcy before reaching the payout region. As we increase the initial cash reserves, we observe the opposite effect. The closer the constrained firm is to its cash target, the more likely it is to start paying out dividends. Looking at the diagonal, where the firm always starts with the cash stock halfway from its cash target, we observe that the payout probability increases with volatility. This indicates that the effect of initial cash reserves is more pronounced than the increase in volatility.

Table 2 illustrates the simulated bankruptcy probabilities.

Table 2: Bankruptcy probability.[b]

	Initial cash level, x						
Volatility, σ	0.0659	0.3428	0.7129	1.5180	2.3956	3.3331	Cash target, $\bar{x}$
0.0283	19.946%	0.023%	0.035%	0.026%	0.023%	0.021%	0.0132
0.1417	75.281%	7.431%	0 %	0 %	0 %	0 %	0.6855
0.2834	83.585%	30.259%	1.548 %	0 %	0 %	0 %	1.4257
0.5669	88.110%	50.797%	15.557%	0.040%	0 %	0 %	3.0360
0.8503	89.639%	59.878%	27.609%	1.288%	0.003%	0 %	4.7911
1.1338	90.552%	65.198%	36.034%	4.316%	0.048%	0 %	6.6662

[b] Bankruptcy probabilities of the constrained firm based on 100,000 simulation runs for the following set of the parameter values: $a = 5$, $b = 0.5$, $c = 0$, $\rho = 0.05$.

Here, we observe a non-monotonic effect of volatility on the bankruptcy probability of the constrained firm. This is because the probability of going bankrupt before the cash target $\bar{x}$ is reached, is affected both directly (through change in volatility) and indirectly (through the resulting increase in cash target). A larger value of σ implies both larger shocks as well as a higher cash target that is farther away from the initial cash reserves level. Therefore, for a given initial cash reserve level, the probability that the negative shocks will drive the firm into bankruptcy before $\bar{x}$ is reached becomes larger. On the contrary, once the firm reaches its cash target, its probability of going bankrupt is primarily affected by an indirect effect of an increase in cash target, implying a lower bankruptcy probability. However, on the diagonal the bankruptcy probability decreases with σ. This indicates that the effect of increased volatility is dominated by the increased cash target, which is further away from zero.

6 Conclusion

This paper presents a dynamic model of liquidity management that integrates the effect of product market competition between two firms that are symmetric in the product market, but are asymmetric in terms of their financial strength. We focus on the incentives of the financially strong firm to engage in predation in the product market and analyze the impact of such predatory behavior on dividend payouts, production strategies, dynamics of cash reserves and stock returns. Our model provides a theoretical link between observed empirical phenomena that have been discussed in isolation in the finance literature.

We explain the rationale behind cash hoarding by the presence of uncertainty over future profits, which supports the precautionary motive, and the possibility of predation, which adds a strategic "deep pockets" dimension to the problem. When profits evolve stochastically, a negative liquidity shock can lead a cash-constrained firm to bankruptcy. In this setting, a financially strong firm may have an incentive to engage in aggressive competition that could drive the opponent out of the market. This effect further increases the incentives to accumulate cash and postpone dividend payouts. This predatory effect has several interesting implications for both the product and financial markets. First, we find that the level of predation is time varying (through the predator's quantity policy's dependence on the prey's cash level) and heavily dependent on market conditions. Second, our model predicts that stock return volatility is time varying for both firms, and is discontinuous for the predator. Third, in our model, the marginal value of cash also varies over time for the constrained firm, depending on its financial position, and can exhibit non-monotonic behavior.

Our model provides several promising directions for future research. One potential extension is to consider a setting with two financially constrained firms and to investigate the conditions under which predation is optimal. Another possibility is to allow for the opportunity to invest and (or) raise debt financing, and investigate how these affect the problem. Lastly, it would be interesting to incorporate the financiers' perspective into the framework in which they can optimally choose which firm to inject their capital into. This could add to the literature on capital structure decisions of firms with limited financial resources, which explicitly considers the optimal contract provided by financiers to the firm (Biais *et al.*, 2007; DeMarzo and Sannikov, 2006; Miao and Rivera, 2016). Our model is a first step in this direction since it offers insights into the consequences of financiers choice to support only one firm.

References

Aguerrevere, F. L. (2009). Real options, product market competition, and asset returns. *The Journal of Finance*, 64, 957–983.

Akyildirim, E., I. E. Güney, J.-C. Rochet, and H. M. Soner (2014). Optimal dividend policy with random interest rates. *Journal of Mathematical Economics*, 51, 93 – 101.

Alimov, A. (2014). Product market competition and the value of corporate cash: Evidence from trade liberalization. *Journal of Corporate Finance*, 25, 122 – 139.

Almeida, H., M. Campello, and M. S. Weisbach (2004). The cash flow sensitivity of cash. *The Journal of Finance*, 59, 1777–1804.

Babenko, I., O. Boguth, and Y. Tserlukevich (2016). Idiosyncratic cash flows and systematic risk. *The Journal of Finance*, 71, 425–456.

Bates, T. W., C.-H. Chang, and J. D. Chi (2018). Why has the value of cash increased over time? *Journal of Financial and Quantitative Analysis*, 53, 749–787.

Bates, T. W., K. M. Kahle, and R. M. Stulz (2009). Why do U.S. firms hold so much more cash than they used to? *The Journal of Finance*, 64, 1985–2021.

Benoit, J. P. (1983). Entry with exit: An extensive form treatment of predation with financial constraints. *IMSSS Technical Report 405*, Stanford University, Stanford, California, Unites States of America.

Biais, B., T. Mariotti, G. Plantin, and J.-C. Rochet (2007). Dynamic security design: Convergence to continuous time and asset pricing implications. *The Review of Economic Studies*, 74, 345–390.

Bolton, P. and D. S. Scharfstein (1990). A theory of predation based on agency problems in financial contracting. *American Economic Review*, 80, 93–106.

Bonatti, A., G. Cisternas, and J. Toikka (2016). Dynamic Oligopoly with Incomplete Information. *The Review of Economic Studies*, 84, 503–546.

Campello, M., J. R. Graham, and C. R. Harvey (2010). The real effects of financial constraints: Evidence from a financial crisis. *Journal of Financial Economics*, 97, 470 – 487. The 2007-8 financial crisis: Lessons from corporate finance.

Chi, J. D. and X. Su (2016). Product market threats and the value of corporate cash holdings. *Financial Management*, 45, 705–735.

Cole, C. J. (2014). Stockpiling cash: How much is enough? *Journal of Corporate Accounting & Finance*, 26, 29–32.

Décamps, J.-P., T. Mariotti, J.-C. Rochet, and S. Villeneuve (2011). Free cash flow, issuance costs, and stock prices. *The Journal of Finance*, 66, 1501–1544.

Décamps, J.-P. and S. Villeneuve (2007). Optimal dividend policy and growth option. *Finance and Stochastics*, 11, 3–27.

Della Seta, M. (2011). Cash holdings and competition. *Ssrn working paper*, Tilburg University, Tilburg, the Netherlands.

DeMarzo, P. M. and Y. Sannikov (2006). Optimal security design and dynamic capital structure in a continuous-time agency model. *The Journal of Finance*, 61, 2681–2724.

Denis, D. J. and V. Sibilkov (2010). Financial constraints, investment, and the value of cash holdings. *The Review of Financial Studies*, 23, 247–269.

Faulkender, M. and R. Wang (2006). Corporate financial policy and the value of cash. *The Journal of Finance*, 61, 1957–1990.

Fresard, L. (2010). Financial strength and product market behavior: The real effects of corporate cash holdings. *The Journal of Finance*, 65, 1097–1122.

Galbraith, J. K. (1952). *American Capitalism; The concept of countervailing power*. Houghton Miffin Company.

Grenadier, S. R. (2002). Option Exercise Games: An Application to the Equilibrium Investment Strategies of Firms. *The Review of Financial Studies*, 15, 691–721.

GRYGLEWICZ, S. (2011). A theory of financial decisions with liquidity and solvency concerns. *Journal of Financial Economics*, 99, 365–384.

GU, L. (2016). Product market competition, r&d investment, and stock returns. *Journal of Financial Economics*, 119, 441 – 455.

HAUSHALTER, D., S. KLASA, AND W. F. MAXWELL (2007). The influence of product market dynamics on a firm's cash holdings and hedging behavior. *Journal of Financial Economics*, 84, 797 – 825.

HOBERG, G., G. PHILLIPS, AND N. PRABHALA (2014). Product market threats, payouts, and financial flexibility. *The Journal of Finance*, 69, 293–324.

HØJGAARD, B. AND M. TAKSAR (1999). Controlling risk exposure and dividends payout schemes: Insurance company example. *Mathematical Finance*, 9, 153–182.

HOU, K. AND D. T. ROBINSON (2006). Industry concentration and average stock returns. *The Journal of Finance*, 61, 1927–1956.

HUGONNIER, J., S. MALAMUD, AND E. MORELLEC (2014). Capital supply uncertainty, cash holdings, and investment. *The Review of Financial Studies*, 28, 391–445.

IRVINE, P. J. AND J. PONTIFF (2008). Idiosyncratic Return Volatility, Cash Flows, and Product Market Competition. *The Review of Financial Studies*, 22, 1149–1177.

JEANBLANC-PICQUÉ, M. AND A. N. SHIRYAEV (1995). Optimization of the flow of dividends. *Russian Mathematical Surveys*, 50, 257–277.

JIANG, Z. AND M. PISTORIUS (2012). Optimal dividend distribution under markov regime switching. *Finance and Stochastics*, 16, 449–476.

JOSKOW, P. L. AND A. K. KLEVORICK (1979). A framework for analyzing predatory pricing policy. *The Yale Law Journal*, 89, 213–270.

KEYNES, J. M. (1936). *The General Theory of Employment, Interest and Money*. MacMillan, London.

KOVENOCK, D. AND G. M. PHILLIPS (1997). Capital structure and product market behaviour: An examination of plant exit and investment decisions. *The Review of Financial Studies*, 10, 767–803.

LAMBRECHT, B. M. (2001). The impact of debt financing on entry and exit in a duopoly. *The Review of Financial Studies*, 14, 765–804.

LYANDRES, E. AND B. PALAZZO (2016). Cash holdings, competition, and innovation. *Journal of Financial and Quantitative Analysis*, 51, 1823–1861.

MCGEE, J. (1958). Predatory price cutting: The Standard Oil (N.J.) case. *The Journal of Law and Economics*, 1, 137–169.

MIAO, J. (2005). Optimal capital structure and industry dynamics. *The Journal of Finance*, 60, 2621–2659.

MIAO, J. AND A. RIVERA (2016). Robust contracts in continuous time. *Econometrica*, 84, 1405–1440.

MORELLEC, E., B. NIKOLOV, AND F. ZUCCHI (2014). Competition, cash holdings and financing decisions. *Swiss Finance Institute Research Paper 13-72*, École polytechnique fédérale de Lausanne, Swiss Finance Institute, Lausanne, Switzerland.

MORELLEC, E. AND A. ZHDANOV (2019). Product Market Competition and Option Prices. *The Review of Financial Studies*, 32, 4343–4386.

PHILIPPON, T. (2019). *The Great Revearsal. How America gave up on free markets*. The Belknap Press of the Harvard University Press, Cambridge, Mass.

PINKOWITZ, L., R. M. STULZ, AND R. WILLIAMSON (2015). Do U.S. Firms Hold More Cash than Foreign Firms Do? *The Review of Financial Studies*, 29, 309–348.

POVEL, P. AND M. RAITH (2004). Financial constraints and product market competition: ex ante vs. ex post incentives. *International Journal of Industrial Organization*, 22, 917 – 949.

RADNER, R. AND L. SHEPP (1996). Risk vs. profit potential: A model for corporate strategy. *Journal of Economic Dynamics and Control*, 20, 1373 – 1393.

Telser, T. (1966). Cutthroat competition and the long purse. *The Journal of Law and Economics*, 9, 259–277.

Zingales, L. (1998). Survival of the fittest or the fattest? exit and financing in the trucking industry. *The Journal of Finance*, 53, 905–938.

A Proofs of propositions

Proof of Proposition 1. *The derivations of the optimal cash target $\tilde{x}_q$, as well as the value function, v_2, directly follow from Jeanblanc-Picqué and Shiryaev (1995).*

The value function v_1 can be derived by using the fact that in the payout region, Firm 2's cash reserves stay constant in expectation, implying that for $x > \tilde{x}_q$, it holds that $v_1'(x) = 0$. Furthermore, for $x \leq 0$, Firm 1 becomes a monopolist, which yields a payoff of $\frac{\pi_M}{\rho}$.

Thus, on $(0, \tilde{x}_q)$, v_1 is the solution of the ODE

$$\frac{1}{2}\sigma^2 v_1'' + \pi_2 v_1' - \rho v_1 + \pi_1 = 0, \tag{21}$$

with the boundary conditions $v_1'(\tilde{x}-) = 0$ and $v_1(0+) = \frac{\pi_M}{\rho}$.

Let $\Delta_2 = \sqrt{\left(\frac{\pi_2}{\sigma^2}\right)^2 + \frac{2\rho}{\sigma^2}}$. Then the general solution of (21) is given by $A_1 \mathrm{e}^{-\left(\frac{\pi_2}{\sigma^2}+\Delta_2\right)x} + A_2 \mathrm{e}^{-\left(\frac{\pi_2}{\sigma^2}-\Delta_2\right)x} + \frac{\pi_1}{\rho}$ where A_1 and A_2 are constants determined by the following system of equations:

$$\begin{cases} A_1 + A_2 + \frac{\pi_1}{\rho} = \frac{\pi_M}{\rho} \\ -\left(\frac{\pi_2}{\sigma^2} + \Delta_2\right) A_1 \mathrm{e}^{-\left(\frac{\pi_2}{\sigma^2}+\Delta_2\right)\tilde{x}_q} - \left(\frac{\pi_2}{\sigma^2} - \Delta_2\right) A_2 \mathrm{e}^{-\left(\frac{\pi_2}{\sigma^2}-\Delta_2\right)\tilde{x}_q} = 0. \end{cases} \tag{22}$$

A particular solution is given by π_1/ρ, so that the resulting value function is given by

$$v_1(x) = \begin{cases} \frac{\pi_M}{\rho} & \text{if } x \leq 0 \\ \frac{\pi_1}{\rho} + \mathrm{e}^{-\frac{\pi_2}{\sigma^2}x} \frac{(\pi_M - \pi_1)\left(\Delta_2 \cosh\left(\Delta_2(x-\tilde{x}_q)\right) + \frac{\pi_2}{\sigma^2}\sinh\left(\Delta_2(x-\tilde{x}_q)\right)\right)}{\rho\left(\Delta_2 \cosh\left(\Delta_2\tilde{x}_q\right) - \frac{\pi_2}{\sigma^2}\sinh\left(\Delta_2\tilde{x}_q\right)\right)} & \text{if } 0 < x < \tilde{x}_q \\ \frac{\pi_1}{\rho} + \mathrm{e}^{-\frac{\pi_2}{\sigma^2}\tilde{x}_q} \frac{\Delta_2(\pi_M - \pi_1)}{\rho\left(\Delta_2 \cosh\left(\Delta_2\tilde{x}_q\right) - \frac{\pi_2}{\sigma^2}\sinh\left(\Delta_2\tilde{x}_q\right)\right)} & \text{if } x \geq \tilde{x}_q. \end{cases} \tag{23}$$

Note that for $0 < x < \tilde{x}_q$, it holds that $\sinh\left(\Delta_2(x - \tilde{x}_q)\right) < 0$, which together with the fact that $\frac{\pi_2}{\sigma^2} < \Delta_2$, yields

$$v_1'(x) = \mathrm{e}^{-\frac{\pi_2}{\sigma^2}x} \frac{(\pi_M - \pi_1)\left(-\frac{\pi_2}{\sigma^2}+\Delta_2\right)\left(\frac{\pi_2}{\sigma^2}+\Delta_2\right)\sinh\left(\Delta_2(x-\tilde{x}_q)\right)}{\rho\left(\Delta_2 \cosh\left(\Delta_2\tilde{x}_q\right) - \frac{\pi_2}{\sigma^2}\sinh\left(\Delta_2\tilde{x}_q\right)\right)} < 0, \quad \text{and} \tag{24}$$

$$v_1''(x) = \mathrm{e}^{-\frac{\pi_2}{\sigma^2}x} \frac{(\pi_M - \pi_1)\left(-\frac{\pi_2}{\sigma^2}+\Delta_2\right)\left(\frac{\pi_2}{\sigma^2}+\Delta_2\right)\left(\Delta_2 \cosh\left(\Delta_2(x-\tilde{x}_q)\right) - \frac{\pi_2}{\sigma^2}\sinh\left(\Delta_2(x-\tilde{x}_q)\right)\right)}{\rho\left(\Delta_2 \cosh\left(\Delta_2\tilde{x}_q\right) - \frac{\pi_2}{\sigma^2}\sinh\left(\Delta_2\tilde{x}_q\right)\right)} > 0. \tag{25}$$

□

Proof of Proposition 2. *The proof proceeds in several steps. First, we verify that the functions $\bar{q}_1$ and $\bar{q}_2$ solve $\sup_{q_1 \in \mathscr{M}_1} \mathscr{L}^{q_1, \bar{q}_2}$ and $\sup_{q_2 \in \mathscr{M}_2} \mathscr{L}^{\bar{q}_1, q_2}$, respectively. Then we show that our conditions on φ_1 and φ_2 imply that $\mathscr{H}_1^{\bar{q}_2}\varphi_1 = \mathscr{H}_2^{\bar{q}_1}\varphi_2 = 0$, after which we show that $\bar{q}_1$ is a best response to $(\bar{q}_2, \bar{Z}_2)$ and vice versa.*

In the remainder, for any $\zeta = (q_i, Z_i)_{i=1,2} \in \mathscr{Z}$ and $x > 0$, define

$$\tau_{(0,x)} \triangleq \inf \left\{ t \geq 0 \,\middle|\, X^{\zeta}(t) \notin (0, x) \right\}.$$

1. *First note that*

$$\bar{q}_1(x) = \arg\sup_{q_1 \geq 0} \left\{ \mathscr{L}^{q_1, \bar{q}_2} \varphi_1(x) + \pi(q_1, \bar{q}_2(x)) \right\}, \quad \textit{and}$$
$$\bar{q}_2(x) = \arg\sup_{q_2 \geq 0} \left\{ \mathscr{L}^{\bar{q}_1, q_2} \varphi_1(x) \right\}.$$

Since φ_1 is C^2 on $(0, \bar{x})$ it holds that $\bar{q}_1$ and $\bar{q}_2$ are, indeed, continuous functions. Furthermore, on $(0, \bar{x})$ it holds that $\varphi_1' < 0$ and $\varphi_1'' > 0$. [This is the case, because if $\varphi_1' \geq 0$, then the (static) profit of Firm 2 is larger than the static Cournot profit, while Firm 1's profit is smaller than the static Cournot profit. This cannot be an equilibrium since the unconstrained firm can always get higher instantaneous profits, which reduce the instantaneous profits of the constraint firm and its survival probability. Hence, $\varphi_1' < 0$. It then follows from the HJB equation that $\varphi_1'' > 0$.] Also, since $\varphi_1'(\bar{x}) = 0$, it holds that $\bar{q}_1(\bar{x}) = \bar{q}_2(\bar{x}) = q_D$, i.e., at $\bar{x}$ firms produce the Cournot quantity. In addition, since $\varphi_2''(\bar{x}-) = 0$, we know from the HJB equation that

$$\varphi_2(\bar{x}) = \frac{\pi_D}{\rho}.$$

Finally, it is easily seen that $\pi_D / \rho \leq \varphi_1(\bar{x}) \leq \pi_M / \rho$.

2. *Next we show that $\mathscr{H}_1^{\bar{q}_2} \varphi_1 = \mathscr{H}_2^{\bar{q}_1} \varphi_2 = 0$. The fact that $\mathscr{H}_1^{\bar{q}_2} \varphi_1 = 0$ follows from the previous step and condition 2(a). As $\bar{Z}_2$ imposes a reflecting barrier of $X^{\bar{\zeta}}$ at $\bar{x}$, it holds, in combination with condition 3(b), that $\varphi_2' = 1$ on $[\bar{x}, \infty)$. Together with condition 4(b) this implies that $\varphi_2'' = 0$. Given $\bar{q}_1$ and $\bar{q}_2$ and $\varphi_1'(\bar{x}) = 0$, we know that $\pi(\cdot) = \pi_D$ on $[\bar{x}, \infty)$. This implies that on $[\bar{x}, \infty)$ it holds that*

$$\begin{aligned}
\mathscr{L}^{\bar{q}_1, \bar{q}_2} \varphi_2(x) &= \frac{1}{2} \sigma^2 \varphi_2''(x) + \pi(\bar{q}_2(x), \bar{q}_1(x)) \varphi_2'(x) - \rho \varphi_2(x) \\
&= \frac{1}{2} \sigma^2 \varphi_2''(x) + \pi_D \varphi_2'(x) - \rho \varphi_2(x) \\
&= \pi_D \varphi_2'(x) - \rho [(x - \bar{x}) + \varphi_2(\bar{x}) \\
&= \rho \left[\frac{\pi_D}{\rho} - (x - \bar{x}) + \frac{\pi_D}{\rho} \right] \leq 0.
\end{aligned}$$

Together with the assumption that $\varphi_2' > 1$ on $(0, \bar{x})$, we see that

$$\mathscr{H}^{\bar{q}_1} \varphi_2 = \max \{ \sup_{q_2 \in \mathscr{M}_2} \mathscr{L}^{\bar{q}_1, q_2} \varphi_2, 1 - \varphi_2' \} = 0.$$

3. *We now show that $\bar{\zeta}_1$ is a best response to $\bar{\zeta}_2$. It is easily seen that $\bar{Z}_1$ is a dominant dividend policy for Firm 1, because it has no incentive to hoard cash. So, in what follows we always assume that $Z_1 = \bar{Z}_1$. Given $(\bar{q}_2, \bar{Z}_2) \in \mathscr{M}_2 \times \mathscr{D}_2$, the process $X^{\zeta_1, \bar{\zeta}_2}$ has a reflecting barrier*

at $\bar{x}$, for any $q_1 \in \mathscr{M}_1$. Let $\bar{q}_1^{\bar{\zeta}_i} \in \mathscr{Q}_i$ be the admissible quantity policy associated with the quantity function $\bar{q}_i$, $i = 1, 2$. Note that $\mathsf{P}(\tau_{(0,\bar{x})} < \infty) = 1$. Applying Ito's lemma and taking expectations gives[3]

$$
\begin{aligned}
\varphi_1(x) =& \mathsf{E}_x \left[e^{-\rho \tau_{(0,\bar{x})}} \varphi_1(X^{\bar{\zeta}}(\tau_{(0,\bar{x})})) \right] - \mathsf{E}_x \left[\int_0^{\tau_{(0,\bar{x})}} \mathscr{L}^{\bar{q}_1, \bar{q}_2} \varphi_1(X^{\bar{\zeta}}(t)) dt \right] \\
=& \mathsf{E}_x \left[e^{-\rho \tau_{(0,\bar{x})}} \varphi_1(X^{\bar{\zeta}}(\tau_{(0,\bar{x})})) \right] \\
&+ \mathsf{E}_x \left[\int_0^{\tau_{(0,\bar{x})}} \pi \left(\bar{q}_1(X^{\bar{\zeta}}(t)), \bar{q}_2(X^{\bar{\zeta}}(t)) \right) dt \right] \\
&- \mathsf{E}_x \left[\int_0^{\tau_{(0,\bar{x})}} \left(\mathscr{L}^{\bar{q}_1, \bar{q}_2} \varphi_1(X^{\bar{\zeta}}(t)) + \pi \left(\bar{q}_1(X^{\bar{\zeta}}(t)), \bar{q}_2(X^{\bar{\zeta}}(t)) \right) \right) dt \right] \\
=& \mathsf{E}_x \left[e^{-\rho \tau_{(0,\bar{x})}}, X^{\bar{\zeta}}(\tau_{(0,\bar{x})})) = 0 \right] \frac{\pi_M}{\rho} \\
&+ \mathsf{E}_x \left[e^{-\rho \tau_{(0,\bar{x})}}, X^{\bar{\zeta}}(\tau_{(0,\bar{x})})) = \bar{x} \right] \varphi_1(\bar{x}) \\
&+ \mathsf{E}_x \left[\int_0^{\tau_{(0,\bar{x})}} \pi \left(\bar{q}_1(X^{\bar{\zeta}}(t)), \bar{q}_2(X^{\bar{\zeta}}(t)) \right) dt \right] \\
=& V_1(x; \xi_1^{\bar{\zeta}_1}, \xi_2^{\bar{\zeta}_2}).
\end{aligned}
$$

Now let $\zeta_1 = (q_1, \bar{Z}_1)$, where $q_1 \in \mathscr{M}_1$ is an arbitrary quantity function with corresponding admissible quantity process $q_1^{\zeta_1} \in \mathscr{Q}_1$. Applying Ito's lemma and taking expectations gives that

$$
\begin{aligned}
V_1(x; \xi_1^{\bar{\zeta}_1}, \xi_2^{\bar{\zeta}_2}) =& \varphi_1(x) \\
\geq& \mathsf{E}_x \left[e^{-\rho \tau_{(0,\bar{x})}} \varphi_1(X^{\zeta_1, \bar{\zeta}_2}(\tau_{(0,\bar{x})})) \right] \\
&+ \mathsf{E}_x \left[\int_0^{\tau_{(0,\bar{x})}} \pi \left(q_1(X^{\zeta_1, \bar{\zeta}_2}(t)), \bar{q}_2(X^{\zeta_1, \bar{\zeta}_2}(t)) \right) dt \right] \\
&- \mathsf{E}_x \left[\int_0^{\tau_{(0,\bar{x})}} \left(\mathscr{L}^{q_1, \bar{q}_2} \varphi_1(X^{\zeta_1, \bar{\zeta}_2}(t)) \right. \right. \\
&\left. \left. + \pi \left(q_1(X^{\zeta_1, \bar{\zeta}_2}(t)), \bar{q}_2(X^{\zeta_1, \bar{\zeta}_2}(t)) \right) \right) dt \right] \\
=& \mathsf{E}_x \left[e^{-\rho \tau_{(0,\bar{x})}} \varphi_1(X^{\zeta_1, \bar{\zeta}_2}(\tau_{(0,\bar{x})})) \right] \\
&+ \mathsf{E}_x \left[\int_0^{\tau_{(0,\bar{x})}} \pi \left(q_1(X^{\zeta_1, \bar{\zeta}_2}(t)), \bar{q}_2(X^{\zeta_1, \bar{\zeta}_2}(t)) \right) dt \right] \\
=& \mathsf{E}_x \left[e^{-\rho \tau_{(0,\bar{x})}}, X^{\zeta_1, \bar{\zeta}_2}(\tau_{(0,\bar{x})})) = 0 \right] \frac{\pi_M}{\rho} \\
&+ \mathsf{E}_x \left[e^{-\rho \tau_{(0,\bar{x})}}, X^{\zeta_1, \bar{\zeta}_2}(\tau_{(0,\bar{x})})) = \bar{x} \right] \varphi_1(\bar{x}) \\
&+ \mathsf{E}_x \left[\int_0^{\tau_{(0,\bar{x})}} \pi \left(q_1(X^{\zeta_1, \bar{\zeta}_2}(t)), \bar{q}_2(X^{\zeta_1, \bar{\zeta}_2}(t)) \right) dt \right] \\
=& V_1(x; \xi_1^{\zeta_1}, \xi_2^{\bar{\zeta}_2}).
\end{aligned}
$$

[3] Henceforth, for a random variable X and event $A \in \mathscr{F}$, we denote $\mathsf{E}[X, A] \triangleq \mathsf{E}[X|A]P(A)$.

4. *Finally, we show that $\bar{\zeta}_2$ is a best response to $\bar{\zeta}_1$. The strategy $(\bar{q}_2, \bar{Z}_2) \in \mathscr{M}_2 \times \mathscr{D}_2$ ensures that that the process $X^{\zeta_1,\bar{\zeta}_2}$ has a reflecting barrier at $\bar{x}$ for any $\zeta_1 \in \mathscr{Z}_1$. Applying Ito's lemma and taking expectations, we find that*

$$\begin{aligned}
\varphi_2(x) &= \mathsf{E}_x\left[e^{-\rho\tau_{(0,\bar{x})}}\varphi_2(X^{\bar{\zeta}_1,\bar{\zeta}_2}(\tau_{(0,\bar{x})}))\right] - \mathsf{E}_x\left[\int_0^{\tau_{(0,\bar{x})}} \mathscr{L}^{\bar{q}_1,\bar{q}_2}\varphi_2(X^{\bar{\zeta}_1,\bar{\zeta}_2}(t))dt\right] \\
&= \mathsf{E}_x\left[e^{-\rho\tau_{(0,\bar{x})}}\varphi_2(X^{\bar{\zeta}_1,\bar{\zeta}_2}(\tau_{(0,\bar{x})}))\right] \\
&= \mathsf{E}_x\left[e^{-\rho\tau_{(0,\bar{x})}}, X^{\bar{q}_1,\bar{q}_2,\bar{Z}_2}(\tau_{(0,\bar{x})})) = \bar{x}\right]\frac{\pi_D}{\rho} \\
&= V_1(x;\xi_1^{\bar{\zeta}_1},\xi_2^{\bar{\zeta}_2}).
\end{aligned}$$

Now let $\zeta_2(q_2, Z_2) \in \mathscr{M}_2 \times \mathscr{D}_2$ be an arbitrary admissible strategy with corresponding admissible quantity process $q_2^{\zeta_2} \in \mathscr{Q}_2$. Let Z_2^c denote the continuous part of Z_2. Fix $T > 0$ and define $\tau \triangleq T \wedge \tau_{(0,\infty)}$. Since $\mathscr{L}^{\bar{q}_1,q_2}\varphi_2 \leq 0$ and $\varphi_2' \geq 1$, it follows from Ito's lemma and after taking expectations that for all $x > 0$

$$\begin{aligned}
\mathsf{E}_x\left[e^{-\rho\tau}\varphi_2(X^{\bar{q}_1,q_2,Z_2}(\tau))\right] =&\varphi_2(x) + \mathsf{E}_x\left[\int_0^{\tau} e^{-\rho t}\mathscr{L}^{\bar{q}_1,q_2}\varphi_2\left(X^{\bar{\zeta}_1,\zeta_2}(t)\right)dt\right] \\
&- \mathsf{E}_x\left[\int_0^{\tau} e^{-\rho t}\varphi_2'\left(X^{\bar{\zeta}_1,\zeta_2}(t)\right)dZ_2^c(t)\right] \\
&- \mathsf{E}_x\left[\sum_{0\leq t\leq\tau}\left\{\varphi_2\left(X^{\bar{\zeta}_1,\zeta_2}(t+)\right) - \varphi_2\left(X^{\bar{\zeta}_1,\zeta_2}(t)\right)\right\}\right] \\
\leq&\varphi_2(x) - \mathsf{E}_x\left[\int_0^{\tau} e^{-\rho t}dZ_2^c(t)\right] \\
&- \mathsf{E}_x\left[\sum_{0\leq t\leq\tau}\left\{\varphi_2\left(X^{\bar{\zeta}_1,\zeta_2}(t+)\right) - \varphi_2\left(X^{\bar{\zeta}_1,\zeta_2}(t)\right)\right\}\right] \\
=&\varphi_2(x) - \mathsf{E}_x\left[\int_0^{\tau} e^{-\rho t}dZ_2(t)\right].
\end{aligned}$$

Note that

$$\begin{aligned}
\mathsf{E}_x\left[e^{-\rho\tau}\varphi_2(X^{\bar{\zeta}_1,\zeta_2}(\tau))\right] &= \mathsf{E}_x\left[e^{-\rho\tau}\varphi_2(X^{\bar{\zeta}_1,\zeta_2}(\tau)), T > \tau_{(0,\infty)}\right] \\
&\to 0, \quad \textit{as } T \to \infty.
\end{aligned}$$

So,

$$V_2(x;\xi_1^{\bar{\zeta}_1},\xi_2^{\zeta_2}) = \mathsf{E}_x\left[\int_0^{\tau} e^{-\rho t}dZ_2(t)\right] \leq \varphi_2(x) = V_2(x;\xi_1^{\bar{\zeta}_1},\xi_2^{\bar{\zeta}_2}).$$

$\square$

Proof of Proposition 3. *For $\varphi_1 \in C^1$, with $\varphi_1' \neq 3$, and $\bar{x} > 0$ define the functions $\bar{\pi}_i : (0, \bar{x}) \to \mathbb{R}$,*

$i = 1, 2$, by

$$\bar{\pi}_1(x) \triangleq \frac{(a-c)^2}{b}\frac{1-\varphi_1'(x)}{(3-\varphi_1'(x))^2}, \quad \text{and} \quad \bar{\pi}_2(x) \triangleq \frac{(a-c)^2}{b}\frac{1}{(3-\varphi_1'(x))^2}.$$

1. *Note that for any $\bar{x} > 0$ the HJB equation of Firm 1 on $(0, \bar{x})$ can be written as*

$$\begin{aligned} &\frac{1}{2}\sigma^2\varphi_1''(x) + \bar{\pi}_2(x)\varphi_1'(x) - \rho\varphi_1(x) + \bar{\pi}_1(x) = 0 \\ \iff &[\frac{1}{2}\sigma^2\varphi_1''(x) - \rho\varphi_1(x)][3-\varphi_1'(x)]^2 + \frac{(a-c)^2}{b} = 0. \end{aligned}$$

The required boundary conditions are $\varphi_1'(\bar{x}-) = \varphi_1''(\bar{x}-) = 0$. By defining

$$\psi_1 \triangleq (3-\varphi_1')^2, \quad \text{and} \quad \nu_1 \triangleq \varphi_1',$$

This can be written as

$$\begin{cases} \frac{1}{2}\sigma^2\nu_1'\psi_1 - \rho\varphi_1\psi_1 + \frac{(a-c)^2}{b}, & \varphi_1(0) = \frac{\pi_M}{\rho} \\ \psi_1 = (3-\varphi_1')^2, & \psi_1(\bar{x}) = 9 \\ \nu_1 = \varphi_1', & \nu_1(\bar{x}) = 0 \end{cases},$$

which has a unique solution on $[0, \bar{x}]$. Furthermore, note that $\varphi_1 \geq \pi_D/\rho$, because Firm 1 can always guarantee to earn π_D by producing the Cournot quantity q_D. Since $\varphi_1' < 0$ and $\bar{\pi}_2(\cdot) \leq \pi_D$ for the same reason, it follows from the HJB equation that $\varphi_1'' \geq 0$.

2. *For any $\tilde{x} > 0$ there exists a unique solution to the Neumann problem*

$$\frac{1}{2}\sigma^2\varphi_2''(x) + \pi_2(x)\varphi_2'(x) - \rho\varphi_2(x) = 0, \quad \varphi_2(0) = 0, \quad \varphi_2(\tilde{x}) = \frac{\pi_D}{\rho}.$$

Since $\pi_D/\rho > 0$, it holds for $\tilde{x}$ small enough that $\varphi_2'(\tilde{x}) > 1$ on $(0, \tilde{x})$. By increasing $\tilde{x}$ there will be a unique $\bar{x}$ for which the Neumann problem has a solution with $\varphi_2'(\bar{x}) = 1$ and $\varphi_2' > 1$ on $(0, \bar{x})$. At $\bar{x}$ it then holds that $\varphi_2''(\bar{x}) = 0$ and, thus, that $\varphi_2''(\bar{x}-) = 0$. □

Proof of Proposition 4. *Recall from the proof of Proposition 2 that Firm 1's value function is decreasing. Then from the HJB it follows that $\varphi_1'' > 0$.*

For the constrained firm, $\varphi_2' > 1$ by the definition of the continuation region. The sign of φ_2'' is ambiguous. Specifically, $\varphi_2'' > 0$ if and only if

$$\frac{1}{2}\sigma^2\varphi_2''(x) = -\pi(q_2(x), q_1(x))\varphi_2'(x) + \rho\varphi_2(x) > 0. \tag{26}$$

This leads to the following condition for the convexity of $\varphi_2(x)$

$$\pi(q_2(x), q_1(x))\varphi_2'(x) < \rho\varphi_2(x). \tag{27}$$

This inequality holds in a neighborhood of 0 and a neighborhood of $\bar{x}$, implying that φ_2 is concave *in these regions. Note that there may be regions away from 0 and $\bar{x}$ where φ_2 is convex.* $\square$

Proof of Proposition 5. *For Firm 1, because the myopic strategy is admissible, its value can never exceed the equilibrium value, $V_1 \geq v_1$.*

For Firm 2, note that in the continuation region it holds that

$$\frac{1}{2}\sigma^2\varphi_2''(x) + \frac{(a-c)^2}{b(3-\varphi_1'(x))^2}\varphi_2'(x) - \rho\varphi_2(x) = 0. \tag{28}$$

Let $\nu(x)$ satisfy

$$\frac{1}{2}\sigma^2\nu''(x) + \pi_D\nu'(x) - \rho\nu(x) = 0. \tag{29}$$

Then

$$\frac{1}{2}\sigma^2\nu''(x) + \frac{(a-c)^2}{b(3-\varphi_1'(x))^2}\nu'(x) + \rho\nu(x) < 0, \quad \text{all } 0 < x < \bar{x}. \tag{30}$$

Note that ν is supermartingale, since

$$\mathsf{E}_x\left[\nu(x_\tau)\right] = \nu(x) + \mathsf{E}_x\left[\int_0^\tau \left(\mathscr{L}_2\nu(X(t)) - \rho\nu(X(t))\right)dt\right] \leq \nu(x). \tag{31}$$

Thus for all $x \in (0, \bar{x})$ it holds that $\varphi_2(x) < \nu(x)$. Recall that the cash target under myopic strategy given by

$$\tilde{x}_D = \frac{1}{2\Delta_D}\ln\left(\frac{\sigma^2\Delta_D^2 + \Delta_D\pi_D - \rho}{\sigma^2\Delta_D^2 - \Delta_D\pi_D - \rho}\right), \quad \text{with} \quad \Delta_D = \sqrt{\left(\frac{\pi_D}{\sigma^2}\right)^2 + \frac{2\rho}{\sigma^2}};$$

cf. Proposition 1. Then it directly follows that $\varphi_2(\tilde{x}_D) < \nu(\tilde{x}_D) = \frac{\pi_D}{\rho} = \varphi_2(\bar{x})$ and $\tilde{x}_D < \bar{x}$. This also implies that for $x \in (0, \bar{x})$ it holds that $\varphi_2(x) < v_2(x)$, where $v_2(x)$ is Firm 2's value function of the myopic strategy and, thus, satisfies (29).

As $V_2(\bar{x}) = v_2(\tilde{x}_D)$, it holds that $V_2(x) \leq v_2(x)$. $\square$

Proof of Proposition 6. *Comparing the equilibrium profit for the unconstrained firm,*

$$\bar{\pi}_1(x) = \frac{(a-c)^2\left(1-\varphi_1'(x)\right)}{b\left(3-\varphi_1'(x)\right)^2},$$

to $\pi_D = \frac{(a-c)^2}{9b}$, it is easy to obtain that $\tilde{\pi}_1(x) - \pi_D > 0$ if and only if $-3 < \varphi_1'(x) < 0$. $\square$

Proof of Proposition 7. *See the proof of Proposition 5.*

Proof of Proposition 8. *The return volatilities can be expressed as*

$$\sigma_i(x) = \left|\sigma\frac{V_i'(x)}{V_i(x)}\right| \tag{32}$$

for $i = 1, 2$.

Differentiating w.r.t. x, for $x > 0$ yields

$$\sigma_i'(x) = \sigma \frac{V_i''(x)V_i(x) - (V_i'(x))^2}{(V_i(x))^2} \frac{\sigma_i(x)}{|\sigma_i(x)|} \tag{33}$$

1. First, we prove that $\sigma_2'(x) < 0$.

For Firm 2, it holds that $V_2'(x) > 0$ and $\sigma \frac{V_2'(x)}{V_2(x)} > 0$, so that

$$\sigma_2'(x) = \sigma \frac{V_2''(x)V_2(x) - (V_2'(x))^2}{(V_2(x))^2}. \tag{34}$$

From the HJB equation for Firm 2, it holds that

$$\frac{1}{2}\sigma^2 V_2''(x) + \bar{\pi}_2(x)V_2'(x) - \rho V_2(x) = 0, \quad 0 < x < \bar{x}, \tag{35}$$

(recall that $\bar{\pi}_2(x) = \frac{(a-c)^2}{b(3-V_1'(x))^2}$), and $V_2''(x)$ can be expressed as

$$V_2''(x) = \frac{2}{\sigma^2}\left(\rho V_2(x) - \bar{\pi}_2(x)V_2'(x)\right) \tag{36}$$

Then, we can write the following

$$\begin{aligned} V_2''(x)V_2(x) - (V_2'(x))^2 &= \frac{2\rho}{\sigma^2}(V_2(x))^2 - \frac{2\bar{\pi}_2(x)}{\sigma^2}V_2(x)V_2'(x) - (V_2'(x))^2 \\ &= \frac{2\rho}{\sigma^2}\left(V_2(x) + V_2'(x)\frac{-\bar{\pi}_2(x) + \sqrt{\bar{\pi}_2(x)^2 + 2\rho\sigma^2}}{2\rho}\right)\left(V_2(x) - V_2'(x)\frac{\bar{\pi}_2(x) + \sqrt{\bar{\pi}_2(x)^2 + 2\rho\sigma^2}}{2\rho}\right) \end{aligned} \tag{37}$$

The first term in (37) is positive, as $V_2'(x) > 0$. Then

$$\operatorname{sign}(\sigma_2'(x)) = \operatorname{sign}\left(V_2(x) - V_2'(x)\frac{\bar{\pi}_2(x) + \sqrt{\bar{\pi}_2(x)^2 + 2\rho\sigma^2}}{2\rho}\right). \tag{38}$$

Define $f_2 : (0, \bar{x}) \to \mathbb{R}$ by

$$f_2(x) = \frac{2\rho\sigma}{\bar{\pi}_2(x) + \sqrt{\bar{\pi}_2(x)^2 + 2\rho\sigma^2}}.$$

Then

$$\begin{aligned} V_2(x) - V_2'(x)\frac{\bar{\pi}_2(x) + \sqrt{\bar{\pi}_2(x)^2 + 2\rho\sigma^2}}{2\rho} &= V_2(x) - V_2'(x)\frac{\sigma}{f_2(x)} \\ &= \frac{1}{f_2(x)}V_2(x)\left(f_2(x) - \sigma\frac{V_2'(x)}{V_2(x)}\right) \\ &= \frac{1}{f_2(x)}V_2(x)\left(f_2(x) - \sigma_2(x)\right). \end{aligned} \tag{39}$$

We, therefore, find that:

- *for $x \in (0+, \bar{x})$ such that $f_2(x) > \sigma_2(x)$, it holds that $\sigma_2'(x) > 0$, and*
- *for $x \in (0+, \bar{x})$ such that $f_2(x) < \sigma_2(x)$, it holds that $\sigma_2'(x) < 0$.*

Note that $f_2(0) = \sqrt{2\rho} > 0$ and $f_2'(x) < 0$ as $\bar{\pi}_2'(x) > 0$. In addition, $\sigma_2(\bar{x}) = \frac{2\rho}{\pi_D}\sigma > \frac{2\rho\sigma}{\pi_D + \sqrt{\pi_D^2 + 2\rho\sigma^2}} = f_2(\bar{x})$. Lastly, $\sigma_2(0+) = +\infty$, as $V_2(0+) = 0$ and $1 < V_2'(0+) < +\infty$. Thus, it must hold that $f_2(x) < \sigma_2(x)$ and $\sigma_2'(x) < 0$, for all $x \in (0, \bar{x})$.

2. Second, we prove that $\sigma_1'(x) < 0$ and, thus, $|\sigma_1'(x)| < 0$. For Firm 1, it holds that $V_1'(x) < 0$ and $\sigma \frac{V_1'(x)}{V_1(x)} < 0$, so that

$$\sigma_1'(x) = -\sigma \frac{V_1''(x)V_1(x) - (V_1'(x))^2}{(V_1(x))^2} \tag{40}$$

From the HJB equation for Firm 1, it holds that

$$\frac{1}{2}\sigma^2 V_1''(x) + \bar{\pi}_2(x)V_1'(x) - \rho V_1(x) + \bar{\pi}_1(x) = 0, \quad 0 < x < \bar{x}, \tag{41}$$

where $\bar{\pi}_2(x) = \frac{(a-c)^2}{b(3-V_1'(x))^2}$, $\bar{\pi}_1(x) = \frac{(a-c)^2(1-V_1'(x))}{b(3-V_1'(x))^2}$, and $V_1''(x)$ can be expressed as

$$V_1''(x) = \frac{2}{\sigma^2}\left(\rho V_1(x) - \frac{(a-c)^2}{b(3-V_1'(x))^2}\right) \tag{42}$$

We can, therefore, see that

$$V_1''(x)V_1(x) - (V_1'(x))^2 = \frac{2\rho}{\sigma^2}(V_1(x))^2 - \frac{2(a-c)^2}{\sigma^2 b(3-V_1'(x))^2}V_1(x) - (V_1'(x))^2 \tag{43}$$

$$= \frac{2\rho}{\sigma^2}\left(V_1(x) - V_1'(x)\frac{-\psi(x) + \sqrt{\psi(x)^2 + 2\rho\sigma^2}}{2\rho}\right)\left(V_1(x) + \varphi_1'(x)\frac{\psi(x) + \sqrt{\psi(x)^2 + 2\rho\sigma^2}}{2\rho}\right),$$

where $\psi(x) = -\frac{1}{V_1'(x)}\frac{(a-c)^2}{b(3-V_1'(x))^2} > 0$, because $V_1'(x) < 0$. The first term in (43) is positive, so that

$$\operatorname{sign}(\sigma_1'(x)) = -\operatorname{sign}\left(V_1(x) + \varphi_1'(x)\frac{\psi(x) + \sqrt{\psi(x)^2 + 2\rho\sigma^2}}{2\rho}\right). \tag{44}$$

Define $f_1 : (0, \bar{x}) \to \mathbb{R}$ by

$$f_1(x) = \frac{2\rho\sigma}{\psi(x) + \sqrt{\psi(x)^2 + 2\rho\sigma^2}}.$$

Then we see that

$$
\begin{aligned}
V_1(x) + \varphi_1'(x) \frac{\psi(x) + \sqrt{\psi(x)^2 + 2\rho\sigma^2}}{2\rho} &= V_1(x) + \varphi_1'(x) \frac{\sigma}{f_1(x)} \\
&= \frac{1}{f_1(x)} V_1(x) \left(f_1(x) + \sigma \frac{V_1'(x)}{V_1(x)} \right) \\
&= \frac{1}{f_1(x)} V_2(x) \left(f_1(x) - \sigma_1(x) \right), \qquad (45)
\end{aligned}
$$

for all $0 < x < \bar{x}$.

Therefore, since $f_1(x) > 0$ *and* $-\sigma_1(x) > 0$, *we conclude that*

- *for* $x \in (0+, \bar{x})$ *such that* $f_1(x) > \sigma_1(x)$, *it holds that* $\sigma_1'(x) < 0$, *and*
- *for* $x \in (0+, \bar{x})$ *such that* $f_1(x) < \sigma_1(x)$, *it holds that* $\sigma_1'(x) > 0$.

Note that $f_1(0+) > 0$ *and* $f_1'(x) < 0$ *as* $\varphi_1'(x) < 0$ *and, as a result,* $\psi(x) > 0$. *In addition,* $-\sigma_1(\bar{x}) = f_1(\bar{x}) = 0$. *Thus, it must hold that* $f_1(x) > \sigma_1(x)$ *and* $\sigma_1'(x) < 0$, *for all* $x \in (0, \bar{x})$. $\square$