Investment and Financing Decisions with Learning-Curve Technology

Abstract

The learning curve has a significant impact on production cost (hence corporate profit) in a number of industries. While it is well recognized in the Economics literature and its effect on operating costs and production decisions has been widely studied, its effect on corporate investment has been largely unexplored. To our knowledge, there is one paper that examines this issue, but it is limited to unlevered firms. We therefore examine a *levered* firm's optimal investment and financing choices when using learning-curve technology. The main findings are as follows. The effect of leverage on the investment decision depends on the level of debt. Using the optimal debt level will result in earlier investment, whereas the investment size might be smaller or larger depending on the speed of learning; however, investment overall (taking into account both timing and size) will be higher than an unlevered firm, and the difference an increasing function of learning speed. The optimal leverage ratio is an increasing function of learning speed, but with a borrowing constraint it is, in general, initially increasing and subsequently decreasing in learning speed. Moreover, it is a decreasing function over a wider range for a more stringent borrowing constraint, for decreasing-returns-to-scale technology and for a less volatile product market.

JEL Classification Code: G31.

Keywords: Learning curve; Real-option model; Investment trigger; Investment scale; Leverage ratio.

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1. Introduction

Learning-curve technology plays an important role in a number of industries, such as chemicals (Lieberman, 1983), semiconductors (Irwin and Klenow, 1994), aircraft manufacturing (Benkard, 2000), shipbuilding (Thornton and Thompson, 20001), Ethanol (Goldemberg et al., 2004), Photovoltaic Cells and Electric/hybrid cars (Seta et al., 2012). After the idea of a "learning curve" was introduced by Wright (1936), a large body of research has studied the learning curve and its role in production analysis, in various industrial sectors such as electronic, automotive, software and chemicals (Anzanello and Fogliatto, 2011). With learning-curve technology, the average production cost declines with production, because workers and managers absorb new technological information as they become more experienced (and more effective) at their jobs. As a result of this "learning," the production cost per unit of output falls as the cumulative output rises, resulting in higher profits and firm values. This is the "learning-curve effect" studied by Majd and Pindyck (1989) and Seta et al. (2012).[1]

With learning-curve technology, unit operating cost falls as the cumulative output rises; the rate at which the cost declines is given by the speed or intensity of learning. Since cost reduction implies higher profit, the learning curve has a significant impact on the valuation and competitive position of the company; for instance, the learning curve seems to have made Ethanol competitive in Brazil (Goldemberg et al., 2004). However, there is great heterogeneity in the rate or speed of learning across industries (Argote and Epple, 1990, Gruber, 1992), hence there will be significant cross-sectional differences in learning-curve effects across various industries.

Naturally, the learning curve can be expected to impact investment decisions. For instance, it can be argued that the large investments made in green technology were partly driven by learning effects (of course, they could also be partly driven by government incentives such as support prices, investment subsidies and tax breaks, since many governments are keen to encourage corporate investment in such industries). The effect of the

[1] Although there are other reasons that contribute to the decline in average production cost, such as economies of scale, the learning curve effect is an important factor because of its large magnitude. For instance, in the chemical industry, for each doubling of plant scale (economies of scale) the average production cost falls by 11% whereas for each doubling of cumulative output (learning-curve effect) the average production cost falls by 27% (Lieberman, 1984).

learning curve on corporate investment decisions should be of interest to governments, economists and corporations. However, there is only one paper on the effect of the learning curve on investment, and that is the real-option model of Seta, Gryglewicz and Kort (2012, SGK hereafter).[2]

Real-option models in general focus on the timing of investment (Dixit and Pindyck, 1994). For learning-curve technology, however, it is important to look at both timing and size of investment, because a larger operating scale hastens learning and the resulting cost reduction; hence SGK (2012) examine both optimal timing and scale of investment. The learning curve will provide an incentive to invest earlier (more time to learn) and also to invest on a larger scale (expedited learning).[3] SGK (2012) show that the speed of learning has a significant effect on both the timing and the size (or scale) of investment; the investment trigger is a decreasing function, and the investment size an inverted-U shaped function, of the speed of learning. Compared to the traditional (no-learning) case, the firm will make a delayed and larger investment if the learning speed is low, but an accelerated and smaller investment if the learning speed is high. That is, the firm will exploit the scale option if learning is slow and the timing option if learning is fast.

The SGK (2012) study leaves two questions unanswered:

(i) If the learning curve causes investment to be earlier but smaller (or delayed but larger), as in SGK (2012), what is the *overall* effect on investment?

(ii) How would the learning curve affect investment when the firm uses debt financing?

The first question arises because early (late) investment implies a positive (negative) effect, while smaller (larger) investment size implies a negative (positive) effect. Since timing and size have conflicting effects, the overall effect on investment is not clear. This is of interest to governments and policymakers, who would like to encourage corporate investment, particularly in knowledge-based industries (which are more likely to use learning-curve technology). To address this issue, we use a composite measure of investment, the EPVI (see

[2] There are papers that examine other aspects of learning-curve technology, such as the effect of the learning curve on production policy (Majd and Pindyck, 1989), pricing policy (Cabral and Riordan, 1994), market structure and performance (Ghemawat and Spence, 1985) or competitive strategy (Spence, 1981). However, these papers do not study the important effect that the learning curve might have on corporate investment.

[3] SGK (2012) call them the “timing option” and “scale option” respectively.

Sarkar, 2019), that incorporates both timing and size effects. This helps us analyze the overall effect of the learning curve on investment, which was not possible in the SGK (2012) model because it considered timing and size effects separately.

The second question arises because SGK (2012) study an unlevered firm while most firms in practice use some leverage. Since corporate investment and financing policies are inter-related (Dotan and Ravid, 1985, Mauer and Triantis, 1994, Dammon and Senbet, 1988), it is likely that a levered firm's investment decision will differ from that of a unlevered firm.

We will therefore try to address the following research questions:

- How does the levered firm's investment decision differ from the unlevered firm's?
- How does the levered firm's investment decision depend on the speed of learning?
- How does the speed of learning affect the firm's leverage ratio?

We extend the SGK (2012) model to identify the investment decision of a *levered* firm that uses learning-curve technology; in addition to timing and size, we also look at a composite measure of investment (EPVI) that takes into account both size and timing (discussed in detail in Section 2.4). Also, we identify the optimal leverage ratio and examine how it is affected by the learning curve.[4]

Our paper brings together the learning-curve-investment literature (Seta et al., 2012) and the dynamic investment-financing literature (Hackbarth and Sun, 2017, Shibata and Nishihara, 2012, Sundaresan et al., 2015). We show that debt financing can have a large impact on the investment decision, with the exact effect depending on the level of debt. A firm using the optimal debt level will invest earlier than an unlevered firm, but the investment size can be smaller, larger or the same as an unlevered firm, depending on the speed of learning. Investment overall (taking into account both timing and size) with the optimal leverage ratio is always higher than an unlevered firm, and the difference is an increasing function of learning speed. The optimal leverage ratio is an

[4] Deb and Balasubramanian (2016) empirically examine the effect of learning curve on capital structure, but they do not analyze the investment decision.

increasing function of the speed of learning. However, under a borrowing constraint (which is very likely for such firms, see Deb and Balasubramanian, 2016), the leverage ratio is an inverted-U-shaped function of learning speed.

The rest of the paper is organized as follows. Section 2 describes the model, and derives the optimal investment and financing policies with learning-curve technology. Section 3 presents and discusses the numerical results from the model. Section 4 concludes.

2. Model Setup

SGK (2012) examine the investment decision of an unlevered firm that uses learning-curve technology. We extend their model to a levered firm, thus our starting point is the SGK model.

2.1. Model Basics

A firm holds an option to build a production facility with learning-curve technology. Its investment decision consists of the timing (or investment trigger) and the size (or investment scale). The investment scale is given by the amount of capital K, which can be acquired at a cost of \$i per unit, so the total investment cost is \$iK. The production capacity is given by the transformation $q = K^{\eta}$ (where $\eta < 1$ is the returns-to-scale parameter).[5]

The output price is given by x, which evolves randomly as a lognormal process:[6]

$$dx = \mu x dt + \sigma x dZ, \tag{1}$$

[5] This transformation is consistent with a Cobb-Douglas production function with decreasing returns to scale, and is widely used in the production and contingent-claim literature, e.g., Bar-Ilan and Strange (1999), Bertola and Caballero (1994), Chen et al. (2018), Carlson et al. (2004), Tserlukevich (2008), Aguerrevere (2009), Hackbarth and Johnson (2015). In some of these papers, the price is derived endogenously as a function of stochastic demand; however, their revenue function is similar to our reduced-form expression. Bertola and Caballero (1994) show that this reduced form of revenue function is consistent with empirical data.

[6] The SGK (2012) model has a downward-sloping demand curve, so the firm is not a price-taker. However, they state (their footnote 5) "The results presented are not driven by the model specification. In particular, all the main results are also present in another popular specification in these types of models: a price taking firm with decreasing returns to scale technology (price is an exogenous diffusion process and the rate of production with capital K is K^{α}, $\alpha < 1$)." We abstract from competition and demand effects and assume a price-taking firm with decreasing returns to scale, in order to focus on the joint investment-leverage decision.

where μ and σ are the expected growth rate and volatility, respectively, of the process, and Z is a standard Wiener process. The production project is infinitely-lived, and will operate at full capacity; thus the production quantity will be $q = (K^{\eta})$ units of the output per unit time.[7] The variable operating cost per unit is given by the time-varying cost c_t, which exhibits learning-curve effects as in SGK (2012):

$$c_t = c_0 e^{-\gamma Q_t}, \tag{2}$$

where c_0 is the starting or initial cost, γ is the speed or intensity of learning, and Q_t is the cumulative output by time t. That is, because of the learning curve, the unit production cost falls with cumulative production. Since capacity is fully utilized, the cumulative output is given by $Q_t = (K^{\eta})t$.

The firm's profits are taxed at a constant corporate tax rate of τ $(0 < \tau < 1)$, and all cash flows are discounted at a constant interest rate of r. The after-tax profit flow from the project can then be expressed as:

$$\pi_t = (1-\tau)(x_t - c_t)K^{\eta} \tag{3}$$

2.2. Unlevered Firm

As in SGK (2012), the (post-investment) project value is given by:

$$V(x,Q) = \mathbb{E}\int_t^T (1-\tau)(x_s - c_0 e^{-\gamma Q_s})K^{\eta}\, e^{-r(s-t)} ds$$

which simplifies to:

$$V(x,Q) = (1-\tau)K^{\eta}\left(\frac{x}{r-\mu} - \frac{c_0 e^{-\gamma Q}}{r+\gamma K^{\eta}}\right) \tag{4}$$

The optimal investment trigger, as a function of capital, is $x^*(K)$:

$$x^*(K) = (r-\mu)\frac{\alpha}{\alpha-1}\left(\frac{c_0}{r+\gamma K^{\eta}} + \frac{iK^{1-\eta}}{1-\tau}\right) \tag{5}$$

[7] In real life, it might be possible for the firm to produce at reduced rates (leaving some of the capacity idle) when demand is low. However, as Seta et al. (2012) explain, in many industries firms make production plans before the actual realization of market demand, and may find it difficult to produce below capacity due to fixed costs. Also, even when firms can keep some capacity idle, a temporary suspension of production is often costly, because of maintenance costs needed to avoid deterioration of the equipment.

Finally, the optimal investment scale K^* is the solution to the equation:

$$\frac{1}{\alpha-1}\frac{c_0}{r+\gamma K^\eta}+\left(\frac{\alpha}{\alpha-1}-\frac{1}{\eta}\right)\frac{I}{K^{\eta-1}(1-\tau)}+\frac{\gamma c_0 K^\eta}{(r+\gamma K^\eta)^2}=0 \qquad (6)$$

As in SGK (2012), an all-equity firm choosing both optimal investment timing and optimal investment size will solve equations (5) and (6) simultaneously. This gives the optimal investment decision for an unlevered firm. We examine the levered firm next.

2.3. Levered Firm

In this section, we extend the SGK (2012) model to identify the optimal investment decision of a levered firm with learning-curve technology. Accordingly, we assume that the investment is (partially) financed with perpetual debt, which will pay a constant coupon at a rate of $$b$ per unit time in perpetuity (unless the firm declares bankruptcy). For a levered firm, we must consider the possibility of bankruptcy. We therefore assume that when conditions deteriorate sufficiently (or x falls to a low enough level, say x_b) the company declares bankruptcy, as is standard in the corporate finance literature (Goldstein, et al., 2001, Leland, 1994). At bankruptcy, bondholders take over the assets of the company after incurring fractional bankruptcy cost of ω, and shareholders exit with zero payoff.

Since the production cost c_t is dependent on the cumulative output Q_t, the bankruptcy trigger will also depend on Q_t; the bankruptcy trigger is therefore written as $x_b(Q_t)$. This implies equity and debt values will be a function of Q as well as x; let the equity value be $E(x, Q)$ and the debt value $D(x, Q)$.

2.3.1. Equity Valuation

As shown in Appendix A, equity value E(x, Q) will be the solution to the following partial differential equation (PDE):

$$0.5\sigma^2 x^2 E_{xx}(x,Q)+\mu x E_x(x,Q)+\left(K^\eta\right)E_Q(x,Q)+(1-\tau)[K^\eta(x-c_0e^{-\gamma Q})-b]=rE(x,Q) \qquad (7)$$

The equity value must also satisfy the following boundary conditions.

(i) At the bankruptcy trigger $x_b(Q)$: When the firm declares bankruptcy, the payoff to equity holders is zero. This gives the value-matching condition:

$$E(x_b(Q),Q) = 0, \ \forall Q \tag{8}$$

Also, x_b is endogenously chosen by the firm. For it to be optimal, it has to satisfy the smooth-pasting condition:

$$E_x(x_b(Q),Q) = 0, \ \forall Q \tag{9}$$

(ii) As $x \to \infty$: When the output price x becomes very large, the risk of bankruptcy becomes negligible, hence equity value is simply the project value less the (riskless) bond value, or:

$$\lim_{x\to\infty} E(x,Q) = V(x,Q) - b/r, \ \forall Q \tag{10}$$

where the project value V(x, Q) is given by equation (4).

(iii) As $Q \to \infty$: As shown in Appendix B, when the cumulative output reaches very high levels ($Q \to \infty$), we get the boundary condition:

$$\lim_{Q\to\infty} E(x,Q) = (1-\tau)\left(\frac{K^{\eta}x}{r-\mu} - \frac{b}{r}\right) + Ax^{\beta} \tag{11}$$

where $A = -\dfrac{(1-\tau)\left(K^{\eta}\right)\left(x_{b\infty}\right)^{1-\beta}}{\beta(r-\mu)}$, $x_{b\infty} = \dfrac{b(1-\mu/r)}{K^{\eta}(1-1/\beta)}$, and $\beta = \dfrac{1}{2} - \dfrac{\mu}{\sigma^2} - \sqrt{\left(\dfrac{1}{2} - \dfrac{\mu}{\sigma^2}\right)^2 + \dfrac{2r}{\sigma^2}}$.

The equity value is obtained by solving equation (7) subject to the above boundary conditions (equations (8) – (11)). Since there is no analytical solution, we use numerical methods to solve for equity value (Appendix C).

2.3.2. Debt Valuation

Similarly, it can be shown that debt value D(*x*, *Q*) is the solution to the following PDE:

$$0.5\sigma^2 x^2 D_{xx}(x,Q) + \mu x D_x(x,Q) + \left(K^{\eta}\right) D_Q(x,Q) + b = rD(x,Q)$$

subject to the following boundary conditions:

(i) At the default trigger $x_b(Q)$: $D(x_b(Q),Q) = (1-\omega)V(x_b(Q),Q),\ \forall Q$ (value-matching condition).

(ii) As $x \to \infty$: when the demand parameter becomes very large, the risk of bankruptcy becomes negligible, hence the debt becomes equivalent to a risk-free bond: $D(x, Q) \to b/r$.

(iii) As $Q \to \infty$: As shown in Appendix B, when the cumulative output reaches very high levels, the boundary condition is: $\lim_{Q\to\infty} D(x,Q) = \frac{b}{r} + Bx^{\beta}$,where $B = \left[(1-\tau)(1-\omega)\left(K^{\eta} x_{b\infty} /(r-\mu)\right) - b/r\right](x_{b\infty})^{-\beta}$.

The above PDE is solved numerically, along with the boundary conditions, for the debt value D(x,Q).

We have shown above how to compute equity value E(x,Q) and debt value D(x,Q) for a levered firm after investment. Since we are interested in the investment decision, we next look at the option to invest and how to exercise this option (i.e., how to invest) optimally.

2.3.3. Option to Invest and the Optimal Investment Decision

The investment decision can be viewed as the exercise of an (American) option to invest. Prior to investment, shareholders have this (perpetual) option to invest; the value of the option will be a function of just the state variable x, say F(x). Then it can be shown that the option value is given by:

$$F(x) = H\, x^{\alpha}, \tag{12}$$

where H is a constant to be determined by boundary conditions, and $\alpha = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}$ (note that $\alpha > 1$). The firm value prior to investment is just the option value Hx^{α}. Thus the constant H can be used as a time-independent proxy for the ex-ante (pre-investment) firm value or shareholder value.

When the option is exercised (i.e., the investment is made), say, at $x = x^*$, shareholders issue debt and receive the proceeds $D(x^*,Q)$. The remaining investment cost is paid by the shareholders; thus the shareholders have to invest an amount $\{iK - D(x^*,0)\}$, since Q = 0 at investment. Then the shareholders' net payoff at investment is $E(x^*, 0) - \{iK - D(x^*, 0)\}$, or $\{E(x^*, 0) + D(x^*, 0) - iK\}$. The value-matching condition for the option exercise decision is:

$$F(x^*) = E(x^*, 0) + D(x^*, 0) - iK \tag{13}$$

For the option to be exercised optimally, the smooth-pasting condition must also be satisfied (see Dixit and Pindyck, 1994):

$$F'(x^*) = E'(x^*, 0) + D'(x^*, 0) \tag{14}$$

In addition to choosing the best time to invest (or the optimal investment trigger x^*), the firm also chooses the investment scale or capacity K. The optimal capacity is given by K^*:

$$K^* = \arg\max_{K} \{E(x,0) + D(x,0) - iK\} \tag{15}$$

We now have the optimal investment decision, (x^*, K^*), with an exogenously-specified debt level. From the above discussion, it is clear that the optimal investment policy will depend on the debt level. Then the question of interest is: what is the appropriate debt level, or what level of debt is the firm likely to use? Most of the literature seems to agree that the appropriate debt level is the optimal level (Leland, 1994, Mauer and Sarkar, 2005, etc), since managers are rational and hence likely to optimize. We therefore identify the optimal debt level next.

2.3.4. Optimal Debt Level

The optimal debt level will maximize the total value of the firm (Leland, 1994, Mauer and Sarkar, 2005); that is, the firm must choose b so as to maximize $\{E(x^*, 0)+D(x^*, 0)\}$. Then the optimal debt level is given by:

$$b^* = \arg\max_{b} \{E(x^*,0) + D(x^*,0)\} \tag{16}$$

and the corresponding (optimal) leverage ratio:

$$Lev^* = D(x^*,0)/[E(x^*,0) + D(x^*,0)] \tag{17}$$

It is clear from the above discussion that the optimal debt level and optimal leverage ratio will depend on the speed of learning.

The optimal investment-financing decision (x^*, K^* and b^*) and the resulting value H are obtained by solving equations (13), (14), (15) and (16) simultaneously. There is no analytical solution to this system of equations, hence it is solved using numerical methods.

2.3.5. With Borrowing Constraint

In practice, the firm is likely to face a limit on how much it can borrow (or how much debt financing it can use). Such an external borrowing constraint might arise from asymmetric information, credit rationing, agency problems such as risk-shifting, etc (Deb and Balasubramanian, 2016, Koussis and Martzoukous, 2012, Shibata and Nishihara, 2012). Debt constraints have already been introduced in real-option models. Shibata and Nishihara (2012), for instance, study the investment timing decision and the leverage decision with an external borrowing constraint of $qI, where q is a friction parameter and $I is the amount to be invested in the project.

The borrowing constraint will be particularly relevant for a firm using learning-curve technology, because a significant fraction of the firm's value comes from future earnings growth caused by the decline in operating cost as the firm moves down the learning curve. Future earnings growth resulting from learning cannot be claimed, ex ante, hence such a firm will have relatively fewer tangible assets to pledge as collateral (Deb and Balasubramanian, 2016). Since learning-curve firms have relatively more intangible assets, they are more likely to be subject to a borrowing constraint.

To incorporate the borrowing constraint, we follow Shibata and Nishihara (2012), and assume the firm can borrow a fraction q of the investment amount, thus the borrowing constraint is: $D(x^*,0) \leq qiK^*$ (since Q = 0 at investment), and the firm's capital structure decision becomes:

$$b^{**} = \arg\max_{b} \left\{E(x^*,0) + D(x^*,0)\right\} \text{ s.t. } D(x^*,0) \leq qiK \tag{18}$$

where b^{**} is the constrained-optimal debt level, and the resulting leverage ratio is the constrained-optimal leverage ratio. Because the financing package is different, the investment decision might also be different from the unconstrained optimal-leverage investment decision studied above.

2.4. The Expected Present Value of Investment (EPVI)

As SGK (2012) show, earlier investment is generally smaller and delayed investment is larger in size. (This is consistent with the other studies (with no learning curve), e.g., Bar-Ilan and Strange, 1999, Huberts et al., 2012). This makes it difficult to gauge the overall effect of the learning curve on investment, as discussed in the introductory section, because the timing and size effects act in opposite directions. What is needed is an overall measure of investment that takes into account both the size and timing of the investment, because delayed investment is equivalent to smaller investment (or underinvestment) in present-value terms (Lukas and Thiergart, 2019). Thus, from the SGK (2012) results, we are unable to say whether the *overall* effect of the learning curve on investment is negative or positive.

We therefore use the composite measure *Expected Present Value of Investment (EPVI)*, which takes into account both the size and timing of investment.[8] The EPVI is just the expected present value of the amount of investment (to be) made by the company.[9] The company will invest \$$iK^*$ when x rises to x^* (or at the first passage time of x to x^*), the expected present value of which is given by: $iK^*\left(x_0/x^*\right)^{\alpha}$, as shown by Leland (1994, footnote 16). This depends on the current price x_0. We ignore the dependence on price (which is stochastic) and focus on the time-independent part that depends on exogenous parameters. Therefore, we normalize the expression to $x_0 = 1$, to obtain the time-independent measure:

$$\mathrm{EPVI} = \frac{iK}{\left(x^*\right)^{\alpha}} \tag{19}$$

This will be our measure of overall investment.

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[8] This measure, or some variant of it, has recently been used by Lukas and Thiergart (2019), Nishihara et al. (2019) and Sarkar (2019).

[9] *Expected*, because the time of investment is not known with certainty (see Leland, 1994).

3. Numerical Results

3.1. Parameter Values

Since we use numerical methods to identify the optimal investment and financing decisions, the input parameter values need to be specified. We use parameter values similar to Seta et al. (2012): expected demand growth rate μ = 1%, demand volatility σ = 10%, discount rate r = 5%, returns-to-scale parameter η = 0.5, initial production cost c_0 = 1, and unit price of capital i = 5. In addition, the two financing-related variables tax rate (τ) and bankruptcy cost (ω) are taken from Leland (1994), hence τ = 15% and ω = 50%. Finally, we use a base-case value of the speed of learning of γ = 0.1; this is for illustrative purposes, since we repeat the computation with a range of values of γ.

3.2. With Exogenously Specified Debt Level

With the base-case parameter values specified above, we computed the investment decision and the resulting EPVI for an unlevered firm (b = 0) and a levered firm (b = 1), using the procedure of Sections 2.3.3 and 2.4. The results are given below.

Unlevered (b = 0): x^* *= 1.08,* K^* *=5. 23; and EPVI= 21.1224.*

Levered (b = 1): x^* *= 0.86,* K^* *= 3. 47; and EPVI= 26.4909.*

From these results, it is clear that debt financing can have a significant impact on investment timing, investment size, and investment overall (EPVI). Below we repeat the numerical computations with different debt levels and different learning speeds to get a better idea of the effect of debt financing on investment with a learning curve.

Figure 1 shows how the investment decision varies with the debt level, for the “no-learning” case (γ = 0) and the “learning” case with three different learning speeds, γ = 0.01, 0.1 and 0.2. From parts (a) and (b) of Figure 1, we see that both optimal investment trigger x^* and optimal investment size K^* are initially decreasing and subsequently increasing in debt level. For instance, with $\gamma = 0.1$, x^* falls from 1.048 to 0.86 and then rises to 1.86, and K^* falls from 4.69 to 3.47, and then rises to 22.97, as debt level goes from 0 (unlevered) to 1 and then to 8.

Figure 1 about here

Thus, as the debt level is increased, the investment is initially smaller and accelerated, and subsequently larger and delayed. This leverage effect can be explained by the well-known trade-off theory of capital structure, according to which debt financing has two effects – tax benefits and bankruptcy costs. Because of the valuable tax shield associated with leverage, debt financing makes the project more attractive, resulting in earlier investment (lower x^*). However, earlier investment results in smaller size K^*, as shown by Bar-Ilan and Strange (1999), Hagspiel et al. (2016), and Huberts et al. (2015). The tax effect will then result in smaller x^* and K^* (earlier and smaller investment) as debt level is increased. On the other hand, the bankruptcy cost associated with debt has the opposite effect; it makes the project less attractive, resulting in delayed investment (higher x^*) and hence larger size K^*. Thus the bankruptcy-cost effect results in larger x^* and K^* (delayed and larger investment) as debt level is increased.

Because of the two opposing effects, the overall impact of debt financing on the investment decision is difficult to identify unambiguously, and will depend on which of the two dominates. For low debt level, the probability of bankruptcy is small, hence the tax effect dominates, and the overall effect of increasing the debt level is earlier but smaller investment. For high debt level, probability of bankruptcy is high, hence the bankruptcy cost effect dominates, and the overall effect of a higher debt level is delayed but larger investment. Thus, both trigger and size are initially decreasing and subsequently increasing in debt level.

From Figure 1(a) and (b), we also note the following: (i) the investment trigger with debt financing can be higher or lower than the unlevered firm; (ii) the investment size with debt financing can be higher or lower than the unlevered firm; and (ii) in both cases, the difference between levered and unlevered firm can be quantitatively significant.

The EPVI (from equation (19)) is shown in part (c) of Figure 1. We note that EPVI is insensitive to debt financing for no learning ($\gamma = 0$) and very slow learning ($\gamma = 0.01$). Thus, for very low learning rates, while debt financing does affect the timing and size of investment, it has no significant effect on investment overall. In other cases, EPVI is initially an increasing function and subsequently a decreasing function of debt level. Moreover, EPVI can be quite sensitive to debt financing; with $\gamma = 0.1$, for instance, EPVI rises from 20.68 to 25.49 and then falls to 21.48 as debt level goes from 0 to 3 and then to 8.

The intuition behind the inverted-U shape of the EPVI curve is as follows. A smaller (larger) x^* results in a larger (smaller) EPVI, everything else remaining the same; hence the U-shaped relationship between x^* and debt level will translate to an EPVI curve that is *inverted-U-shaped* in debt level. Similarly, a smaller (larger) K^* results in a smaller (larger) EPVI, so a U-shaped K^* curve will lead to a U-shaped EPVI curve. Thus, we have the two opposing effects on EPVI – the timing (x^*) effect and scale (K^*) effect. Recall that $\mathrm{EPVI} = iK \big/ \left(x^*\right)^{\alpha}$; since $\alpha > 1$, the timing (x^*) effect will be stronger than the scale (K^*) effect. Thus the net effect on EPVI will be closer to the former; this explains why EPVI is generally inverted-U-shaped in debt level. For very slow learning, the scale effect becomes relatively more important (SGK, 2012), hence the inverted-U shape is weaker.

Figure 2(d) shows the ex-ante firm value (H) as a function of debt level. As mentioned in the introductory section, this is what the shareholders/manager would like to maximize. As expected from the trade-off theory, this is initially rising and subsequently falling in debt level, giving an optimal debt level.[10] In the rest of the paper, we focus on the EPVI rather than the value.

Result 1. The effect of debt financing, in the presence of a learning curve, can be summarized as follows:

- Investment trigger and investment size can be higher or lower with debt financing than with no debt financing, and the difference can be quantitatively significant.
- Both investment trigger and size are U-shaped functions of debt level, while both investment overall (EPVI) and firm value are, in general, inverted-U-shaped functions of debt level.

[10] For $\gamma = 0$ and 0.01, the optimal debt level exceeds b = 8 and hence does not show up in the graph.

It is clear from the above results that the effect of the learning curve on investment depends on the level of debt. This leads to the question: what would be an appropriate level of debt? It is common in the literature to assume that managers, being rational decision-makers, choose the debt level optimally, i.e., to maximize total firm value (Leland, 1994, Mauer and Sarkar, 2005). Therefore, we next look at the numerical results when the firm optimizes jointly the investment and financing decisions under learning-curve technology.

3.3. With the Optimal Debt Level

In addition to the optimal investment decision (x^*, K^*) discussed above, the optimal debt level b^* and the corresponding leverage ratio are also computed from equations (16) and (17). Thus, both investment (timing and size) and capital structure are chosen optimally in this section.

3.3.1. Investment

The optimal leverage ratio (hence the optimal investment size and timing as well) will depend on the speed of learning γ. The numerical results with the base-case parameter values and $\gamma = 0.1$ are shown below.

Optimal financing: $b^* = 1.66$, leverage ratio = 71.63%;

Optimal investment: $x^* = 0.97$, $K^* = 5.14$, EPVI = 27.62.

For comparison with the unlevered firm, we give below the results for $b = 0$.

Optimal unlevered investment: $x^* = 1.08$, $K^* = 5.23$, EPVI = 21.12.

While the investment size is only slightly smaller for the optimally-levered firm, the investment trigger is significantly lower, 1.08 versus 0.97. The overall effect on investment, measured by the EPVI, is significantly higher for the optimally-levered firm, likely because of the earlier investment. From the above results, it is clear that, for $\gamma = 0.1$, the optimally-levered firm's investment decision will be substantially different from the unlevered firm. To see how this difference varies with γ, the computations were repeated with different values of γ, and the results shown in Figure 2.

Figure 2 about here

Figure 2 shows a comparison of unlevered and optimally-levered firms' investment decisions over a range of learning speed (γ = 0 to 0.3). The optimally-levered firm has a lower investment trigger x^* than the unlevered firm over the entire range of γ, and in both cases the trigger is a decreasing function of γ. Thus, the optimally-levered firm will invest earlier than the unlevered firm, and learning speed has a similar effect on investment timing in both cases. Also, as discussed above, the difference between the two can be significant.

The effect on investment size K^* is more ambiguous, and the optimally-levered firm's investment size can be larger, smaller or approximately equal to that of the unlevered firm, depending on the speed of learning. For low learning speed ($\gamma < 0.01$), the investment size is larger with optimal leverage. For intermediate learning speed ($0.01 < \gamma < 0.1$), the unlevered firm's investment size will be larger, and for faster learning ($\gamma > 0.1$) there is virtually no difference between the two.

The learning speed γ affects investment size K^* differently in the two cases; K^* is an inverted-U-shaped function of γ for unlevered, but a decreasing function of γ for optimally-levered. This means the optimally-levered firm will not rely on the "scale option," unlike the unlevered firm (SGK, 2012). For rapid learning (large γ), the difference between unlevered and optimally-levered size is small, e.g., for $\gamma = 0.15$, we have $K^* = 3.6$ and 3.5 for unlevered and optimally-levered, respectively. But for slow learning, optimally-levered size can be significantly smaller than unlevered size, e.g., for $\gamma = 0.01$, $K^* = 34.62$ and 25.18 for unlevered and optimally-levered, respectively. Thus, the optimally-levered firm's investment decision can be significantly different from the unlevered firm's investment studied by SGK (2012).

As discussed in the previous paragraph, for an unlevered firm the "scale option" dominates for low γ. This is because with slow learning the firm needs more capacity in order to produce more (to achieve faster learning). This outweighs the negative effect of smaller x^* (which causes small K^*); thus the net effect is a larger K^*, that is, the scale option dominates. However, in the optimally-levered firm, the scale option does not dominate even for very small γ. This is because optimal leverage ratio increases with γ (Section 3.3.2), which results in higher bankruptcy risk and makes the firm less willing to invest in large capacity. This causes K^* to go down, hence K^* is a decreasing function of γ, thus the scale option does not dominate.

Figure 2(c) shows the EPVI of unlevered and optimally-levered firms as functions of γ. There are three points worth noting: (i) for both, EPVI is an increasing function of γ (but increasing at a decreasing rate); (ii) the optimally-levered EPVI is higher than unlevered EPVI over the entire range of γ; and (iii) the difference between the two is an increasing function of γ. Clearly, therefore, using the optimal amount of debt financing will have a positive effect on investment overall.

Thus the investment decision by an optimally-levered firm can be significantly difference from an unlevered firm. These differences persist for a wide range of parameter values (not reported here, for brevity). The results of this section are summarized in Result 2 below.

Result 2. *Comparison of unlevered and optimally-levered firms:*

- The optimally-levered firm's investment trigger is lower, but its investment size can be larger, smaller or the same, depending on the speed of learning; in both cases, the difference might be substantial.
- The effect of learning speed on investment trigger is the same, but the effect on investment size can be quite different.
- The optimally-levered firm's overall investment (measured by EPVI) is higher, and the difference increases with learning speed.

3.3.2. Financing

The optimal debt level b^* and corresponding leverage ratio are computed using equations (16) and (17). Figure 3 shows the optimal debt level b^* and the optimal leverage ratio as functions of learning speed γ. The optimal debt level is a decreasing (at a declining rate) function of learning speed; this is not surprising since debt level generally moves in the same direction as investment size (since smaller investment can only support a lower debt level).

Figure 3 about here

The optimal leverage ratio, on the other hand, is an increasing (at a declining rate) function of learning speed. This is because operating costs will fall with a learning curve, resulting in higher project value per dollar invested; therefore the project can support more debt per dollar of assets, as a result of which the optimal leverage ratio is higher. The effect of γ in both cases flattens out as γ is increased, because the learning effect gets smaller as γ is increased, resulting in a flattening of the optimal debt level and leverage ratio curves. This gives:

Result 3. The optimal leverage ratio is an increasing function of learning speed γ.

As mentioned at the beginning of Section 3.3.1., for $\gamma = 0.1$, the optimal debt level (from Section 2.3.2) is 1.66 and the corresponding leverage ratio is 71.63%; this results in a debt value (at issue) of \$31.71. The investment cost is $iK^* = 5(5.14) = \$25.70$. That is, when x rises to x^* the firm will optimally borrow \$31.71 from bondholders and invest \$25.70 in the project. The optimal financing policy thus requires the firm to borrow an amount exceeding the investment amount. This might make theoretical sense because of the growth in future profits as the firm moves down the learning curve. In real life, however, this is not feasible, particularly since the future growth in profits represents intangible assets; hence it is likely that there will be a constraint on the amount the firm can borrow, as discussed in Section 2.3.5. In the next section, we look at the investment and financing decisions with a borrowing constraint.

3.4. With a Borrowing Constraint

3.4.1. Base Case Results

We start with a friction parameter of q = 1 (as in Deb and Balasubramanian, 2016), i.e., a borrowing constraint of $D(x^*,0) \leq iK^*$. With this constraint and the base-case parameters, we get the following results for $\gamma = 0.1$:

$$b^* = 1.30, \text{ leverage ratio} = 59.51\%, x^* = 0.96, K^* = 5.12 \text{ and EPVI} = 28.5848.$$

Comparing the above numbers with those of the unconstrained optimization in Section 3.3.1, we note first that the financing decision is quite different, with both debt level and leverage ratio being significantly lower than the unconstrained optimal case. This is not surprising, given that there is a constraint on borrowing. The investment decision, however, is very close to the unconstrained. The computations are repeated for different learning speed and the results shown in Figure 4 (investment) and Figure 5 (financing).

Figures 4 and 5 about here

Figure 4 shows the investment outcomes (x^*, K^* and EPVI) for unlevered, optimally-levered, and constrained-levered (constraint of $D(x^*,0) \leq iK^*$) firms. For all three outcomes, the difference between optimally-levered and constrained-levered is negligible over the entire range of learning speeds. This is also true for all other parameter values examined (not reported here). Thus the borrowing constraint makes no significant difference to the investment decision.

Figure 5 shows the financing decision with the borrowing constraint, along with the financing decision of the optimally-levered firm. Part (a) shows the debt level as a function of γ, and part (b) the leverage ratio. As expected, both debt level and leverage ratio are now considerably lower than the unconstrained case. In both cases, the debt level is a decreasing function of γ. However, unlike the unconstrained case, the leverage ratio now is *initially an increasing and subsequently a decreasing* (inverted-U-shaped) function of γ. The borrowing constraint has a negative effect on leverage ratio, the magnitude of which increases with γ.

The reason for this inverted-U-shaped effect is as follows. A higher γ means a larger fraction of firm value is intangible, which will have a more negative effect on debt on borrowing ability. Thus, the speed of learning will have two opposing effects in the constrained optimization problem – a higher γ will raise the leverage ratio because it increases value, but it will also have a negative effect because it increases the asset intangibility. Thus we cannot unambiguously sign the learning-speed-leverage-ratio relationship. However, intangibility will be less important for small γ, hence the second effect will be less relevant and the overall relationship should be positive. For large γ, the intangibility aspect becomes important, hence the second effect dominates and leverage ratio is a decreasing function of learning speed. That is, the leverage ratio will be an inverted-U shaped function of γ. We now state:

Result 4. With a borrowing constraint:

- the investment decision is not significantly different from the (unconstrained) optimally-levered firm;
- the leverage ratio is an inverted-U-shaped function of the learning speed γ.

We repeated the computations with different borrowing constraints and different parameter values, to establish the robustness of the leverage ratio – learning relationship. These are discussed below.

3.4.2. Some Comparative Static Results for Constrained Borrowing

In the above computations, we had assumed $q = 1$ (as in Deb and Balasubramanian, 2016). However, creditors will generally not lend the entire amount needed to invest in the project; for instance, Shibata and Nishihara (2012) use $q = 0.2$ and 0.6 in their numerical illustrations. With a more realistic (tighter) borrowing constraint ($q < 1$), the "intangibility effect" mentioned above will be stronger while the "value effect" will be unchanged, hence the leverage ratio should start falling earlier (at a smaller value of γ). Thus the downward-sloping portion of the curve will be more prominent, because the curve will be downward-sloping over a wider range of γ.

Figure 6(a) shows the results with $q = 0.4$; it can be noted that leverage ratio starts falling much earlier (at $\gamma = 0.05$ instead of 0.13). Thus, with a tighter borrowing constraint, we are more likely to observe a negative relationship between learning speed and leverage ratio.

<u>Figure 6 about here</u>

Figure 6(b) and (c) show the results with different values of returns-to-scale parameter η and demand volatility σ (but retaining the earlier assumption of $q = 1$). With a higher η the option value (or firm value, since prior to investment the firm has only the option to invest) will be higher, hence the "value effect" mentioned above will be more dominant relative to the "intangibility effect;" thus the leverage ratio will be rising for a longer stretch as γ is increased. For higher returns-to-scale, therefore, leverage ratio starts falling later; as shown in Figure 6(b), when $\eta = 0.55$ the leverage ratio starts falling at $\gamma = 0.2$ (instead of 0.13). Also, higher volatility increases the value, so it has the same effect as a higher η (above). Thus, the leverage ratio should start falling later with higher demand volatility. This is indeed the case, as shown in Figure 6(c) where the leverage ratio starts falling at $\gamma = 0.17$ with a higher volatility of $\sigma = 15\%$. This gives:

Result 5. With a borrowing constraint, we are more likely to observe a negative relationship between learning speed (γ) and leverage ratio with a tighter constraint, smaller returns-to-scale and lower demand volatility.

There is only one empirical study, to our knowledge, on the effect of the learning curve on capital structure, Deb and Balasubramanian (2019). They find that leverage ratio is negatively related to learning intensity. However, their finding does not necessarily contradict the inverted-U-shaped relationship predicted by our model, because their study does not consider possible non-linearity in the relationship. In order to appropriately test our model's implication regarding leverage ratio, the regressions would have to include the squared term of learning speed in the list of explanatory variables.

4. Conclusion

This paper extends the SGK (2012) investment model with learning curve to the case of a levered firm. To take into account both the size and timing of investment, we use a composite measure of overall investment (the expected present value of investment or EPVI). We show that debt financing can make a significant difference to the investment decision, but the exact effect depends on the level of debt used.

With the optimal debt level, investment is always made earlier than the unlevered firm; investment size, however, can be (i) larger than the unlevered firm if the speed of learning is low enough, (ii) smaller than unlevered for higher learning speed, or (iii) roughly equal to unlevered for high learning speed.

We also identify the optimal capital structure with a learning curve. The optimal debt ratio is an increasing function of the speed of learning. However, in the presence of a borrowing constraint, leverage ratio is an inverted-U-shaped function of learning speed, the downward-sloping part of which becomes more prominent when the borrowing constraint is tighter, when the technology is more decreasing-returns-to-scale and when demand volatility is lower.

Our model has some implications for leverage ratios with a learning curve, and these can be tested empirically. Also, as in SGK (2012), our model does not allow the firm to change its output rate. The model can be extended to allow the firm to raise or lower its output rate in response to demand fluctuations. Finally, we examine lumpy investment as in SGK (2012); the model can be extended to study incremental investment, where the firm can increase capacity gradually as demand rises.

Appendix A. Derivation of Partial Differential Equation (7)

We are interested in the value of equity, or E(x, Q). The two state variables evolve as follows: $dx = \mu x dt + \sigma x dZ$ and $dQ = K^{\eta} dt$, and the cash flow to shareholders is: $(1-\tau)\left[\left(x - c_0 e^{-\gamma Q}\right)\left(K^{\eta}\right) - b\right]$ per unit time.

Let dE be the change in equity value over the next instant dt. Then, $dE = E_x\, dx + 0.5\, E_{xx}\, (dx)^2 + E_Q\, dQ$, from Ito's lemma (the other terms drop out because $(dt)^2 = 0$, $(dQ)^2 = 0$, and $(dt\, dQ) = 0$). Substituting for dx and dQ, we get: $dE = E_x\, (\mu x\, dt + \sigma x\, dz) + 0.5\, E_{xx}\, (\sigma^2 x^2\, dt) + E_Q\, K^{\eta}\, dt$. Taking expectations (and keeping in mind that $E(dz) = 0$), we get the expected value of dE:

$$E(dE) = [E_x\, \mu x + 0.5\, E_{xx}\, (\sigma^2 x^2) + E_Q\, K^{\eta}]\, dt.$$

This is the expected capital gain from holding the equity over the next instant dt. There is also a cash inflow (or dividend) of $(1-\tau)\left[\left(x - c_0 e^{-\gamma Q}\right)\left(K^{\eta}\right) - b\right]dt$, over this time period. Thus, the total instantaneous return is:

$$\frac{0.5\sigma^2 x^2 E_{xx} + \mu x E_x + \left(K^{\eta}\right)E_Q + (1-\tau)\left[\left(x - c_0 e^{-\gamma Q}\right)\left(K^{\eta}\right) - b\right]}{E}$$

From the Local Expectations Hypothesis, this is equal to the risk-free rate r. Setting the above return equal to r, we get: $0.5\sigma^2 x^2 E_{xx} + \mu x E_x + K^{\eta} E_Q + (1-\tau)[\left(K^{\eta}\right)(x - c e^{-\gamma Q}) - b] = rE$, which is equation (7) of the paper.

Appendix B. Boundary Conditions for Q → ∞.

When the cumulative output is very large, or $Q \to \infty$, the production cost becomes very small, i.e., $c = c_0 e^{-\gamma Q} \to 0$, hence the profit stream $\pi \to (1-\tau)[K^{\eta} x - b]$; that is, the problem becomes time-independent. Then the equity and debt values are given by: $E(x) = (1-\tau)\left(\frac{K^{\eta} x}{r-\mu} - \frac{b}{r}\right) + Ax^{\beta}$ and $D(x) = b/r + Bx^{\beta}$, where the constants A and B are obtained from the boundary conditions, and $\beta = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}$.

Boundary conditions:

At the bankruptcy trigger $x_{b\infty}$, there are two boundary conditions for equity value:

Value-matching: $E(x_{b\infty}) = 0$, and Smooth-pasting: $E'(x_{b\infty}) = 0$.

Also at bankruptcy trigger $x_{b\infty}$, there is one condition for debt value:

Value-matching: $D(x_{b\infty}) = (1-\omega)(1-\tau)\left(\dfrac{K^{\eta} x_{b\infty}}{r-\mu}\right)$.

The three boundary conditions are solved for the three unknowns A, B and $x_{b\infty}$, giving:

$$A = -\frac{(1-\tau)\left(K^{\eta}\right)\left(x_{b\infty}\right)^{1-\beta}}{\beta(r-\mu)}, \quad B = \left[(1-\tau)(1-\omega)\left(\frac{K^{\eta} x_{b\infty}}{r-\mu}\right) - \frac{b}{r}\right]\left(x_{b\infty}\right)^{-\beta} \text{ and } x_{b\infty} = \frac{b(1-\mu/r)}{K^{\eta}(1-1/\beta)}.$$

Appendix C. Numerical methods to solve PDE

We use finite difference method to solve the equation since it has been proven to be quite efficient and stable in dynamic corporate finance modelling (e.g. Carverhill and Anderson 2012). Specifically, we apply Crank-Nicolson discretization scheme coupled with Successive-Over-Relaxation (SOR) to ensure computational stability. The mesh grid is constructed as (x_i, t_j) with i = 1, 2….n and j = 1, 2….m, where

$$x_i = 0, \Delta x, 2\Delta x \ldots \ldots n\Delta x$$

$$t_j = 0, \Delta t, 2\Delta t \ldots \ldots m\Delta t$$

$$V_x = \frac{1}{2}\left(\frac{V_{i+1,j-1} - V_{i-1,j-1}}{2\Delta x} + \frac{V_{i+1,j} - V_{i-1,j}}{2\Delta x}\right)$$

$$V_{xx} = \frac{1}{2}\left(\frac{V_{i+1,j-1} - 2V_{i,j-1} + V_{i-1,j-1}}{(\Delta x)^2} + \frac{V_{i+1,j} - 2V_{i,j} + V_{i-1,j}}{(\Delta x)^2}\right)$$

$$V_t = \frac{V_{i,j} - V_{i,j-1}}{\Delta t} \text{ and } V = \frac{V_{i,j} + V_{i,j-1}}{2}$$

In general, the smooth pasting conditions, for bankruptcy timing, can be realized by searching the first time that the equity equal or larger than the zero (for instance eq. 9) along the discretization scheme. For reader's replication, we adopt a mesh grid of, x ∊ [0, 30] with 1200 steps, and T ∊ [0, 200] with 1000 steps, the scheme is selected to balance off cumbersome CPU time and results precision. The only challenge in the computation probably comes from the time horizon *T* because we have to reinterpret the PDE as function of calendar time *t* to solve optimal capital *K*. To compare with SGK (2012) we choose infinite length T. Another reason is that previous studies have shown finite project life T has great impact on investment decisions (Gryglewicz, Huisman, Kort 2008). Thus the question is that how large T can be considered as close as much to inifinite? To give a best estimation, we firstly solve SGK(2012) investment decision with finite T, and then we compare how large T is good? For example, the solutions to $v = \mathbb{E}\int_t^T (1-\tau)(x - \varphi K - ce^{-\gamma Q_s})K\, e^{-r(s-t)} ds$ are

$$V(T,t) = (1-\tau)\left[\frac{xK}{r-\mu}\left(1 - e^{-(r-\mu)(T-t)}\right) - \frac{\varphi K^2}{r}\left(1 - e^{-r(T-t)}\right) - \frac{ce^{-\gamma Q}K}{r+\gamma K}\left(1 - e^{-(r+\gamma K)(T-t)}\right)\right]$$

Note for the finite-horizon case, we implicitly assume the scrap value of assets at *T* is zero. We emphasize this since it will also be applied in the levered case

The investment timing and size are solved with

$$x^*(T,0) = \frac{r-\mu}{1-e^{-(r-\mu)T}} \frac{\alpha}{\alpha-1} \left[\frac{\varphi K}{r}(1-e^{-rT}) + \frac{c}{r+\gamma K}\left(1-e^{-(r+\gamma K)T}\right) + \frac{i}{1-\tau}\right]$$

And the optimal size is solved with

$$\frac{i}{(1-\tau)c} = (\alpha-2)\frac{\varphi K^*}{cr}(1-e^{-rT}) - \frac{r+\alpha\gamma K^*}{(r+\gamma K^*)^2}\left(1-e^{-(r+\gamma K)T}\right) + (\alpha-1)\frac{\gamma T}{r+\gamma K}e^{-(r+\gamma K)T}$$

And the investment option is

$$f(T,0) = \frac{1-\tau}{\alpha}\frac{x^* K}{r-\mu}\left(\frac{x}{x^*}\right)^{\alpha}\left(1-e^{-(r-\mu)T}\right)$$

Conclusion: it discloses that T = 200 has an enough approximation of infinite solution with error tolerance on the magnitude of 10^{-5}. Thus in the numerical scheme, we assume T = 200.

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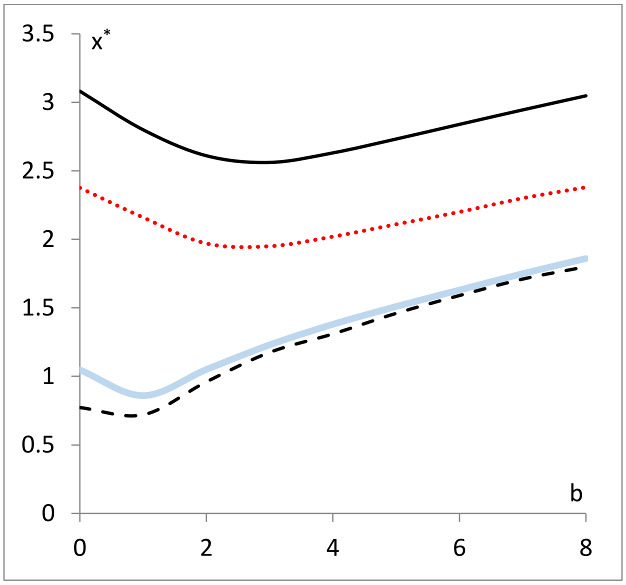
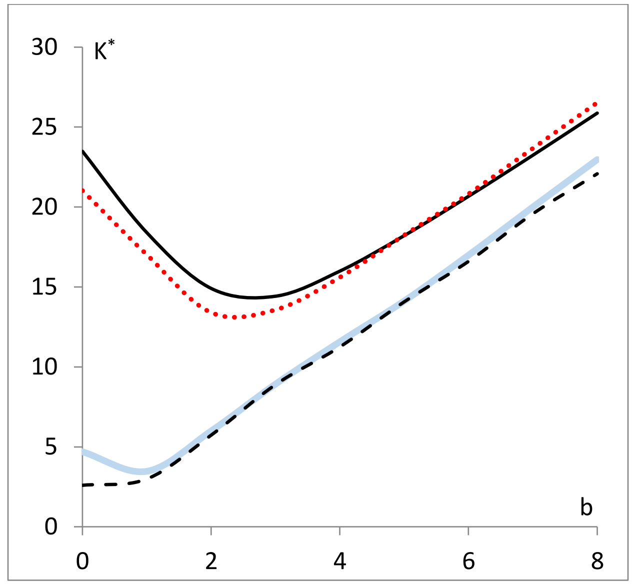


(a)



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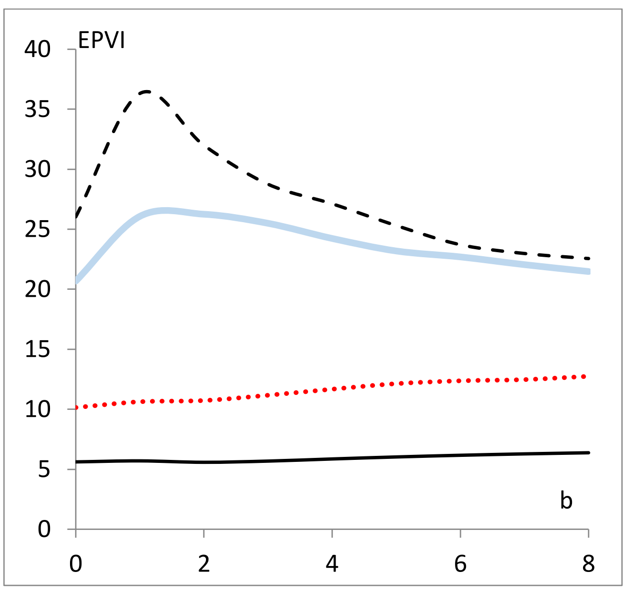

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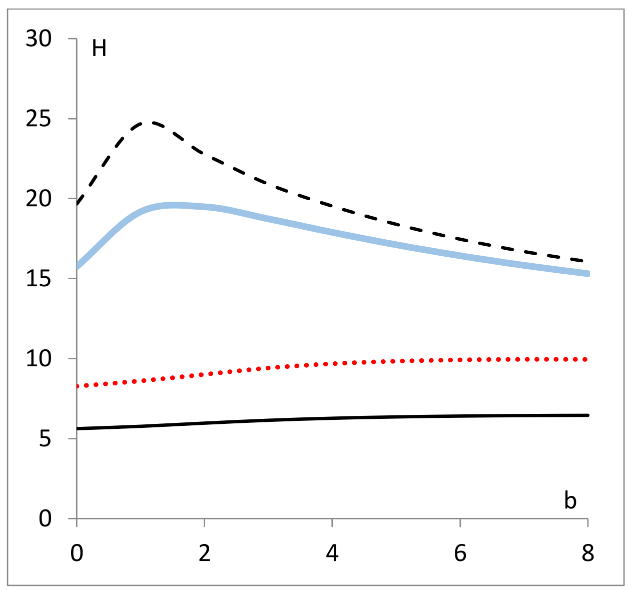
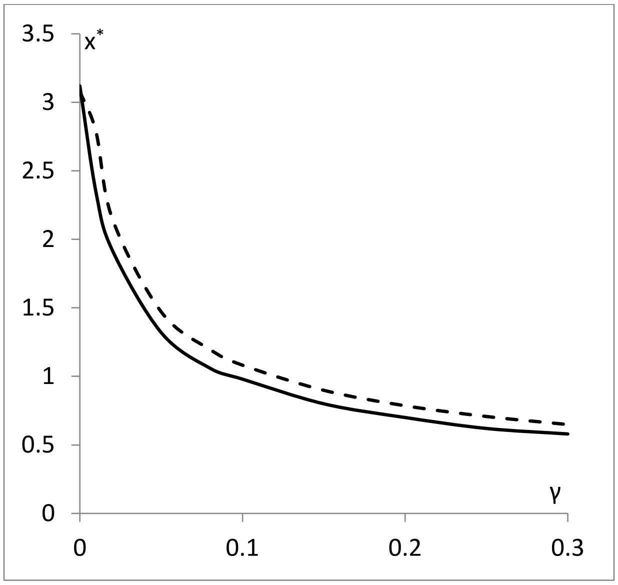
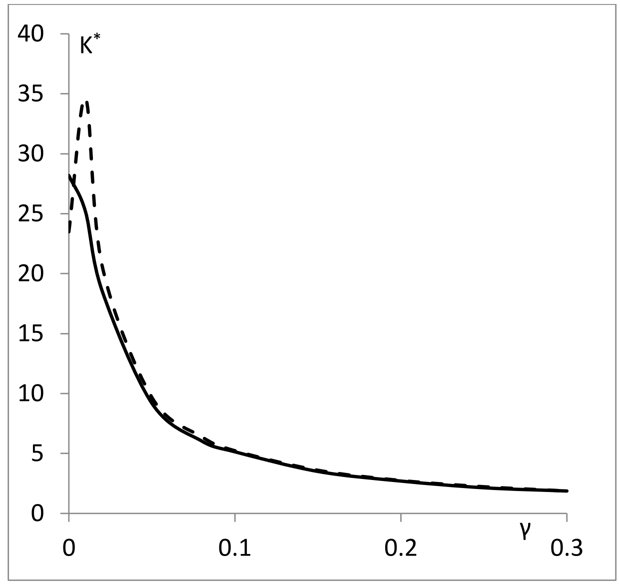


(d)

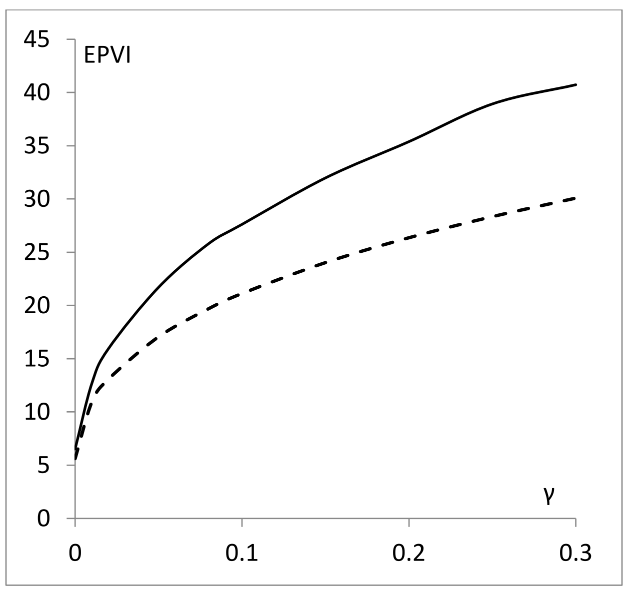
Figure 1. The effect of debt level b on optimal trigger (x^*), optimal size (K^*), overall investment (EPVI), and project value (H) for four learning speeds: $\gamma = 0$ (thin black line), 0.01 (red dotted line), 0.1 (thick blue line) and 0.2 (broken black line). The base-case parameter values are used: $\mu = 1\%$, $\sigma = 10\%$, $r = 5\%$, $\eta = 0.5$, $c_0 = 1$, $i = 5$, $\tau = 15\%$ and $\omega = 50\%$.



(a)



(b)



(c)

Figure 2. The effect of learning speed (γ) on optimal investment trigger (x^*), optimal size (K^*) and overall investment (EPVI), for unlevered firm (broken line) and optimally-levered firm (solid line). The base-case parameter values are used: $\mu = 1\%$, $\sigma = 10\%$, $r = 5\%$, $\eta = 0.5$, $c_0 = 1$, $i = 5$, $\tau = 15\%$ and $\omega = 50\%$.

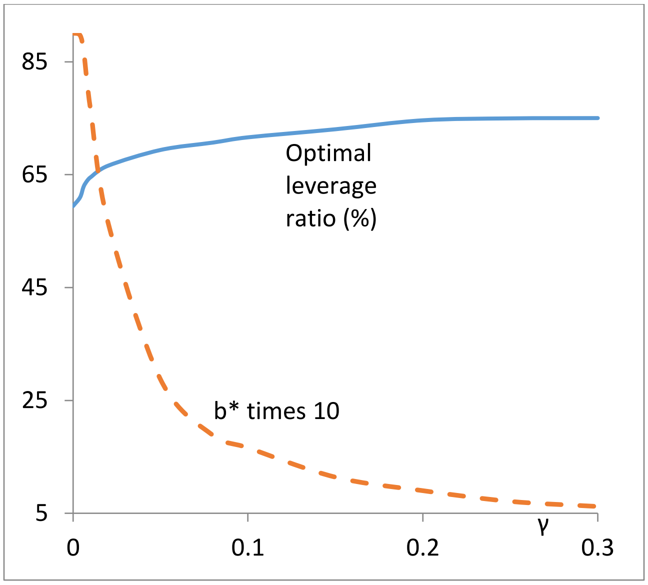


Figure 3. *Optimal financing.* Effect of learning speed (γ) on (10 times) the optimal debt level (broken line) and optimal leverage ratio (solid line). The base-case parameter values are used: $\mu = 1\%$, $\sigma = 10\%$, $r = 5\%$, $\eta = 0.5$, $c_0 = 1$, $i = 5$, $\tau = 15\%$ and $\omega = 50\%$.

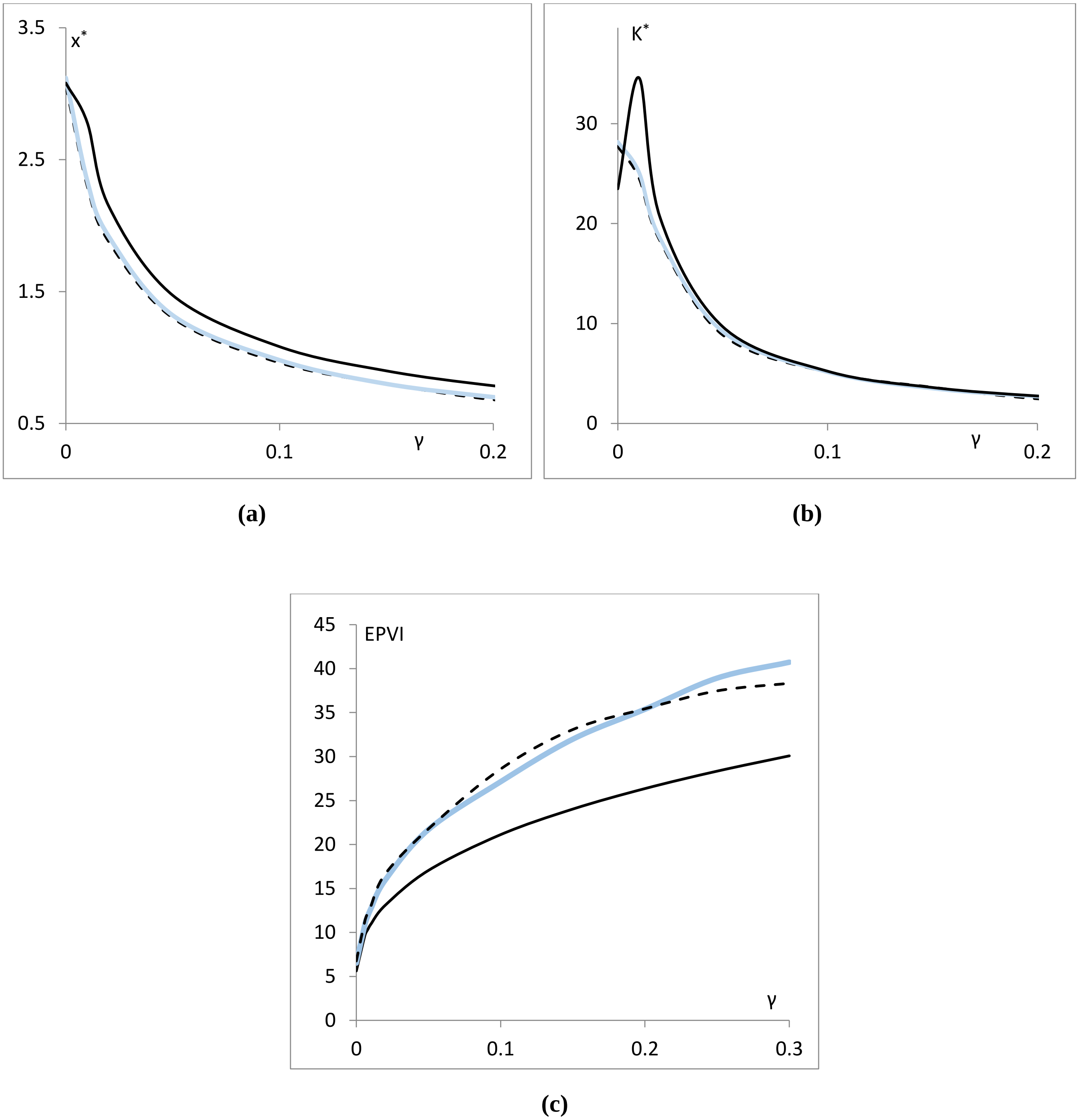
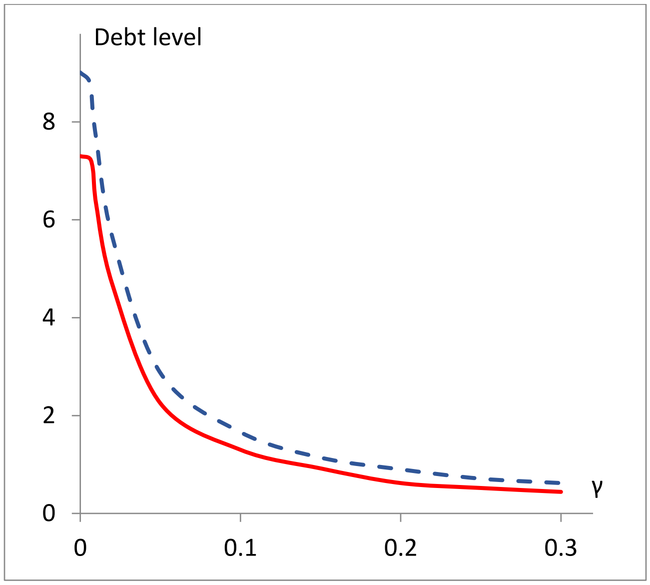
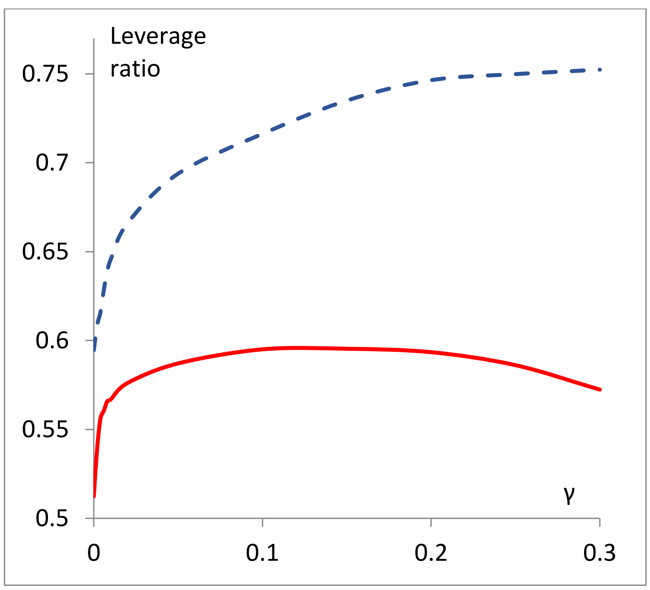


Figure 4. The effect of learning speed (γ) on optimal investment trigger (x^*), optimal investment scale (K^*) and investment overall (EPVI), for the three cases: unlevered (thin black line), optimally-levered (thick blue line) and constrained-optimal (broken black line). The base-case parameter values are used: $\mu = 1\%$, $\sigma = 10\%$, $r = 5\%$, $\eta = 0.5$, $c_0 = 1$, $i = 5$, $\tau = 15\%$ and $\omega = 50\%$, and the borrowing constraint is $D \leq iK$.

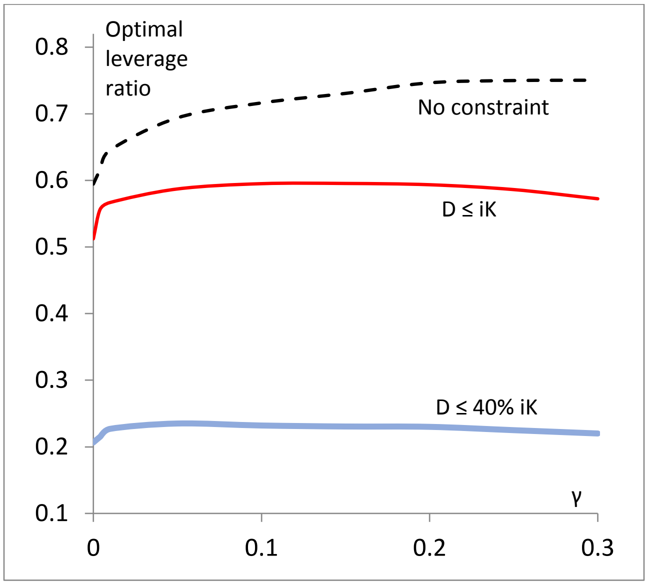


(a)

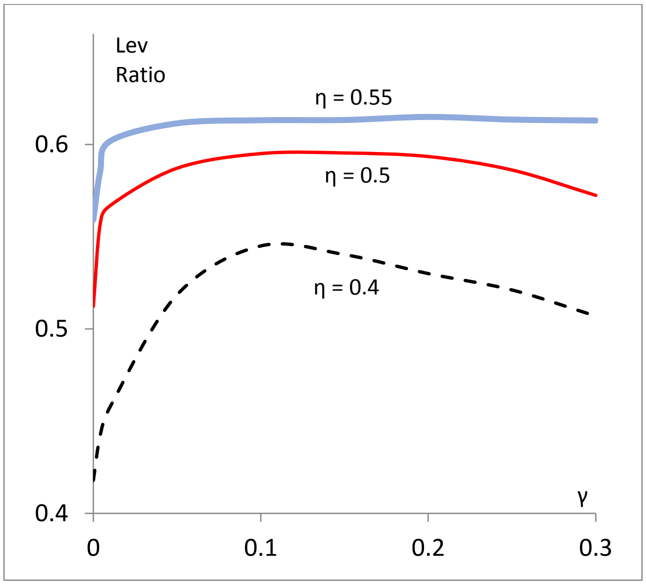


(b)

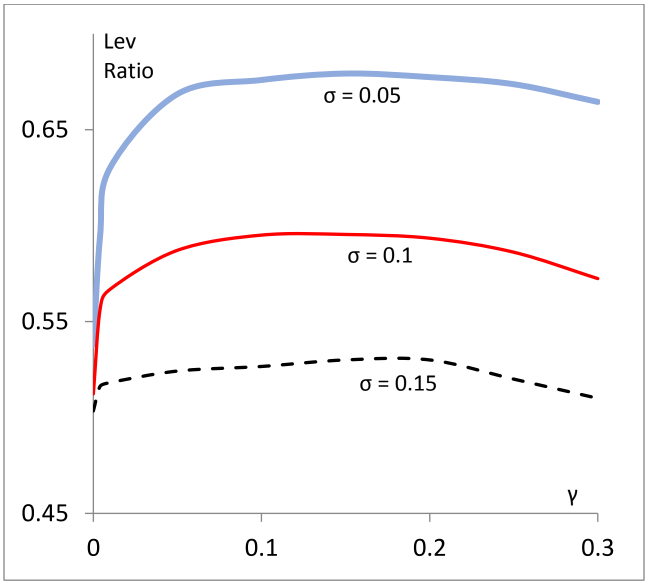
Figure 5. *Optimal financing versus borrowing-constrained financing.* Effect of learning speed (γ) on the financing decision, for an optimally-levered firm versus a firm that is borrowing-constrained ($D \leq iK$). Part (a) shows the debt level and part (b) shows the leverage ratio. In both cases, the broken blue line is the optimally-levered firm, and the solid red line the firm facing the borrowing constraint. The base-case parameter values are used: $\mu = 1\%$, $\sigma = 10\%$, $r = 5\%$, $\eta = 0.5$, $c_0 = 1$, $i = 5$, $\tau = 15\%$ and $\omega = 50\%$.



(a)



(b)



(c)

Figure 6. Part (a) shows leverage ratio for unconstrained borrowing and two cases of constrained borrowing ($D \leq iK$ and $D \leq 40\% iK$). Part (b) shows constrained leverage ratio ($D \leq iK$) for $\eta = 0.4$, 0.5 and 0.55. Part (c) shows constrained leverage ratio ($D \leq iK$) for $\sigma = 5\%$, 10 and and 15%. The base-case parameter values are used: $\mu = 1\%$, $\sigma = 10\%$, $r = 5\%$, $\eta = 0.5$, $c_0 = 1$, $i = 5$, $\tau = 15\%$ and $\omega = 50\%$.