Innovation and Patent Litigation with Financial Constraints: American versus English Rule *

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Abstract

We take a first theoretical step toward understanding how legal systems influence corporate innovation by examining the impact of financial constraints on firms' strategies in innovation disputes and R&D decisions. In the presence of capital market frictions, we develop a compound real options model to examine product firms' sequential decisions during and before patent litigation. We show that the English rule (i.e.," loser pays") and the American rule (i.e., "each litigant pays its own cost") interact differently with firms' financial constraints, causing the English rule to shift the effective bargaining power from the patent-owning firm (the "incumbent") to the alleged infringing firm (the "challenger"). We discover that, under the English rule, (1) the litigation thresholds are more sensitive to a key product market characteristic (i.e., gain-to-loss ratio); (2) royalty rates in an ex-post settlement (i.e., settlement after the filing of a lawsuit) are lower; (3) an opposite effect of the incumbent's winning probability and a stronger effect of product market volatility on settlement likelihood; (4) firms have lower innovation incentives, all compared to the American rule. Our model also generates new testable implications regarding IP litigation with financing considerations.

Keywords: *patent litigation risk; American rule; English rule; financial constraints; real options*

*We thank participants at the 2021 AEA poster session.

1 Introduction

How capital market frictions affect corporate innovation has been an important question for financial economists (e.g., Brown et al., 2009; Li, 2011; Malamud and Zucchi, 2019; Lin, 2021). Meanwhile, recent empirical studies in finance document that changes in legal rules affect innovation incentives for corporations (e.g., Lee et al., 2019; Appel et al., 2019; Caskurlu, 2019; Mezzanotti et al., 2016). We set an ambitious goal and take one of the first theoretical steps trying to understand how legal systems affect corporate innovation with the presence of financial market frictions. In a nutshell, we show through patent litigation setting that financial constraints can give firm a competitive disadvantage or advantage, depending on its position in the competition. Because financial constraints impact firms differently under various legal systems through the legal cost allocation rules, legal systems indirectly affect firms' relative competitive positions and the aforementioned effective bargaining power shift consequently leads to distinct corporate innovation incentives.

In this paper, we examine how capital market frictions interact with legal systems in affecting firms' patent litigation strategies and R&D decisions. In particular, we compare likely outcomes of patent litigation and R&D investment under two typical cost allocation rules (the "American rule" and the "English rule") of legal systems, while the two firms involved are production firms (that is, not patent trolls[1]) and are subject to financial constraints. Under the English rule (interchangeable with "the UK system"), which is widely used in many countries and is also referred to as "the loser pays rule", the party who loses in court pays the other party's legal costs. In contrast, the American rule (interchangeable with "the US system") states that each party is generally responsible to pay its own legal costs. Using a dynamic model, we argue that compared to the American rule, the English rule shifts the negative impact of firms' financial constraints towards the less financially constrained firm, and therefore changes firms' incentives to litigate, settle, and innovate. We believe our findings are not only important to the patent litigation literature in the law and economics, they also showcase the significant role played by financial constraints on firms' incentive to innovate through the new channel of patent litigation.

We build a dynamic model with complete information to study firms' strategies related to patent litigation, including innovation, (arguable) infringement, litigation, settlement, and non-settlement. Two all-equity firms compete in the same product market, and make decisions to maximise firm values. An *Incumbent* makes innovation efforts and it becomes the owner of a patent and uses the technology in production upon a successful innovation. A *Challenger* can decide to use a technology similar to that patent

[1]We do not consider patent trolls (see e.g., Cohen et al., 2016), as they are very different from the typical product companies in nature and deserve separate discussions.

in its production, and may arguably infringe the patent in the process. After the entry of the Challenger, the market structure changes from a monopoly of the Incumbent to a duopoly. For simplicity, we assume both firms have no other sources of income except the flow profits from productions that are linear to the uncertain market demand, which is captured by a Geometric Brownian Motion. Under standard conditions, firms' optimal strategies are equivalent to threshold strategies based on the market demand level. For example, the Incumbent's optimal litigation strategy is to file a lawsuit against the Challenger once the market demand goes up to a certain level conditional on the Challenger entering the market, or the two firms settle in equilibrium during litigation if the market demand goes down to a certain point and a patent lawsuit had started. Both firms incur ongoing litigation costs. Meanwhile, the legal judgement follows an exogenous Poisson process, and the probability that one firm wins the ruling against the other is common knowledge. If it is not worthwhile for the parties to settle, the lawsuit may stop due to either a withdrawal from judgement by the Incumbent or an exit by the Challenger. The incumbent can also threaten to start the litigation and induce the Challenger to exit the market immediately, when the market demand drops to Challenger's exit level before litigation starts.

Taking the American rule as a base case, the English rule has direct consequences on firm values through altering the firms' payoffs upon court judgement. We argue that with the consideration of financial constraints, the English rule (1) makes litigation more expensive for the Incumbent relative to the Challenger, regardless of each party's winning probability; and/or (2) is more detrimental for the Incumbent if he loses the lawsuit. The first effect is due to the asymmetry between the Incumbent and the Challenger of their cash flows upon losing a lawsuit (i.e. financial constraints). When the Incumbent loses, he pays the Challenger's legal costs if he remains a going-concern. However, when the Challenger loses, she has to leave the market, thus the Incumbent can hardly ever recover his legal costs due to the Challenger's financial constraints. The second effect is related to the Incumbent and takes place when the Incumbent cannot afford the Challenger's legal costs upon losing the lawsuit due to his own financial constraints. Thus the Incumbent has to liquidate and exit the market, leaving the Challenger to become the new monopolist.

Our first major finding from the baseline model, which begins with the occurrence of the arguable infringement, is that the litigation threshold is more sensitive to a key product market characteristic which reflects the ratio between the gain from the infringement for the alleged infringer and the lost profit for the patent owner ("gain-to-loss ratio") under the English rule than under the American rule. This is because the litigation thresholds are affected by the litigation outcomes, which are driven by the gain-to-loss ratio. Furthermore, the two legalsystems have distinct litigation costs allocation rules, which contributes to the discrepancy. If the non-settlement outcomes are different in two systems (i.e., the Challenger exits under the American rule but the Incumbent withdraws under the English rule), the litigation threshold is accelerated under the English rule. However, if the non-settlement outcomes are the same in two systems (i.e., the Incumbent withdraws under both rules), the litigation threshold is delayed under the English rule. One interpretation of the first result is that under the English rule, the higher effective litigation costs for the Incumbent relative to the Challenger makes the likelihood of withdrawal by the Incumbent once litigation starts is more prevalent. This makes the Incumbent less likely to threaten the challenger out of the market with patent litigation, and thus reduces the Incumbent's value of waiting before starting a lawsuit, making the Incumbent less willing to wait (to litigate). The intuition for the latter is as follows: the English rule means litigation becomes more expensive for the Incumbent, thus the Incumbent has lower litigation incentives compared to the American rule. As a result, the litigation threshods change significantly when the gain-to-loss ratio varies under the English rule.

A second result found in the model is that the royalty rate in a settlement agreement between the two firms is higher under the American rule than under the English rule. The royalty rate is the proportion of the Challenger's future flow profit paid to the Incumbent, provided that a settlement is successful. Intuitively, the effective bargaining power during litigation shifts from the Challenger to the Incumbent, as a result of the aforementioned two effects of the English rule, leading to the lower royalty rate received by the Incumbent under the English rule.

A third and striking result is that the winning probability of the Incumbent p has opposite effects on the likelihood of settlement in the two legal systems. Our model suggests that if it becomes more likely that the Incumbent would win in the court, for example, because the patent approval becomes more stringent or more regulatory hurdles are put in place to prove infringement, then firms become less willing to settle under the American rule but become more willing to settle under the English rule. To see what causes the startling difference, it is crucial to recognize that there are two possible forces that prevent ex-post settlement from happening. Firstly, the Challenger can resist settlement, and a higher p reduces the likelihood of this happening. By not settling, the Challenger could become the new monopolist if the Incumbent liquidates upon judgement under the English rule, or the Challenger can keep sharing the market profits if the Incumbent withdraws from the litigation. Two, the Incumbent may be unwilling to settle, and increases in p increases the likelihood of such events. By not settling, the Incumbent can restore its monopoly if the Challenger has to exit the market at some point. As a result, when the Challenger's rejection of the settlement offer (due to the Incumbent's later withdrawal) is the main reason for non-settlement, as under the English rule, then settlement likelihood increases with p. On the contrary, when the Incumbent's refusal to offer settlement (due to the Challenger's later exit) matters more for non-settlement, as in the American rule, then settlement likelihood decreases with p.

The last result is that we show that whilst product market volatility σ reduces settlement likelihood in two legal rules, the effect of σ is more significant under the English rule than under the American rule due. This is because, under English rule, the Incumbent is more financially constrained and thus more willing to settle when market volatility is low.

With our extended model that includes firms' infringement and innovation decisions, we find that the Incumbent has less incentive to innovate under the English rule compared with its counterpart in the American rule. This is consistent with the aforementioned mechanism that the English rule shifts the negative impact of financial constraints on patent litigation towards the Incumbent, which then reduces the Incumbent's incentive to invest in R&D in the first place.

Our model generates a list of testable implications regarding patent litigation. For example, the litigation rate is higher/lower under the English rule than the American rule if the infringing products are substitutes/compliments to the Incumbent products, where litigation rate is defined as the probability of reaching the litigation threshold in a given period of time. Our model also implies that policies that increase the winning probability of the plaintiff in a patent infringement lawsuit reduce settlement rate in the American rule but increase the settlement rate under the English rule, provided that we define settlement rate similarly.

The closest work to ours is Aoki and Hu (1999a) which investigates how the legal system affects the possibility of settlement and litigation, using a game theoretic model. Like us, they show that the relative litigation cost is a key determinant for the likelihood of settlement before litigation. However, they do not model settlement during litigation or the potential possibility that the Incumbent may liquidate due to its inability to pay for the Challenger's legal costs if the Incumbent loses. In contrast, our model focus on the role played by the financial constraints by looking at the possible consequences of the liquidation of firms.

This paper contributes to the study of patent litigation in the law and economics literature. (e.g., Landes, 1971; Ordover, Rubinstein et al., 1983; Bebchuk, 1984; Reinganum and Wilde, 1986; Aoki and Hu, 1999b; Lanjouw and Schankerman, 2001; Bessen and Meurer, 2006). None of the existing models (e.g., Meurer, 1989; Aoki and Hu, 1999a; Llobet, 2003) incorporate each party's (in)ability to pay legal costs. Our model not only captures the patent litigation dynamics more thoroughly than prior papers, more importantly, it also emphasizes the effect of financial constraints on firms' litigation strategies and innovation decisions. We argue that the different cost allocation rules in legal systems cause a different degree of financial constraints for firms facing the risk of litigation.

This paper also builds on the real options models with multiple players (such as Lambrecht, 2001; Lambrecht and Perraudin, 2003; Pawlina and Kort, 2006; Smit and Trigeorgis, 2006; Azevedo and Paxson, 2014; Bustamante, 2015) and on sequential games (such as Jeon, 2015). However, prior papers in this literature often either do not take into account the sequential nature of firms' interactions or the strategic nature of the dynamic games with uncertainty. This paper also relates to the finance theory literature which shows financial constraints affect firms' real decisions (e.g., Bolton and Scharfstein, 1990; Hugonnier, Malamud, and Morellec, 2015; Bolton, Wang, and Yang, 2019), especially in a setting with competition (Ma, Mello, and Wu, 2020). As far as we are aware, we are the first ones to look at the patent litigation setting and carefully model the strategic interactions of the duopoly.

The paper proceeds as follows. Section 2 presents the baseline dynamic model, including baseline model analysis during litigation and before litigation. Section 3 extends the model to examine firms' innovation and arguable infringement decisions. Section 4 presents comparative statics of the model, among other findings. Section 5 concludes.

2 The Basic Model

We build a compound real options model. Two risk neural all-equity firms operate in product markets and make their decisions based on firm value maximization. Both firms generate random profits in their product markets using some patentable technology. The two firms' profits from selling their products are both driven by the underlying stochastic market demand $\{x_t \geq 0\}$, with its distribution regarded as common knowledge. The market demand follows a Geometric Brownian Motion characterized by the constant growth rate μ ($\mu < r$, r being the risk free rate), the volatility σ and an initial level x_0. Using $\{W_t\}_{t\geq 0}$ to denote a Wiener process, we can write the demand process as:

$$dx_t = \mu x_t dt + \sigma x_t dW_t \tag{1}$$

The Incumbent ("I") is also the owner of a patent of the technology used in its production. The Challenger ("C") has allegedly infringed the patent in its production. Figure 1 depicts the game tree of the model, which starts after the alleged infringement. I and C may reach a settlement without resorting to the legal system ("ex-ante settlement"). Alternatively, I can litigate against C. Once the two firms enter litigation, they pay ongoing litigation costs (C_l^I and C_l^C, with superscripts representing the firm) in each period until one of the following four outcomes happen: (1) the two firms settle ("ex-post settlement"); (2) I withdraws from the litigation; (3) C exits the market and thus drops out of the lawsuit; (4) the court rules which firm wins and which firm loses. We investigate firms' strategies that lead to the first three outcomes, and those strategies are equivalent to threshold strategies, meaning firms take actions when market demand drops to certain levels. Meanwhile, we model the court ruling process as exogenous, and its timing follows a Poisson distribution. Thus we cannot say for sure whether an outcome will definitely realize or not, because it may happen before or after the court ruling, thus we term it "the likely outcome".

[Insert Figure 1 here.]

We assume, for simplicity, that neither firm has other sources of cash flow or cash reserves. Before the alleged infringement, I earns a flow monopoly profit of $\pi_1 x_t$. After C enters the market by allegedly infringing I's patent, I and C split the market with flow duopoly profits of $\pi_2^I x_t$ and $\pi_2^C x_t$ respectively. Upon the judgement, if the Incumbent wins the lawsuit, then C is forced to leave the market and I regains its monopoly flow profit $\pi_1^I x_t$. If C wins the lawsuit, the status quo of duopoly remains.

We model settlement between the firms by following Lukas et al. (2012). Settlement takes the form of royalty payments from C to I, and proportional to C's flow profit. The proportion is termed as "royalty rate" hereafter, and denoted as θ_a and θ_p for ex-ante and ex-post settlements respectively. If two firms settle, I proposes a royalty rate, and C decides when to accept the settlement offer, if ever. C's decision is equivalent to a trigger strategy of accepting the royalty rate proposal from I once demand x_t falls to settlement threshold (x_p for ex-post settlement and x_a for ex-ante settlement). We assume a one-time cost of settlement for each firm and denote these as C_a^I, C_a^C, C_p^I, and C_p^C. In addition, we model the court ruling as an independent Poisson process with the rate parameter λ. Thus the expected duration of the litigation process is $\frac{1}{\lambda}$. With probability p which is common knowledge, the court rules in favor of I; with probability $1-p$, C wins the case.

Our main focus of the legal differences is the default cost allocation rule between the plaintiff and the defendant. Under the English rule, the party that loses in the litigation is required to pay the winning party's legal expenses. Under the American rule, both parties bear their own legal costs. This difference has two implications regarding how outcomes differ under the two systems when financial constraints are considered.[2] First, if C is ruled to win the lawsuit (i.e., C's infringement cannot be proved or I's patent is regarded invalid), I needs to pay for C's litigation cost under the English rule whilst the market remains a

[2] The model is simplified to make the analysis tractable, and the settling captures the effect of financial constraints without unnecessary complexity. The model can be generalized to model firms having other sources of revenue, as well as to incorporate ruling on damage payment transfer.

duopoly. We call such an outcome the "modified status quo", as opposed to the status quo in the American rule. In contrast, when I wins, the outcome is identical in the two systems, that is, C leaves and I recovers monopoly. There is no actual transfer of litigation cost from C to I under the English rule because C is unable to make payments if it loses. Therefore, similar to the American rule, I's present value of the expected future profit if winning the lawsuit is $\frac{\pi_1^I x_t}{r-\mu}$. Since C has no other revenues to pay I's litigation costs, I's payoff from the court ruling is still $\frac{\pi_1^I x_t}{r-\mu}$. Second, it is only under the English rule that we may have the scenario in which I may not have enough money to pay for C's litigation costs when I loses. This is most relevant if I is financially constrained, e.g., I is a small start-up company that has very limited financial resources. When I is not able to pay for C's legal costs in the English rule, we assume I may liquidate with no scrap value. As a result, C becomes the new monopolist in the market and earns π_1^C. This outcome never happens in the American rule.

The first implication aforementioned leads to a simple variation on the payoff of the firms if I loses. Denote the expected discounted value of litigation cost for I and C as

$$H_l^I \equiv E\int_0^\tau e^{-rt} C_l^I dt = \frac{C_l^I}{r+\lambda}, \quad H_l^C \equiv \frac{C_l^C}{r+\lambda}, \tag{2}$$

which is the same as the litigation cost under the American rule. Given that I's present value of the expected future profit if losing the lawsuit is $\frac{\pi_2^I x_t}{r-\mu}$, and I's payoff from the court ruling is $\frac{\pi_2^I x_t}{r-\mu} - \mathbb{1}_{\text{UK}} \cdot H_l^C$, where $\mathbb{1}_{\text{UK}}$ is an indicator function for the English rule (i.e., $\mathbb{1}_{\text{UK}} = 0$ for the US system). Similarly, C's expected payoff in that case is $\frac{\pi_2^C x}{r-\mu} + \mathbb{1}_{\text{UK}} \cdot H_l^C$. The second aforementioned implication means that we need to consider two possibilities in the analysis when I loses, depending on whether I's net present value at the time of losing the judgement is negative or not (i.e, $\frac{\pi_2^I x_\tau}{r-\mu} - \mathbb{1}_{\text{UK}} \cdot H_l^C \lesseqgtr 0$). If the demand condition at the time of judgement is higher than I's "liquidation threshold" $\bar{x}$ i.e., $x_\tau \geq \bar{x}$ where

$$\bar{x} = \frac{\mathbb{1}_{\text{UK}} \cdot H_l^C (r-\mu)}{\pi_2^I}, \tag{3}$$

then I can pay for C's litigation cost as required and keep operating as a going-concern, which is always the case under the American rule (see from $\mathbb{1}_{\text{UK}} = 0$). We call such case "I remains a going-concern". However, if the demand condition is too low (i.e., $x_\tau < \bar{x}$), I has to liquidate because there aren't sufficient funds to pay for C's litigation cost. We call such case as "I may liquidate".

We use backward induction to solve the model, and start by obtaining the general forms of value functions during litigation from HJB equations used in stochastic models (Section 2.1). Such value functions are denoted as (V^I, v^I, V^C, v^C), where "V" represents *firm value* if I remains a going-concern, "v" represents *firm value* if I may liquidate and the superscripts "I" and "C" represent *the relevant firm.* We then use boundary conditions and smooth pasting conditions to figure out the firm values depending on firms' strategies, such as ex-post settlement (i.e., $(V_p^I, v_p^I, V_p^C, v_p^C)$), withdrawal by the Incumbent (i.e., $(V_w^I, v_w^I, V_w^C, v_w^C)$), or exit by the Challenger (i.e., $(V_e^I, v_e^I, V_e^C, v_e^C)$). Given these, we figure out firms' optimal strategies during litigation by comparing values from different outcomes provided any set of parameter values. Finally we find firm values if they were to settle ex-ante (i.e., $(V_a^I, v_a^I, V_a^C, v_a^C)$) or I were to litigate. (i.e., $(V_l^I, v_l^I, V_l^C, v_l^C)$). In Section 4, we analyse quantitatively the optimal strategies of the two firms considering their strategic interactions and obtain the potential corresponding outcomes for any given parameter set.

2.1 During Litigation

Firms' strategies and values during litigation depend on whether upon court ruling, the Incumbent remains in operation or not. For what follows, we separate the analysis for two cases at court ruling: in Case A, the Incumbent can afford to pay the Challenger's litigation cost if it loses in judgement for any potential value of market demand, thus the Incumbent remains a going-concern regardless of winning or losing; in Case B, the Incumbent cannot always afford to pay the Challenger's litigation cost if it loses in judgement, thus may have to liquidate and exit the market if the demand level when judgement occurs is too low, i.e. if the Incumbent's present value of continuing in operations is less than the cost it has been ordered to pay by the court.

We solve the model by noticing that firm values when using strategies in litigation depend on the specific strategy and the legal system. For example, if I withdraws, then the payoff for each firm remains the status quo (in both the US and the UK). If C exits, the market restores to I monopoly (in both the US and the UK). Therefore, we use the value matching conditions for both firms and the smooth pasting conditions for the acting firm, on the firm values at the (to-be-determined) action threshold, in order to figure out the arbitrary constants. From there, we can solve for firm values in different outcomes as well as the strategy thresholds. Whether ex-post settlement during litigation is the equilibrium outcome or not depends on whether it gives higher firm values to both firms, comparing with alternative outcomes. To distinguish between the two cases, we use X_d to represent the action thresholds during litigation if I remains a going concern and x_d to represent the action threshold during litigation if there is the possibility that I will liquidate following judgement.

We proceed the analysis by separating firms' strategies during litigation into "non-settlement" vs "ex-post settlement", with the former further separated into "I withdraws" vs "C exits". In analysing the two non-settlement strategies, we proceed by first only considering withdrawal by the Incumbent, then we only consider exit by the Challenger, after that, we figure out which of the two (I withdraws vs C exits) will be the non-settlement outcome by looking at the chronological order of withdrawal and exit using the approach in Lambrecht (2001), detailed in Appendix 6.

Ex-post settlement takes the form of I proposing a royalty rate (Θ_p if I remains a going-concern and θ_p if I may liquidate) of C's future profits at the start of the litigation, and C deciding the time to settle. C's decision is equivalent to determining the ex-post settlement threshold (X_p/x_p). We can thus investigate the settlement timing as well as the settlement terms without having to model the bargaining process explicitly.

2.1.1 Case A: The Unconstrained Incumbent ("I Remains a Going-Concern")

We first look at the simpler case (i.e, Case A) when the Incumbent remains a going-concern regardless of when judgement occurs. In our setting, all lawsuits under the American rule fall into Case A. However, Case A does not hold under the English rule if the Incumbent cannot pay for C's litigation cost in all cases in the event of losing the lawsuit. Intuitively, a high level of market demand triggers I's litigation against C, and afterwards, lower market demand triggers firms leaving the lawsuit. To help with the model analysis, we transform Case A to the following sufficient condition on market demand during litigation: by the time that firms take actions, the demand has not yet fallen to I's liquidation threshold $\bar{x}$ (i.e. $x_d \geq \bar{x}$, where $d = \{w, e, p\}$ representing "I withdraws", "C exits", "ex-post settlement").

Here is why "I remains a going-concern" if we have $x_d \geq \bar{x}$. Recall that in our model, the court ruling happens at a random time independent of the market demand fluctuations. We separate all possible scenarios during litigation by whether the court rules before firms take actions. The court ruling either (1) happens before firms' strategies during litigation (i.e., $\tau \leq t_d$), or (2) would have happened after firms' strategies, i.e., $\tau > t_d$. The condition $x_d \geq \bar{x}$ in Scenario (1), provided that x starts high at litigation, implies a high enough firm value for I to cover its liability of C's litigation cost at the time of ruling (i.e., $x_\tau > x_d \geq \bar{x}$). The condition $x_d \geq \bar{x}$ in Scenario (2) implies that the lawsuit ends before I would have liquidated. Therefore, regardless of how the court rules, I does not liquidate in all possible scenarios if $x_d \geq \bar{x}$. We also interpret Case A as I being financially unconstrained.

In Case A during litigation (i.e., $x_d \geq \bar{x}$), before firms take any action ($x > x_d$), I's value function (V^I) satisfies the following HJB equation:

$$rV^I dt = \mathbb{E}\mathcal{D}V^I + (\pi_2^I x - C_l^I)dt + \lambda\Big[p \cdot \frac{\pi_1^I x}{r-\mu} + (1-p) \cdot \Big(\frac{\pi_2^I x}{r-\mu} - \mathbb{1}_{\text{UK}} \cdot H_l^C\Big) - V^I\Big]dt \tag{4}$$

where $\mathbb{E}\mathcal{D}V^I = (\frac{1}{2}\frac{\partial^2 V^I}{\partial x^2}x^2\sigma^2 + \mu x\frac{\partial V^I}{\partial x})dt$. Similarly, the value function of the Challenger (V^I) during litigation satisfies:

$$rV^C = \frac{1}{2}\frac{\partial^2 V^C}{\partial x^2}x^2\sigma^2 + \frac{\partial V^C}{\partial x}\mu x + (\pi_2^C x - C_l^C) + \lambda\Big[(1-p)\Big(\frac{\pi_2^C}{r-\mu}x + \mathbb{1}_{\text{UK}}\cdot H_l^C\Big) - V^C\Big], \forall x \geq \bar{x}. \quad (5)$$

With the boundary condition that $\lim_{x\to\infty} V^I(x) = -(H_l^I + \frac{\mathbb{1}_{\text{UK}}\cdot H_l^C(1-p)\lambda}{r+\lambda}) + \frac{\pi_2^I x}{r-\mu} + \delta p(\pi_1^I - \pi_2^I)x$ and $\lim_{x\to\infty} V^C(x) = -(H_l^C - \frac{\mathbb{1}_{\text{UK}}\cdot H_l^C(1-p)\lambda}{r+\lambda}) + (\frac{1}{r-\mu} - p\delta)\pi_2^C x$, where $\delta = \frac{\lambda}{(r-\mu)(r+\lambda-\mu)}$, and writing the value functions slightly differently, we obtain the following result.

Proposition 1. *In Case A (i.e., $X_d \geq \bar{x}$ which applies to all of the lawsuits in the US and some of the lawsuits in of UK), the Incumbent remains a going-concern, and the value functions during litigation follow*

$$V_d^I(x) \;=\; B_d^I x^{\beta_\lambda} - \bar{H}_l^I + \frac{\pi_2^I x}{r-\mu} + \delta p(\pi_1^I - \pi_2^I)x, \quad (6)$$

$$V_d^C(x) \;=\; B_d^C x^{\beta_\lambda} - \bar{H}_l^C + (\frac{1}{r-\mu} - p\delta)\pi_2^C x, \quad (7)$$

for $x > X_d$ where $\delta = \frac{\lambda}{(r-\mu)(r+\lambda-\mu)}$, $\beta_\lambda = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{(\frac{1}{2} - \frac{\mu}{\sigma^2})^2 + \frac{2(r+\lambda)}{\sigma^2}}$.

$$\bar{H}_l^I = H_l^I + \frac{\mathbb{1}_{\text{UK}}\cdot H_l^C(1-p)\lambda}{r+\lambda}, \quad \bar{H}_l^C = H_l^C - \frac{\mathbb{1}_{\text{UK}}\cdot H_l^C(1-p)\lambda}{r+\lambda} \quad (8)$$

are the expected effective expenditure on litigation with $H_l^C = \frac{C_l^I}{r+\lambda}$ and $H_l^I = \frac{C_l^C}{r+\lambda}$. B_d^I and B_d^C are two arbitrary constants to be determined depending on firms' strategies. The subscripts $d \in \{w, e, p\}$, and they represent "I withdraws", "C exits", "ex-post settlement".

The cost-shifting after judgment under the English rule increases the Incumbent's costs and lower the Challenger's costs. I use Δ to denote the fee difference under two legal rules and can be expressed as

$$\Delta = \frac{H_l^C(1-p)\lambda}{(r+\lambda)}, \quad \bar{H}_l^I = H_l^I + \Delta, \quad \bar{H}_l^C = H_l^C - \Delta \quad (9)$$

The difference under the two legal systems is more significant when p is low, because $\frac{\partial\Delta}{\partial p} = -\frac{\lambda H_l^C}{(r+\lambda)} < 0$. Fee shifting only occurs in the model if the Challenger wins, which occurs with probability $1-p$, so the lower p the greater the expected fee shift from Challenger to Incumbent. Thus, when the probability of winning for the Incumbent (i.e., p) is low, the difference between two distinct legal systems is more obvious.

We then discuss firms' strategies with no option to settle and with the option to settle ex-post in Case A. It should be noted that in the absence of an ex-post settlement, the Incumbent may choose to withdraw from the lawsuit, whereas the Challenger may choose to exit the market due to its inability to afford litigation costs. Except for the difference in anticipated litigation costs as a result of the different cost allocation rule, the analysis is comparable to the case under the American rule.

Withdrawal by the Incumbent

Upon I's withdrawal during litigation in both cases, the discounted future profits are $\frac{\pi_2^I X_w}{r-\mu}$ for I, and $\frac{\pi_2^C X_w}{r-\mu}$ for C, which indicates the status quo is maintained. For $x \leq X_w$, the payoff function when exercising the withdrawal option can be expressed as

$$\begin{aligned} \tilde{V}_w^I(x) = V_d^I = \frac{\pi_2^I x}{r-\mu}, \\ \tilde{V}_w^C(x) = V_d^C = \frac{\pi_2^C x}{r-\mu}. \end{aligned} \tag{10}$$

Applying the value matching and smooth pasting conditions on I's value function, and the value matching condition on C's value function, all at the withdraw threshold, we get the following result.

Corollary 1. *The firm values with the Incumbent's option to withdraw during litigation V_w^I and V_w^C in Case A ("I remains a going-concern") follow Proposition 1, the withdraw threshold is $X_w = \frac{\beta_\lambda \bar{H}_l^I}{(\beta_\lambda - 1)p\delta(\pi_1 - \pi_2^I)}$ and the arbitrary constants are*

$$B_w^I = \{\bar{H}_l^I - p(\pi_1 - \pi_2^I)\delta X_w\} X_w^{-\beta_\lambda}, \quad B_w^C = \{\bar{H}_l^C + p\pi_2^C \delta X_w\} X_w^{-\beta_\lambda}. \tag{11}$$

Equation (11) shows that the arbitrary constant B_w^I increases with $\bar{H}_l^I$. Since under the English rule, the litigation cost is higher than that under the American rule, i.e., $\bar{H}_l^I > H_l^I$, this constant is higher under the English rule, causing a higher impact on the option value for the Incumbent if the Incumbent withdraws first.

Exit by the Challenger

Upon the exit by C during litigation in Case A, C's firm value falls to zero and I's expected present value of future monopoly profits is $\frac{\pi_1 X_e}{r-\mu}$. For $x \leq X_e$, the payoff function of exercising the exit option can be written as

$$\tilde{V}_e^I(x) = V_m^I = \frac{\pi_1^I x}{r-\mu}, \tag{12}$$

$$\tilde{V}_e^C = 0. \tag{13}$$

By applying the value matching and the smooth pasting conditions on C, and the value matching condition on I, at the exit threshold, we get the relevant value functions and the strategy threshold:

Corollary 2. *The firm values with the Challenger's option to exit during litigation* V_e^I *and* V_e^C *follow Proposition 1, the exit threshold is* $X_e = \frac{\beta_\lambda \bar{H}_l^C}{(\beta_\lambda - 1)(\frac{1}{r-\mu} - p\delta)\pi_2^C}$ *and*

$$B_e^I = \{\bar{H}_l^I + (\frac{1}{r-\mu} - p\delta)(\pi_1 - \pi_2^I)X_e\}X_e^{-\beta_\lambda}, \quad B_e^C = \{\bar{H}_l^C - (\frac{1}{r-\mu} - p\delta)\pi_2^C X_e\}X_e^{-\beta_\lambda}. \tag{14}$$

Equation (14) shows that the arbitrary constant B_e^C increases with $\bar{H}_l^C$. Since in the English rule, the litigation cost is lower than that in the US system, i.e., $\bar{H}_l^C < H_l^C$, this constant is lower under the English rule, causing a smaller impact on the option value for the Challenger if the Challenger exits first.

2.1.2 Summary of Non-Settlement: I Withdraws vs C Exits

We follow Lambrecht (2001) in determining the order of withdrawal or exit and the calculation of these thresholds. The explanation of the approach and the exact conditions are provided in Appendix 6. We use "I withdraws first" to represent the case if I first withdraws before C exits. "C exits first" stands for the case when C exits before I withdraws from the litigation.

The value of not settling during litigation in Case A $(V_{ns}^I(x), V_{ns}^C(x))$ where the subscript "ns" represent "no-settlement" are

$$(V_{ns}^I(x), V_{ns}^C(x)) = \begin{cases} (V_w^I(x), V_w^C(x)), & \text{if I withdraws first} \\ (V_e^I(x), V_e^C(x)). & \text{if C exits first} \end{cases} \tag{15}$$

The constants in the value functions in Case A are

$$(B_{ns}^I, B_{ns}^C) = \begin{cases} (B_w^I, B_w^C), & \text{if I withdraws first} \\ (B_e^I, B_e^C), & \text{if C exits first} \end{cases} \tag{16}$$

with the action threshold

$$X_{ns} = \begin{cases} X_w, & \text{if I withdraws first} \\ X_e. & \text{if C exits first} \end{cases} \tag{17}$$

Ex-post Settlement

The majority of patent infringement lawsuits end up with settlements between the two parties. We examine whether these observable settlements, which correspond to "ex-post settlement" in the model, happen under different conditions in the US vs UK legal systems and why. Without asymmetric information, our model helps focus on firms' financial incentive to settle, that is, to save future litigation costs. We examine the conditions for *possible occurrence* of settlement as opposed to C exits or I withdraws, via comparing value functions for firms with different options. However, given that the judgement follows a Poisson process in our model, we do not have predictions regarding whether settlement happens definitely or not.

Once ex-post settlement occurs during litigation, the Incumbent can recoup some of his losses by collecting a fraction of the Challenger's ongoing profit $\Theta_p \pi_2^C x$. Thus, for any $x \leq X_p$, the value functions of both parties after ex-post settlement can be expressed as follows,

$$\begin{aligned} \tilde{V}_p^I(x) &= \frac{\pi_2^I + \Theta_p \pi_2^C}{r-\mu} x - C_p^I, \\ \tilde{V}_p^C(x) &= \frac{(1-\Theta_p)\pi_2^C}{r-\mu} x - C_p^C, \end{aligned} \tag{18}$$

By backward induction, we proceed with two steps. Step One, given any royalty rate proposed by I (i.e. Θ_p), we apply the value matching and smooth pasting conditions on C's value functions, and value matching condition on the I's value functions at the settlement threshold (i.e., $X_p(\Theta_p)$), to get the settlement threshold as well as arbitrary constants in value functions as functions of the royalty rate ($B_p^I(\Theta_p)$, $B_p^C(\Theta_p)$). Step Two, we maximise the Incumbent's value with the option to settle with respect to Θ_p to get the optimal royalty rate for I, which I proposes the settlement with.

Depending on the Challenger's acceptance threshold, there are two ex-post settlement strategies. One is called "later ex-post settlement" and the other is called "immediate settlement". For brevity, we show firms' later ex-post settlement strategy below. The value functions in this case are the same with that in American rule, but with different anticipated costs ($\bar{H}$). We also describe the immediate settlement in Appendix 6.

Corollary 3. *In Case A, the firm values with the ex-post settlement option during litigation (V_p^I and V_p^C) follow Proposition 1, the settlement threshold is* $X_p(\Theta_p) = \frac{\beta_\lambda(\bar{H}_l^C - C_p^C)}{(\beta_\lambda - 1)(\frac{\Theta_p}{r-\mu} - p\delta)\pi_2^C}$ *and*

$$\begin{aligned} B_p^I(\Theta_p) &= \Big[\bar{H}_l^I - C_p^I + \Big(\frac{\Theta_p \pi_2^C}{r-\mu} - p\delta(\pi_1 - \pi_2^I)\Big) X_p\Big] X_p^{-\beta_\lambda}, \\ B_p^C(\Theta_p) &= \Big[\bar{H}_l^C - C_p^C + \Big(p\delta - \frac{\Theta_p}{r-\mu}\Big)\pi_2^C X_p\Big] X_p^{-\beta_\lambda}, \end{aligned} \tag{19}$$

$$\Theta_p^* = p\delta(r-\mu)\Big(1 - \frac{1}{\frac{\beta_\lambda}{\beta_\lambda - 1} + \frac{1}{\bar{\Gamma}}}\Big) + \frac{p\delta(r-\mu)}{\Phi}\Big(\frac{1}{\frac{\beta_\lambda}{\beta_\lambda - 1} + \frac{1}{\bar{\Gamma}}}\Big) \geq p\delta(r-\mu), \tag{20}$$

where we define "modified relative cost saving" $\bar{\Gamma} = \frac{\bar{H}_l^C - C_p^C}{\bar{H}_l^I - C_p^I}$ *and "gain-to-loss ratio"* $\Phi = \frac{\pi_2^C}{\pi_1 - \pi_2^I}$.

The inequality in Eq. (20) is to ensure $X_p \geq 0$, which is equivalent to $\Phi \leq 1$. From Eq. (20), $\frac{\partial \Theta_p^*}{\partial \bar{\Gamma}} = p\delta(r-\mu)(\frac{1}{\Phi} - 1)\frac{\Gamma^{-2}}{\frac{\beta}{\beta-1} + \frac{1}{\bar{\Gamma}}} > 0$. Given the expression of $\bar{H}_l^i, i \in \{\mathrm{I}, \mathrm{C}\}$ in Eq. (8), $\Gamma_{\mathrm{UK}} < \Gamma_{\mathrm{US}}$, we can thus get the following result:

Theorem 1. *The royalty rate in an ex-post settlement is higher in the US than in the UK legal system (i.e.,* $\Theta_{p,UK}^* < \Theta_{p,US}^*$.*).*

Intuitively, the lower royalty rate under the English rule is a result of the shift of willingness to settle from C to I. C has less incentive to settle during litigation under the UK system. Because of the cost allocation rule in the UK and C's financial constraints, C recovers its litigation cost if it wins but does not have to pay for I's litigation cost when it loses. For the same reason, I has a higher incentive to settle in the UK system.

This result demonstrates the importance of considering financial constraints when we examine the impact of different legal systems on litigation outcomes. Intuitively, I's effective bargaining power is weakened under the English rule because of C's financial constraints. We will revisit the royalty rate in ex-post settlement again when we incorporate the possibility of I liquidating due to its financial constraints.

2.1.3 Summary of Strategies During Litigation

Although this royalty level Θ_p^* can maximise the $V_p^I(X_p)$, if settlement occurs each party will agree to the ex-post settlement only if their value including the option to settle is higher than the value to them of not settling during litigation.

If both firms have incentives to settle, that is, finding themselves better off settling than not settling, ex-post settlement is the litigation outcome. On the one hand, C only accepts I's royalty rate offer if by agreeing to settle and settling at its optimal threshold, C gets a higher firm value than not settling, that is $V_p^C(x, \Theta_p) \geq V_{ns}^C(x)$ which implies that, Θ_p^* has an upper bound (Θ_p^{Cmax}) which arises from C's participation constraint:

$$\Theta_p^{Cmax} = \begin{cases} p\delta(r-\mu)\{1 + (\bar{h}_l\bar{\gamma}_c/\Phi)^{1-\frac{1}{\beta_\lambda}}[\bar{h}_l(1-\beta_\lambda)/\Phi - \beta_\lambda]^{\frac{1}{\beta_\lambda}}\}, & \text{if I withdraws} \\ (1 - p\delta(r-\mu))\bar{\gamma}_c^{1-\frac{1}{\beta_\lambda}} + p\delta(r-\mu), & \text{if C exits} \end{cases} \tag{21}$$

where

$$\bar{h}_l = \frac{\bar{H}_l^C}{\bar{H}_l^I}, \quad \bar{\gamma}_c = \frac{\bar{H}_l^C - C_p^C}{\bar{H}_l^C}. \tag{22}$$

On the other hand, I offers ex-post settlement only if its firm value including the ex-post settlement option is higher than the value of not settling during litigation, that is $V_p^I(x, \Theta_p) \geq V_{ns}^I(x)$. This implies that there is a range of $\Theta_p \in [\Theta_p^{Imin}, \Theta_p^{Imax}]$ such that I is better off settling than not-settling, the range depending on the order of I's withdrawal vs C's exit. If I withdraws first (i.e $B_{ns}^I = B_w^I$), I's participation constraint (i.e., $V_p^I(x, \Theta_p) \geq V_{ns}^I(x)$) becomes

$$[\frac{1-\beta_\lambda}{\bar{\Gamma}}(\frac{\Theta_p^I}{r-\mu} - p\delta) - \beta_\lambda(\frac{\Theta_p^I}{r-\mu} - p\delta/\Phi)](\frac{\Theta_p^I}{r-\mu} - p\delta)^{\beta_\lambda - 1} - (\bar{\gamma}_c \bar{h}_l)^{\beta_\lambda - 1}(p\delta/\Phi)^{\beta_\lambda} \geq 0. \tag{23}$$

and if C exits first (i.e $B_{ns}^I = B_e^I$), it becomes

$$\begin{aligned}\Big[\frac{1}{\bar{\Gamma}}(\frac{\Theta_p^I}{r-\mu} - p\delta) + \frac{\beta_\lambda}{\beta_\lambda - 1}(\frac{\Theta_p^I}{r-\mu} - \frac{p\delta}{\Phi})\Big](\frac{\Theta_p^I}{r-\mu} - p\delta)^{\beta_\lambda - 1} \\ -(\bar{\gamma}_c)^{\beta_\lambda - 1}(\frac{1}{r-\mu} - p\delta)^{\beta_\lambda}(\frac{1}{\bar{h}_l} + \frac{\beta_\lambda}{(\beta_\lambda - 1)\Phi}) \geq 0.\end{aligned} \tag{24}$$

Lemma 1. *For ex-post settlement to be feasible, $\Theta_p^{Imin} \leq \min\{\Theta_p^{Cmax}, \Theta_p^{Imax}\}$ with Θ_p^{Cmax} defined in Eq. (21). Θ_p^{Imax} and Θ_p^{Imin} are the solutions for Eq. (23) if I withdraws first, and are the solutions for Eq. (24) if C exits first.*

We define the maximum royalty levels in ex-post settlement as θ_p^{max}, which are determined by the Challenger or the Incumbent, that is

$$\Theta_p^{max} = min\{\Theta_p^{Cmax}, \Theta_p^{Imax}\} \tag{25}$$

If conditions $\Theta_p^{Imin} \leq \Theta_p^{max}$ and $\Theta_p^* \in [\Theta_p^{Imin}, \Theta_p^{max}]$ hold, firms settle at the optimal royalty rate Θ_p^* defined in Eq. (20) in Corollary 3. Otherwise, if only $\Theta_p^{Imin} \leq \Theta_p^{max}$ holds, but the optimal royalty rate Θ_p^* expressed in Eq. (20) is higher than Θ_p^{max}, the Incumbent is willing to decrease the optimal royalty rate Θ_p^* to Θ_p^{max}. In other words, ex-post settlement will not be the litigation outcome if $\Theta_p^{Imin} \leq \Theta_p^{max}$ is not satisfied, or if we cannot solve for Θ_p^{Imin} and Θ_p^{Imax}, i.e., there is no solution for equation (23) or (24).

For completion of the analysis, we also consider the possibility that firms may find it optimal to settle immediately after litigation starts. The details are in Appendix 6.

Therefore, firms value during litigation in Case A are

$$V_d^I = \begin{cases} V_p^I, & \text{if ex-post settlement occurs} \\ V_{ns}^I, & \text{if no ex-post settlement} \end{cases} \tag{26}$$

$$V_d^C = \begin{cases} V_p^C, & \text{if ex-post settlement occurs} \\ V_{ns}^C, & \text{if no ex-post settlement} \end{cases} \tag{27}$$

where $V_{ns}^i, i \in \{textI, C\}$ follows the expression in Eq. (15).

It is also possible that the court rules before the demand shock drops to the settlement threshold, in which case there will be no settlement even if both firms would be willing to settle.

2.1.4 Case B: The Constrained Incumbent ("I May Liquidate")

In the above analysis we have assumed that before the final judgement occurs, the Incumbent is always able to pay the Challenger's litigation cost if he loses. We now go on to consider the Case B that incorporates the possibility that judgement occurs at a value of x for which the Incumbent will not be able to pay the Challenger's legal fees in full if the Challenger wins the case.

When we examine the effect of different cost allocation rules on litigation outcomes, it is important to realize that the fee shifting rule in the UK leads to a new possibility regarding the outcome of the lawsuit beyond the ones in the US: I may liquidate ("Case B"). As a result of having to pay for C's litigation cost if I loses in judgement under the UK system, it becomes possible that I cannot afford the required payment and thus has to liquidate. Because litigation happens at high market demand, and firms take actions during litigation when demand drops afterwards, we can transform this case to the following necessary condition in the model analysis: firms' non-settlement threshold during litigation is lower than I's litigation threshold, i.e., $x_d < \bar{x}$. where $d = \{w, e, p\}$ representing "I withdraws", "C exits", "ex-post settlement".

To see why $x_d < \bar{x}$ gives us "I may liquidate", we separate all possible scenarios during litigation by focusing on the random court ruling time relative to when firms take actions, and the demand level at the ruling relative to I's liquidation threshold. Court rules either (1) before firms take their actions whilst at a demand lower than I's liquidation threshold (i.e., $\tau < t_d$ and $x_\tau < \bar{x}$), or (2) before firms take their actions whilst at a demand higher than I's liquidation threshold (i.e., $\tau < t_d$ and $x_\tau \geq \bar{x}$), or (3) court ruling would have happened after firms' action(s) (i.e., $\tau > t_d$). When imposing $x_d < \bar{x}$, I liquidates in Scenario (1) if the court rules against I, but remains a going-concern in the other two scenarios during litigation. We call Case B "I may liquidate", and we interpret Case B as the case if we consider the Incumbent facing financial constraints (although the constraints are not binding in Scenario (2) or (3)).

Given $x_d < \bar{x}$, there are two ranges of the market demand x that are relevant during litigation: 1) the demand is above I's liquidation threshold, i.e., $(x_d <)\bar{x} \leq x$ and 2) the demand is in between firms action threshold and I's liquidation threshold $x_d < x < \bar{x}$. For 1) the HJB equations of Eq. (4) and Eq. (5) as well as their boundary conditions for $x \to \infty$ are still valid. For 2), we revise the firms' payoffs in the event of I losing the lawsuit and C becoming the new monopoly. As a result, I's firm value during litigation for $(x_d <)x < \bar{x}$ satisfies the following ordinary HJB equation:

$$rv^I = \frac{1}{2}\frac{\partial^2 v^I}{\partial x^2}x^2\sigma^2 + \mu x\frac{\partial v^I}{\partial x} - C_l^I + \pi_2^I x + \lambda\Big(p\frac{\pi_1^I x}{r-\mu} - v^I\Big). \tag{28}$$

Similarly, C's firm value during litigation if $(x_s <)x < \bar{x}$ (i.e., v^C) satisfies

$$rv^C = \frac{1}{2}\frac{\partial^2 v^C}{\partial x^2}x^2\sigma^2 + \mu x\frac{\partial v^C}{\partial x} - C_l^C + \pi_2^C x + \lambda\Big(\frac{(1-p)\pi_1^C}{r-\mu}x - v^C\Big). \tag{29}$$

One notable difference in Case A and Case B is that C's value becomes $\frac{\pi_1^C x}{r-\mu}$ if I liquidates, which indicates C becomes the new monopolist. This is because the value transferred from I to C after judgement is $\min\{H_L^C, \frac{\pi_2^I x}{r-\mu}\}$. If I cannot pay in full, I liquidates and C becomes a creditor so effectively takes over the firm and this is what allows C to become a monopolist. When $x < \bar{x}$, the Incumbent is unable to pay the full amount of the Challenger's cost and so there are additional value matching and smooth pasting conditions for v_d^I and v_d^C at $\bar{x}$ during litigation.

We use the following proposition to summarize the value functions in the case of "I may liquidate".

Proposition 2. *In Case B (which only applies to some of the lawsuits in the UK), I may liquidate, and the firm values during litigation v^I and v^C follow:*

$$v_d^I = \begin{cases} a_d^I x^{\alpha_\lambda} + b_d^I x^{\beta_\lambda} - H_l^I + \frac{\pi_2^I x}{r-\mu+\lambda} + p\delta\pi_1^I x, & \text{if} \quad x < \bar{x} \\ \check{B}_d^I x^{\beta_\lambda} - \bar{H}_l^I + \frac{\pi_2^I x}{r-\mu} + \delta p(\pi_1^I - \pi_2^I)x. & \text{if} \quad x > \bar{x} \end{cases} \tag{30}$$

$$v_d^C = \begin{cases} a_d^C x^{\alpha_\lambda} + b_d^C x^{\beta_\lambda} - H_l^C + \frac{\pi_2^C x}{r-\mu+\lambda} + \pi_1^C(1-p)\delta x, & \text{if} \quad x < \bar{x} \\ \check{B}_d^C x^{\beta_\lambda} - \bar{H}_l^C + (\frac{1}{r-\mu} - p\delta)\pi_2^C x. & \text{if} \quad x > \bar{x} \end{cases} \tag{31}$$

for $x > x_d$, where β_λ, δ, H_L^I, H_L^C, $\bar{H}_L^I$, $\bar{H}_L^C$ are defended as in Case A. $\alpha_\lambda = \frac{1}{2} + \frac{\mu}{\sigma^2} + \sqrt{(\frac{1}{2} - \frac{\mu}{\sigma^2})^2 + \frac{2(r+\lambda)}{\sigma^2}}$.

Two arbitrary constants are the same for all cases i.e. withdrawal, exit and settlement by solving the value-matching and smoothing pasting conditions, i.e., $a^I = \frac{1}{\beta_\lambda - \alpha_\lambda}\big[(\beta_\lambda - 1)(1-p)\pi_2^I \delta \bar{x} - \beta_\lambda(\bar{H}_l^I - H_l^I)\big]\bar{x}^{-\alpha_\lambda}$ *and* $a^C = \frac{1}{\beta_\lambda - \alpha_\lambda}\big[(\beta_\lambda - 1)(1-p)(\pi_2^C - \pi_1^C)\delta\bar{x} - \beta_\lambda(\bar{H}_l^C - H_l^C)\big]\bar{x}^{-\alpha_\lambda}$*. The other arbitrary constants* b_d^I*,* B_d^I*,* b_d^C*,* B_d^C *depend on firms' strategies. The subscripts* $d \in \{w, e, p\}$*, representing "I withdraws", "C exits", "ex-post settlement". Note that* B_d^I *and* B_d^C *will be different from the values in Case A.*

When we compare the value functions of firms in Proposition 1 and Proposition 2, we discover that firms have an additional option value $a_d^i x^{\alpha_\lambda}$. Therefore, firms would optimally choose their strategies in Case B first by considering the additional values. If there are no feasible strategies can be found in this case, i.e. $x_d > \bar{x}$, the firms decide their strategies in Case A as discussed in Section 2.1.1. Note if we cannot find any feasible strategies in two cases, that is, there is neither action threshold in Case A that satisfies the condition $X_d > \bar{x}$ nor action threshold in Case B that satisfies the condition $x_d < \bar{x}$, firms choose to make the strategy at $\bar{x}$ in terms of maximising values with respect to threshold choice.

We then analyse firms' strategies of withdrawal, exit and ex-post settlement in Case B. The methods to derive the action thresholds and the value of arbitrary constants are similar with that in Case A, but we have additional value-matching and smooth-pasting conditions at $\bar{x}$.

Withdrawal by the Incumbent

If the Incumbent liquidates because the Incumbent is unable to pay the Challenger's cost in full when he loses, the Challenger thus earns monopoly profit. However, the Incumbent can choose to withdraw from litigation at x_w if the litigation procedure is too long. We therefore obtain the same payoff function as in Eq. (10) for any $x \leq x_w$. With the proof in Appendix 6, we get the following result.

Corollary 4. *The firm values with the Incumbent's option to withdraw during litigation* v_w^I *and* v_w^C *in Case B ("I may liquidate") follow Proposition 2, the withdraw threshold* x_w *solves*

$$\big[(\beta_\lambda - 1)(1-p)\pi_2^I \delta\bar{x} - \beta_\lambda(\bar{H}_l^I - H_l^I)\big](\frac{x_w}{\bar{x}})^{\alpha_\lambda} - \beta_\lambda h_l^I + (\beta_\lambda - 1)\delta(p\pi_1^I - \pi_2^I)x_w = 0, \tag{32}$$

and the arbitrary constants are

$$\begin{aligned}
b_w^I &= \big[\frac{\alpha_\lambda}{\alpha_\lambda - \beta_\lambda}H_l^I - \frac{\alpha_\lambda - 1}{\alpha_\lambda - \beta_\lambda}(p\pi_1^I - \pi_2^I)\delta x_w\big]x_w^{-\beta_\lambda}, \\
b_w^C &= \big[H_l^C - ((1-p)\pi_1^C - \pi_2^C)\delta x_w - a^C x_w^{\alpha_\lambda}\big]x_w^{-\beta_\lambda}, \\
\check{B}_w^I &= \big[\bar{H}_l^I - H_l^I + (p-1)\pi_2^I \bar{x}\delta + a_w^I \bar{x}^{\alpha_\lambda} + b_w^I \bar{x}^{\beta_\lambda}\big]\bar{x}^{-\beta_\lambda}, \\
\check{B}_w^C &= \big[\bar{H}_l^C - H_l^C + (1-p)(\pi_2^I + \pi_1^C - \pi_2^C)\delta\bar{x} + a_w^C \bar{x}^{\alpha_\lambda} + b_w^C \bar{x}^{\beta_\lambda}\big]\bar{x}^{-\beta_\lambda}.
\end{aligned} \tag{33}$$

If the withdraw threshold x_w calculated in this case is higher than $\bar{x}$, the Incumbent can pay full damages at this threshold if the Challenger wins, so he withdraws at the threshold in Corollary 1. However, if the threshold in Corollary 1 is also outside its range of validity i.e. if it is lower than $\bar{x}$, he will optimally choose to withdraw at $\bar{x}$.

Note that the arbitrary constants $\check{B}_w^i, i \in \{\mathrm{I}, \mathrm{C}\}$ in Eq. (33) are different with B_w^i in Case A as expressed in Eq. (11) as the range of the market demand x for the two cases are different, making firms have different option values.

Exit by the Challenger

If the Incumbent liquidates because the Incumbent is unable to pay the Challenger's cost in full when he loses, Challenger thus earns monopoly profit. However, the Challenger can still choose to exit from litigation at x_e if the litigation becomes too costly. The payoff functions for any $x \leq x_e$ are the same in Eq. (12). With the proof in Appendix 6, we have

Corollary 5. *The firm values with the Challenger's option to exit during litigation v_e^I and v_e^C follow Proposition 2, with the exit threshold x_e satisfies*

$$\big[(\beta_\lambda - 1)(\pi_2^C - \pi_1^C)(1-p)\delta\bar{x} - \beta_\lambda(\bar{H}_l^C - H_l^C)\big](\frac{x_e}{\bar{x}})^{\alpha_\lambda} - \beta_\lambda H_l^C + (\beta_\lambda - 1)((1-p)\pi_1^C\delta + \frac{\pi_2^C}{r-\mu+\lambda})x_e = 0 \tag{34}$$

and the arbitrary constants are

$$\begin{aligned}
b_e^I &= \big[H_l^I + ((1-p)\frac{\pi_2^I}{r-\mu} + \frac{p\pi_1^I - \pi_2^I}{r-\mu+\lambda})x_e - a^I x_e^{\alpha_\lambda}\big]x_e^{-\beta_\lambda} \\
b_e^C &= \frac{1}{\alpha_\lambda - \beta_\lambda}\big[\alpha_\lambda H_l^C - (\alpha_\lambda - 1)((1-p)\pi_1^C\delta + \frac{\pi_2^C}{r-\mu+\lambda})x_e\big]x_e^{-\beta_\lambda}, \\
\check{B}_e^I &= \big[\bar{H}_l^I - H_l^I + (p-1)\delta\pi_2^I\bar{x} + a^I\bar{x}^{\alpha_\lambda} + b_e^I\bar{x}^{\beta_\lambda}\big]\bar{x}^{-\beta_\lambda}, \\
\check{B}_e^C &= \big[\bar{H}_l^C - H_l^C - ((p-1)\pi_2^I - \pi_1^C)\delta\bar{x} + a^C\bar{x}^{\alpha_\lambda} + b_e^C\bar{x}^{\beta_\lambda}\big]\bar{x}^{-\beta_\lambda}
\end{aligned} \tag{35}$$

If the exit threshold x_e calculated in this case is higher than $\bar{x}$, the Challenger will not wait to exit until the Incumbent cannot pay full damages if the Challenger wins. Instead, the Challenger exits at the threshold expressed in Corollary 2. If the exit threshold in Corollary 2 is also lower than $\bar{x}$, she will exit at $\bar{x}$.

Similarly, the arbitrary constants $\check{B}_e^i, i \in \{\mathrm{I}, \mathrm{C}\}$ in Eq. (35) differ from B_e^i in Case A as expressed in Eq. (14) because the range of market demand x for the two cases differs, resulting in firms having different option values.

In Case B, the value functions v_{ns}^I, v_{ns}^C and constants are determined by two litigants' reservation threshold based on Lambrecht (2001), which is the same as in Case A expressed in Section 2.1.2 and detailed in Appendix 6.

Ex-post Settlement

To avoid the high ongoing litigation cost, both parties can choose to settle. We follow the same method in Lukas and Welling (2012). The Incumbent is the party who offers a settlement contract with an optimal royalty level and the Challenger can choose to reject or accept this offer at her optimal timing. With the proof in Appendix 6, we show firms' values with the later ex-post settlement option below

Corollary 6. *In Case B, the firm values with the later ex-post settlement option during litigation (v_p^I and v_p^C) follow Proposition 2, the settlement threshold $x_p(\theta_p)$ satisfies the following equation*

$$\begin{aligned}&\big[(\beta_\lambda-1)(\pi_2^C-\pi_1^C)(1-p)\delta\bar{x}-\beta_\lambda(\bar{H}_l^C-H_l^C)\big](\frac{x_p}{\bar{x}})^{\alpha_\lambda}\\ &-\beta_\lambda(H_l^C-C_p^C)+(\beta_\lambda-1)(((1-p)\pi_1^C-\pi_2^C)\delta+\frac{\theta_p^*\pi_2^C}{r-\mu})x_p=0\end{aligned}\tag{36}$$

and

$$\begin{aligned}b_p^I(\theta_p^*)&=\big[H_l^I-C_p^I+((\pi_2^I-p\pi_1^I)\delta+\frac{\theta_p^*\pi_2^C}{r-\mu})x_p-a^Ix_p^{\alpha_\lambda}\big]x_p^{-\beta_\lambda},\\ b_p^C(\theta_p^*)&=\frac{1}{\alpha_\lambda-\beta_\lambda}\big[\alpha_\lambda(H_l^C-C_p^C)+(\alpha_\lambda-1)(((p-1)\pi_1^C+\pi_2^C)\delta-\frac{\theta_p^*\pi_2^c}{r-\mu})x_p\big]x_p^{-\beta_\lambda},\\ \check{B}_p^I(\theta_p^*)&=\big[\bar{H}_l^I-H_l^I+(p-1)\delta\pi_2^I\bar{x}+a^I\bar{x}^{\alpha_\lambda}+b_p^I\bar{x}^{\beta_\lambda}\big]\bar{x}^{-\beta_\lambda},\\ \check{B}_p^C(\theta_p^*)&=\big[\bar{H}_l^C-H_l^C-((p-1)\pi_2^I-\pi_1^C)\delta\bar{x}+a^C\bar{x}^{\alpha_\lambda}+b_p^C\bar{x}^{\beta_\lambda}\big]\bar{x}^{-\beta_\lambda},\end{aligned}\tag{37}$$

where θ_p^* is the optimal royalty rate determined by I, i.e.

$$\begin{aligned}\theta_p^*(x_p)=-\frac{r-\mu}{\beta_\lambda\pi_2^Cx_p}\big(&[(1-p)[\alpha_\lambda(\pi_2^C-\pi_1^C)-(\beta_\lambda-1)\pi_2^I]\delta\bar{x}\\ &-\beta_\lambda[\frac{\alpha_\lambda}{\beta_\lambda-1}(\bar{H}_l^C-H_l^C)-(\bar{H}_l^I-H_l^I)]](\frac{x_p}{\bar{x}})^{\alpha_\lambda}\\ &+[(1-p)\pi_1^C-\pi_2^C+(\beta_\lambda-1)(\pi_2^I-p\pi_1^I)]\delta x_p+\beta_\lambda(H_l^I-C_p^I)\big).\end{aligned}\tag{38}$$

If the ex-post settlement threshold x_p calculated in this case is higher than $\bar{x}$, the Incumbent still can pay full damages at this threshold if the Challenger wins, so both parties will settle later at the threshold in Corollary 3. However, if the threshold in Corollary 3 is lower than $\bar{x}$, then firms settle at $\bar{x}$ with a new optimal royalty that maximises I's firm value with the option to settle ex-post, which is $\theta_p^*(\bar{x})=\frac{\bar{x}^{1-\alpha_1}\pi_2^C}{r-\mu}$.

Similar with the case that the Incumbent can pay full costs, ex-post settlement will not occur if $b_p^I(\theta_p^*) < b_{ns}^I$ or $b_p^C(\theta_p^*) < b_{ns}^C$ because the arbitrary constants $a^i, i \in \{\text{I, C}\}$ are independent of the settlement threshold, which indicates the value of not settling is higher than the value of settling for both parties in Case B. It is also possible that the court rules before the demand shock drops to the settlement threshold, in which case there will be no settlement even if both firms agree to settle.

We can also derive the boundaries of optimal ex-post settlement royalty derived from each party's value functions via the same feasibility conditions, i.e.

$$\tilde{v}_p^I(\theta_p^I, x) \geq v_{ns}^I(x), \tag{39}$$

$$\tilde{v}_p^C(\theta_p^C, x) \geq v_{ns}^C(x). \tag{40}$$

Due to the complexity of these expressions, we numerically check the upper and lower bounds of firms' optimal strategies during litigation in Case B using MATLAB. For completion of the analysis, we also consider the possibility that firms may find it optimal to settle immediately after litigation starts. The details are in the Appendix 6.

Summary of Strategies During Litigation

Similar with Case A, firms' make their strategies during litigation depending on their firm values in different strategies, i.e. by comparing the value of non-settlement and the value of settling either ex-post or immediately, i.e. as soon as litigation commences.

We follow Lambrecht (2001) in determining the order of withdrawal or exit and the calculation of these thresholds. The explanation of the approach and the exact conditions are provided in Appendix 6. We use "I withdraws first" to represent the case if I first withdraws before C exits. "C exits first" stands for the case when C exits before I withdraws from the litigation.

The value of not settling during litigation in Case A $(v_{ns}^I(x), v_{ns}^C(x))$ where the subscript "ns" represent "no-settlement" are

$$(v_{ns}^I(x), v_{ns}^C(x)) = \begin{cases} (v_w^I(x), v_w^C(x)), & \text{if I withdraws first} \\ (v_e^I(x), v_e^C(x)). & \text{if C exits first} \end{cases} \tag{41}$$

with the action threshold

$$x_{ns} = \begin{cases} x_w, & \text{if I withdraws first} \\ x_e. & \text{if C exits first} \end{cases} \tag{42}$$

We numerically compare firms' value of settling v_p^i and firms' value of not settling v_{ns}^i in MATLAB, and summarize firms' value during litigation as follows

$$v_d^I = \begin{cases} v_p^I, & \text{if ex-post settlement occurs} \\ v_{ns}^I, & \text{if no ex-post settlement} \end{cases} \tag{43}$$

$$v_d^C = \begin{cases} v_p^C, & \text{if ex-post settlement occurs} \\ v_{ns}^C. & \text{if no ex-post settlement} \end{cases} \tag{44}$$

Summary of Case A and Case B

In summary, firms determine their strategies of non-settlement and ex-post settlement once litigation starts by comparing their firm values with the options to make these strategies in each case. Compared to the American rule, the possibility of liquidation for I affects firms' strategies during litigation, therefore we have an additional Case B under the English rule, which complicates the analysis.

Taking into account the expected effective litigation costs during litigation, firms optimally choose their strategies during litigation in different cases depending on the market demand, i.e. $X_d > \bar{x}$ in Case A and $x_d < \bar{x}$ in Case B. We then show the method of choosing the strategies in either case A or B under the English rule in three different scenarios.

(1) If both conditions are satisfied, firms optimally choose the Case B version of their non-settlement strategies with firms' value (v_d^i), since this gives rise to a higher firm value at $\bar{x}$ and hence for all x. If the non-settlement threshold is in Case B, and the incumbent can choose his ex-post settlement strategy by comparing his firm values with the option to settle ex-post in Case A and B (V_p^i versus v_p^i) and then choose the case with higher firm value.

(2) If neither condition is satisfied, i.e. if the action threshold X_d is lower than $\bar{x}$ and if x_d is higher than $\bar{x}$, firms choose their non-settlement threshold at $\bar{x}$ to achieve the maximum firm values based on the value-matching condition.

(3) It is also possible that we only find one Case is relevant in each strategy, i.e. one of the conditions

$X_d > \bar{x}$ and $x_d < \bar{x}$ is satisfied. if we only find one Case is relevant when determining the non-settlement strategies, to accommodate various scenarios, firms make their non-settlement decisions based on their order of exit and withdrawal, that is the case with a higher reservation threshold, detailed in Appendix 6. For example, it is possible that we only find a valid withdraw threshold in Case A and a valid exit threshold in Case B, i.e. $X_e < \bar{x}$, $X_w > \bar{x}$, $x_e < \bar{x}$ and $x_w > \bar{x}$. If I's reservation threshold is higher than C's reservation threshold, the non-settlement strategy is withdrawal in Case A. Moreover, if ex-post settlement is feasible in one case by comparing the value of settling and the value of not settling in the same case, then firms settle in the relevant case. If we find the valid ex-post settlement threshold and non-settlement threshold in different case, we compare the value of B_d^i and $\breve{B}_d^i$ for $x \geq \bar{x}$, that exists in both cases, to determine the feasibility of ex-post settlement.

Therefore, firms' value during litigation depending on their strategies (i.e. ex-post settlement or non-settlement) and cases (i.e. A or B). The incumbent chooses the optimal royalty rate that maximises his value function if he offers settlement subject to the constraint that settlement has to be worthwhile for both parties i.e. the constraints on each party's value function. The payoff to litigation for the two firms can be written as

$$(V_d^I(x), v_d^I(x)) = \begin{cases} (V_p^I(x), v_p^I(x)), & \text{if settle ex-post} \\ (V_{ns}^I(x), v_{ns}^I(x)). & \text{if no settlement} \end{cases} \tag{45}$$

$$(V_d^C(x), v_d^C(x)) = \begin{cases} (V_p^C(x), v_p^C(x)), & \text{if settle ex-post} \\ (V_{ns}^C(x), v_{ns}^C(x)). & \text{if no settlement} \end{cases} \tag{46}$$

Knowing firms' strategies during litigation, we then analyse firms' strategies before litigation, i.e. the incumbent's litigation strategy and both firms' ex-ante settlement strategy.

2.2 Before Litigation

In this section, we analyse firms' strategies before litigation starts. In this stage, the Incumbent can choose whether to litigate or settle ex-ante, while the Challenger can choose whether to accept the ex-ante settlement if ex-ante settlement is offered by the Incumbent. Firms' strategies in this case will not be affected by the possibility of liquidation for I, but the values after litigation occurs will affect firms' values with the option to litigate and these litigation option values will impact firms' ex-ante settlement strategies.

Before litigation, two firms share the market and earn the duopoly profit from the product. Therefore, firms' value functions V_{bl}^I and V_{bl}^C in Case A and v_{bl}^I and v_{bl}^C in Case B satisfy the HJB equations that can be written as

$$r(V,v)_{bl}^I dt = \mathbb{E}\mathcal{D}(V,v)_{bl}^I + \pi_2^I x dt, \tag{47}$$

$$r(V,v)_{bl}^C dt = \mathbb{E}\mathcal{D}(V,v)_{bl}^C + \pi_2^C x dt, \tag{48}$$

where $\mathbb{E}\mathcal{D}(V,v)^i = (\frac{1}{2}\frac{\partial^2 (V,v)^i}{\partial x^2}x^2\sigma^2 + \mu x\frac{\partial (V,v)^i}{\partial x})dt$ for firm $i, i \in \{I, C\}$.

The boundary conditions to solve the above value function depend on the litigation outcome discussed in the above section 2.1. Specifically, if the Incumbent would withdraw first during litigation in the absence of any settlement offer, we have the boundary conditions $V_{bl}^I(0) = 0$ and $V_{bl}^C(0) = 0$, while if the Challenger would exit first in the absence of an ex-post settlement offer, $V_{bl}^I(0) = 0$ and $V_{bl}^C(0) = 0$ do not hold but the value-matching conditions and smooth-pasting condition at $(X,x)_e$ should apply, i.e. $V_{bl}^I((X,x)_e) = \frac{\pi_1^I}{r-\mu}, V_{bl}^C((X,x)_e) = 0$ and $\frac{\partial V_{bl}^C((X,x)_e)}{\partial (X,x)_e} = 0$. Therefore, when the Challenger exits first once the litigation starts, the Incumbent thus has the option to force the Challenger out of the market with the threat of patent litigation, which we call the "forcing out option". I then analyse firms' strategies before litigation, that is litigation or ex-ante settlement, in the two scenarios, i.e. I withdraws first during litigation (Case 1) and C exits first during litigation (Case 2) due to the different value functional forms.

2.2.1 Case 1: I Withdraws First During Litigation

With boundary conditions $V_{bl}^I(0) = 0$ and $V_{bl}^C(0) = 0$, we have

Proposition 3. *If the Incumbent would withdraw first during litigation in the absence of settlement, then the value functions before litigation follow*

$$(V_{bl}^I(x), V_{bl}^C(x)) = (\frac{\pi_2^I x}{r-\mu} + A_{bl}^I x^\alpha, \frac{\pi_2^C x}{r-\mu} + A_{bl}^C x^\alpha),\ \textit{In Case A} \tag{49}$$

$$(v_{bl}^I(x), v_{bl}^C(x)) = (\frac{\pi_2^I x}{r-\mu} + a_{bl}^I x^\alpha, \frac{\pi_2^C x}{r-\mu} + a_{bl}^C x^\alpha),\ \textit{In Case B} \tag{50}$$

where $\alpha = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{(\frac{1}{2} - \frac{\mu}{\sigma^2})^2 + \frac{2r}{\sigma^2}} > 1,$*. The arbitrary constants* A_{bl}^I*,* A_{bl}^C*,* a_{bl}^I*,* a_{bl}^C *are determined by whether firms settle ex-ante or enter litigation.*

2.2.2 The Incumbent Litigates

We first investigate the Incumbent's litigation strategy assuming litigation occurs. The firms' values after starting the lawsuit are $(V_d^i, v_d^i), i \in$ {I, C} discussed in Section 2.1. Applying value matching and smooth-pasting conditions between the value functions during litigation, i.e., (V_d^I, V_d^C) or (v_d^I, v_d^C) and value functions with the option to litigate (V_l^I and V_l^C), we can derive the litigation threshold and value functions in Case A and B. For brevity, the following Corollary 7 shows the litigation strategies if settlement does not occur immediately once litigation commences. This includes withdrawal, exit and later ex-post settlement possibilities. The strategies that followed by immediate settlement can be found in Appendix 6.

Corollary 7. *If the Incumbent would withdraw first during litigation in the absence of ex-ante settlement, the firm values before litigation with I's option to litigate follow Eq. (49) and (50) with*

$$
\begin{aligned}
a_l^I &= \begin{cases} \{(p\pi_1^I - \pi_2^I)\delta x_l + a^I x_l^{\beta_\lambda} + b^I x_l^{\alpha_\lambda} - H_l^I\} x_l^{-\alpha}, & x_l \leq \bar{x} \\ \{p(\pi_1 - \pi_2^I)\delta x_l + \check{B}^I x_l^{\beta_\lambda} - \bar{H}_l^I\} x_l^{-\alpha}. & x_l > \bar{x} \end{cases} \\
a_l^C &= \begin{cases} \{((1-p)\pi_1^C - \pi_2^C)\delta x_l + a^C x_l^{\beta_\lambda} + b^C x_l^{\alpha_\lambda} - H_l^C\} x_l^{-\alpha}, & x_l \leq \bar{x} \\ \{-p\delta\pi_2^C x_l + \check{B}^C x_l^{\beta_\lambda} - \bar{H}_l^C\} x_l^{-\alpha}. & x_l > \bar{x} \end{cases} \\
A_l^I &= \{p(\pi_1 - \pi_2^I)\delta X_l + B^I X_l^{\beta_\lambda} - \bar{H}_l^I\} X_l^{-\alpha}, \\
A_l^C &= \{-p\delta\pi_2^C X_l + B^C X_l^{\beta_\lambda} - \bar{H}_l^C\} X_l^{-\alpha}.
\end{aligned}
$$

where B^i, $\check{B}^i$, a^i and b^i are defined in Section 2.1. The litigation thresholds in Case A (X_l) and Case B (x_l)satisfy

$$(\alpha - 1)X_l(\pi_1 - \pi_2^I)p\delta + (\alpha - \beta_\lambda)B^I X_l^{\beta_\lambda} - \alpha\bar{H}_l^I = 0, \tag{51}$$

$$\begin{cases} (\alpha - 1)x_l(p\pi_1^I - \pi_2^I)\delta + (\alpha - \beta_\lambda)b^I x_l^{\beta_\lambda} + (\alpha - \alpha_\lambda)a^I x_l^{\alpha_\lambda} - \alpha H_l^I = 0 & \text{for} \quad x_l \leq \bar{x} \\ (\alpha - 1)x_l(\pi_1 - \pi_2^I)p\delta + (\alpha - \beta_\lambda)\check{B}^I x_l^{\beta_\lambda} - \alpha\bar{H}_l^I = 0 & \text{for} \quad x_l > \bar{x} \end{cases} \tag{52}$$

In general, the litigation threshold is higher than the various thresholds during litigation which include exit threshold, withdraw threshold, and ex-post settlement threshold, i.e., $(X_l, x_l) > \max\{(X_e, x_e), (X_w, x_w), (X_p, x_p)\}$. This means that the Incumbent waits until the demand condition is sufficiently high before litigating, and then the parties wait until the demand condition deteriorates to certain extent before trying to settle or to drop the case. We also allow the litigation threshold to equal the ex-post settlement threshold, i.e., $(X_l, x_l) = (X_p, x_p)$. In this case, ex-post settlement may occur immediately when the Incumbent chooses to litigate. Details can be found in Appendix 6.

2.2.3 Ex-ante Settlement

With the proof in Appendix 6, we can show

Corollary 8. *If the Incumbent would withdraw first during litigation in the absence of settlement, then the* *firm values before litigation follow Eq. (49) and (50) with*

$$A_a^I = \{\frac{\theta_a^* \pi_2^C X_a}{r-\mu} - C_a^I\} X_a^{-\alpha_\lambda}, \quad A_a^C = -\{\frac{\theta_a^* \pi_2^C X_a}{r-\mu} + C_a^C\} X_a^{-\alpha_\lambda}. \tag{53}$$

$$a_a^I = \{\frac{\theta_a^* \pi_2^C x_a}{r-\mu} - C_a^I\} x_a^{-\alpha_\lambda}, \quad a_a^C = -\{\frac{\theta_a^* \pi_2^C x_a}{r-\mu} + C_a^C\} x_a^{-\alpha_\lambda}. \tag{54}$$

The ex-ante settlement threshold is the same as litigation threshold ($X_a = X_l$, $x_a = x_l$), and the royalty rate in ex-ante settlement is

$$\theta_a^* = \theta_a^{max} = \begin{cases} -\frac{r-\mu}{X_l \pi_2^C}(A_l^C X_l^{\alpha_\lambda} + C_a^C), & \text{in Case A} \\ -\frac{r-\mu}{X_l \pi_2^C}(a_l^C X_l^{\alpha_\lambda} + C_a^C), & \text{in Case B.} \end{cases} \tag{55}$$

Ex-ante settlement happens if $\theta_a^* \geq \theta_a^{min}$ where $\theta_a^{min} = \frac{r-\mu}{X_l \pi_2^C}(A_l^I X_l^{\alpha_\lambda} + C_a^I)$ in Case A, and $\theta_a^{min} = \frac{r-\mu}{X_l \pi_2^C}(a_l^I X_l^{\alpha_\lambda} + C_a^I)$ in Case B.

2.2.4 Case 2: C Exits First During Litigation

We model the possibility for the Incumbent to force the Challenger out of the market after the arguable infringement by a threat of litigation. It happens if the market demand before litigation commences drops to the Challenger's exit threshold before it rises to the Incumbent's litigation threshold. This is relevant if the Challenger would exit first during litigation in the absence of settlement, but not relevant if the Incumbent would withdraw first during litigation in the absence of settlement. With such consideration,

Proposition 4. *The firm the firm $i, i \in \{I, C\}$'s values before litigation in Case 2 are:*

$$V_{bl}^i(x) = \frac{\pi_2^i x}{r-\mu} + A_{bl}^i x^\alpha + B_{bl}^i x^\beta, \text{ In Case A} \tag{56}$$

$$v_{bl}^i(x) = \frac{\pi_2^i x}{r-\mu} + a_{bl}^i x^\alpha + b_{bl}^i x^\beta, \text{ In Case B} \tag{57}$$

where $\alpha = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{(\frac{1}{2} - \frac{\mu}{\sigma^2})^2 + \frac{2r}{\sigma^2}}$, and $\beta = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{(\frac{1}{2} - \frac{\mu}{\sigma^2})^2 + \frac{2r}{\sigma^2}}$. The arbitrary constants A_{bl}^i, B_{bl}^i, a_{bl}^i, b_{bl}^i are determined by whether firms settle ex-ante or enter litigation.

The Incumbent Litigates

We can derive firms' value with the option to litigate by the Incumbent via value-matching and smooth-pasting conditions at litigation threshold $(X, x)_l$ and exit threshold $(X, x)_e$.

Corollary 9. *If the Challenger would exit first during litigation in the absence of settlement, the firm values before litigation with I's option to litigate follow Eq. (56) and (57), we obtain the value of arbitrary* *constants in value functions and the litigation threshold that can be written as (For brevity, we only show the litigation strategy that is followed by non-immediate settlement and leave the litigation strategy that is followed by immediate settlement in the appendix.):*

$$
\begin{aligned}
A_l^C =& \frac{1}{{X_l}^{\beta-\alpha} - X_e^{\beta-\alpha}}(B^C {X_l}^{\beta_\lambda-\alpha} - p\delta\pi_2^C {X_l}^{1-\alpha} - H_l^C {x_l}^{-\alpha}),\\
B_l^C =& \frac{1}{{X_l}^{\alpha-\beta} - X_e^{\alpha-\beta}}(B^C {X_l}^{\beta_\lambda-\beta} - p\delta\pi_2^C {X_l}^{1-\beta} - H_l^C {X_l}^{-\beta}),\\
A_l^I =& \frac{1}{X_l^{\beta-\alpha} - X_e^{\beta-\alpha}}\big[p\delta(\pi_1-\pi_2^I)X_l^{1-\alpha} - \frac{\pi_1-\pi_2^I}{r-\mu}X_e^{1-\alpha} + B^I X_l^{\beta_\lambda-\alpha} - \bar{H}_l^I X_l^{-\alpha}\big],\\
B_l^I =& \frac{1}{X_l^{\alpha-\beta} - X_e^{\alpha-\beta}}\big[p\delta(\pi_1-\pi_2^I)X_l^{1-\beta} - \frac{\pi_1-\pi_2^I}{r-\mu}X_e^{1-\beta} + B^I X_l^{\beta_\lambda-\beta} - \bar{H}_l^I X_l^{-\beta}\big],
\end{aligned}
\tag{58}
$$

$$
\begin{aligned}
a_l^I &= \begin{cases} \frac{1}{x_l^{\beta-\alpha}-x_e^{\beta-\alpha}}\big[a^I x_l^{\alpha_\lambda-\alpha} + b^I x_l^{\beta_\lambda-\alpha} - H_l^I x_l^{-\alpha} + (p\pi_1^I - \pi_2^I)\delta x_l^{1-\alpha} - \frac{p\pi_1^I-\pi_2^I}{r-\mu}x_e^{1-\alpha}\big], & x_l \le \bar{x}\\ \frac{1}{x_l^{\beta-\alpha}-x_e^{\beta-\alpha}}\big[p\delta(\pi_1-\pi_2^I)x_l^{1-\alpha} - \frac{\pi_1-\pi_2^I}{r-\mu}x_e^{1-\alpha} + \check{B}^I x_l^{\beta_\lambda-\alpha} - \bar{H}_l^I x_l^{-\alpha}\big], & x_l > \bar{x} \end{cases}\\
b_l^I &= \begin{cases} \frac{1}{x_l^{\alpha-\beta}-x_e^{\alpha-\beta}}\big[a^I x_l^{\alpha_\lambda-\beta} + b^I x_l^{\beta_\lambda-\beta} - H_l^I x_l^{-\beta} + (p\pi_1^I - \pi_2^I)\delta x_l^{1-\beta} - \frac{p\pi_1^I-\pi_2^I}{r-\mu}x_e^{1-\beta}\big], & x_l \le \bar{x}\\ \frac{1}{x_l^{\alpha-\beta}-x_e^{\alpha-\beta}}\big[p\delta(\pi_1-\pi_2^I)x_l^{1-\beta} - \frac{\pi_1-\pi_2^I}{r-\mu}x_e^{1-\beta} + \check{B}^I x_l^{\beta_\lambda-\beta} - \bar{H}_l^I x_l^{-\beta}\big], & x_l > \bar{x} \end{cases}\\
a_l^C &= \begin{cases} \frac{1}{x_l^{\alpha_\lambda-\alpha}-x_e^{\beta-\alpha}}\big[a^C x_l^{\alpha_\lambda-\alpha} + b^C x_l^{\beta_\lambda-\alpha} - H_l^C x_l^{-\alpha} + (\pi_1^C(1-p) - \pi_2^C)\delta x_l^{1-\alpha} + \frac{\pi_2^C}{r-\mu}x_e^{1-\alpha}\big], & x_l \le \bar{x}\\ \frac{1}{{x_l}^{\beta-\alpha}-x_e^{\beta-\alpha}}(\check{B}^C {x_l}^{\beta_\lambda-\alpha} - p\delta\pi_2^C {x_l}^{1-\alpha} - H_l^C {x_l}^{-\alpha}), & x_l > \bar{x} \end{cases}\\
b_l^C &= \begin{cases} \frac{1}{x_l^{\alpha-\beta}-x_e^{\alpha-\beta}}\big[a^C x_l^{\alpha_\lambda-\beta} + b^C x_l^{\beta_\lambda-\beta} - H_l^C x_l^{-\beta} + (\pi_1^C(1-p) - \pi_2^C)\delta x_l^{1-\beta} + \frac{\pi_2^C}{r-\mu}x_e^{1-\beta}\big], & x_l \le \bar{x}\\ \frac{1}{{x_l}^{\alpha-\beta}-x_e^{\alpha-\beta}}(\check{B}^C {x_l}^{\beta_\lambda-\beta} - p\delta\pi_2^C {x_l}^{1-\beta} - H_l^C {x_l}^{-\beta}). & x_l > \bar{x} \end{cases}
\end{aligned}
\tag{59}
$$

The litigation thresholds in Case A and Case B (X_l, x_l) satisfy (60) and (61) below and a^i, b^i, B^i and $\check{B}^i$ $(i \in \{I, C\})$ are defined in Section 2.1.

$$
\begin{aligned}
&\big[(1-\beta)(\frac{X_e}{X_l})^\alpha - (1-\alpha)(\frac{X_e}{X_l})^\beta\big]p\delta(\pi_1-\pi_2^I) + \big[(\beta_\lambda-\beta)(\frac{X_e}{X_l})^\alpha - (\beta_\lambda-\alpha)(\frac{X_e}{X_l})^\beta\big]B^I X_l^{\beta_\lambda-1}\\
&-(\alpha-\beta)\frac{\pi_1-\pi_2^I}{r-\mu}\frac{X_e}{X_l} + (\beta(\frac{x_e}{X_l})^\alpha - \alpha(\frac{X_e}{X_l})^\beta)H_l^I X_l^{-1} = 0.
\end{aligned}
\tag{60}
$$

$$
\begin{cases}
\beta a_l^I x_l^{\beta-1} + \alpha b_l^I x_l^{\alpha-1} - \alpha_\lambda a^I x_l^{\alpha_\lambda-1} - \beta_\lambda b^I x_l^{\beta_\lambda-1} - (p\pi_1^I - \pi_2^I)\delta = 0, & \text{for} \quad x_l \le \bar{x}\\
\big[(1-\beta)(\frac{x_e}{x_l})^\alpha - (1-\alpha)(\frac{x_e}{x_l})^\beta\big]p\delta(\pi_1-\pi_2^I) + \big[(\beta_\lambda-\beta)(\frac{x_e}{x_l})^\alpha - (\beta_\lambda-\alpha)(\frac{x_e}{x_l})^\beta\big]\check{B}^I x_l^{\beta_\lambda-1} & \\
-(\alpha-\beta)\frac{\pi_1-\pi_2^I}{r-\mu}\frac{x_e}{x_l} + (\beta(\frac{x_e}{x_l})^\alpha - \alpha(\frac{x_e}{x_l})^\beta)H_l^I x_l^{-1} = 0. & \text{for} \quad x_l > \bar{x}
\end{cases}
\tag{61}
$$

2.2.5 Ex-ante Settlement

Similarly, the two firms can also settle ex-ante to avoid costly litigation. Applying value-matching conditions and optimality conditions at both x_e and x_l, we derive the following:

Corollary 10. *we have the firm values before litigation with ex-ante settlement option follow Eq. (56) and (57) if the Challenger would exit first during litigation in the absence of settlement, with*

$$A_a^I = \frac{1}{X_a^{\beta-\alpha} - X_e^{\beta-\alpha}}\big[\theta_a\frac{\pi_2^C}{r-\mu}X_a^{1-\alpha} - \frac{\pi_1^I - \pi_2^I}{r-\mu}X_e^{1-\alpha} - C_a^I X_a^{-\alpha}\big], \tag{62}$$

$$B_a^I = \frac{1}{X_a^{\alpha-\beta} - X_e^{\alpha-\beta}}\big[\theta_a\frac{\pi_2^C}{r-\mu}X_a^{1-\beta} - \frac{\pi_1^I - \pi_2^I}{r-\mu}X_e^{1-\beta} - C_a^I X_a^{-\beta}\big],$$

$$A_a^C = \frac{1}{X_a^{\beta-\alpha} - X_e^{\beta-\alpha}}\big[-\theta_a\frac{\pi_2^C}{r-\mu}X_a^{1-\alpha} - C_a^C X_a^{-\alpha}\big], \tag{63}$$

$$B_a^C = \frac{1}{X_a^{\alpha-\beta} - X_e^{\alpha-\beta}}\big[-\theta_a\frac{\pi_2^C}{r-\mu}X_a^{1-\beta} - C_a^C X_a^{-\beta}\big].$$

$$a_a^I = \frac{1}{x_a^{\beta-\alpha} - x_e^{\beta-\alpha}}\big[\theta_a\frac{\pi_2^C}{r-\mu}x_a^{1-\alpha} - \frac{\pi_1^I - \pi_2^I}{r-\mu}x_e^{1-\alpha} - C_a^I x_a^{-\alpha}\big], \tag{64}$$

$$b_a^I = \frac{1}{x_a^{\alpha-\beta} - x_e^{\alpha-\beta}}\big[\theta_a\frac{\pi_2^C}{r-\mu}x_a^{1-\beta} - \frac{\pi_1^I - \pi_2^I}{r-\mu}x_e^{1-\beta} - C_a^I x_a^{1-\beta}\big],$$

$$a_a^C = \frac{1}{x_a^{\beta-\alpha} - x_e^{\beta-\alpha}}\big[-\theta_a\frac{\pi_2^C}{r-\mu}x_a^{1-\alpha} - C_a^C x_a^{-\alpha}\big], \tag{65}$$

$$b_a^C = \frac{1}{x_a^{\alpha-\beta} - x_e^{\alpha-\beta}}\big[-\theta_a\frac{\pi_2^C}{r-\mu}x_a^{1-\beta} - C_a^C x_a^{-\beta}\big].$$

The Challenger optimally waits until litigation threshold to accept ex-ante settlement, and the royalty rates are

$$\Theta_a = \Theta_a^{Cmax} = -\frac{r-\mu}{\pi_2^C x_l}\big[A_l^C x_l^\beta + B_l^C x_l^\alpha + C_a^C\big] \quad \textit{in Case A} \tag{66}$$

$$\theta_a = \theta_a^{Cmax} = -\frac{r-\mu}{\pi_2^C x_l}\big[a_l^C x_l^\beta + b_l^C x_l^\alpha + C_a^C\big] \quad \textit{in Case B} \tag{67}$$

The feasibility of ex-ante settlement is determined by both firms' willingness of to settle ex-ante, therefore,

Corollary 11. *Firms reach an agreement of ex-ante settlement to avoid litigation when both firms' values of settling ex-ante are higher than the value of litigation, therefore, firms' optimal strategy before litigation is ex-ante settlement if*

1. *In Case A,* $\Theta_a \geq \Theta_a^{Imin}$*, where* $\Theta_a^{Imin} = \frac{r-\mu}{\pi_2^C x_l}\big[A_l^I x_l^\beta + B_l^I x_l^\alpha + C_a^I\big]$.

2. *In Case B,* $\theta_a \geq \theta_a^{Imin}$*, where* $\theta_a^{Imin} = \frac{r-\mu}{\pi_2^C x_l}\left[a_l^I x_l^\beta + b_l^I x_l^\alpha + C_a^I\right]$.

The relevant optimal ex-ante settlement royalties are defined in Eq. (66) and arbitrary constants are listed in Eq. (58) and Eq. (59).

2.2.6 Summary of Firm Strategies Before Litigation in Case 1 and 2

According to Décamps et al. (2006), the decision maker will select the project which generates the highest net expected discounted profit.

Similar to ex-post settlement, ex-ante settlement before litigation is possible when both parties' value of settling is higher than the value of litigation. That is, $V_a^I \geq V_l^I$ in Case A or $v_a^I \geq v_l^I$ in Case B and $V_a^C \geq V_l^C$ in Case A or $v_a^C \geq v_l^C$ in Case B.

So the value before litigation for both parties are

$$(V_{bl}^I(x), v_{bl}^I(x)) = \begin{cases} (V_a^I(x), v_a^I(x)), & \text{if settle ex-ante} \\ (V_l^I(x), v_l^I(x)). & \text{if litigates} \end{cases} \tag{68}$$

$$(V_{bl}^C(x), v_{bl}^C(x)) = \begin{cases} (V_p^C(x), v_p^C(x)), & \text{if settle ex-ante} \\ (V_l^C(x), v_l^C(x)). & \text{if litigates} \end{cases} \tag{69}$$

The above analysis indicates that before litigation, companies' values under the English rule have the same functional forms with those under the American rule, but the values of arbitrary constants in the value functions are different. The possibility of I's liquidation will influence the values under the English rule, whereas this is not possible under the American rule. This further demonstrates that while differences in legal systems appear to solely effect firms' actions once litigation begins, they can have an indirect impact on firms' strategies prior to litigation, as well as prior to infringement or innovation, which we will examine in the next section.

3 Model Extension - Infringement and Innovation

The game tree in Figure 1 starts after the arguable infringement. In this section, we discuss the extended model with simplifying assumptions which studies two earlier decisions: (1) the Challenger's technology adoption decision (i.e., the arguable infringement); and (2) the Incumbent's R&D decision which leads to its patenting of the technology that is then used in its production.

In two cases of "I remains a going-concern" and "I may liquidate" discussed as above, for firm $i, i \in \{\mathrm{I}, \mathrm{C}\}$, we use $(V,v)^i_g$ and $(V,v)^i_r$ to denote the values when the Challenger has the option to infringe but has not yet entered the market and the values before the Incumbent has entered the market and so the Incumbent has the option to innovate, respectively. The corresponding action thresholds are denoted by x_g and x_r. To distinguish between the firm's value with no forcing out option when the Incumbent withdraws first during litigation and the firm's value with forcing out option when the Challenger exits first during litigation, we use $(A,a)^i_g$ and $(A,a)^i_r$ to denote the arbitrary constants in $(V,v)^i_g$ and $(V,v)^i_r$ if " I withdraws first", where there is no forcing out option, and $(\hat{A},\hat{a})^i_g$ and $(\hat{A},\hat{a})^i_r$ to denote the arbitrary constants in $(V,v)^i_g$ and $(V,v)^i_r$ if "C exits first", which generates a forcing out option for the incumbent.

3.1 The (Alleged) Infringement Decision

Suppose the Challenger pays a fixed cost C_g in arguably infringing the Incumbent's patent and enters the market. Before infringement, C does not earn profits whilst I earns monopoly profit. We can get the following result regarding firm values before the arguable infringement decision:

Proposition 5. *The value functions before infringement for Case A (V^I_g, V^C_g) and Case B (v^I_g, v^C_g) are*

$$((V,v)^I_g,(V,v)^C_g) = \begin{cases} (\frac{\pi^I_1}{r-\mu}x + (A,a)^I_g x^\alpha, (A,a)^C_g x^\alpha) & \textit{if I withdraws first,} \\ (\frac{\pi^I_1}{r-\mu}x + (\hat{A},\hat{a})^I_g x^\alpha, (\hat{A},\hat{a})^C_g x^\alpha) & \textit{if C exits first.} \end{cases} \tag{70}$$

where the arbitrary constants $((A,a)^i_g$, $(\hat{A},\hat{a})^i_g)$, $i \in \{I, C\}$, are determined by the infringement option of the Challenger based on the litigation outcomes in two cases if the Incumbent starts litigation later on.

Since in this stage, the Challenger can only decide whether to infringe, through value-matching and smooth-pasting conditions, we can determine the arbitrary constants $(A,a)^C_g$ and $(\hat{A},\hat{a})^C_g$ and the infringement threshold denoted by $(x,\hat{x})_g$, which is the same in Case A and B but different in the withdrawal and exit case due to the forcing out option.

Corollary 12. *Before infringement happens, the value functions of the two firms V^I_g and V^C_g follow Eq. (70). The arbitrary constants in the value functions depend on which firm leaves the lawsuit if litigation happens and no settlement occurs:*

Case 1: If the Incumbent withdraws first during litigation,

$$(A^I_g, a^I_g) = ([\frac{\pi^I_2 - \pi_1}{r-\mu}x_g + A^I_{bl}x^\alpha_g]x^{-\alpha}_g, [\frac{\pi^I_2 - \pi_1}{r-\mu}x_g + a^I_{bl}x^\alpha_g]x^{-\alpha}_g)$$

$$(A^C_g, a^C_g) = ([\frac{\pi^C_2}{r-\mu}x_g + A^C_{bl}x^\alpha_g - C_g]x^{-\alpha}_g, [\frac{\pi^C_2}{r-\mu}x_g + a^C_{bl}x^\alpha_g - C_g]x^{-\alpha}_g),$$

where the infringement threshold is $x_g = \frac{\alpha(r-\mu)C_g}{(\alpha-1)\pi_2^C}$.

Case 2: If the Challenger exits first during litigation,

$$(\hat{A}_g^I, \hat{a}_g^I) = ([\frac{\pi_2^I - \pi_1}{r-\mu}\hat{x}_g + A_{bl}^I x_g^\alpha + B_{bl}^I \hat{x}_g^\beta]\hat{x}_g^{-\alpha}[\frac{\pi_2^I - \pi_1}{r-\mu}\hat{x}_g + a_{bl}^I \hat{x}_g^\alpha + b_{bl}^I \hat{x}_g^\beta]\hat{x}_g^{-\alpha})$$

$$(\hat{A}_g^C, \hat{A}_g^C) = ([\frac{\pi_2^C}{r-\mu}\hat{x}_g + A_{bl}^C \hat{x}_g^\alpha + B_{bl}^C \hat{x}_g^\beta - C_g]\hat{x}_g^{-\alpha}, [\frac{\pi_2^C}{r-\mu}\hat{x}_g + a_{bl}^C \hat{x}_g^\alpha + b_{bl}^C \hat{x}_g^\beta - C_g]\hat{x}_g^{-\alpha}),$$

where the infringement threshold $\hat{x}_g$ *satisfies*

$$(\alpha-1)\frac{\pi_2^C \hat{x}_g}{r-\mu} + (\alpha-\beta)(B,b)_{bl}^C \hat{x}_g^\beta - \alpha C_g = 0.$$

If we do not consider the possibility that the Incumbent can force the Challenger out of the market, after the arguable infringement, by a threat of litigation, then there is no difference between the above cases a) and b). Comparing the infringement threshold with and without the forcing out options, we find that the Incumbent's option to force the Challenger out of the market by a threat of litigation delays the Challenger's infringement and lowers the Challenger's incentive to infringe. Intuitively, a firm's willingness to continue to pay for the litigation cost during litigation also affects firms' decisions before litigation. The Challenger becomes more reluctant to use similar technology of the Incumbent in its products, if the Incumbent can later use the threat of litigation to force the Incumbent out of the market. The Challenger optimally waits until the market demand is higher before it arguably infringes, comparing with the absence of the Incumbent's forcing out option.

Furthermore, the impact of the different legal system on the Challenger's infringement incentives in Case A is obvious. Because of the different effective litigation costs during litigation, i.e. $\bar{H}_l^C \leq H_l^C$, the Challenger's value with the option to infringe is higher under the English rule, compared to that under the American rule.

3.2 The Innovation Decision

We simply assume that the Incumbent invests in R&D with the cost of C_r, if it innovates successfully and gets the patent which gets commercialized and used in its production with the monopoly profit π_1^I.

Proposition 6. *The Incumbent's value function before innovation is*

$$(V_r^I, v_r^I) = \begin{cases} ([\frac{\pi_1}{r-\mu}x_r + A_g^I x_r^\alpha - C_r](\frac{x}{x_r})^\alpha, [\frac{\pi_1}{r-\mu}x_r + a_g^I x_r^\alpha - C_r](\frac{x}{x_r})^\alpha) & \textit{I withdraws first,} \\ ([\frac{\pi_1}{r-\mu}x_r + \hat{A}_g^I x_r^\alpha - C_r](\frac{x}{x_r})^\alpha, [\frac{\pi_1}{r-\mu}x_r + \hat{a}_g^I x_r^\alpha - C_r](\frac{x}{x_r})^\alpha) & \textit{C exits first.} \end{cases} \tag{71}$$

The innovation threshold when I withdraws first is $x_r = \frac{\alpha(r-\mu)C_r}{(\alpha-1)\pi_1}$.

We compare the Incumbent's incentive to innovate with and without the option to force out the Challenger via the threat to litigate after its arguable infringement. Not surprisingly, the Incumbent's forcing out option leads to higher values of the arbitrary constants in the innovation stage, which we interpret as a higher incentive for the Incumbent to invest in R&D. However, the innovation thresholds are the same for the Incumbent with or without the forcing out option. We leave the comparison of innovation incentive between American rule and English rule to the numerical exercise to Section 4.

4 Comparative Statics using Quantitative Analysis

We examine how the litigation outcomes vary with product market characteristics (i.e., the gain-to-loss ratio defined in Corollary 3, and market demand volatility defined as σ in Eq. (1)), litigation process characteristics (i.e., the relative cost saving defined in Corollary 3), and characteristics of patent approval process (probability of patent validity p). Our model offers new insights on the likelihood of both ex-post and ex-ante settlement, and the term of settlement represented by the royalty rate. We list the benchmark parameter values in Table 1.

[Insert Table 1 here.]

4.1 Litigation Risk and Rate Inferred from Litigation Thresholds

A low litigation threshold x_l in our model maps to high unconditional litigation risk and a high litigation rate. Because the Incumbent starts the litigation when the demand raises to the litigation threshold, a low (high) litigation threshold represents a more (less) aggressive litigation strategy. Investigating litigation risk is important because the possibility of being involved in non-troll patent litigation imposes a high risk for firms selling innovative products, and on a macro level, litigation rate is relevant for social welfare and thus for policy makers.

We show in Figure 2 the thresholds for litigation (solid lines) and ex-post settlements (dotted lines) with respect to the gain-to-loss ratio (Φ) under two legal systems separately.[3] The black (red) lines represent the American (English) rule. The overlap of the litigation and settlement threshold indicates a settlement that occurs immediately after the Incumbent starts the litigation.

[Insert Figure 2 here.]

[3] At the benchmark parameter set, we are at Case A ("I remains a going-concern") for the full range of $\Phi \in (0, 1]$ under the English rule.

Evident from the figure, the litigation threshold x_l increases monotonically with respect to Φ under the English rule if Φ except there is a kink when firms' optimal strategy is switched from not settling to immediate settlement, and it surpasses x_l under the American rule from below as $\Phi \uparrow$. As the products of the two firms become more complementary (e.g., the market profit does not decline much after the alleged infringement), the Incumbent under the English rule delays litigation, and more so than the Incumbent under the American rule. Here is our interpretation from looking at the more competitive end of the spectrum (i.e.,$\Phi \downarrow$): a lower Φ indicates a high probability that C exits during litigation, as it is harder for C to keep up with the litigation cost with its low profit. This possibility is higher under the American rule because the Incumbent's expected effective litigation cost is lower without the fee-shifting feature and thus he is less financially constrained. In this case, the Incumbent has a higher forcing out option value before litigation starts when Φ is low. In order to exercise the forcing out option, the Incumbent is more willing to wait until a higher litigation threshold under the American rule. When settlement is optimal if Φ is high, exercising the forcing out option is not optimal under both systems. The Incumbent responds to the optimal settlement strategy under both legal rules by starting litigation earlier (i.e.,$x_l \downarrow$) due to the increases of litigation incentives. This strategic effect is enhanced due to the loose financial constraints for the Incumbent under the American rule, leading to more aggressive litigation by the Incumbent (i.e., $x_l \downarrow\downarrow$).

4.2 Settlement Rate Inferred from Settlement Thresholds

Figure 2 shows that when both parties have the same expected cost saving from settlement (i.e., $\frac{H_l^C - C_s^C}{H_l^I - C_s^I} = 1$), settlement is feasible when the gain-to-loss ratio is high enough for both legal rules. However, under the American rule, settlement occurs with a slightly wider range of gain-to-loss ratios due to the high incentives to avoid litigation costs for both parties. In general, under both legal rules, C exits first if I litigates when the gain-to-loss ratio is low and the settlement is feasible when the gain-to-loss ratio is high. Furthermore, when I withdraws first when the gain-to-loss ratio is high, settlement takes the form of immediate settlement in both legal rules, making the litigation and settlement thresholds the same.

A high gain-to-loss ratio allows C to stay longer in the litigation and is able to pay the royalty required if firms agree to settle. In order to avoid the high ongoing litigation fee, both parties prefer to settle, either ex-ante or ex-post when the gain-to-loss ratio is high. When the gain-to-loss ratio is high enough, firms settle ex-ante instead of ex-post to avoid litigation costs. This is true for both legal rules. In addition, the fee-shifting feature combined with I's large loss of profit due to infringement when Φ is small gives I a higher incentive to drive C out of the market instead of settling. Due to the fee-shifting feature under the English rule, litigation occurs on a narrower range of parameters when the settlement is possible (i.e., the gain-to-loss ratio is high enough). Thus the Incumbent under the English rule delays litigation if its loss due to the alleged infringement is not that huge. However, once litigation starts, the Incumbent is willing to settle sooner when the gain-to-loss ratio is higher because of C's increased ability to pay.

Under the American rule, the settlement is feasible when the gain-to-loss ratio is high and the gap between the litigation thresholds and later ex-post settlement thresholds decreases when gain-to-loss ratio increases. However, firms reach ex-post or ex-ante settlement agreement on a wider range of parameter values under the American rule because C's effective litigation cost (H_l^C) is higher than that under the English rule, and thus the Challenger is more likely to accept the the settlement offer. Moreover, because the effective litigation cost for I (H_l^I) is lower under the American rule, the Incumbent does not have a strong incentive to avoid filing the lawsuit even when its loss is large. This leads to a lower litigation threshold and settlement threshold.

4.3 Royalty Rates in Settlement

Figure 3 compares the royalty rates in settlements, assuming the same litigation costs for the two firms. The black (red) lines plot royalty rate in settlement under the American (English) rule, and the solid (dashed) lines plot ex-post (ex-ante) settlement royalty rate. Note in this base case, I withdraws first in the most of regions, so settlement occurs immediately.

[Insert Figure 3 here.]

The royalty rates in ex-post settlement decrease with the gain-to-loss ratio under both legal rules, except the middle region under the American rule, where the settlement occurs immediately when C exits first. As Φ increases, the Incumbent loses fewer profits due to the Challenger's arguable infringement (i.e. $\pi_1^I - \pi_2^I \downarrow$), it requires a lower royalty rate from the Challenger in a settlement. However, under the American rule when both parties know C exits first and settlement occurs immediately, the Challenger's ability to pay royalty rate increases with Φ, thus increasing the royalty rate. Furthermore, under the English rule, royalty rates are lower than under the American rule. This is because the Incumbent is more financially constrained under the English rule due to the higher expected effective litigation cost, which increases his incentive to settle, lowering his required royalty rate.

4.4 Outcomes: Settlement vs Continuing to Litigate

We use Figure 4 to show the likely outcomes in patent litigation with respect to both the relative cost saving (i.e., Γ) and the gain-to-loss ratio (i.e., Φ) under the two legal systems. The fee-shifting feature under the English rule effectively increases I's expected litigation costs, thus increases the Incumbent's withdraw threshold, and makes I-withdraw more likely during litigation, as opposed to C-exit. This shift is caused by both of the two implications of the default cost allocation rules, i.e., (1) I gets less in its won lawsuit due to C's financial constraints and (2) I may liquidate because of its own financial constraints.

[Insert Figure 4 here.]

The green (blue) areas mark the likely ex-post (ex-ante) settlement and the lighter blue area shows the ex-ante settlement that is followed by immediate ex-post settlement. The general patterns of settlements are similar in the American and the English rules. The likelihood of settlement depends crucially on the gain-to-loss ratio. Only when the Challenger's alleged infringement has a relatively small impact on the Incumbent's market profit (i.e., Φ is high), settlements between the two firms are possible. This is because on the one hand, the Incumbent does not have a strong incentive to litigate when his lose is small and on the other hand, the Challenger generates enough profits to pay for a royalty payment if the gain-to-loss ratio is high.

The main distinction between the outcome plots for the two legal systems is that the settlement regions under the English rule are relatively insensitive to the relative cost saving, except when the Incumbent's cost savings are much greater than the Challenger's. In the US, the Incumbent is only likely to withdraw when it faces relatively high cost savings/litigation cost (low Γ), whereas the Challenger is more likely to exit in non-settlement when the Challenger's cost savings/litigation cost was relatively high (high Γ). However, under the English rule, the Incumbent is likely to withdraw on a much wider range of relative cost saving, and only when a very small gain-to-loss ratio combined with a very high relative cost saving, the Challenger's exit becomes relevant. This distinction arises because of two direct consequences of the fee-shifting feature in the English rule. First, the Incumbent's expected litigation costs become higher whereas the Challenger's expected litigation costs are reduced. Second, the Incumbent's expected litigation costs depend on both its own and its opponent's costs, the relative size of the cost for each firm is less important.

This further induces another distinction between firms' settlement strategies under two legal rules. Due to I's financial constraints, firms are more willing to settle immediately under English rule. This is attributed to the reason that the I withdraws first region is much larger under English rule than it is under American rule. When I withdraws first, firms prefer to settle immediately; however, when C exits first, the Incumbent has more comparative advantages in patent litigation and is willing to delay the settlement threshold.

4.5 The Incumbent's Winning Probability p

We show the likely litigation outcomes with two levels of p in Figure 5 (for the English rule) and in Figure 6 (for the American rule). These graphs show two robust patterns that we find in more numerical exercises: (1) under both legal systems, a higher p pushes down the boundary that separates I-withdraw and C-exit, such that there is a larger region $(\Phi \times \Gamma)$ in which C-exit absent of ex-post settlement; (2) p increases settlement likelihood under the English rule but decreases settlement likelihood under the American rule.

[Insert Figure 5 and 6 here.]

The intuition for the first observation is that a lower winning possibility increases the Incumbent's expected litigation costs, and makes the Incumbent effectively more financially constrained. This effect leads to I-withdraw on a larger range of gain-to-loss ratios, as opposed to C-exit. Figure 7 reveals the reason for the second observation, which compares the two scenarios that prevent ex-post settlement from happening: the Incumbent is unwilling to make an offer, and the Challenger rejects a settlement offer. This figure shows that although the impacts of p on the likelihood of ex-post settlement are opposite under two legal systems, the rejection region by the Challenger, which is represented by the solid lines shaded area, decreases with p in both legal systems. Meanwhile, the no-offering region by the Incumbent, which is represented by the dashed lines shaded area, increases with p in both legal systems. Under the English rule, the Challenger is more likely to reject the ex-post settlement offer because the likelihood of the Incumbent withdraws first increases significantly compared with the US system, resulting in a low likelihood of settlement if p is small. Meanwhile, the no-offering region is not large enough to offset the effect of decreased rejection region when is high, leading to a high likelihood of ex-post settlement under the English rule.

[Insert Figure 7 here.]

4.6 Product Market Volatility σ

We show the effect of product market volatility σ on the likely outcomes under the English rule in Figure 8. σ reduces the settlement region (in blue), pushing it towards the higher end of the gain-to-loss ratio. On the one hand, the Incumbent's incentives to continue in the litigation increase with σ. On the other hand, his the option value with settlement with a given royalty rate decreases with σ. As a result of both forces, the Incumbent becomes less willing to settle but more willing to continue paying the litigation fee and staying in the litigation when market volatility is higher.

[Insert Figure 8 here.]

We compare how settlement regions change with respect to σ under the American rule (black shaded area) vs. the English rule (red shaded area) in Figure 9. This figure confirms that, as under the English rule, settlement regions under the American rule also shrink when σ is higher. Some differences remain: (1) there is much less change on settlement likelihood around the boundary between I-withdraw and C-exit under the American rule, and (2) the settlement regions are not as affected by σ under the American rule. In general, the Incumbent has stronger incentives to settle when the market volatility is low and these incentives are enhanced under the English rule because the Incumbent is more financially constraints during litigation.

[Insert Figure 9 here.]

4.7 Innovation Incentives for the Incumbent

Table 2 lists the arbitrary constants in the Incumbent's value function before its innovation decision is made (i.e. A_r^I), and presents them in heat maps. The relative magnitude of the numbers in Panel (a) and (b) reflects the relative magnitude of the Incumbent's firm values before innovation under the two systems. We list the ratios between the value of the arbitrary constants under the English rule and that under the American rule in Panel (c) (i.e. $\frac{A_r^{I,UK}}{A_r^{I,US}}$). A ratio higher than 1 indicates the Incumbent has a higher incentives to innovate under the American rule, otherwise the ratio is less than 1.

[Insert Table 2 here.]

The innovation incentive for the patent holder is lower if the gain-to-loss ratio is higher. This holds in both legal systems, as shown in Panel (a) and (b). A high gain-to-loss ratio is associated with settlement being the likely outcome. Although the patent holding firm can save litigation costs from settling, it also gives up the possibility of restoring the monopoly profit in a potential favorable court ruling. The second force dominates, which lowers the Incumbent's firm value and reduces the innovation incentive. Meanwhile, Panel (c) shows the ratio is less than 1, indicating a lower innovation incentives under the English rule. This can be explained by the patent holder being more financially constrained under the English rule due to the fee-shifting feature, lowering the Incumbent's incentive to engage in R&D. In addition, we find that the ratio is lower when the relative cost saving is high and the gain-to-loss ratio is higher than 0.2 (lighter blue region). This is the region where C-exit under the American rule but I-withdraw under the English rule, absent of ex-post settlement. C-exit (as oppose to I-withdraw) is the likely outcome on a wider range of parameter values under the American rule because the Incumbent is less financially constrained. The likely outcome of C-exit gives the Incumbent the opportunity to utilize the forcing out option, and further reduces the Challenger's infringement incentives. As a result, the Incumbent's innovation incentives becomes higher.

5 Conclusion

The law and economics literature has studied IP litigation for five decades, before it is picked up more recently by financial economists. The latest interest is probably due to the corporate sector's new emphasis on intangible assets management, the raised awareness of IP infringement/litigation risk, and the increasing legal bills faced by firms. Our study takes a first theory step towards answering the open question of how legal systems affect corporate innovation, and through the lens of financial constraints.

In this paper, we explore the likely outcomes for non-troll patent litigation, i.e., patent litigation between two product firms, and compare them under the American rule (i.e., each party pays its own legal costs) and the English rule (i.e., the loser pays all legal costs). We focus on the impact of the cost allocation rules through firms' financial constraints, and show that the English rule shifts the negative impact of financial constraints towards the Incumbent, leading to our findings and testable implications on royalty rates in settlement, settlement likelihood, litigation timing, and innovation incentives. For example, the probability of ruling in favor of the patent owning Incumbent reduces the likelihood that two firms settle under the American rule, but its effect is opposite under the English rule.

Our contribution to the literature on corporate innovation roots in our more comprehensive way of modelling the strategic interaction between the two firms involved in patent litigation. These include, among others, recognizing the possibilities that: (1) the patent owning Incumbent can force the Challenger out of the market with a threat of patent litigation; (2) the Challenger, as well as the Incumbent, may drop out of the lawsuit because it is not worthwhile to keep financing the litigation cost; (3) the Incumbent may liquidate upon an unfavourable court ruling, because it cannot afford to pay the Challenger's legal costs. We also use a rigorous approach to model immediate settlement when the ex-post settlement threshold (i.e., the firms settle during litigation when the market demand drops to this level) turns out to be higher than the litigation threshold (i.e., the Incumbent starts the litigation when the demand goes up to this level).

In our analysis, we assumed that the decision to infringe or innovate is made at a certain level of market demand and that firms cannot choose not to infringe or innovate. Although we can generate some implications in this simplified setting, we are limited in our understanding of the impact of financial constraints imposed by different legal rules on firms' incentives to innovate. What would happen if we included abandonment options in the value of firms? We could then extend our game by taking into account the probability of failure in R&D for the Incumbent and the probability of infringement for the Challenger. We expect that the main results will remain robust in the new setting, but we leave this for future research due to the complexity.

References

Aoki, R., and J. Hu. 1999a. A cooperative game approach to patent litigation, settlement, and allocation of legal costs. *Working Papers 216, Department of Economics, The University of Auckland* .

Aoki, R., and J.-L. Hu. 1999b. Licensing vs. litigation: the effect of the legal system on incentives to innovate. *Journal of Economics & Management Strategy* 8:133–160.

Appel, I., J. Farre-Mensa, and E. Simintzi. 2019. Patent trolls and startup employment. *Journal of Financial Economics* .

Azevedo, A., and D. Paxson. 2014. Developing real option game models. *European Journal of Operational Research* 237:909–920.

Bebchuk, L. A. 1984. Litigation and settlement under imperfect information. *The RAND Journal of Economics* 15:404–415.

Bessen, J. E., and M. J. Meurer. 2006. Patent litigation with endogenous disputes. *American Economic Review* 96:77–81.

Bolton, P., and D. S. Scharfstein. 1990. A theory of predation based on agency problems in financial contracting. *The American economic review* pp. 93–106.

Bolton, P., N. Wang, and J. Yang. 2019. Investment under uncertainty with financial constraints. *Journal of Economic Theory* 184:104912.

Brown, J. R., S. M. Fazzari, and B. C. Petersen. 2009. Financing innovation and growth: Cash flow, external equity, and the 1990s R&D boom. *The Journal of Finance* 64:151–185.

Bustamante, M. C. 2015. Strategic investment and industry risk dynamics. *The Review of Financial Studies* 28:297–341.

Caskurlu, T. 2019. A dark side of Going Public: Patent lawsuits. Tech. rep., Working Paper, Amsterdam Business School.

Cohen, L., U. G. Gurun, and S. D. Kominers. 2016. The growing problem of patent trolling. *Science* 352:521–522.

Décamps, J.-P., T. Mariotti, and S. Villeneuve. 2006. Irreversible investment in alternative projects. *Economic Theory* 28:425–448.

Hugonnier, J., S. Malamud, and E. Morellec. 2015. Capital supply uncertainty, cash holdings, and investment. *The Review of Financial Studies* 28:391–445.

Jeon, H. 2015. Patent infringement, litigation, and settlement. *Economic Modelling* 51:99–111.

Lambrecht, B., and W. Perraudin. 2003. Real options and preemption under incomplete information. *Journal of Economic dynamics and Control* 27:619–643.

Lambrecht, B. M. 2001. The impact of debt financing on entry and exit in a duopoly. *The Review of Financial Studies* 14:765–804.

Landes, W. M. 1971. An economic analysis of the courts. *The Journal of Law and Economics* 14:61–107.

Lanjouw, J. O., and M. Schankerman. 2001. Characteristics of patent litigation: a window on competition. *RAND Journal of Economics* pp. 129–151.

Lee, J., S. Oh, and P. Suh. 2019. Inter-firm Patent Litigation and Innovation Competition. *SSRN working paper* .

Li, D. 2011. Financial constraints, R&D investment, and stock returns. *The Review of Financial Studies* 24:2974–3007.

Lin, D. 2021. R&D Investment under Financial Constraints and Competition. *working paper* .

Lin, D., D. Liu, and A. E. Whalley. 2021. The Impact of Product Market Characteristics on Firms' Strategies in Patent Litigation. *working paper* .

Llobet, G. 2003. Patent litigation when innovation is cumulative. *International Journal of Industrial Organization* 21:1135–1157.

Lukas, E., J. J. Reuer, and A. Welling. 2012. Earnouts in mergers and acquisitions: A game-theoretic option pricing approach. *European Journal of Operational Research* 223:256–263.

Lukas, E., and A. Welling. 2012. Negotiating M&As under uncertainty: The influence of managerial flexibility on the first-mover advantage. *Finance Research Letters* 9:29–35.

Ma, L., A. S. Mello, and Y. Wu. 2020. First-mover advantage, time to finance, and cash holdings. *Journal of Corporate Finance* 62:101584.

Malamud, S., and F. Zucchi. 2019. Liquidity, innovation, and endogenous growth. *Journal of Financial Economics* 132:519–541.

Meurer, M. J. 1989. The settlement of patent litigation. *The RAND Journal of Economics* pp. 77–91.

Mezzanotti, F., et al. 2016. Roadblock to innovation: The role of patent litigation in corporate R&D. *Unpublished manuscript. Harvard University* .

Ordover, J. A., A. Rubinstein, et al. 1983. *On Bargaining, Settling, and Litigating: A Problem in Multistage Games with Imperfect Information.* CV Starr Center for Applied Economics, New York University, Faculty of Arts and Science, Department of Economics.

Pawlina, G., and P. M. Kort. 2006. Real options in an asymmetric duopoly: Who benefits from your competitive disadvantage? *Journal of Economics & Management Strategy* 15:1–35.

Reinganum, J. F., and L. L. Wilde. 1986. Settlement, litigation, and the allocation of litigation costs. *The RAND Journal of Economics* pp. 557–566.

Smit, H. T., and L. Trigeorgis. 2006. Real options and games: Competition, alliances and other applications of valuation and strategy. *Review of Financial Economics* 15:95–112.

Figure 1: Strategies and Payoffs of the Incumbent ("I") and the Challenger ("C")

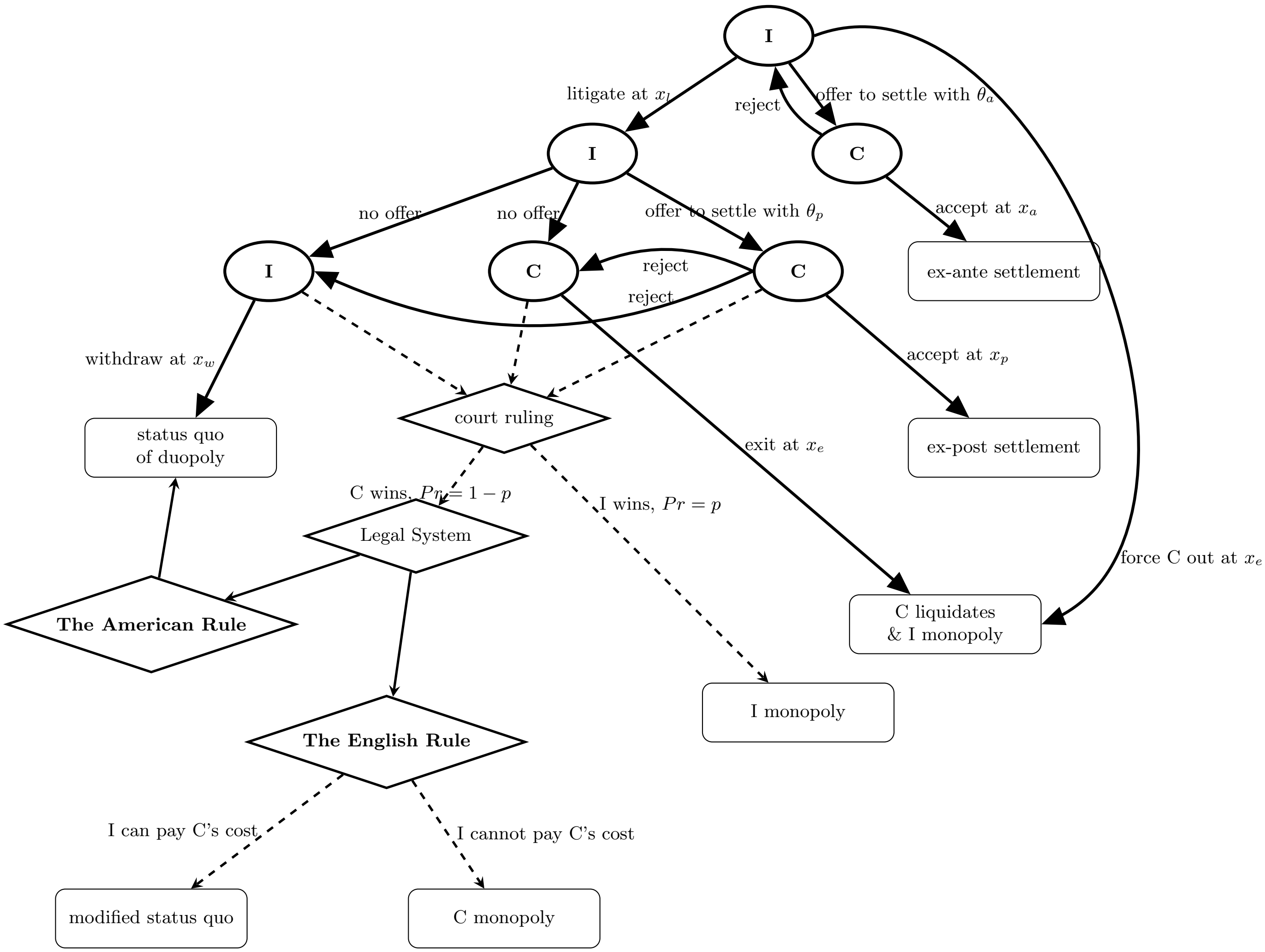


Figure 2: Litigation and Settlement Thresholds with Respect to Φ

The solid lines represent the litigation thresholds if litigation were to happen, and the dotted lines represent ex-post settlement threshold if ex-post settlement were to happen. Black lines represent the American rule and the red lines represent the English rule. When the solid and the dashed lines overlap, the firms are likely to settle ex-post immediately, which we then count as a special form of ex-ante settlement. The gain-to-loss ratio is defined as $\Phi = \dfrac{\pi_2^C}{\pi_1^I - \pi_2^I}$.

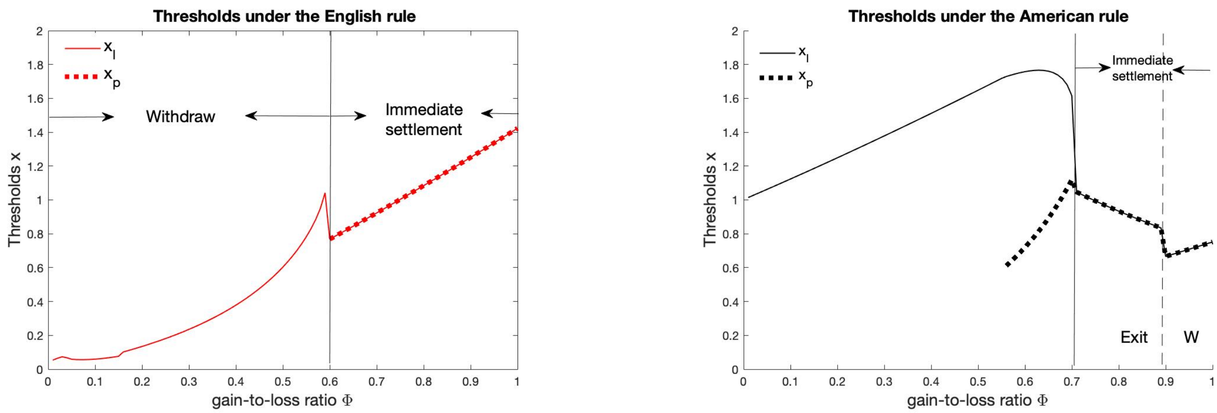


Figure 3: Settlement Royalty Rates with Respect to Φ

The solid line represents the optimal royalty rates for ex-post settlement, if ex-post settlement happens. The dashed lines are the optimal royalty rates in ex-ante settlement, if ex-ante settlement happens. Black lines represent the American rule and the red lines represent the English rule. When the solid and the dashed lines overlap, the firms are likely to settle ex-post immediately, which we then count as a special form of ex-ante settlement. Otherwise, the ex-post settlement does not happen immediately after litigation starts. The gain-to-loss ratio is defined as $\Phi = \dfrac{\pi_2^C}{\pi_1^I - \pi_2^I}$.

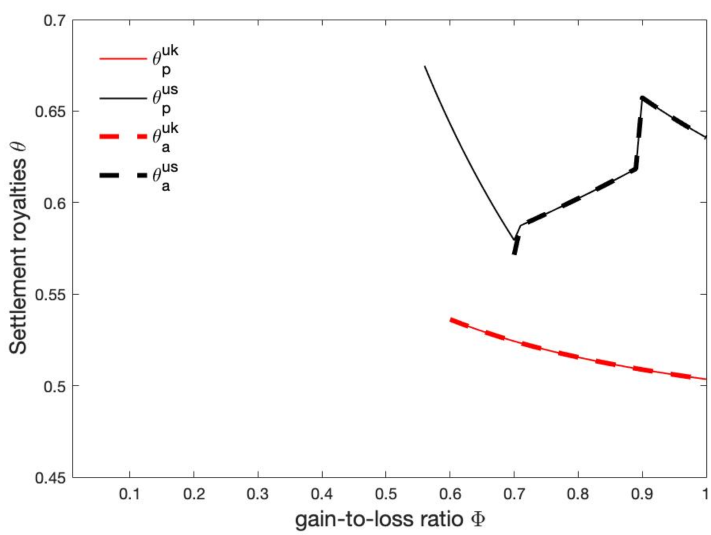


Figure 4: Likely outcomes in Patent Litigation

This graph shows the likely litigation outcomes at the baseline parameter values under the English rule versus the American rule. The green area is the feasible region for ex-post settlement. The blue area is the possible region for ex-ante settlement , while the immediate settlement region is represented by light blue regions. The black dashed line represents the boundary between regions where the Challenger exits first and where the Incumbent withdraws first. The black dotted line shows the boundary between the regions where the Incumbent remain as a going-concern upon judgement (Case A) and may liquidate (Case B). The top figure shows likely outcomes under the English rule and the bottom figure shows likely outcomes under the American rule. We vary the value of H_l^I when changing the relative cost saving $\Gamma = \frac{H_l^C - C_p^C}{H_l^I - C_p^I}$, and vary the value of π_1^l when changing the gain-to-loss ratio $\Phi = \dfrac{\pi_2^C}{\pi_1^I - \pi_2^I}$.

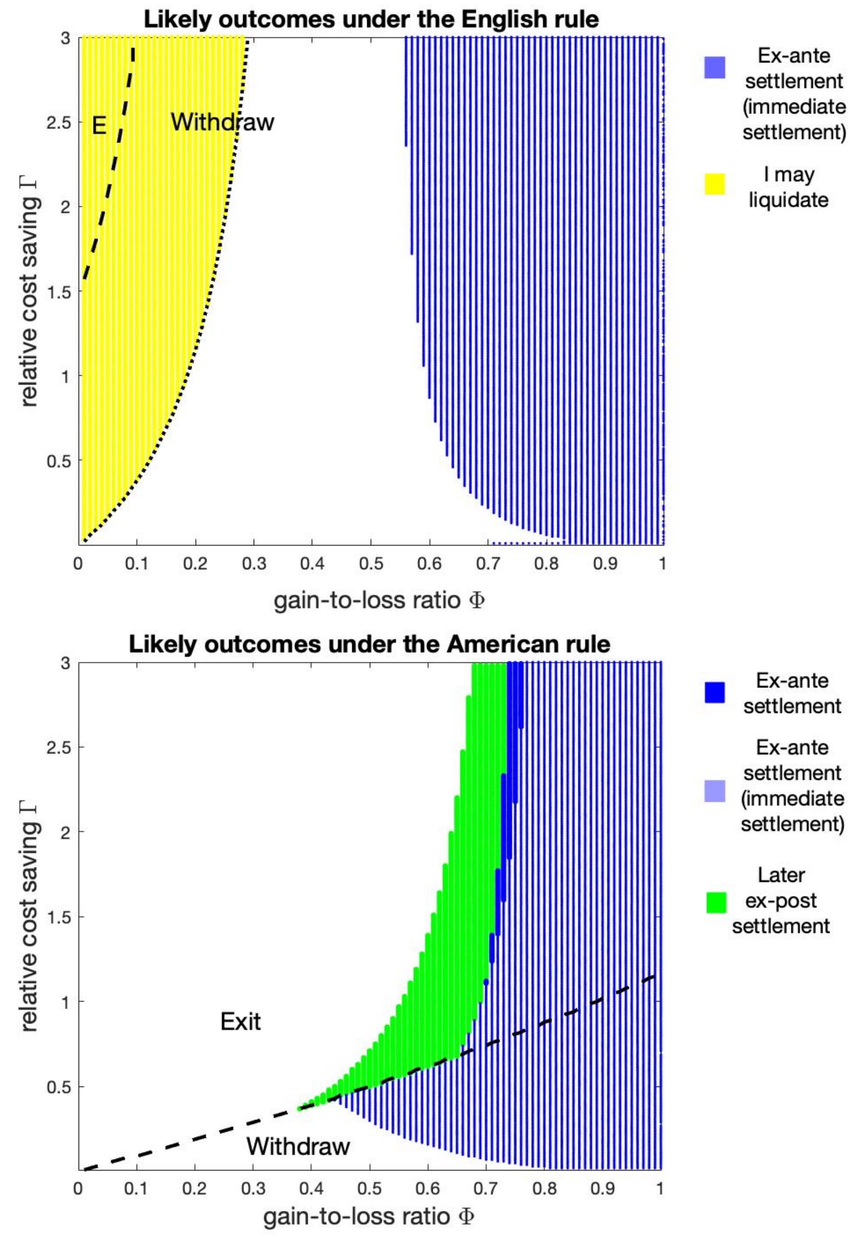
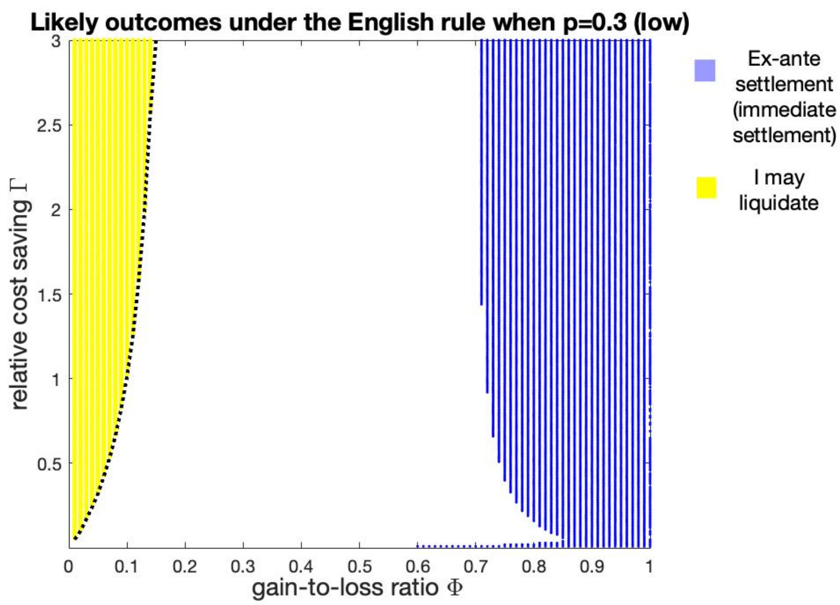


Figure 5: Likely outcomes with Different P - English Rule

This graph shows the likely litigation outcomes with respect to gain-to-loss ratio and relative cost saving under the English rule. The top plot has the winning probability of the Incumbent $p = 0.3$ and the bottom plot has $p = 0.7$. See the explanation of Figure 4 regarding the different colored regions and different styles of lines on the graph. The relative cost saving is defined as $\Gamma = \frac{H_l^C - C_p^C}{H_l^I - C_p^I}$, and the gain-to-loss ratio is defined as $\Phi = \frac{\pi_2^C}{\pi_1^I - \pi_2^I}$.



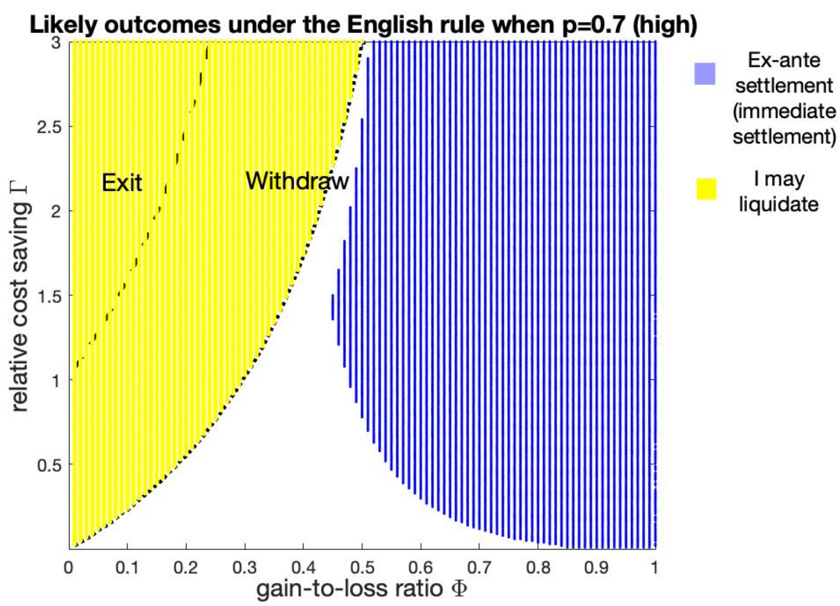
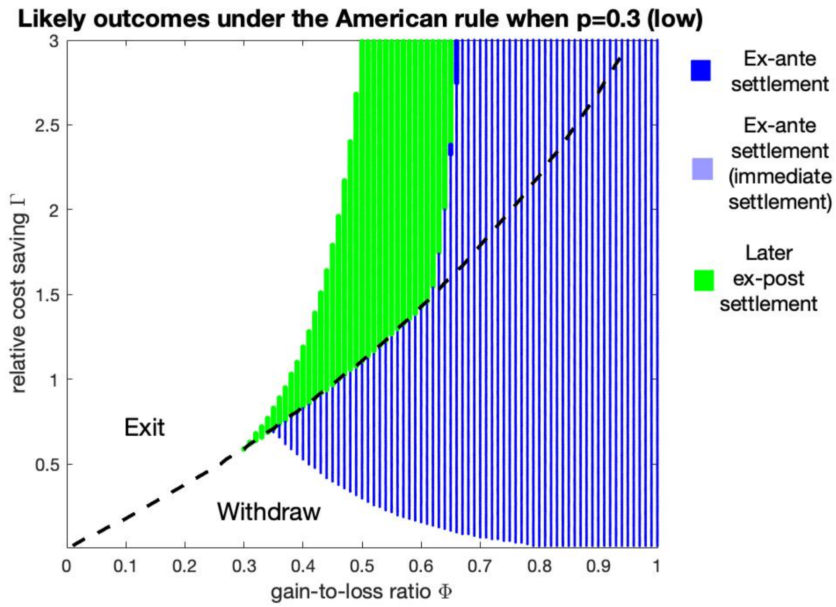


Figure 6: Likely outcomes with Different P - American Rule

The green area is the possible region for ex-post settlement. The blue area is the possible region for ex-ante settlement , while the immediate settlement region is represented by light blue regions. The black dashed line represents the boundary between regions where the challenger exits first and where the incumbent withdraws first. The relative cost saving is defined as $\Gamma = \frac{H_l^C - C_p^C}{H_l^I - C_p^I}$, and the gain-to-loss ratio is defined as $\Phi = \dfrac{\pi_2^C}{\pi_1 - \pi_2^I}$.



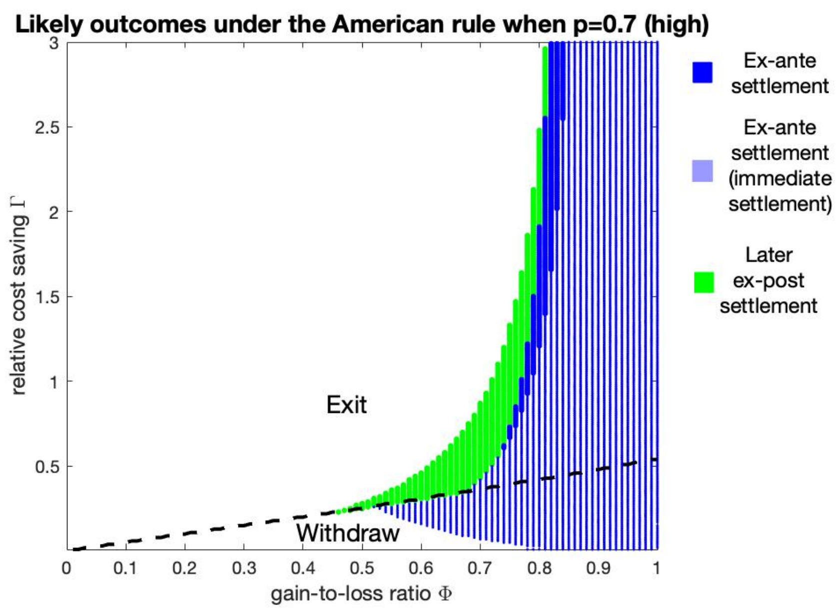
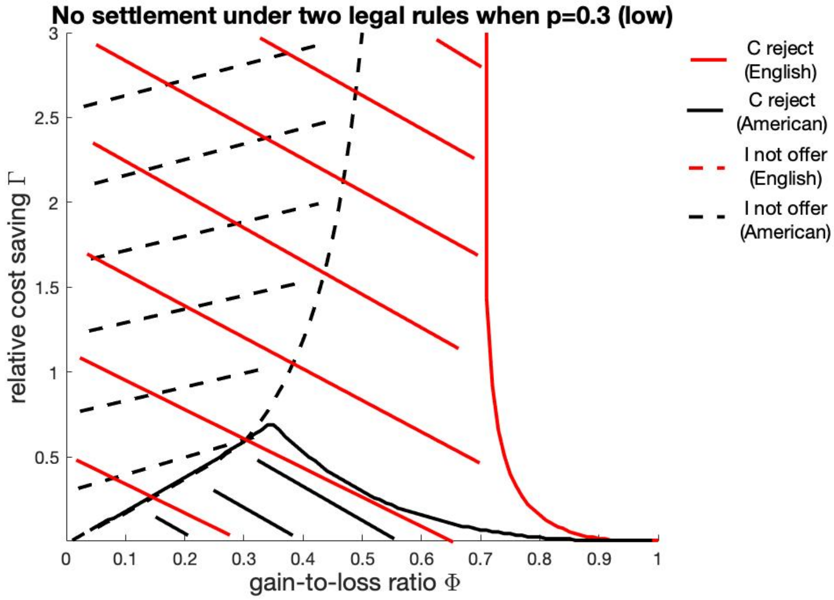


Figure 7: The Cause of Non-Settlement: Comparison of the Two Legal Rules

This graph shows whether the non-settlement was caused by (1) the Incumbent is not willing to offer settlement (dashed-lined areas), or (2) the Challenger rejects the settlement offer (solid lined areas). Red represents the English rule and black represents the American rule. The relative cost saving is defined as $\Gamma = \frac{H_l^C - C_p^C}{H_l^I - C_p^I}$, and the gain-to-loss ratio is defined as $\Phi = \frac{\pi_2^C}{\pi_1^I - \pi_2^I}$.



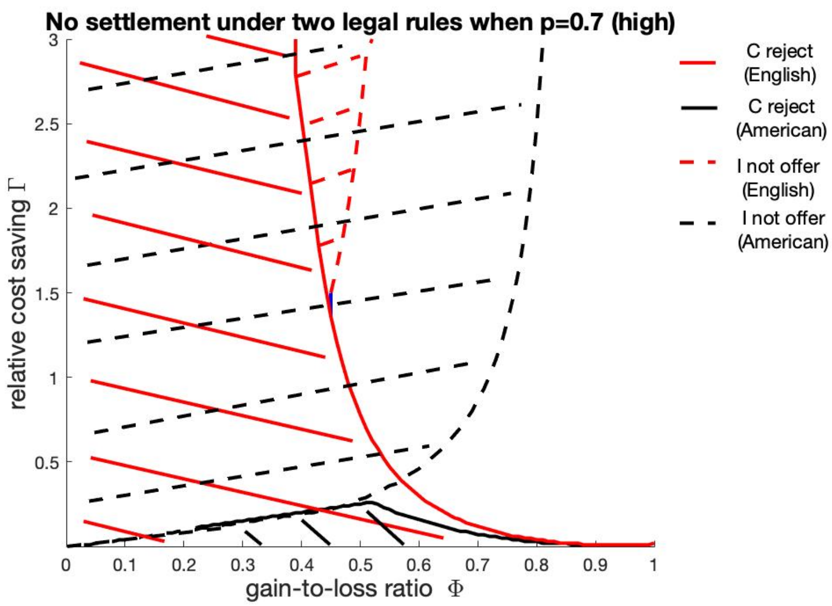


Figure 8: Likely outcomes with Different σ - English Rule

This graph shows the likely litigation outcomes with respect to gain-to-loss ratio and relative cost saving under the English rule. The top plot has product market volatility $\sigma = 0.3$ and the bottom plot has $\sigma = 0.5$. See the explanation of Figure 4 regarding the different colored regions and different styles of lines on the graph. The relative cost saving is defined as $\Gamma = \frac{H_l^C - C_p^C}{H_l^I - C_p^I}$, and the gain-to-loss ratio is defined as $\Phi = \dfrac{\pi_2^C}{\pi_1^I - \pi_2^I}$.

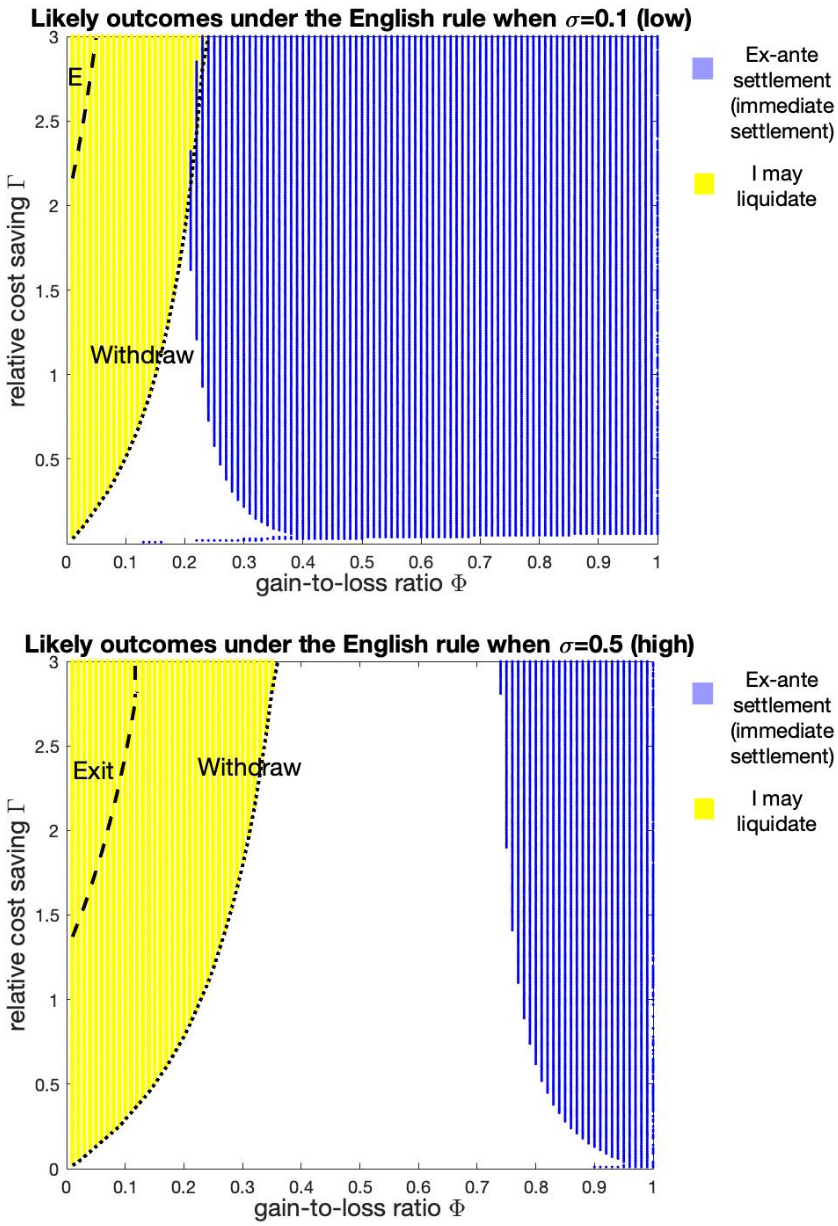


Figure 9: The Cause of Settlement: Comparison of the Two Legal Rules

These two graphs show the settlement likelihood regions under the two systems with low market volatility and high market volatility. Red represents the English rule and black represents the American rule. The relative cost saving is defined as $\Gamma = \frac{H_l^C - C_p^C}{H_l^I - C_p^I}$, and the gain-to-loss ratio is defined as $\Phi = \dfrac{\pi_2^C}{\pi_1^I - \pi_2^I}$.

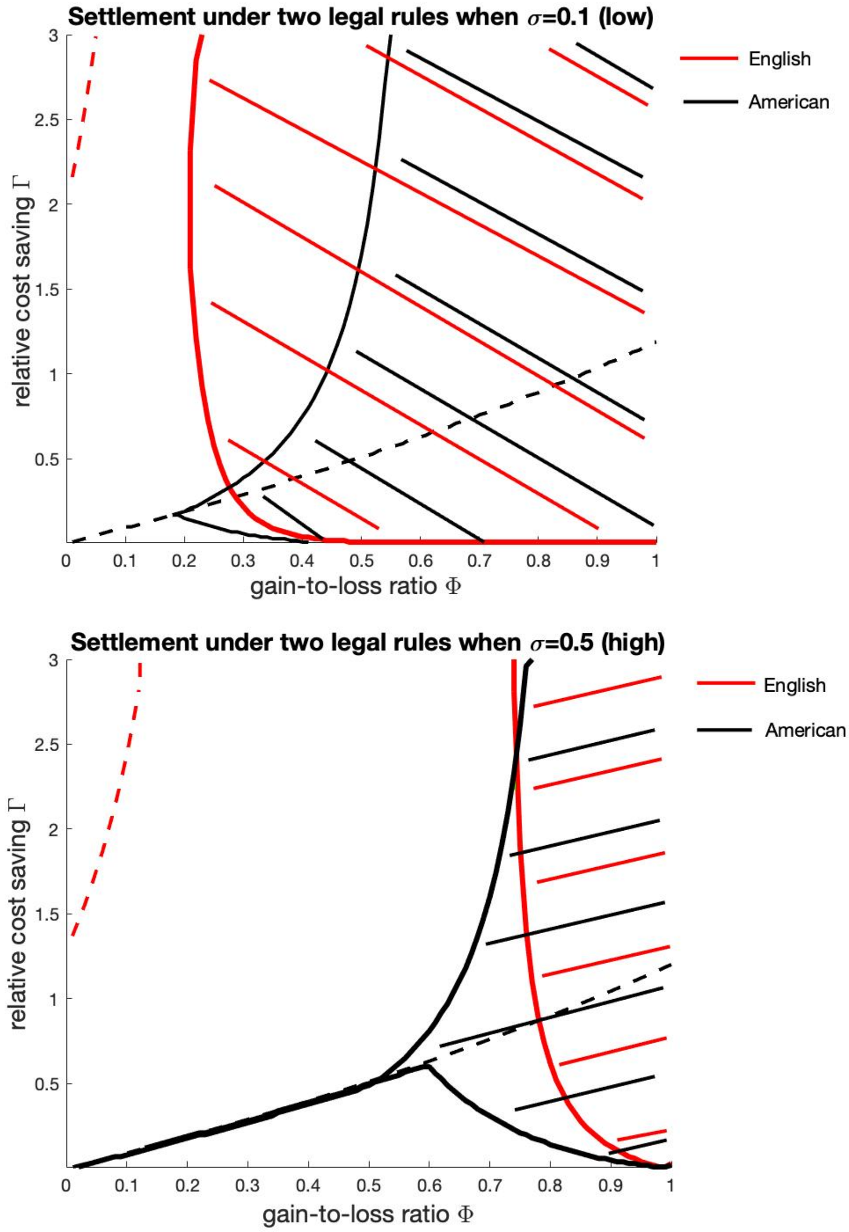


Table 1: Parameter Values in the Benchmark Case

Parameter	Value
Basics	
Risk free rate	$r = 0.05$
Arrival rate of court ruling	$\lambda = \frac{1}{2.5}$
Arrival rate of R&D success	$\lambda_r = \frac{1}{1.5}$
Probability of patent validity	$p = 0.5$
Growth rate/volatility of the demand shock	$\mu = 0.02, \sigma = 0.3$
I and C's monopoly profit multiplier (profit = $\pi_1 x$)	$\pi_1^{I,C} = 1.2$
Duopoly profit multipliers	$\pi_2^I = 0.7, \pi_2^C = 0.3$
Flow litigation costs	$C_l^I = 1, C_l^C = 1$
One-time settlement costs	$C_s^I = 0.5, C_s^C = 0.5$, where $s = a, p$
C's one-time infringement cost	$C_g^C = 1.5$
I's one-time innovation cost	$C_r^C = 3$
Ratios	
gain-to-loss ratio	$\Phi = \frac{\pi_2^C}{\pi_1 - \pi_2^I} = 0.6$
relative cost saving	$\Gamma = \frac{H_l^C - C_s^C}{H_l^I - C_s^I} = 1$
Other Greeks	
	$\delta = \frac{1}{r-\mu} - \frac{1}{r+\lambda-\mu} = 31.01$
	$\omega = \delta(r - \mu) = 0.93$
	$\beta_\lambda = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{(\frac{1}{2} - \frac{\mu}{\sigma^2})^2 + \frac{2(r+\lambda)}{\sigma^2}} = -2.90$
	$\alpha_\lambda = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{(\frac{1}{2} - \frac{\mu}{\sigma^2})^2 + \frac{2(r+\lambda)}{\sigma^2}} = 3.45$
	$\beta = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{(\frac{1}{2} - \frac{\mu}{\sigma^2})^2 + \frac{2r}{\sigma^2}} = -0.81$
	$\alpha = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{(\frac{1}{2} - \frac{\mu}{\sigma^2})^2 + \frac{2r}{\sigma^2}} = 1.37$

Table 2: Innovation Incentives under the Two Legal Rules

These tables list the values of arbitrary constant for the Incumbent's value function with the option to innovate, i.e., the arbitrary constant of V_r^I or v_r^I and the ratio between the values of arbitrary constants for the Incumbent's value function with the option to innovate under the English rule and the American rule.

(a)

The innovation incentives for the incumbent under the English rule

relative cost saving \ gain-to-loss ratio	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
3	156.9	54.37	44.01	37.88	34.36	31.93	30.1	28.68	27.41	29.09
2.8	155.9	53.62	43.93	37.81	34.31	31.89	30.06	28.65	27.39	29.04
2.6	154.7	52.79	43.84	37.74	34.25	31.84	30.02	28.62	27.36	28.99
2.4	153.4	51.87	43.74	37.66	34.18	31.79	29.98	28.59	27.34	28.94
2.2	151.8	50.84	43.62	37.57	34.11	31.73	29.93	28.55	27.31	28.87
2	149.9	57.27	43.49	37.46	34.02	31.66	29.87	28.5	27.27	28.8
1.8	147.6	57.06	43.33	37.34	33.92	31.58	29.81	28.44	27.23	28.72
1.6	144.7	56.81	43.15	37.19	33.8	31.48	29.73	28.38	27.18	28.62
1.4	141.1	56.51	42.94	37.02	33.66	31.37	29.63	28.3	27.12	28.5
1.2	136.3	56.16	42.68	36.82	33.49	31.23	29.52	28.21	27.05	28.35
1	129.9	55.72	42.35	36.56	33.28	31.05	29.38	28.09	26.96	28.17
0.8	120.8	55.17	41.94	36.23	33.01	30.83	29.19	27.94	26.84	27.94
0.6	109.6	54.45	41.4	35.79	32.64	30.53	28.94	27.74	26.68	27.63
0.4	106	53.43	40.62	35.15	32.11	30.09	28.58	27.44	26.45	27.18
0.2	103.5	51.81	39.35	34.11	31.23	29.35	27.97	26.94	26.06	26.46

(b)

The innovation incentives for the incumbent under the American rule

relative cost saving \ gain-to-loss ratio	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
3	164.2	79.74	56.78	46.51	40.79	37.17	34.71	33.26	32.58	32.13
2.8	164.2	79.71	56.75	46.49	40.76	37.15	34.69	33.26	32.58	32.13
2.6	164.1	79.68	56.72	46.46	40.74	37.13	34.66	33.26	32.58	32.13
2.4	164.1	79.64	56.68	46.42	40.71	37.1	34.64	33.26	32.58	32.13
2.2	164	79.59	56.64	46.38	40.67	37.06	34.61	33.26	32.58	32.13
2	164	79.54	56.59	46.34	40.63	37.02	34.58	33.26	32.58	32.13
1.8	163.9	79.47	56.53	46.28	40.57	36.97	34.53	33.26	32.58	32.13
1.6	163.8	79.39	56.46	46.21	40.51	36.92	34.49	33.26	32.58	32.13
1.4	163.7	79.28	56.36	46.13	40.43	36.85	34.43	33.26	32.58	32.13
1.2	163.5	79.15	56.24	46.02	40.34	36.76	34.35	33.26	32.58	32.13
1	163.3	78.96	56.08	45.88	40.21	36.65	34.25	33.26	31.11	30.38
0.8	163	78.7	55.85	45.68	40.03	36.5	34.25	31.25	30.39	29.72
0.6	162.5	78.29	55.51	45.38	39.78	32.83	31.39	30.36	29.58	28.97
0.4	161.6	77.56	54.92	44.9	33.42	31.5	30.22	29.31	28.62	28.08
0.2	159.3	75.94	42.52	35.35	31.63	29.87	28.76	27.97	27.39	26.93

(c)

The comparision of innovation incentives for the incumbent under two rules (English/American)

relative cost saving \ gain-to-loss ratio	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
3	0.9554	0.6818	0.7751	0.8143	0.8423	0.859	0.8672	0.8624	0.8414	0.9055
2.8	0.9495	0.6727	0.7741	0.8134	0.8416	0.8584	0.8667	0.8616	0.8407	0.9041
2.6	0.9427	0.6626	0.7729	0.8123	0.8407	0.8577	0.8661	0.8606	0.84	0.9026
2.4	0.9347	0.6513	0.7716	0.8112	0.8397	0.857	0.8655	0.8596	0.8391	0.9008
2.2	0.9253	0.6388	0.7701	0.8099	0.8386	0.8561	0.8648	0.8583	0.8382	0.8988
2	0.914	0.7201	0.7684	0.8084	0.8374	0.8551	0.864	0.8569	0.8371	0.8965
1.8	0.9004	0.718	0.7665	0.8068	0.836	0.854	0.8631	0.8553	0.8358	0.8939
1.6	0.8834	0.7156	0.7643	0.8049	0.8344	0.8528	0.862	0.8534	0.8342	0.8907
1.4	0.8618	0.7128	0.7618	0.8026	0.8325	0.8513	0.8608	0.8511	0.8324	0.887
1.2	0.8335	0.7096	0.7588	0.8	0.8303	0.8494	0.8593	0.8483	0.8302	0.8825
1	0.7951	0.7057	0.7553	0.7969	0.8276	0.8472	0.8576	0.8447	0.8664	0.9274
0.8	0.741	0.7011	0.751	0.7932	0.8245	0.8446	0.8522	0.894	0.8831	0.94
0.6	0.6743	0.6955	0.7458	0.7886	0.8204	0.9298	0.9221	0.9138	0.902	0.9536
0.4	0.6558	0.6889	0.7396	0.7828	0.9609	0.955	0.9456	0.9364	0.9242	0.9678
0.2	0.6498	0.6822	0.9256	0.9648	0.9873	0.9825	0.9726	0.9632	0.9517	0.9823

Table 3: Notations for Model Solutions

We use the capital letters/greeks to represent the notations in Case A and small letters/greeks to represent the notations in Case B. Since all possible strategies in Case A are also possible in Case B, for simplicity, we only show the capital letters that representing notations in Case A in the following table and do not list the corresponding counterparties in Case B. We also list the notations that do not exist in Case A but only exist in Case B below.

Notation	**Interpretation**
Thresholds	
X_l	litigation threshold (by I)
X_a / X_p	ex-ante /ex-post settlement threshold
X_e / X_w	C's exit threshold / I's withdraw threshold
Royalty rates	
Θ_a / Θ_p	ex-ante / ex-post settlement royalty rate
Θ^{Imin} / Θ^{Imax}	I's minimum / maximum royalty rate for feasible settlement
Θ^{Cmax}	C's maximum royalty rate for feasible settlement
Value functions	
V_{bl}	before litigation and includes option values
V_l	with I's litigation option
V_a	with the ex-ante settlement option
V_{dl}	during litigation and includes option values
V_p / V_{ns}	with / without the ex-post settlement option
V_e / V_w	when C exits / I withdraws first
$\tilde{V}$	when the option has exercised
Arbitrary constants in the value functions	
A_{bl}, B_{bl}	before litigation
A_l / B_l	with I's litigation option
A_a / B_a	with an ex-ante settlement option
B_s	during litigation
B_p / B_{ns}	with / without an ex-post settlement option
B_e / B_w	when C exits / I withdraws first
Arbitrary constants only exist in Case B	
a	during litigation
a_p / a_{ns}	with / without an ex-post settlement option
a_e / a_w	when C exits / I withdraws first

6 Appendix: Proofs

The Method To Determine the Order of Withdrawal and Exit in Case a and B

Proof. We calculate the "reservation threshold" as in Lambrecht (2001) for each firm in Case A and B, i.e. the value for which firm is "indifferent between leaving first at their optimal exit/withdraw threshold and waiting until the rival leaves". We use x_I to denote this threshold for the Incumbent in two cases. This can be derived by setting the Incumbent's value including its withdrawal option equal to its value after the Challenger has exited, i.e.,

$$V_w^I(x_I) = V_m^I(x_I), \tag{72}$$

where $V_m^I(x) = \frac{\pi_1}{r-\mu}x$.

Therefore, in Case A, x_I is a solution of

$$(\pi_1 - \pi_2^I)(p\delta - \frac{1}{r-\mu})x_I + \frac{\bar{H}_l^I}{1-\beta_\lambda}(\frac{x_I}{X_w})^{\beta_\lambda} - \bar{H}_l^I = 0 \tag{73}$$

In Case B, x_I is a solution of

$$(\pi_1^I(p\delta - \frac{1}{r-\mu}) + \frac{\pi_2^I}{r-\mu+\lambda})x_I + a^I x_I^{\alpha_\lambda} + b_w^I {x_I}^{\beta_\lambda} - H_l^I = 0 \tag{74}$$

We use x_C to denote the reservation threshold for the Challenger, which can be derived by settling the Challenger's value during litigation including its exit option equal to its value after the Incumbent has withdrawn.

$$V_e^C(x_C) = V_d^C(x_C), \tag{75}$$

where $V_d^C(x) = \frac{\pi_2^C}{r-\mu}x$.

Therefore, in Case A, x_C is a solution of

$$p\delta\pi_2^C x_C - \frac{\bar{H}_l^C}{1-\beta_\lambda}(\frac{x_C}{x_e})^{\beta_\lambda} + \bar{H}_l^C = 0, \tag{76}$$

In Case B, x_C is a solution of

$$\delta(\pi_1^C(1-p) - \pi_2^C)x_C + a^C x_C^{\alpha_\lambda} + b_e^C {x_C}^{\beta_\lambda} - H_l^C = 0, \tag{77}$$

If $x_I \geq x_C$, which both calculated in Case B, the Incumbent withdraws first. Note that if both Case A and B are feasible, i.e. the calculated action thresholds satisfy $X_d > \bar{x}$ or $x_d < \bar{x}$, firms optimally choose their threshold strategies. However, if only one Case A or B is feasible for the non-settlement strategies, the non-settlement strategies are determined by the comparison of reservation thresholds. For example, if $x_I \geq x_C$ and x_I is calculated in Case A, whereas x_C is calculated in Case B, then we are in the scenario of Case A and I withdrawal. □

The proof of Proposition 2

Proof. We get the value functions of $x \geq \bar{x}$ in Proposition 2 using the same approach as in the proof of Proposition 1. We get the general forms of firm values of $x_s < x < \bar{x}$ in Proposition 2 from Eq. (28) and (29). To derive the expressions for a^I and a^C in the proposition, we use the value-matching and smooth-pasting conditions for v^I at $\bar{x}$ and v^C at $\bar{x}$ during litigation respectively, i.e.,

$$a_d^I \bar{x}^{\alpha_\lambda} + b_d^I \bar{x}^{\beta_\lambda} - H_l^I + \frac{\pi_2^I \bar{x}}{r-\mu+\lambda} + p\delta\pi_1^I \bar{x} = B_d^I \bar{x}^{\beta_\lambda} - \bar{H}_l^I + \frac{\pi_2^I \bar{x}}{r-\mu} + \delta p(\pi_1^I - \pi_2^I)\bar{x} \tag{78}$$

$$a_d^C\bar{x}^{\alpha_\lambda}+b_d^C\bar{x}^{\beta_\lambda}-H_l^C+\frac{\pi_2^C\bar{x}}{r-\mu+\lambda}+\pi_1^C(1-p)\delta\bar{x}=B_d^C\bar{x}^{\beta_\lambda}-\bar{H}_l^C+(\frac{1}{r-\mu}-p\delta)\pi_2^C\bar{x} \tag{79}$$

$$\alpha_\lambda a_d^I\bar{x}^{\alpha_\lambda-1}+\beta_\lambda b_d^I\bar{x}^{\beta_\lambda-1}+\frac{\pi_2^I}{r-\mu+\lambda}+p\delta\pi_1^I=\beta_\lambda B_d^I\bar{x}^{\beta_\lambda-1}+\frac{\pi_2^I}{r-\mu}+\delta p(\pi_1^I-\pi_2^I) \tag{80}$$

$$\alpha_\lambda a_d^C\bar{x}^{\alpha_\lambda-1}+\beta_\lambda b_d^C\bar{x}^{\beta_\lambda-1}+\frac{\pi_2^C}{r-\mu+\lambda}+\pi_1^C(1-p)\delta=\beta_\lambda B_d^C\bar{x}^{\beta_\lambda-1}+(\frac{1}{r-\mu}-p\delta)\pi_2^C \tag{81}$$

Then we can get the expression of a^I by $\alpha_\lambda\cdot$ (78) - $\bar{x}\cdot$ (80), and get the expression of a^C by $\alpha_\lambda\cdot$ (79) - $\bar{x}\cdot$ (81). □

The proof of Corollary 4

Proof. We get the results for Case A in Corollary 1 by applying (1) the value matching and smooth pasting condition for I's value function V^I at the withdraw threshold x_w, and (2) the value matching condition for C's value function V^C at x_w. For Case B in Corollary 4, we get the results using the following boundary condition at the withdraw threshold x_w when $x<\bar{x}$:

$$v_w^I(x_w)=\frac{\pi_2^I x_w}{r-\mu}.\quad \text{(Value-matching at } x_w) \tag{82}$$

$$\frac{\partial v_w^I(x_w)}{\partial x_w}=\frac{\pi_2^I}{r-\mu}.\quad \text{(Smooth-pasting at } x_w) \tag{83}$$

$$v_w^C(x_w)=\frac{\pi_2^C x_w}{r-\mu}.\quad \text{(Value-matching at } x_w) \tag{84}$$

Together with Eq. (78), (79), (80) and (81), i.e.,

$$v_w^I(\bar{x})=\bar{V}_w^I(\bar{x}).\quad \text{(Value-matching at } \bar{x}) \tag{85}$$

$$v_w^C(\bar{x})=\bar{V}_w^C(\bar{x}).\quad \text{(Value-matching at } \bar{x}) \tag{86}$$

$$\frac{\partial v_w^I(\bar{x})}{\partial\bar{x}}=\frac{\partial\bar{V}_w^I(\bar{x})}{\partial\bar{x}}.\quad \text{(Smooth-pasting at } \bar{x}) \tag{87}$$

$$\frac{\partial v_w^C(\bar{x})}{\partial\bar{x}}=\frac{\partial\bar{V}_w^C(\bar{x})}{\partial\bar{x}}.\quad \text{(Smooth-pasting at } \bar{x}) \tag{88}$$

□

The proof of Corollary 5

Proof. We get the results on Case A in Corollary 2 by applying (1) the value matching and smooth pasting condition for C's value function V^C at the withdraw threshold x_e, and (2) the value matching condition for I's value function V^I at x_w. In Case B, when $x<\bar{x}$, we obtain the results through the following boundary condition at the exit threshold x_e:

$$v_{dle}^I(x_e)=\frac{\pi_1^I x_e}{r-\mu}.\quad \text{(Value-matching at } x_e) \tag{89}$$

$$v_{dle}^C(x_e)=0.\quad \text{(Value-matching at } x_e) \tag{90}$$

$$\frac{\partial v_{dle}^I(x_e)}{\partial x_e}=0.\quad \text{(Smooth-pasting at } x_e) \tag{91}$$

Together with Eq (78),(79),(80) and (81), i.e.,

$$v_{dle}^I(\bar{x})=\bar{V}_{dle}^I(\bar{x}).\quad \text{(Value-matching at } \bar{x}) \tag{92}$$

$$v_{dle}^C(\bar{x})=\bar{V}_{dle}^C(\bar{x}).\quad \text{(Value-matching at } \bar{x}) \tag{93}$$

$$\frac{\partial v^I_{dle}(\bar{x})}{\partial \bar{x}} = \frac{\partial \bar{V}^I_{dle}(\bar{x})}{\partial \bar{x}}. \quad \text{(Smooth-pasting at } \bar{x}) \tag{94}$$

$$\frac{\partial v^C_{dle}(\bar{x})}{\partial \bar{x}} = \frac{\partial \bar{V}^C_{dle}(\bar{x})}{\partial \bar{x}}. \quad \text{(Smooth-pasting at } \bar{x}) \tag{95}$$

□

The proof of Corollary 6

Proof. When $x < \bar{x}$, we have the following boundary condition at the ex-post settlement threshold x_p:

$$v^I_{dlp}(x_p) = \frac{(\pi^I_2 + \theta_p \pi^C_2)x_p}{r-\mu} - C^I_p. \quad \text{(Value-matching at } x_p) \tag{96}$$

$$v^C_{dlp}(x_p) = \frac{(1-\theta_p)\pi^C_2 x_p}{r-\mu} - C^C_p. \quad \text{(Value-matching at } x_p) \tag{97}$$

$$\frac{\partial v^I_{dlp}(x_p)}{\partial x_p} = \frac{(1-\theta_p)\pi^C_2}{r-\mu}. \quad \text{(Smooth-pasting at } x_p) \tag{98}$$

Solving above conditions and equations (78),(79),(80) and (81), i.e.,

$$v^I_{dlp}(\bar{x}) = \bar{V}^I_{dlp}(\bar{x}). \quad \text{(Value-matching at } \bar{x}) \tag{99}$$

$$v^C_{dlp}(\bar{x}) = \bar{V}^C_{dlp}(\bar{x}). \quad \text{(Value-matching at } \bar{x}) \tag{100}$$

$$\frac{\partial v^I_{dlp}(\bar{x})}{\partial \bar{x}} = \frac{\partial \bar{V}^I_{dlp}(\bar{x})}{\partial \bar{x}}. \quad \text{(Smooth-pasting at } \bar{x}) \tag{101}$$

$$\frac{\partial v^C_{dlp}(\bar{x})}{\partial \bar{x}} = \frac{\partial \bar{V}^C_{dlp}(\bar{x})}{\partial \bar{x}}. \quad \text{(Smooth-pasting at } \bar{x}) \tag{102}$$

we have the values of ex-post settlement threshold and arbitrary constants in Corollary 6.

Taking the first derivative of equation (36) with respect to θ^*_p, we have

$$\begin{aligned} \frac{\partial x_p}{\partial \theta_p} =& - x_p \frac{\pi^C_2}{r-\mu}\{\alpha_\lambda\big[(\pi^C_2 - \pi^C_1)(1-p)\delta\bar{x} - \frac{\beta_\lambda}{\beta_\lambda - 1}(\bar{H}^C_l - H^C_l)\big]\bar{x}^{-\alpha_\lambda} x_p^{\alpha_\lambda - 1} \\ &+ ((1-p)\pi^C_1 - \pi^C_2)\delta + \frac{\theta_p \pi^C_2}{r-\mu}\}^{-1} \end{aligned} \tag{103}$$

In order to get the optimal θ^*_p that maximises the Incumbent's value with the option to settle ex-post (i.e, v^I_p), we can maximise the constant b^I_p only with respect to θ_p, since a^I is constant in all three possibilities during litigation, and derive the optimal royalty payment. Therefore, we have

$$\begin{aligned} \frac{\partial b^I_p}{\partial \theta_p} =& \{(-\beta_\lambda)(H^I_l - C^I_p)x_p^{-1} + (1-\beta_\lambda)[(\pi^I_2 - p\pi^I_1)\delta + \frac{\theta_p \pi^C_2}{r-\mu}] \\ &+[(\beta_\lambda - 1)(1-p)\pi^I_2\delta\bar{x} - \beta_\lambda(\bar{H}^I_l - H^I_l)](\frac{x_p}{\bar{x}})^{\alpha_\lambda} x_p^{-1}\}\frac{\partial x_p}{\partial \theta_p} x_p^{-\beta_\lambda} + \frac{\pi^C_2}{r-\mu} x_p^{1-\beta_\lambda} = 0 \end{aligned} \tag{104}$$

Substituting $\frac{\partial x_p}{\partial \theta_p}$, we obtain

$$\begin{aligned}
\{\beta_\lambda(H_l^I - C_p^I) + (\beta_\lambda - 1)[(\pi_2^I - p\pi_1^I)\delta + \frac{\theta_p \pi_2^C}{r-\mu}]x_p \\
-[(\beta_\lambda - 1)(1-p)\pi_2^I \delta\bar{x} - \beta_\lambda(\bar{H}_l^I - H_l^I)](\frac{x_p}{\bar{x}})^{\alpha_\lambda}\}x_p^{-\beta_\lambda}\frac{\pi_2^C}{r-\mu} \\
+\frac{\pi_2^C}{r-\mu}x_p^{-\beta_\lambda}\{\alpha_\lambda\big[(\pi_2^C - \pi_1^C)(1-p)\delta\bar{x} - \frac{\beta_\lambda}{\beta_\lambda - 1}(\bar{H}_l^C - H_l^C)\big](\frac{x_p}{\bar{x}})^{\alpha_\lambda} \\
+[((1-p)\pi_1^C - \pi_2^C)\delta + \frac{\theta_p^* \pi_2^C}{r-\mu}]x_p\} = 0
\end{aligned} \tag{105}$$

Simplifying equation (105), we have

$$\begin{aligned}
\big[(1-p)[\alpha_\lambda(\pi_2^C - \pi_1^C) - (\beta_\lambda - 1)\pi_2^I]\delta\bar{x} - \beta_\lambda[\frac{\alpha_\lambda}{\beta_\lambda - 1}(\bar{H}_l^C - H_l^C) - (\bar{H}_l^I - H_l^I)]\big](\frac{x_p}{\bar{x}})^{\alpha_\lambda} \\
-[(1-p)\pi_1^C - \pi_2^C + (\beta_\lambda - 1)(\pi_2^I - p\pi_1^I)]\delta x_p + \beta_\lambda(H_l^I - C_p^I) + \beta_\lambda\frac{\theta_p^* \pi_2^C}{r-\mu}x_p = 0
\end{aligned} \tag{106}$$

Therefore, the optimal royalty payment rate is:

$$\begin{aligned}
\theta_p^*(x_p) = -\frac{r-\mu}{\beta_\lambda \pi_2^C x_p}\big([(1-p)[\alpha_\lambda(\pi_2^C - \pi_1^C) - (\beta_\lambda - 1)\pi_2^I]\delta\bar{x} \\
-\beta_\lambda[\frac{\alpha_\lambda}{\beta_\lambda - 1}(\bar{H}_l^C - H_l^C) - (\bar{H}_l^I - H_l^I)]](\frac{x_p}{\bar{x}})^{\alpha_\lambda} \\
+[(1-p)\pi_1^C - \pi_2^C + (\beta_\lambda - 1)(\pi_2^I - p\pi_1^I)]\delta x_p + \beta_\lambda(H_l^I - C_p^I)\big)
\end{aligned} \tag{107}$$

We assume $k = \frac{x_p}{\bar{x}} < 1$, so the above equation (107) can be rewritten as

$$\begin{aligned}
\theta_p^*(k) = -\frac{r-\mu}{\beta_\lambda \pi_2^C k\bar{x}}\big([(1-p)[\alpha_\lambda(\pi_2^C - \pi_1^C) - (\beta_\lambda - 1)\pi_2^I]\delta\bar{x} \\
-\beta_\lambda[\frac{\alpha_\lambda}{\beta_\lambda - 1}(\bar{H}_l^C - H_l^C - (\bar{H}_l^I - H_l^I)]]k^{\alpha_\lambda} \\
+[(1-p)\pi_1^C - \pi_2^C + (\beta_\lambda - 1)(\pi_2^I - p\pi_1^I)]\delta k\bar{x} + \beta_\lambda(H_l^I - C_p^I)\big)
\end{aligned} \tag{108}$$

Substituting θ_p^* into equation (36), the ratio k satisfies the following equation:

$$\begin{aligned}
\big[(\beta_\lambda - 1)(\pi_2^C - \pi_1^C)(1-p)\delta\bar{x} - \beta_\lambda(\bar{H}_l^C - H_l^C)\big]k^{\alpha_\lambda} \\
-\beta_\lambda(H_l^C - C_p^C) + (\beta_\lambda - 1)\big[((1-p)\pi_1^C - \pi_2^C)\delta k\bar{x} \\
-\frac{1}{\beta_\lambda k\bar{x}}\big([(1-p)[\alpha_\lambda(\pi_2^C - \pi_1^C) - (\beta_\lambda - 1)\pi_2^I]\delta\bar{x} \\
-\beta_\lambda[\frac{\alpha_\lambda}{\beta_\lambda - 1}(\bar{H}_l^C - H_l^C) - (\bar{H}_l^I - H_l^I)]]k^{\alpha_\lambda} \\
+[(1-p)\pi_1^C - \pi_2^C + (\beta_\lambda - 1)(\pi_2^I - p\pi_1^I)]\delta k\bar{x} + \beta_\lambda(H_l^I - C_p^I)\big)\big] = 0
\end{aligned} \tag{109}$$

Solving k, we get the ex-post settlement threshold $x_p = k\bar{x}$ and the ex-post settlement royalty rate $\theta_p^*(k)$ defined in formula (108) and thus obtain the optimal royalty rate in Corollary 6.

□

The proof of Corollary 8

Proof. Case A: If both parties have the ex-ante settlement option before litigation, we have the following value-matching conditions on firms' value functions before litigation at ex-ante settlement threshold X_a:

$$V_a^I(X_a) = \frac{(\pi_2^I + \theta_a \pi_2^C)X_a}{r-\mu} - C_a^I, \tag{110}$$

$$V_a^C(X_a) = \frac{(1-\theta_a)\pi_2^C X_a}{r-\mu} - C_a^C. \tag{111}$$

Since the Incumbent's value of settling increases with the royalty rate, so the optimal royalty rate chosen by the Incumbent is the maximum royalty rate such that the Challenger's value of settling is higher than the value of not settling. The minimum royalty rate is derived by setting the Incumbent's firm value equal to its the value of not settling.

Case B: If both parties have the ex-ante settlement option before litigation, we have the following value-matching conditions on firms' value functions before litigation at ex-ante settlement threshold x_a:

$$v_a^I(x_a) = \frac{(\pi_2^I + \theta_a \pi_2^C)x_a}{r-\mu} - C_a^I, \tag{112}$$

$$v_a^C(x_a) = \frac{(1-\theta_a)\pi_2^C x_a}{r-\mu} - C_a^C. \tag{113}$$

We derive the royalty rate and settlement condition in Case B following the same approach as in Case A.

Using optimality method, we have

$$\frac{\partial V_a^C}{\partial x_a} = \frac{\partial A_a^C}{\partial x_a} x_a^\alpha + \frac{\partial B_a^C}{\partial x_a} x_a^\beta > 0, \quad \frac{\partial v_a^C}{\partial x_a} = \frac{\partial a_a^C}{\partial x_a} x_a^\alpha + \frac{\partial b_a^C}{\partial x_a} x_a^\beta > 0 \tag{114}$$

Since $x_a \leq x_l$, the optimal acceptance threshold for ex-ante settlement is x_l. □

Detailed Proof of Immediate Settlement Under the English Rule

Proof. As discussed in Section 2.1.2 and 2.1.4, without considering firms value of not settling, the Incumbent offers an ex-post settlement contract with a royalty rate that can maximise his value with the option to settle during litigation (Note here, though the Incumbent maximises his value with respect to royalty rate with any x, the derived royalty rate will be affected by the settlement threshold because it is incorporated into the option value), while the Challenger decides when to accept the settlement offer (i.e., acceptance threshold/settlement threshold) with any royalty rate (Note, we then finalize the settlement threshold by taking into account the royalty rate chosen by the Incumbent). Basically, higher royalty rates are associated with lower acceptance thresholds.

In some cases, with certain royalty rate chosen by the Incumbent, the Challenger chooses to accept the settlement offer immediately after the case has been filed (i.e. the time that the Incumbent decides to litigate). This is because when the market demand reaches the litigation threshold, it is already optimal for the Challenger to accept settlement. Knowing this, the Incumbent, therefore, needs to choose a royalty rate that can maximise his value by taking into account the Challengers decreased acceptance threshold/settlement threshold.

Considering the not settling case, the settlement is only feasible when the values of settling are higher than the values of not settling for both parties. Therefore, the settlement royalty rate should be higher than the minimum royalty rate that the Incumbent is willing to offer (θ_p^{Imin}) and lower than the maximum royalty rate that the Challenger is willing to accept (θ_p^{Cmax}). We show the details of royalty rates in immediate settlement for the two cases i.e. whether or not the Incumbent is able to pay full litigation costs below.

Case A: The Unconstrained Incumbent ("I Remains a Going-Concern")

In this case, the value functions for both the Incumbent and the Challenger if firms choose to settle at the settlement threshold for $x \geq X_p^{ns}$ can be expressed as follows (Note if $x < X_p^{ns}$, firms choose to withdraw or exit):

$$V_p^I = \begin{cases} (\frac{\pi_2^I}{r-\mu} + p\delta(\Pi_1 - \pi_2^I))x + B_p^I x^{\beta_\lambda} - \bar{H}_l^I, & \text{if} \quad x \geq X_p(\Theta_p^*) \\ \frac{\pi_2^I + \bar{\theta}_p \pi_2^C}{r-\mu} x - C_p^I, & \text{if} \quad x \in [X_p^-, X_p^*(\Theta_p^*)) \\ (\frac{\pi_2^I}{r-\mu} + p\delta(\Pi_1 - \pi_2^I))x + \bar{A}_p^I x^{\alpha_\lambda} + \bar{B}_p^I x^{\beta_\lambda} - \bar{H}_l^I. & \text{if} \quad x \in (X_p^{ns}, x < X_p^-), \end{cases} \tag{115}$$

$$V_p^C = \begin{cases} (\frac{1}{r-\mu} - p\delta)\pi_2^C x + B_p^C x^{\beta_\lambda} - \bar{H}_l^C, & \text{if} \quad x \geq X_p(\Theta_p^*) \\ \frac{(1-\bar{\theta}_p)\pi_2^C}{r-\mu} x - C_p^C, & \text{if} \quad x \in [X_p^-, X_p^*(\Theta_p^*)) \\ (\frac{1}{r-\mu} - p\delta)\pi_2^C x + \bar{A}_p^C x^{\alpha_\lambda} + \bar{B}_p^C x^{\beta_\lambda} - \bar{H}_l^C. & \text{if} \quad x \in (X_p^{ns}, x < X_p^-), \end{cases} \tag{116}$$

where B_p^I, B_p^C and $X_p(\Theta_p^*)$ follow the expression in Corollary 3.

These value functions show that settlement occurs immediately for x between $X_p^-$and $X_p^*(theta_p^*)$ with a royalty rate $\bar{theta}_p(x)$. For $x < X_p^-$ or $x > X_p^*$, settlement does not occur immediately but only when the demand level first reaches the Challenger's optimal threshold.

Applying the value-matching and smooth-pasting conditions at X_p^- and X_p^{ns}, we have

$$\begin{aligned}
\bar{A}_p^I &= \frac{1}{(X_p^-)^{\alpha_\lambda-\beta_\lambda} - (X_p^{ns})^{\alpha_\lambda-\beta_\lambda}}\big[(\frac{\theta_p\pi_2^C}{r-\mu} - C_p^I)(X_p^-)^{-\beta_\lambda} + \bar{H}_l^I((X_p^-)^{-\beta_\lambda} - (X_p^{ns})^{-\beta_\lambda}) \\
&\quad - p\delta(\pi_1^I - \pi_2^I)((X_p^-)^{1-\beta_\lambda} - (X_p^{ns})^{1-\beta_\lambda}) - I_e\frac{\pi_1^I - \pi_2^I}{r-\mu}(x_p^e)^{1-\beta_\lambda}\big], \\
\bar{B}_p^I &= \frac{1}{(X_p^-)^{\beta_\lambda-\alpha_\lambda} - (X_p^{ns})^{\beta_\lambda-\alpha_\lambda}}\big[(\frac{\theta_p\pi_2^C}{r-\mu} - C_p^I)(X_p^-)^{-\alpha_\lambda} + \bar{H}_l^I((X_p^-)^{-\alpha_\lambda} - (X_p^{ns})^{-\alpha_\lambda}) \\
&\quad - p\delta(\pi_1^I - \pi_2^I)((X_p^-)^{1-\alpha_\lambda} - (X_p^{ns})^{1-\alpha_\lambda}) - I_e\frac{\pi_1^I - \pi_2^I}{r-\mu}(x_p^e)^{1-\alpha_\lambda}\big], \\
\bar{A}_p^C &= \frac{1}{(X_p^-)^{\alpha_\lambda-\beta_\lambda} - (X_p^{ns})^{\alpha_\lambda-\beta_\lambda}}\big[-(\frac{\theta_p\pi_2^C}{r-\mu} + C_p^C)(X_p^-)^{-\beta_\lambda} + \bar{H}_l^C((X_p^-)^{-\beta_\lambda} - (X_p^{ns})^{-\beta_\lambda}) \\
&\quad + p\delta\pi_2^C((X_p^-)^{1-\beta_\lambda} - (X_p^{ns})^{1-\beta_\lambda}) + I_e\frac{\pi_2^C}{r-\mu}(x_p^e)^{1-\beta_\lambda}\big], \\
\bar{B}_p^C &= \frac{1}{(X_p^-)^{\beta_\lambda-\alpha_\lambda} - (X_p^{ns})^{\beta_\lambda-\alpha_\lambda}}\big[-(\frac{\theta_p\pi_2^C}{r-\mu} + C_p^C)(X_p^-)^{-\alpha_\lambda} + \bar{H}_l^C((X_p^-)^{-\alpha_\lambda} - (X_p^{ns})^{-\alpha_\lambda}) \\
&\quad + p\delta\pi_2^C((X_p^-)^{1-\alpha_\lambda} - (X_p^{ns})^{1-\alpha_\lambda}) + I_e\frac{\pi_2^C}{r-\mu}(x_p^e)^{1-\alpha_\lambda}\big], \\
X_p^- &= \begin{cases} \frac{(H_p^I - C_p^I)(\alpha_\lambda-\beta_\lambda) + \bar{H}_l^I(\beta_\lambda k^{-\alpha_\lambda} - \alpha_\lambda k^{-\beta_\lambda})}{(p\delta(\pi_1^I-\pi_2^I) - \frac{\theta_p\pi_2^C}{r-\mu})(\alpha_\lambda-\beta_\lambda) + p\delta(\pi_1^I-\pi_2^I)((\beta_\lambda-1)k^{1-\alpha_\lambda} - (\alpha_\lambda-1)k^{1-\beta_\lambda})}, & \text{if I withdraws first} \\ \frac{(H_p^C - C_p^C)(\beta_\lambda-\alpha_\lambda) + \bar{H}_l^C(\alpha_\lambda k^{-\beta_\lambda} - \beta_\lambda k^{-\alpha_\lambda})}{(p\delta - \frac{\theta_p}{r-\mu})\pi_2^C(\alpha_\lambda-\beta_\lambda) + (p\delta - \frac{1}{r-\mu})\pi_2^C((\beta_\lambda-1)k^{1-\alpha_\lambda} - (\alpha_\lambda-1)k^{1-\beta_\lambda})}, & \text{if C exits first} \end{cases}
\end{aligned}$$

We define $k = \frac{X_p^{ns}}{X_p^-} \in (0,1)$ and an indicator function I_e as follows

$$I_e = \begin{cases} 0, & \text{if I withdraws first} \\ 1. & \text{if C exits first} \end{cases} \tag{117}$$

In the middle region, the Incumbent will choose the highest royalty rate that the Challenger accepts immediately. This royalty rate comes from the smooth-pasting conditions when the Challenger decides his optimal acceptance threshold $X_p(\Theta_p^*)$ in the top region or X_p^- in the bottom region.

In the top region ($x \geq X_p(\Theta_p^*)$), the Incumbent chooses the optimal royalty rate Θ_p^* by maximizing V_p^I and the

Challenger decides the optimal acceptance threshold X_p^*. If the value of settling is higher than the value of not settling, i.e. $V_p^I \geq V_{ns}^I$ and $V_p^C \geq V_{ns}^C$ or $\Theta^* \in [\Theta_p^{Imin}, \Theta_p^{Cmax}]$, then firms agree to settle at $X_p(\Theta_p^*)$ with royalty rate Θ_p^*. This case is discussed in Section 2.1.2. We rewrite the expression of $X_p(\Theta_p)$ in Corollary 3 and obtain the royalty payment that Challenger will optimally accept immediately as follows:

$$\bar{\Theta}_p^+(x) = \frac{\beta_\lambda(\bar{H}_l^C - C_p^C)(r-\mu)}{(\beta_\lambda - 1)\pi_2^C x} + p\delta(r-\mu). \tag{118}$$

and

$$\frac{\partial \bar{\Theta}_p^+(x)}{\partial x} = -\frac{\beta_\lambda(\bar{H}_l^C - C_p^C)(r-\mu)}{(\beta_\lambda - 1)\pi_2^C x^2} < 0. \tag{119}$$

Similarly, in the bottom region ($x \in [X_p^{ns}, X_p^-]$), the Incumbent chooses the optimal royalty rate $\bar{\Theta}_p^*$ by maximizing V_p^I and the Challenger decides the optimal acceptance threshold X_p^-. If the value of settling immediately is higher than the value of not settling, i.e. $V_p^I \geq V_{ns}^I$ and $V_p^C \geq V_{ns}^C$ or $\bar{\Theta}_p^* \in [\bar{\Theta}_p^{Imin}, \bar{\Theta}_p^{Cmax}]$, then firms agree to settle at X_p^- with royalty rate $\bar{\Theta}_p^*$.

We have the expressions of $\bar{\Theta}_p^{Imin}$ and $\bar{\Theta}_p^{Cmax}$ which can be expressed as

$$\bar{\Theta}_p^{Cmax}(\bar{X}_p) = \begin{cases} p\delta(r-\mu)\big[1 - (\frac{\bar{X}_p}{X_w})^{\beta_\lambda - 1}\big] + \frac{r-\mu}{\Pi_2^C \bar{X}_p}\big[\bar{H}_l^C - C_p^C - \bar{H}_l^C(\frac{\bar{X}_p}{X_w})^{\beta_\lambda}\big], & \text{Case 1} \\ p\delta(r-\mu)\big[1 - (\frac{\bar{X}_p}{X_e})^{\beta_\lambda - 1}\big] + \frac{r-\mu}{\Pi_2^C X_l}\big[\bar{H}_l^C - C_p^C - \bar{H}_l^C(\frac{\bar{X}_p}{X_e})^{\beta_\lambda}\big] + (\frac{\bar{X}_p}{X_e})^{\beta_\lambda - 1}, & \text{Case 2} \end{cases} \tag{120}$$

and

$$\bar{\Theta}_p^{Imin}(\bar{X}_p) = \begin{cases} \frac{p\delta(r-\mu)}{\Phi}\big[1 - (\frac{\bar{X}_p}{X_w})^{\beta_\lambda - 1}\big] + \frac{r-\mu}{\pi_2^C \bar{X}_p}\big[\bar{H}_l^I(\frac{\bar{X}_p}{X_w})^{\beta_\lambda} - \bar{H}_l^I + C_p^I\big], & \text{Case 1} \\ \frac{p\delta(r-\mu)}{\Phi}\big[1 - (\frac{\bar{X}_p}{X_e})^{\beta_\lambda - 1}\big] + \frac{r-\mu}{\pi_2^C \bar{X}_p}\big[\bar{H}_l^I(\frac{\bar{X}_p}{X_e})^{\beta_\lambda} - \bar{H}_l^I + C_p^I\big] + \frac{1}{\Phi}(\frac{\bar{X}_p}{X_e})^{\beta_\lambda - 1}. & \text{Case 2} \end{cases} \tag{121}$$

The smooth-pasting and value-matching conditions at X_{ns} and X_p^- give the expression of royalty rate that it is optimal to settle immediately

$$\bar{\Theta}_p^-(\bar{X}_p) = \frac{p\delta[\tau_a + \bar{k}(\bar{M} + \bar{N}) + \Phi(M+N) + \bar{k}\Phi n](r-\mu)}{\Phi(M + N + \tau_a)} \tag{122}$$

where $\bar{k} = \frac{X_e}{\bar{X}_p}$, $\tau_a = \frac{n\tau_c + M\tau}{n + M\tau_i}n$, $\tau = \frac{H_l^C - C_p^C}{H_l^I - C_p^I}$, $\tau_c = \frac{H_l^C}{H_l^I - C_p^I}$, $\tau_i = \frac{H_l^I}{H_l^I - C_p^I}$, $n = \alpha_\lambda - \beta_\lambda$, $N = \bar{k}^{\beta_\lambda} - \bar{k}^{\alpha_\lambda}$, $\bar{N} = \bar{k}^{-\beta_\lambda} - \bar{k}^{-\alpha_\lambda}$, $M = \beta\bar{k}^{\alpha_\lambda} - \alpha\bar{k}^{\beta_\lambda}$, and $\bar{M} = \beta\bar{k}^{-\alpha_\lambda} - \alpha\bar{k}^{-\beta_\lambda}$.

We dont find any feasible royalty rates in the bottom region when $x \in [X_p^{ns}, X_p^-]$ numerically, i.e. the value of not settling is always higher than the value of settling in the bottom region.

We then determine the immediate settlement threshold $\bar{x}_p$ with royalty rate $\bar{\Theta}_p^+(\bar{X}_p)$, which is also the litigation threshold x_l by maximizing I's value with the option to settle in either withdrawal or exit case.

When the Incumbent withdraws first in Case 1, we find there are no feasible royalty rates $\bar{\Theta}_p^-(x)$, i.e. $\bar{\Theta}_p^-(x) \geq \Theta_p^{Cmax}$. The maximum royalty rate follows the expressions in Eq. (21). Therefore, the immediate settlement is $\bar{\Theta}_p^+(x)$. When the Incumbent decides his litigation threshold by maximizing V_l^I with respect to X_l, we have

$$\frac{\partial V_l^I}{\partial X_l} = (1-\alpha)p\delta\pi_2^C X_l^{-\alpha} - \alpha(\frac{\beta_\lambda}{\beta_\lambda - 1}(\bar{H}_l^C - C_p^C) - C_p^I)X_l^{-\alpha-1} < 0. \tag{123}$$

Therefore, the incumbent would choose the lowest feasible x, which is defined as X_m. X_m is the x such that $\bar{\Theta}_p^+(X_m) = \Theta_p^{Cmax}(X_m)$, i.e.

$$X_m = [\frac{\bar{H}_l^C - C_p^C}{(1-\beta_\lambda)B_w^C}]^{\frac{1}{\beta_\lambda}}. \tag{124}$$

Therefore, when $x < X_p(\Theta_p^*)$, the Incumbent would choose to litigate at X_m and then settle immediately at X_m with the maximum immediate royalty rate $\bar{\Theta}_p^+(X_m)$, i.e.

$$\bar{\Theta}_p^+(X_m) = \frac{\beta_\lambda(\bar{H}_l^C - C_p^C)^{\beta_\lambda+1}(r-\mu)}{-(\beta_\lambda-1)^2(B_{ns}^C)^{\beta_\lambda}\pi_2^C} + p\delta(r-\mu), \tag{125}$$

and this royalty rate $\bar{\Theta}_p^+(X_m)$ is always greater than $\bar{\Theta}_p^{Imin}(X_m)$, indicating that immediate settlement is feasible.

When the Challenger exits first in Case 2, using the optimality method (i.e. $\frac{\partial V_l^I}{\partial X_l} = 0$), we obtain the expression for the litigation threshold or immediate settlement threshold when $x \in [X_p^-, X_p(\Theta_p^*))$ as follows

$$\bar{X}_l = \frac{(\frac{\beta_\lambda}{\beta_\lambda-1}(\bar{H}_l^C - C_p^C) - C_p^I)(\alpha k_l^\beta - \beta k_l^\alpha)}{p\delta\pi_2^C((\beta_\lambda-1)k_l^\alpha - (\alpha-1)k_l^\beta) - \frac{\pi_1^I - \pi_2^I}{r-\mu}(\beta-\alpha)k_l}, \tag{126}$$

where $k_l = \frac{X_e}{\bar{X}_l}$.

If $\bar{\Theta}_p^+(\bar{X}_l) \in [\Theta_p^{Imin}, \Theta_p^{Cmax}]$, i.e. the values of settling are higher than the values of not settling for both parties, firms settle immediately at $\bar{X}_l$ with royalty rate $\bar{\Theta}_p^+(\bar{X}_l)$.

Case B: The Constrained Incumbent ("I May Liquidate")

Incorporating the possibility that "I may liquidate", the value functions for both the Incumbent and the Challenger if firms choose to settle for any $x \in (x_p^{ns}, \bar{x})$ can be expressed as follows:

$$v_p^I = \begin{cases} a_p^I x^{\alpha_\lambda} + b_p^I x^{\beta_\lambda} - H_l^I + \frac{\pi_2^I x}{r-\mu+\lambda} + p\delta\pi_1^I x, & \text{if} \quad x \in [x_p(\theta_p^*), \bar{x}) \\ \frac{\pi_2^I + \bar{\theta}_p \pi_2^C}{r-\mu} x - C_p^I, & \text{if} \quad x \in [x_p^-, x_p(\theta_p^*)) \\ \bar{a}_p^I x^{\alpha_\lambda} + \bar{b}_p^I x^{\beta_\lambda} - H_l^I + \frac{\pi_2^I x}{r-\mu+\lambda} + p\delta\pi_1^I x. & \text{if} \quad x \in (x_p^{ns}, x_p^-), \end{cases} \tag{127}$$

$$v_p^C = \begin{cases} a_p^C x^{\alpha_\lambda} + b_p^C x^{\beta_\lambda} - H_l^C + \frac{\pi_2^C x}{r-\mu+\lambda} + \pi_1^C(1-p)\delta x, & \text{if} \quad x \in [x_p(\theta_p^*), \bar{x}) \\ \frac{(1-\bar{\theta}_p)\pi_2^C}{r-\mu} x - C_p^C, & \text{if} \quad x \in [x_p^-, x_p(\theta_p^*)) \\ \bar{a}_p^C x^{\alpha_\lambda} + \bar{b}_p^C x^{\beta_\lambda} - H_l^C + \frac{\pi_2^C x}{r-\mu+\lambda} + \pi_1^C(1-p)\delta x. & \text{if} \quad x \in (x_p^{ns}, x_p^-), \end{cases} \tag{128}$$

where a_p^I, a_p^C, b_p^I, b_p^C and $x_p(\theta_p^*)$ follow the expression in Corollary 6. Applying the value-matching and smooth-pasting conditions at x_p^- and x_p^{ns}, we have

$$\begin{aligned} \bar{a}_p^I &= \frac{1}{\beta_\lambda - \alpha_\lambda}[\beta_\lambda H_l^I - (\beta_\lambda - 1)[(p\pi_1^I - \pi_2^I)\delta + I_e \frac{\pi_2^C}{r-\mu+\lambda}]x_p^{ns}](x_p^{ns})^{-\alpha_\lambda}, \\ \bar{b}_p^I &= \frac{1}{\alpha_\lambda - \beta_\lambda}[\alpha_\lambda H_l^I - (\alpha_\lambda - 1)[(p\pi_1^I - \pi_2^I)\delta + I_e \frac{\pi_2^C}{r-\mu+\lambda}]x_p^{ns}](x_p^{ns})^{-\beta_\lambda}, \\ \bar{a}_p^C &= \frac{1}{\beta_\lambda - \alpha_\lambda}\big[\beta_\lambda(H_l^C - C_p^C) - (\beta_\lambda - 1)[((1-p)\pi_1^C - \pi_2^C)\delta x_p^- + \frac{\theta_p \pi_2^C}{r-\mu} x_p^-]\big](x_p^-)^{-\alpha_\lambda} \\ \bar{b}_p^C &= \frac{1}{\alpha_\lambda - \beta_\lambda}\big[\alpha_\lambda(H_l^C - C_p^C) - (\alpha_\lambda - 1)[((1-p)\pi_1^C - \pi_2^C)\delta x_p^- + \frac{\theta_p \pi_2^C}{r-\mu} x_p^-]\big](x_p^-)^{-\beta_\lambda} \end{aligned}$$

Using the similar method, we can obtain the royalty rate in immediate settlement region. Rewriting the expression of the optimal acceptance threshold x_p in Corollary 6, we obtain the royalty rate $\bar{\theta}_p^+(x)$, i.e.,

$$\begin{aligned} \bar{\theta}_p^+(x) = \frac{r-\mu}{\pi_2^C x}\big[&\frac{\beta_\lambda}{\beta_\lambda - 1}[(H_l^C - C_p^C) + (\bar{H}_l^C - C_p^C)(\frac{x}{\bar{x}})^{\alpha_\lambda}] \\ &- [(1-p)\pi_1^C - \pi_2^C]\delta x - [(\pi_2^C - \pi_1^C)(1-p)\delta\bar{x}](\frac{x}{\bar{x}})^{\alpha_\lambda}\big]. \end{aligned} \tag{129}$$

We can also obtain the royalty rate from the bottom region that can be expressed as

$$\bar{\theta}_p^- = \begin{cases} \frac{r-\mu}{\Pi_2^C} \frac{(M+N+nk)\delta[(1-p)\pi_1^C - \pi_2^C] - [k(\bar{M}+\bar{N})-n](p\pi_1^I - \pi_2^I)\delta}{\tau_a + M + N}, & \text{in Case 1} \\ \frac{r-\mu}{\Pi_2^C}\Big[\frac{[(1-p)\pi_1^C - \pi_2^C]\delta[\frac{\alpha_\lambda(\tau_i - \tau k^{\beta_\lambda})}{\beta_\lambda(\tau_i-\tau k^{\alpha_\lambda})(\beta_\lambda-1)k^{\alpha_\lambda}-(\alpha_\lambda-1)k^{\beta_\lambda}}]}{\frac{\alpha_\lambda(\tau_i-\tau k^{\beta_\lambda})}{\beta_\lambda(\tau_i-\tau k^{\alpha_\lambda})}(\beta_\lambda-1)-(\alpha_\lambda-1)} & \\ -(\frac{\pi_2^C}{r-\mu+\lambda} + \pi_1^C(1-p)\delta)k\Big]. & \text{in Case 2} \end{cases} \tag{130}$$

The corresponding threshold x_p^- can be expressed as

$$x_p^- = \begin{cases} \frac{n(H_l^I - C_p^I) + \bar{M}H_l^I}{(\bar{M}+\bar{N})(p\pi_1^I + \pi_2^I)\delta k - n[(p\pi_1^I - \pi_2^I)\delta - \frac{\bar{\theta}_p^- \pi_2^C}{r-\mu}]}, & \text{in Case 1} \\ \frac{\beta_\lambda(H_L^I - (H_l^C - C_p^C)k^{\alpha_\lambda})}{(\beta_\lambda - 1)\Big[[\frac{\bar{\theta}_p^- \pi_2^C}{r-\mu} - ((1-p)\pi_1^C - \pi_2^C)\delta]k^{\alpha_\lambda} + (\frac{\pi_2^C}{r-\mu+\lambda} + \Pi_2^C(1-p)\delta)k\Big]}. & \text{in Case 2} \end{cases} \tag{131}$$

The immediate settlement is feasible if the value of settling immediately is higher than the value of not settling for both parties (i.e. $v_p^I(\bar{\theta}) \geq v_{ns}^I$ and $v_p^C(\bar{\theta}) \geq v_{ns}^C$).

Similarly, we use the optimality method (i.e. $\frac{\partial v_l^I}{\partial x_l} = 0$) to determine the litigation threshold or immediate settlement threshold $x_l/\bar{x}_p$ when $x_l \in (x_p^-, x_p(\theta_p^*))$ in Case 1 (i.e. I withdraws first), which satisfies the following equation:

$$\begin{aligned} &\alpha x_l^{-\alpha-1} C_p^I - \alpha \frac{\beta_\lambda}{\beta_\lambda - 1}(H_l^C - C_p^C)x_l^{-\alpha-1} + (\alpha_\lambda - \alpha)x_l^{\alpha_\lambda-\alpha-1}\frac{\beta_\lambda}{\beta_\lambda-1}(\bar{H}_l^C - H_l^C)\bar{x}^{-\alpha_\lambda} \\ &= (1-\alpha)x_l^{-\alpha}[(1-p)\pi_1^C - \pi_2^C] + (\alpha_\lambda - \alpha)x_l^{\alpha_\lambda-\alpha-1}(\pi_2^C - \pi_1^C)(1-p)\delta(\bar{x})^{1-\alpha_\lambda}. \end{aligned}$$

Therefore, we obtain the optimal immediate settlement royalty that can be expressed as follows

$$\begin{aligned} \bar{\theta}_p^+(x_l) = &\frac{r-\mu}{(\beta_\lambda-1)\pi_2^C x_l}\big[\beta_\lambda(H_l^C - C_p^C) - (\beta_\lambda - 1)[(1-p)\pi_1^C - \pi_2^C]\delta x_l - \\ &[(\beta_\lambda-1)(\pi_2^C - \pi_1^C)(1-p)\delta\bar{x} - \beta_\lambda(\bar{H}_l^C - H_l^C)](\frac{x_l}{\bar{x}})^{\alpha_\lambda}\big]. \end{aligned} \tag{132}$$

Immediate settlement is feasible if both parties' values of settling immediately are higher than the value of not settling (either withdrawal or exit), i.e., $v_p^I \geq v_{ns}^I$ and $v_p^C \geq v_{ns}^C$, we check these conditions numerically in MATLAB.

We do not show the expression of x_l in Case 2 (i.e., C exits first) due to its complexity. Instead, we use MATLAB to check the feasibility conditions for every $x \in (x_p^-, x_p)$ and find that there is no feasible x and corresponding $\bar{\theta}_p(x)$ in our parameter sets. $\square$