Preemption with an endogenously sunk cost

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Abstract

A firm whose capital is relatively divisible invests incrementally to maintain a rival on the brink of entry and enters the product market at a time which is both a myopic optimum and a preemptive equilibrium. This conduct is both the optimal policy of a firm that regulates its rival's entry threat through capital accumulation and the limit of equilibrium outcomes of asymmetric preemption as investment steps become arbitrarily small. If the price of capital follows a stochastic process option value overrides the strategic investment motive so greater volatility delays the firm's jump to completion. An emblematic historical example of strategic commitment is discussed in light of this analysis.

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1 Introduction

Consider for a moment the following scenario. Two entrepreneurs plan to build competing rail lines between a pair of distant cities. These projects are both economically sound as the demand for travel between these destinations is sufficient to sustain a duopoly. Currently the cost of the necessary capital goods –the cost of building either of the proposed rail links– is prohibitively high so it is not viable to build even one. But this fixed cost is expected to decrease steadily over time so that building a first line and then the second will eventually become profitable. The entrepreneurs are identical for all intents and purposes, their routes are of equal length and run through comparable terrain, but their projects differ in one critical respect. Of the two ways to connect the cities, the Northern route that one entrepreneur plans to follow runs through an expanse of uninhabited land, whereas the Southern route selected by the other runs through a more populated region. The farmers whose fields lie along this other route are too few to have an impact on the railroad either as workers or as passengers, but keep a keen eye on their surroundings and gossip enough among themselves that should a section of track be built in their backyards, word of this will quickly travel back to both cities.

Under such conditions one expects economics to explain which entrepreneur likely succeeds in completing a railroad first, and to a large extent it does. The asymmetry in information sets –the only public information available up until the cities have been connected by one railroad or another is the advancement of the line that runs through the populated region– favors the Southern firm by allowing it to make a strategic commitment that the Northern firm cannot. What is less clear, if one takes a dynamic perspective, is the pattern this commitment should follow over time as industry conditions evolve. Here the Southern entrepreneur does not shift product market reaction functions as described in textbook treatments of strategic commitment, but instead uses his ability to make strategic incremental investments to regulate the Northern entrepreneur's entry decision as the price of the capital good drops. The more interesting economic question is not then whether the Southern firm gains an advantage, but rather how it manages exactly to use the greater observability along its route to secure a positional rent. Even if much has been written about both strategic commitment and investment in continuous time, this latter question, that of identifying characteristic properties of a leader's strategic investment over time, has not been as extensively addressed. This question is the main subject of this article. Moreover, understanding firm conduct in this setting is of broader relevance than a fictional railroad scenario, as similar conditions arise under imperfect competition whenever one firm has the capability of dividing investment into finer pieces, or stages, than its rivals, an occurrence which we argue further below is not so rare in practice.[1] We establish that the conduct of leadership under such conditions involves a phase of incremental investment during which the leading firm ratchets up its threat to keep on discouraging preemption up until a specific time is reached at which its level of capital jumps to completion so as to capture a positional rent.

This pattern of capital accumulation can be understood intuitively by means of Figure 1. Time is on the horizontal axis whereas the vertical axis represents levels of capital in the unit interval for one of the firms in a duopoly, hereafter referred to as firm 1, which correspond to possible fractions of the total route that might be built by the Southern firm. Two loci which result from standard arguments involving optimal timing and preemption are drawn in grey. The curve $T^L(\kappa)$ shows optimal monopoly investment times. In a well-behaved model of investment where a unit of capital is required in order for a firm to

[1] Aside from railroads and capacity investment which is discussed further below in the text, this analysis also applies to R&D investments where one firm's research is public whereas the other's is shrouded in secrecy.

start earning a constant profit flow, there is a negative relationship between a firm's existing capital and its optimal investment time. This optimal time results from a trade-off between the costs and benefits of waiting, which is in turn sensitive to the firm's existing capital stock κ. The greater is κ the smaller the investment $1-\kappa$ necessary for product market entry, and therefore the lower is the marginal benefit of delay, leading the firm to invest earlier. The other curve $k^P(t)$ is best understood by adopting the perspective of firm 2 (the Northern firm), which itself holds no capital and observes that its rival has accumulated a stock of magnitude κ. Firm 2 can compare the profitability of investing before firm 1, so as to enjoy a phase of monopoly profit up until a duopoly phase begins once firm 1 joins it in the market, with the profitability of refraining from early investment and entering after firm 1 as a duopolist. The greater is κ however, the earlier is firm 1's optimal duopoly investment time for the same reason as in the monopoly situation above, namely that its marginal benefit from waiting is relatively lower. All else equal therefore, the greater is κ, the shorter the span of time that firm 2 can expect a monopoly profit flow if it enters first, and therefore the weaker its incentive to enter first rather than second. A generally positive relationship should therefore be expected to hold between the levels of capital held by firm 1 and the times at which firm 2 is indifferent between preempting its rival and investing as a follower. In addition to $T^L(\kappa)$ and $k^P(t)$, Figure 1 also depicts the preemption time T^P before which entry is undesirable for any firm, the optimal duopoly entry time T^F at which firm 2 invests regardless of firm 1's capital stock and the capital stock $\overline{\kappa}$ above which no entry time earlier than T^F is desirable for firm 2.

With these elements in mind the optimal path of capital accumulation of firm 1, given its singular ability to invest incrementally, can be described. If the cost of the capital input is relatively high, it should refrain from any investment. As

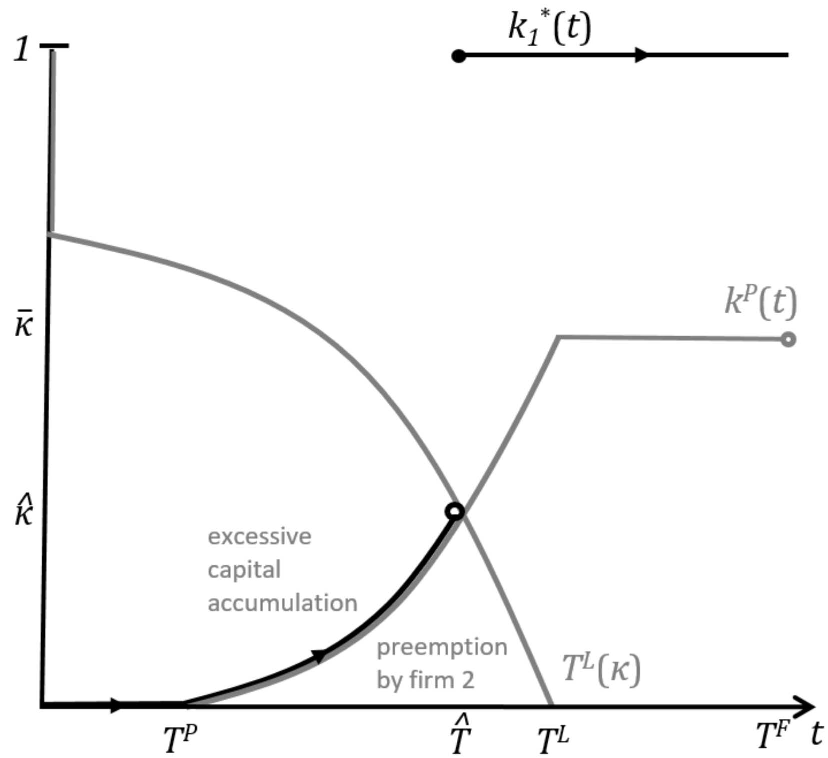

Figure 1: $T^L(\kappa)$ relates optimal monopoly investment times to firm 1's existing capital stock while $k^P(t)$ is the preemptive capital accumulation policy that keeps firm 2 indifferent between investing immediately and waiting until T^F. In black, the policy $k_1^*(t)$ has firm 1 accumulating capital incrementally along the schedule $k^P(t)$ from T^P onward up until $\widehat{T}$ when it jumps to completion whereas firm 2 invests as a follower at T^F.

soon as the price of capital drops sufficiently for market entry to become desirable for its rival (time T^P), firm 1 should start investing. Accumulating capital above the schedule $k^P(t)$ is excessively costly, whereas accumulating below this schedule would allow the rival to enter the market preemptively. Capital accumulation should therefore follow the path indicated by $k_1^*(t)$ which runs along $k^P(t)$ from T^P onward. Naturally firm 1 must eventually complete its investment and enter the product market. A key insight is that this decision must be instantaneously optimal given its current capital stock. At time $\widehat{T}$ therefore, the remaining mass of capital required to operate in the product market is acquired all at once. This time and the associated capital stock $\widehat{\kappa} = k_1^*\left(\widehat{T}\right)$ are characterized by being the only point at which the firm both efficiently maintains its rival indifferent between entering or not and finds it instantaneously optimal to complete its own investment. The industry then functions as a monopoly up until firm 2 enters (time T^F) and as a duopoly thereafter. A noteworthy feature of these industry dynamics is that the entry time $\widehat{T}$ identified here emerges independently in asymmetric games of preemption, in which the capital stock $\widehat{\kappa}$ represents an exogenous degree of fixed cost asymmetry delimiting preemptive and monopolistic investment outcomes.

To cast these ideas formally, this article models investment in continuous time by two firms competing asymmetrically for leadership in a market. Demand and variable cost are stationary and allow both firms to operate profitably. A unit of the capital good is required to operate. Its price is initially high enough for immediate investment to be unprofitable but decreases at a constant rate. Investment is instantaneous and irreversible, capital does not deteriorate and both firms have the same constant discount rate. The economics of the scenario described in the opening paragraph are captured by assuming only firm 1 can invest incrementally. Accordingly its feasible capital accumulation paths are nondecreasing and right-continuous functions from the non-negative reals into a non-trivial subset of the unit interval. Firm 2's investment is binary so its capital accumulation path consists of a single step of unit height.[2]

The strategic investment problem firms face in this environment is approached in two ways. First, to get an intuitive understanding of the dynamics of leader investment, a single-firm decision problem is examined where firm 1 chooses its capital accumulation policy under the constraint that it maintains enough capital for preemptive investment to be unprofitable for its rival. If firm 1's capital is infinitely divisible then its optimal policy is represented by the schedule $k_1^*(t)$ in Figure 1. Second, a non-cooperative foundation for this pattern of investment is obtained by studying an asymmetric preemption race in which firm 1 accumulates its unit of capital in finite increments. From a formal standpoint this game extends existing analyses of preemption to allow for multiple investment rounds. The main result establishes that there is a unique equilibrium outcome, in which firm 1 follows a capital accumulation policy involving incremental investments up until a pivotal stage is reached at which it completes its investment and enters. For arbitrarily small investment increments, the equilibrium outcome of multistage preemption approaches $k_1^*(t)$. The last step in the analysis consists in including a noise term in the input price process so as to model the option value of waiting in an uncertain environment. The delay in initial product market entry is then shown to increase with input price volatility, suggesting that option value remains the driving motive of investment even in the presence of competition.

The model rests on several simplifying assumptions. A first assumption is that product market outputs are fixed. The issue of product market shares generally associated with leadership and sequential capacity choice is thus set aside

[2] Firm 2 might actually have the technical ability to invest incrementally but if firm 1 cannot observe firm 2's capital accumulation prior to product market entry such investment has no strategic effect and its capital may as well be assumed to be binary.

to allow the analysis to focus on dynamic competition. A second assumption is that asymmetry is imposed on the feasible capital accumulation strategies of the two firms. As the objective here is to study the dynamic conduct of a leader in a Stackelberg-like framework, it seems natural to postulate a strategic asymmetry analogous to what is assumed in two-period models of capacity choice, even if the grounds for this assumption can reasonably be questioned in other contexts. Moreover a number of real-world circumstances can lead to such asymmetry and warrant mention. First, there are often technological differences between firms at a given point in time. Through chance or history, one firm may develop a singular ability to make a modular use of an input or build it progressively in-house. In the late 1990s for instance, the internet firm Google opted to accumulate computing capacity by linking together commodity personal computers rather than by procuring a high-end system from an external supplier, allowing the clusters used by its search engine to be scaled incrementally with relative ease (Barroso, Dean and Hölzle [2]). A firm may also be better able to stage investment than a competitor because its relationship with a supplier allows it to make nonrefundable deposit payments, because financial asymmetries allow it more frequent access to input markets, or because it otherwise manages more successfully than its rival to render its capital budgeting fractional, irreversible and public.[3]

Broadly viewed, the analysis in this article contributes to the understanding of the strategic role of investment in the presence of competition. This role is generally formalized by means of the Stackelberg-Spence-Dixit model of capacity choice, and constitutes a fundamental topic in the study of imperfect compe-

[3] Dixit and Pindyck observe ([6], p. 320) "Even investments that appear to involve only a single decision can turn out to be sequential ... each dollar spent gives the firm an option – which it may or may not exercise– to go ahead and spend the next dollar." Smit and Trigeorgis similarly assert that strategic investment is best viewed in practice as a compound option, with projects constituting "links in a chain of interrelated investment decisions, the earlier of which set the path for the ones to follow." ([15], p. xxvii)

tition (Tirole [18]). Accordingly, the central ideas regarding the commitment value of capacity investment are typically derived in the literature in two-stage games even if investment decisions are known to involve richer dynamics.[4]

In order to understand how a firm's capital accumulation is conducted strategically in continuous time, it is natural area to turn first to the dynamic capacity accumulation game literature, such as Jun and Vives [10]. In such models however, firms invest in response to both short and long-run incentives which are difficult to disentangle, whereas a distinctive feature of the introductory railroad scenario is that it clearly separates capital accumulation and dynamic competition on the one hand from product market competition on the other. Mills [12] studies a closely related dynamic capacity accumulation game, showing how the realization of first-mover advantage necessitates strategic threats that are costly precisely because they accelerate investment. Otherwise the analysis conducted here is closer to game-theoretic models of technology adoption. Technology adoption resembles capacity choice, but typically studies the timing of discrete investment decisions. In the extreme case of infinite information lags, a characteristic pattern known as diffusion equilibrium arises (Reinganum [14]). In such an equilibrium the first firm invests at the optimal monopoly time and earns a positional rent until the second firm invests at the optimal duopoly time, after which the industry operates as a duopoly. Assuming the order of firm investments to be predetermined and noting the analogy between investment timing and output choices, the diffusion equilibrium provides a rough dynamic representation of Stackelberg behavior, in which the positional rent the first firm collects during its phase of monopoly profit substitutes for the first-mover advantage a quantity leader obtains due to its greater static market share.

Continuing further within the technology adoption literature, Katz and Shapiro

[4] Dixit is explicit as to the simplifications involved: "It is as if the two players could see through the whole problem and implement the solution immediately." ([4], p. 96)

[11] and Huisman [9] study preemption between asymmetric firms in a deterministic and in a stochastic setting respectively. Although it differs from the introductory example of this article, this interaction is closely related and it is useful to describe it in some detail. Consider the railroad scenario again, but assume instead that both routes run through unpopulated land but that it is common knowledge that firm 1's route is comparatively shorter, perhaps because it had previously built a stretch of track along its route. This means that the investment decisions of both firms are binary, with a lower fixed cost of entry for firm 1. If fixed costs are sufficiently asymmetric, firm 1's choice of investment timing is unconstrained by any rational action of firm 2, and the resulting sequential equilibrium involves the same investment outcomes as the diffusion equilibrium described in the last paragraph. If fixed costs are not too different however, firms may race to enter. In the resulting preemptive equilibrium, the lower cost firm still invests first and obtains a positional rent, but competition affects the timing of its entry. Moreover there exists a critical degree of asymmetry separating equilibria in which a leader firm invests unconstrained at a monopoly threshold from equilibria in which its threshold is preemptive. This critical asymmetry corresponds to a level of capital that emerges naturally in the analysis of this article, as the degree of asymmetry $\widehat{\kappa}$ that the firm with divisible capital chooses to build up to just before completing its investment.

The analysis of noncooperative choice of capital accumulation policies conducted here rests squarely on existing work on preemption games, whose theory is developed by Fudenberg and Tirole [8] in the deterministic case and has been extended to the stochastic case, as well as along many other dimensions (see Azevedo and Paxson [1] for a survey). A recent article by Steg [16] gives a comprehensive account of preemption allowing for asymmetric firms. A slight difference between the present model and the existing literature is that the preemption game is formalized here using a reduced form approach to simultaneous investments. This alternative approach, which assumes efficient rationing, does not alter outcomes on the equilibrium path. It is therefore secondary to the main analysis, but simplifies the exposition while providing economically intuitive payoffs involving differential rents instead of full rent dissipation.

Section 2 presents the main economic assumptions as well as standard terminology and results involving asymmetric preemption. Section 3 discusses the dynamics of leader investment intuitively by studying the optimal capital accumulation policy of a firm that can invest incrementally under the constraint that it regulates its rival's preemption incentive. Section 4 provides a non-cooperative foundation for these investment dynamics by deriving the equilibrium of an investment game in which one firm divides its investment into a finite number of increments. Section 5 incorporates uncertainty by means of a stochastic input price process. Section 6 concludes by discussing an emblematic historical example of strategic commitment in light of the article's analysis.

2 The model

The economic environment in which two firms engage in dynamic competition is described in this section. The main assumptions are presented in Section 2.1, the continuation payoffs that firms obtain when the first firm entry occurs in Section 2.2, and Section 2.3 discusses a standard asymmetric preemption game which is referred to throughout the rest of the article.

2.1 Assumptions

Two firms compete to enter a market which can ultimately accommodate both but is initially profitable for neither. Demand and variable cost are stationary, so active firms earn constant profit flows π^M or π^D depending on whether the product market operates as a monopoly or a duopoly, with $\pi^M \geq 2\pi^D > 0$. Both firms have the same positive and constant discount rate r so capitalized monopoly and duopoly profits are $\Pi^M = \pi^M/r$ and $\Pi^D = \pi^D/r$. A firm must own an entire unit of capital to be active in the product market. The price of a unit of capital $X(t)$ decreases at a constant rate λ so $X(t) = X(0)e^{-\lambda t}$. Investment is instantaneous and irreversible, there are no adjustment costs, and capital does not depreciate. Assume $X(0) \geq \Pi^M$ so firms initially prefer to wait rather than to enter.

Feasible capital stocks are assumed to differ for the two firms. Firm 1 has the ability to accumulate capital incrementally, a costly choice as it raises average investment cost while bringing no immediate product market benefit insofar as profit flows do not begin until the firm holds an entire unit, but which has strategic value nevertheless. Firm 2 on the other hand does not have this ability. The capital accumulation policies that the firms choose are accordingly nondecreasing and right-continuous functions $k_1(t) : R_+ \to \Omega$ where $\{0,1\} \subset \Omega \subseteq [0,1]$ and $k_2(t) : R_+ \to \{0,1\}$. A policy $k_i(t)$ determines firm i's investment over time in the absence of rival entry and defines a planned completion time $T_i = \inf\{t \in R_+ | k_i(t) = 1\}$ or $T_i = \infty$ if $k_i(t) < 1$ for all t. If product market entry occurs any firm that has not entered updates its capital accumulation policy in a continuation phase which is a single-firm decision problem. A firm which enters the product market first is said to be the *leader* and a firm which enters second is said to be the *follower*.

2.2 Payoffs

The continuation payoffs that firm i, $i \in \{1,2\}$, obtains as a leader or as a follower once first entry occurs at time t are denoted $L_i\left(t;\overline{K}_1\right)$ and $F_i\left(t;\overline{K}_1\right)$, and the payoff obtained if both firms enter simultaneously is denoted $M_i\left(t;\overline{K}_1\right)$.

These payoffs are measured in initial currency units and defined for a given level of firm 1's capital stock, denoted $\overline{K}_1$. They represent the discounted profit streams that forward-looking firms expect when their roles as first or second entrant have been determined.

We start with the follower payoffs $F_i\left(t;\overline{K}_1\right)$. These payoffs are obtained by studying the decision problem faced by any remaining firm that updates its capital accumulation policy once its rival has entered at a given time t.

Consider firm 2 first, whose problem is simpler as its capital stock before entry is invariably zero. It chooses a time $t_2 \geq t$ to invest so as to maximize the value of entering as a duopolist,

$$\int_{t_2}^{\infty} \pi^D e^{-rs} ds - X(t)e^{-rt_2} = \left(\Pi^D - X(0)e^{-\lambda t_2}\right) e^{-rt_2}$$

in initial currency units. This function is strictly quasiconcave in t_2 and letting

$$T^F = \frac{1}{\lambda} \ln \left(\frac{\lambda + r}{r} \frac{X(0)}{\Pi^D} \right)$$

denotes its unconstrained maximizer over R_+, firm 2's optimal policy as a follower is to invest at $t_2^* = \max\left\{t, T^F\right\}$. The continuation payoff, $F_2\left(t;\overline{K}_1\right) = \left(\Pi^D - X(0)e^{-\lambda t_2^*}\right) e^{-rt_2^*}$, is independent of $\overline{K}_1$. Because this function is a standard payoff in related models involving symmetric firms, both $\overline{K}_1$ and the firm subscript are dropped hereafter yielding

$$F\left(t\right) = \begin{cases} \frac{\lambda r^{\frac{r}{\lambda}}}{(\lambda+r)^{\frac{\lambda+r}{\lambda}}} \frac{\left[\Pi^D\right]^{\frac{\lambda+r}{\lambda}}}{[X(0)]^{\frac{r}{\lambda}}}, & t < T^F \\ \left(\Pi^D - X(0)e^{-\lambda t}\right) e^{-rt}, & t \geq T^F. \end{cases}$$

Consider firm 1 next, which faces a more involved problem as its capital stock may be positive when its rival enters, so that it holds $\overline{K}_1 \in [0, 1)$ at the onset of the continuation phase. From the moment firm 2 invests onward, any further incremental investment short of the level required for product market entry only raises firm 1's average investment cost without providing any strategic benefit. Since further incremental accumulation is wasteful, firm 1's optimal policy as a follower is therefore to decide at what time $t_1 \geq t$ to acquire the remaining amount of capital it needs to enter. Given $\overline{K}_1$, firm 1 thus maximizes

$$\int_{t_1}^{\infty} \pi^D e^{-rs} ds - \left(1 - \overline{K}_1\right) X(t) e^{-rt_1} = \left(\Pi^D - \left(1 - \overline{K}_1\right) X(0) e^{-\lambda t_1}\right) e^{-rt_1}.$$

Over R_+ this is a strictly quasiconcave function whose unconstrained maximum is attained at

$$T_1^F\left(\overline{K}_1\right) = \begin{cases} \frac{1}{\lambda} \ln\left(\frac{\lambda+r}{r} \frac{\left(1-\overline{K}_1\right) X(0)}{\Pi^D}\right), & \overline{K}_1 \leq 1 - \frac{r}{\lambda+r} \frac{\Pi^D}{X(0)} \\ 0, & \overline{K}_1 > 1 - \frac{r}{\lambda+r} \frac{\Pi^D}{X(0)}. \end{cases}$$

As a follower firm 1 therefore invests at $t_1^* = \max\left\{t, T_1^F(\overline{K}_1)\right\}$. Its continuation payoff is accordingly

$$F_1\left(t; \overline{K}_1\right) = \begin{cases} \frac{\lambda r^{\frac{r}{\lambda}}}{(\lambda+r)^{\frac{\lambda+r}{\lambda}}} \frac{\left[\Pi^D\right]^{\frac{\lambda+r}{\lambda}}}{\left[\left(1-\overline{K}_1\right) X(0)\right]^{\frac{r}{\lambda}}}, & t < T_1^F\left(\overline{K}_1\right) \\ \left(\Pi^D - \left(1 - \overline{K}_1\right) X(0) e^{-\lambda t}\right) e^{-rt}, & t \geq T_1^F\left(\overline{K}_1\right). \end{cases}$$

Observe that the expression of $F_1\left(t; \overline{K}_1\right)$ does not include any investment cost incurred by firm 1 before t, which is sunk when firm roles are determined. This is also the case for the functions $L_1\left(t; \overline{K}_1\right)$ and $M_1\left(t; \overline{K}_1\right)$ defined further below.

Once the follower investment thresholds $T_1^F(\overline{K}_1)$ and T^F have been identified, the leader payoffs $L_i\left(t; \overline{K}_1\right)$, $i \in \{1, 2\}$, can be defined. Again consider firm 2 first. Its continuation payoff from leading at time t is

$$\begin{aligned} L_2\left(t;\overline{K}_1\right) &= \int_t^{T_1^F(\overline{K}_1)} \pi^M e^{-rs} ds + \int_{T_1^F(\overline{K}_1)}^{\infty} \pi^D e^{-rs} ds - X(t) e^{-rt} \\ &= \left(\Pi^M - X(0) e^{-\lambda t}\right) e^{-rt} - \left(\Pi^M - \Pi^D\right) e^{-r \max\left\{t, T_1^F(\overline{K}_1)\right\}} \end{aligned}$$

in initial currency units. The second line expresses the continuation payoff as the sum of two terms, the first corresponding to the net present value of a perpetual monopoly and the second correcting for the reduction in flow profit generated by firm 1's anticipated entry, which occurs at the follower investment threshold $T_1^F(\overline{K}_1)$ obtained above.

Observe that $L_2\left(t;\overline{K}_1\right)$ depends on firm 1's capital accumulation through firm 1's follower investment time, with greater capital accumulation decreasing $T_1^F(\overline{K}_1)$ and thus lowering $L_2\left(t;\overline{K}_1\right)$ over $\left[0, T_1^F(\overline{K}_1)\right]$. This effect of firm 1's prior incremental investment on firm 2's leader payoff plays a central role in the analysis of this article and has an intuitive interpretation. As firm 1 accumulates capital, it decreases the magnitude of the last step required for product market entry. As a result its threat were it to have the follower role becomes more aggressive, since it would find it optimal to enter relatively earlier in a duopoly. This earlier threatened entry in turn reduces the duration of the monopoly phase that firm 2 expects to enjoy by entering first, and thus its payoff from leading.

Consider firm 1 next. If it enters the product market at time t while holding an accumulated capital stock $\overline{K}_1$, the magnitude of its remaining investment step is $1-\overline{K}_1$. Its continuation payoff from leading is therefore

$$\begin{aligned} L_1\left(t;\overline{K}_1\right) &= \int_t^{T^F} \pi^M e^{-rs} ds + \int_{T^F}^{\infty} \pi^D e^{-rs} ds - \left(1-\overline{K}_1\right) X(t) e^{-rt} \\ &= \left(\Pi^M - \left(1-\overline{K}_1\right) X(0) e^{-\lambda t}\right) e^{-rt} - \left(\Pi^M - \Pi^D\right) e^{-r \max\left\{t, T^F\right\}}. \end{aligned}$$

The difference $L_i\left(t;\overline{K}_1\right) - F_i\left(t;\overline{K}_1\right)$ measures firm i's incentive to lead at a given time t rather than to follow. Let $\Theta_i\left(\overline{K}_1\right)=\left\{t\in\left[0,T^F\right]\middle|\, L_i\left(t;\overline{K}_1\right)-F_i\left(t;\overline{K}_1\right)\geq 0\right\}$, $i\in\{1,2\}$, denote the times up until T^F at which leading is individually rational for firm i for a given level of firm 1's capital, $\overline{K}_1$. An essential property of leader and follower payoffs is that firm 1's incentive is at least as large as firm 2's, that is $L_1\left(t;\overline{K}_1\right)-F_1\left(t;\overline{K}_1\right)\geq L_2\left(t;\overline{K}_1\right)-F\left(t\right)$ so that $\Theta_2\left(\overline{K}_1\right)\subseteq\Theta_1\left(\overline{K}_1\right)$.[5]

Finally, the continuation payoffs resulting from simultaneous investments are

$$\begin{aligned} M_1\left(t;\overline{K}_1\right) &= \int_t^\infty \pi^D e^{-rs}ds-\left(1-\overline{K}_1\right)X(t)e^{-rt} \\ &= \left(\Pi^D-\left(1-\overline{K}_1\right)X(0)e^{-\lambda t}\right)e^{-rt} \end{aligned}$$

and

$$\begin{aligned} M\left(t\right) &= \int_t^\infty \pi^D e^{-rs}ds-X(t)e^{-rt} \\ &= \left(\Pi^D-X(0)e^{-\lambda t}\right)e^{-rt} \end{aligned}$$

where the subscript is again omitted for firm 2 because this is a standard payoff in related models.

2.3 Preemption with fixed cost asymmetry

This subsection describes the asymmetric preemption game referred to throughout the rest of the article. Suppose here that firm 1's investment is binary, so its accumulation policy takes the form $\widehat{k}_1\left(t\right):R_+\to\{0,1-\kappa\}$ where $\kappa\in(0,1)$ denotes a positive initial capital stock. The degree of asymmetry between the firms is parameterized by κ, firm 1 having the comparatively lower entry cost

[5] To see intuitively why this holds observe that as $T^F\geq T_1^F(\overline{K}_1)$, the duopoly phase starts later if firm 1 leads and firm 2 follows than if roles are reversed. As duopoly lowers the industry's profit flow, industry profit is therefore at least as large if firm 1 leads and firm 2 follows. Formally,

$$L_1\left(t;\overline{K}_1\right)+F\left(t\right)\geq L_2\left(t;\overline{K}_1\right)+F_1\left(t;\overline{K}_1\right)$$

and rearranging yields the desired result.

$1-\kappa$. Firms simultaneously and non-cooperatively choose contingent capital accumulation policies $\widehat{k}_1(t)$ and $k_2(t)$ (or extended distributions thereof, see Fudenberg and Tirole [8], Steg [16]) that they update if rival entry occurs. Because the feasible capital stocks of both firms are binary, their policies $\widehat{k}_1(t)$ and $k_2(t)$ are identified by the planned completion times T_1 and T_2.

As a benchmark consider the situation of symmetric firms ($\kappa = 0$). Then the indifference conditions of each firm, $L_1(t;0) - F_1\left(T_1^F(0);0\right) = 0$ and $L_2(t;0) - F\left(T^F\right) = 0$, are identical and have a unique solution in $\left(0, T^F\right)$ which is referred to as the preemption time and denoted T^P. The preemption time is the moment at which the investment incentive of both firms becomes positive, and at which positional rents from any subsequent monopoly phase are fully dissipated. The interval $\left(T^P, T^F\right)$ =int($\Theta_i(\kappa)$), $i = 1, 2$, over which the firms race to enter just ahead of one another in an interaction that is similar to the undercutting that occurs in the Bertrand model of price competition is referred to as the preemption range. If capital accumulation policies are chosen non-cooperatively, symmetric equilibrium strategies call for both firms to invest at T^P and the resulting outcome has either firm entering with equal probability whereas its rival follows at T^F.

To describe the equilibrium outcomes of the asymmetric preemption game ($\kappa > 0$), it is useful to first define the myopically optimal time for monopoly investment. Suppose that roles were predetermined with firm 1 assured of leading, so that it simply chose an entry time t to maximize the payoff $L_1(t;\kappa)$ knowing that firm 2 would subsequently enter at t_2^*. Firm 1's optimal completion time in this case would be

$$T^L(\kappa) = \begin{cases} \frac{1}{\lambda}\ln\left(\frac{\lambda+r}{r}\frac{X(0)(1-\kappa)}{\Pi^M}\right), \, \kappa \leq 1 - \frac{r}{\lambda+r}\frac{\Pi^M}{X(0)} \\ 0, \, \kappa > 1 - \frac{r}{\lambda+r}\frac{\Pi^M}{X(0)}. \end{cases}$$

Observe that $dT^L/d\kappa < 0$ for $\kappa < 1 - \left(r\Pi^M / \left(\left(\lambda + r\right) X(0)\right)\right)$ and let $T^L = T^L\left(0\right)$. In addition to this monopoly timing, the characterization of equilibrium involves a critical level of cost asymmetry $\overline{\kappa}$ beyond which firm 2's preemption range vanishes.[6] There are two key continuation payoff configurations and therefore two types of equilibrium, preemptive or sequential, depending on the level of fixed cost asymmetry.

If $\kappa \in (0, \overline{\kappa})$, then firm 2's indifference condition $L_2\left(t; \kappa\right) - F\left(t\right) = 0$ has two solutions $\underline{t}_\kappa$ and $\bar{t}_\kappa$ in $\left(T^P, T_1^F\left(\kappa\right)\right)$, with $\underline{t}_\kappa < \bar{t}_\kappa$, that delimit the preemption range. The first of these is firm 2's preemption time which is a function of firm 1's capital stock and is hereafter denoted $T^P\left(\kappa\right)$. Implicit differentiation of $L_2\left(T^P; \kappa\right) - F\left(T^P\right) = 0$ establishes that $dT^P/d\kappa > 0$ whereas similarly $d\bar{t}_\kappa/d\kappa < 0$, so the preemption range shrinks with κ. In a (preemptive) equilibrium, the outcome resulting from non-cooperative choice of accumulation policies involves investment by firm 1 at $\min\left\{T^L\left(\kappa\right), T^P\left(\kappa\right)\right\}$ and by firm 2 at T^F.

If $\kappa \in [\overline{\kappa}, 1)$, then either $L_2\left(T^L; \overline{\kappa}\right) = F\left(T^L\right)$ whereas $L_2\left(t; \overline{\kappa}\right) < F\left(t\right)$ for all $t < T^F, t \neq T^L$ (if $\kappa = \overline{\kappa}$) or $L_2\left(t; \kappa\right) < F\left(t\right)$ for all $t < T^F$ (if $\kappa > \overline{\kappa}$). As the preemption range is empty, the game is in effect a single firm decision problem. In a (sequential) equilibrium, the outcome resulting from non-cooperative choice of accumulation policies consists of investment by firm 1 at $T^L\left(\kappa\right)$ and by firm 2 at T^F.

As $dT^L/d\kappa < 0$ whereas $dT^P/d\kappa > 0$, the difference $T^L\left(\kappa\right) - T^P\left(\kappa\right)$ is strictly decreasing in κ and there exists a unique level of asymmetry $\widehat{\kappa} \in (0, \overline{\kappa})$ such that $T^L\left(\widehat{\kappa}\right) = T^P\left(\widehat{\kappa}\right)$. Let $\widehat{T}$ denote the corresponding time, which plays

[6] The preemption range vanishes at T^L for $\kappa = \overline{\kappa}$ (see Appendix A.1) where

$$\overline{\kappa} = 1 - \left(\frac{\frac{\Pi^M}{\Pi^D} - 1}{\left(\frac{\Pi^M}{\Pi^D}\right)^{\frac{\lambda + r}{\lambda}} - \frac{\lambda}{\lambda + r}} \right)^{\frac{\lambda}{r}}.$$

an important role in the analysis of the remainder of this article. To simplify exposition, extend $T^P(\kappa)$ to include $T^P(0) = \lim_{\kappa \to 0} T^P(\kappa) = T^P$ and $T^P(\overline{\kappa}) = \lim_{\kappa \to \overline{\kappa}} T^P(\kappa) = T^L$, and furthermore set $T^P(\kappa) = T^F$ for $\kappa > \overline{\kappa}$. Hereafter $T^P(\kappa)$ therefore denotes the first time at which leading becomes profitable for firm 2, for any κ.

In equilibrium, firm 1 invests at $\min\{T^L(\kappa), T^P(\kappa)\}$ (see Appendix A.2):

Proposition 1 *If firm* 1*'s fixed cost is* $1-\kappa$*,* $\kappa \in (0,1)$*, it invests at* $\min\{T^L(\kappa), T^P(\kappa)\}$ *and earns a positive positional rent whereas firm* 2 *invests at* T^F*.*

2.3.1 Preemptive capital accumulation

Return now to the situation studied in the remainder of the article, with capital being divisible for firm 1. Based on Proposition 1 a specific capital accumulation policy, the *preemptive capital accumulation policy*, can be defined. This policy is denoted $k^P(t)$ and describes the path firm 1's capital accumulation should follow over $t \in [0, T^F)$ in order to suppress firm 2's investment incentive.

Observe first that at any $t \leq T^P$ firm 2 is necessarily outside or on the boundary of the preemption range and no capital is required to discourage its entry. At times in (T^P, T^L) firm 2's investment incentive can take on positive values. The preemptive capital accumulation policy is derived in this range by viewing firm 2's indifference condition as an identity. As $T^P(\kappa)$ is a strictly decreasing function over $(0, \overline{\kappa})$ it can be inverted, yielding an explicit form for the level of firm 1's capital stock that leaves firm 2 indifferent between obtaining leader and follower payoffs. For $\kappa = \overline{\kappa}$ firm 2's investment incentive vanishes at T^L and is negative at all other $t < T^F$ whereas for $\kappa > \overline{\kappa}$ this investment incentive is negative at all $t < T^F$, so by holding a capital stock $\overline{\kappa}$ firm 1 shuts out investment by its rival up until the time T^F at which firm 2 invariably enters. The preemptive capital accumulation policy therefore has the specific form

$$k^P(t) = \begin{cases} 0, & t < T^P \\ 1 - \left(\frac{\frac{\Pi^M}{\Pi^D} - 1}{\left(\frac{\Pi^M}{\Pi^D} - \frac{X(t)}{\Pi^D}\right)\left(\frac{\lambda+r}{r}\frac{X(t)}{\Pi^D}\right)^{\frac{r}{\lambda}} - \frac{\lambda}{\lambda+r}} \right)^{\frac{\lambda}{r}}, & T^P \leq t < T^L \\ \overline{\kappa}, & T^L \leq t < T^F. \end{cases}$$

Recall that firm 1's capital accumulation makes its follower entry threat more aggressive by lowering its duopoly investment time $T_1^F(\kappa)$ and hence firm 2's leader payoff $L_2(t;\kappa)$. Over $\left(T^P, T^L\right)$ the intuition underlying $k^P(t)$ is that as the input price decreases, leadership becomes relatively more attractive for firm 2 all else equal, and by following this policy firm 1 offsets the decrease in input price by raising its capital stock just enough to keep its threat sufficiently potent to discourage its rival's entry. More generally any policy $k_1(t)$ that firm 1 chooses which satisfies the no-preemption constraint $k_1(t) \geq k^P(t)$ keeps its rival at bay, allowing it to postpone its own entry beyond the inefficiently early time T^P that preemption otherwise imposes.

3 Gradual leadership in response to an entry threat

This section derives the dynamic pattern of leader investment discussed in the introduction as the optimal policy of a firm under constant threat of entry, leaving the task of explaining how this pattern arises from non-cooperative choices of capital accumulation policies by duopoly firms up to the next section. To simplify matters suppose that capital is perfectly divisible for firm 1 ($\Omega = [0, 1]$). Firm 2 is assumed to enter as soon as this is profitable (for instance because firm

2 is not itself shielded from further entry if it does not instantly secure a place in the industry). Firm 1 therefore has a first-mover advantage in the sense that it can choose when to lead product market entry, provided that firm 2 does not have an incentive invest preemptively.[7] Solving the corresponding constrained maximization problem yields a unique optimal capital accumulation path for firm 1, $k_1^*(t)$, given in the next proposition (where $\mathbf{1}_A$ denotes the indicator function which takes the value 1 if the condition A is true and 0 otherwise).

Proposition 2 *Under an entry threat firm 1's optimal capital accumulation policy is* $k_1^*(t) = k^P(t)\mathbf{1}_{t<\widehat{T}} + \mathbf{1}_{t\geq\widehat{T}}$.

Proof Observe first that up until firm 1 enters the product market, its optimal capital accumulation policy must satisfy the no-preemption constraint $k_1(t) \geq k^P(t)$ with equality, as additional capital accumulation is costly and unnecessary to discourage entry by firm 2. Firm 1's optimal policy must therefore be of the form $k_1(t) = k^P(t)\mathbf{1}_{s<t} + \mathbf{1}_{s\geq t}$ with $t \in \left(T^P, T^F\right)$. Such a policy yields it a leader continuation payoff $L_1\left(t; k^P(t)\right)$ provided that $t < T^F$, so that firm 1's net present value measured in initial currency units is

$$V^P(t) = L_1\left(t; k^P(t)\right) - \int_0^t X(s)e^{-rs}dk^P(s). \tag{1}$$

Because it cannot be desirable to lead before T^P and $L_1\left(t; \overline{K}_1\right)$ is decreasing beyond $T^L\left(\overline{K}_1\right) < T^L$ for $\overline{K}_1 > 0$, firm 1's optimal policy solves $\max_{t\in(T^P,T^L)} V^P(t)$. As $k^P(t)$ and hence $V^P(t)$ are differentiable over $\left(T^P, T^L\right)$, an interior optimum

[7] Firm 1 can ensure itself a payoff arbitrarily close to the leader value $V^P(t)$ defined further below in the text by following a policy for which investment is never individually rational for firm 2 such as $k_1^\delta(t) = k_1^*(t+\delta)$ for small δ.

is described by the first-order condition

$$\frac{\partial L_1\left(t^*; k^P(t^*)\right)}{\partial t} + \frac{\partial L_1\left(t^*; k^P(t^*)\right)}{\partial \overline{K}_1}\left[k^P\left(t^*\right)\right]' - X(t^*)e^{-rt^*}\left[k^P\left(t^*\right)\right]'$$
$$= \frac{\partial L_1\left(t^*; k^P(t^*)\right)}{\partial t} = 0.$$

Evaluating $\partial L_1/\partial t$ yields

$$-r\Pi^M e^{-rt^*} + (\lambda + r)\left(1 - k^P\left(t^*\right)\right) X(0)e^{-(\lambda+r)t^*} = 0. \tag{2}$$

The second derivative is

$$r^2\Pi^M e^{-rt} - (\lambda + r)^2\left(1 - k^P\left(t\right)\right) X(0)e^{-(\lambda+r)t} - (\lambda + r)\left[k^P\left(t\right)\right]' X(0)e^{-(\lambda+r)t},$$

and using the first-order condition to substitute for $r\Pi^M e^{-rt^*}$ gives

$$-(\lambda + r)\left(\lambda\left(1 - k^P\left(t^*\right)\right) + \left[k^P\left(t^*\right)\right]'\right) X(0)e^{-(\lambda+r)t^*} < 0$$

as $\left[k^P\left(t\right)\right]' > 0$ over $\left(T^P, T^L\right)$, so the objective is strictly quasiconcave.

The condition (2) admits a unique solution t^* that satisfies

$$t^* = \frac{1}{\lambda}\ln\left(\frac{\lambda + r}{r}\frac{X(0)\left(1 - k^P\left(t^*\right)\right)}{\Pi^M}\right).$$

Comparing this expression with the definition of $T^L(\kappa)$ in Section 2.3 establishes that t^* is an optimal time to invest for a firm holding a capital stock $\overline{K}_1 = k^P\left(t^*\right)$. Because the no-preemption constraint is satisfied with equality, it is also the case that $t^* = T^P\left(k^P\left(t^*\right)\right)$. t^* therefore corresponds to the critical time $\widehat{T}$ defined in Section 2.3. □

To interpret the optimal policy $k_1^*(t)$ observe that it involves a gradual form of leadership which has firm 1 using preemptive capital accumulation to escalate its threat up until its desired investment time is reached. Because at the margin the strategic investment that firm 1 must undertake to delay its leading entry is exactly offset by a corresponding reduction in the investment required for completion, the first-order condition determining the timing of firm 1's market entry just equates the instantaneous marginal cost of delay $r\Pi^M$ with the instantaneous marginal benefit of delay given $k^P(t^*)$, $(\lambda + r)\left(1 - k^P(t^*)\right) X(t^*)$. At the moment $\widehat{T}$ that firm 1 enters the product market, its investment is therefore instantaneously optimal. Firm 2 subsequently enters the product market as a follower at T^F. Figure 1 illustrates the locus of optimal monopoly investment times, the preemptive capital accumulation policy, and the optimal capital accumulation policy. The leading entry time and corresponding sunk capital $\left(\widehat{T}, \lim_{t\to\widehat{T}^-} k_1^*(t)\right)$ are the only point lying both on the optimal monopoly investment locus $T^L(\kappa)$ and on the no-preemption constraint $k^P(t)$.[8]

Consider an increase in the level of monopoly profit π^M which raises the incentive to lead while leaving the baseline follower payoff unchanged. One would expect a greater first-mover advantage to induce earlier entry by firm 1 in order to lengthen the monopoly phase it enjoys, and therefore to result in more rapid capital accumulation. This intuition must be verified though, since an increase in monopoly profit also raises the cost of preemptive capital accumulation, as firm 1 must compensate for firm 2's greater preemption incentive. Dividing (2) by $re^{-r\widehat{T}}$ and substituting in the expression of $k^P(t)$, the condition becomes

$$-\Pi^M + \Pi^D \left(\frac{\frac{\Pi^M}{\Pi^D} - 1}{\frac{\Pi^M}{\Pi^D} - \frac{X(\widehat{T})}{\Pi^D} - \frac{\lambda}{\lambda+r}\left(\frac{r}{\lambda+r}\frac{\Pi^D}{X(\widehat{T})}\right)^{\frac{r}{\lambda}}} \right)^{\frac{\lambda}{r}} = 0$$

[8] These dynamics contrast with those obtained by Mills [12], where a lower bound on the lag between investment steps induces $\varepsilon-$preemption allowing one of the firms to invest near the monopoly optimum.

after cancelling $X(0)$ terms inside and outside the brackets. Rearranging yields

$$\frac{X\left(\widehat{T}\right)}{\Pi^D} + \frac{\lambda r^{\frac{r}{\lambda}}}{(\lambda+r)^{\frac{\lambda+r}{\lambda}}}\left[\frac{X\left(\widehat{T}\right)}{\Pi^D}\right]^{-\frac{r}{\lambda}} = \frac{\Pi^M}{\Pi^D} - \left(\frac{\Pi^M}{\Pi^D}\right)^{1-\frac{r}{\lambda}} + \left(\frac{\Pi^M}{\Pi^D}\right)^{-\frac{r}{\lambda}}.$$

The right-hand side is an increasing function of Π^M,[9] whereas the derivative of the left-hand side with respect to X is $\left(1-\left(r\Pi^D/\left(\lambda+r\right)X\left(\widehat{T}\right)\right)^{(\lambda+r)/\lambda}\right)/\Pi^D$. This expression is positive, as $X\left(\widehat{T}\right) > X^F$ so $\Pi^D/X\left(\widehat{T}\right) < \left(\lambda+r\right)/r$. Because $dX/d\Pi^M > 0$, it follows that $d\widehat{T}/d\Pi^M < 0$ so greater monopoly profit does indeed result in accelerated capital accumulation and earlier product market entry by firm 1. This comparative static can also be obtained geometrically. Greater monopoly profit raises the marginal cost of waiting for a monopoly firm, lowering its optimal investment threshold at any given level of pre-existing capital so that the locus of optimal investment times $T_1^L(\kappa)$ shifts left. However greater monopoly profit also raises firm 2's incentive to lead, so firm 1 must hold more capital in order to maintain firm 2 indifferent between leading and following, shifting the locus $k^P(t)$ upward. As a result the time $\widehat{T}$ at which product market entry occurs decreases.

The limiting case $\Pi^D = 0$ where neither firm enters as a follower in the continuation phase is of intrinsic interest. In this case the preemptive capital accumulation policy converges to $k^P(t) = \mathbf{1}_{t \geq T^P}$ so firm 1 cannot scale its follower entry threat through incremental investment. The gradual leadership dynamics described above unravel and both firms seek to enter at the breakeven time $(1/\lambda)\ln\left(X(0)/\Pi^M\right)$, earning zero profits. This situation is of broader relevance because the commitment value of strategic investment is closely related

[9] The derivative of this expression has the sign of

$$\left(\frac{\Pi^M}{\Pi^D}\right)^{\frac{\lambda+r}{\lambda}} - \frac{\lambda-r}{\lambda}\frac{\Pi^M}{\Pi^D} - \frac{r}{\lambda},$$

which is zero if $\Pi^M = \Pi^D$ and increasing in Π^M.

to an incumbent firm's ability to deter entry (Dixit [4]). Suppose that the industry environment allows the first entrant to make a complementary technical or regulatory move, which is not prohibitively costly, that renders subsequent entry unprofitable to its rival. Then if firms cannot commit beforehand not to deter subsequent entry upon investment, the follower investment times become $T_1^F = T_2^F = \infty$ and preemptive capital accumulation cannot take place. The role of incremental investment and the leadership dynamics described here therefore hinge on the inability of firm 2 to deter firm 1 as a an incumbent in the product market.

4 Preemption with endogenous asymmetry

Consider once again the entry race described in the introduction, with one of two rival firms able to divide up its investment so as to progressively lower its entry cost. If this firm's feasible investment increments have an arbitrarily small positive lower bound, the equilibrium outcome of such a race is approximated by the gradual leadership policy $k_1^*(t)$ (see Figure 1 and Proposition 2). In order to establish this claim, the entry race between the firms is represented in this section as a noncooperative game in which firms make a sequence of quasi-static decisions pertaining to the timing of their investments. By restricting dynamics to firm decision-making rather than its implementation over time, this approach allows the analysis to focus on the novel issues raised by the endogenous staging of firm 1's investments.

Suppose that firm 1's feasible capital stock levels are $\Omega = \{0, 1/N, ..., 1\}$, N being a large integer. Dynamic competition between both firms determines leading entry into the product market. This process is represented as a multistage game whose outcomes unfold over time. Stages are defined by the number of capital increments accumulated by firm 1, denoted by ξ, with $\xi \in \{0, 1, ..., N-1\}$.

In each stage, firms choose the timing of their next investment conditional upon no rival investment having occurred. If they choose identical times, an efficient rationing rule determines the outcome of their decisions. Once an investment occurs, either the play moves on to a subsequent stage if firm 1 invests and it does not accumulate an entire unit or the game ends if one of the firms has accumulated an entire unit of capital, in which case any firm that has not yet accumulated a unit updates the timing of its next investment.

4.1 The stage game

In a given stage ξ of the investment game, a history h_ξ consists of the timing and magnitude of any prior investments by firm 1. Set $t_0 = 0$ and let t_ξ denote the time of firm 1's last investment for given h_ξ if $\xi > 0$. Stage ξ is thus reached at time t_ξ with firm 1 holding a capital stock $\overline{K}_1 = \xi/N$. The firms choose planned investment times $T_i^\xi \in A_i^\xi(h_\xi)$, $i \in \{1,2\}$ which they are able to revise if they observe a rival investment has occurred, where $A_i^\xi(h_\xi) = [t_\xi, \infty]$ is the set of player i's actions in stage ξ following history h_ξ, with ∞ denoting the action "never invest". If the plans of both firms call for investments at the same moment, it is assumed that input provision is subject to an instantaneous capacity constraint and a tie-breaking rule, described in greater detail further below, determines the outcome of their choices. In all cases the next investment in the industry occurs at $\min\left\{T_1^\xi, T_2^\xi\right\}$.

The last stage, $\xi = N-1$, is particular and therefore discussed separately before the other stages are described.

4.1.1 Stage $\xi = N-1$

In stage $N-1$, investment by any firm necessarily results in product market entry. The firms therefore face the same situation as in the asymmetric preemption game with $\kappa = 1 - (1/N)$ (see Section 2.3 and Appendix A.2). Moreover if N is large enough then $1 - (1/N) > \overline{\kappa}$, which means that the degree of cost asymmetry is sufficient for the game to reduce to a single-firm decision problem involving firm 1. It sets $\widehat{T}_1^{N-1} = \max\left\{t_\xi, T^L\left(1-(1/N)\right)\right\}$ whereas firm 2 chooses $\widehat{T}_2^{N-1} \in \left[\max\left\{t_\xi, T^L\left(1-(1/N)\right)\right\}, \infty\right]$. These choices result respectively in the payoffs $L_1\left(\max\left\{t_\xi, T^L\left(1-(1/N)\right)\right\};\ 1-(1/N)\right)$ and $F\left(\max\left\{t_\xi, T^L\left(1-(1/N)\right)\right\}\right)$.

4.1.2 Stages $\xi < N - 1$

In all other stages $\xi < N - 1$, the plans T_1^ξ and T_2^ξ that the firms adopt must be mapped into outcomes in order to determine continuation payoffs.

If $T_1^\xi \neq T_2^\xi$ the outcomes are straightforward as firm i, $i \in \{1, 2\}$, leads if and only if it has a strictly lower planned investment time than its rival. If $T_1^\xi < T_2^\xi$ so firm 1 leads, it also determines the size of its investment. Let $\widehat{V}_i^{\xi'}$ denote firm i's continuation value if stage $\xi' > \xi$ is next reached. If firm 1 chooses an investment of magnitude $\eta \in \{1, ..., N - \xi\}$, the resulting payoff profile is $\left(\widehat{V}_1^{\xi+\eta} - (\eta/N)X(T_1^\xi)e^{-rT_1^\xi}, \widehat{V}_2^{\xi+\eta}\right)$ if $\eta < N - \xi$ and $\left(L_1\left(T_1^\xi; \xi/N\right), F\left(T_1^\xi\right)\right)$ if $\eta = N - \xi$. Let $\eta^* \in \{1, ..., N - \xi\}$ denote firm 1's optimal investment size (if there are multiple optimum solutions it can be assumed for simplicity that firm 1 chooses the largest). The continuation payoff profile in this subcase, denoted $\left(L_1^\xi\left(T_1^\xi\right), F_2^\xi\left(T_1^\xi\right)\right)$, is therefore $\left(\widehat{V}_1^{\xi+\eta^*} - (\eta^*/N)X(T_1^\xi)e^{-rT_1^\xi}, \widehat{V}_2^{\xi+\eta^*}\right)$ if $\eta^* < N - \xi$ and $\left(L_1\left(T_1^\xi; \xi/N\right), F\left(T_1^\xi\right)\right)$ if $\eta^* = N - \xi$. If $T_1^\xi > T_2^\xi$ so firm 2 leads, the continuation payoff profile is $\left(F_1^\xi\left(T_2^\xi\right), L_2^\xi\left(T_2^\xi\right)\right) = \left(F_1\left(T_2^\xi; \xi/N\right), L_2\left(T_2^\xi; \xi/N\right)\right)$.

If $T_1^\xi = T_2^\xi = T$, static competition between the firms in the input market supersedes dynamic competition. The firm whose investment incentive $L_i^\xi(T) - F_i^\xi(T)$ is comparatively greater prevails but pays a premium amounting to its rival's valuation. It thus earns a differential rent similar in nature to the equilibrium profit of a low-cost Bertrand duopolist. Such a differential rent is consistent with appropriation by upstream suppliers of the common part of the positional rents either firm stands to earn from entry. If the firms have identical incentives, the input is allocated to either with equal probability. Letting $S_i^\xi(T) = L_i^\xi(T) - \max\left\{L_j^\xi(T) - F_j^\xi(T), 0\right\}$ (if $L_i^\xi(T) - F_i^\xi(T) \geq L_j^\xi(T) - F_j^\xi(T)$) or $F_i^\xi(T)$ (if $L_i^\xi(T) - F_i^\xi(T) < L_j^\xi(T) - F_j^\xi(T)$), $i, j \in \{1, 2\}$, $i \neq j$, the continuation payoff profile in this case is accordingly $\left(S_1^\xi(T), S_2^\xi(T)\right)$.[10]

Summarizing, the stage payoffs for $\xi < N - 1$ are

$$V_1^\xi\left(T_1^\xi, T_2^\xi\right) = \begin{cases} L_1^\xi\left(T_1^\xi\right), & T_1^\xi < T_2^\xi \\ S_1^\xi(T), & T_1^\xi = T_2^\xi = T \\ F_1\left(T_2^\xi; \frac{\xi}{N}\right), & T_1^\xi > T_2^\xi \end{cases}$$

and

$$V_2^\xi\left(T_1^\xi, T_2^\xi\right) = \begin{cases} L_2\left(T_2^\xi; \frac{\xi}{N}\right), & T_2^\xi < T_1^\xi \\ S_2^\xi(T), & T_1^\xi = T_2^\xi = T \\ F_2^\xi\left(T_1^\xi\right), & T_2^\xi > T_1^\xi. \end{cases}$$

[10] An alternative approach to simultaneous investments would be to posit the profile $\left(M_1^\xi(T), M_2^\xi(T)\right) = \left(F_1(T; (\xi + \eta^*)/N) - (\eta^*/N)X(T)e^{-rT},\ L_2(T; (\xi + \eta^*)/N)\right)$ (if $\eta^* < N - \xi$) or $(M_1(T; \xi/N), M(T))$ (if $\eta^* = N - \xi$) where $\eta^* \geq 1$ is an optimal investment size for firm 1. However the use of such a payoff raises a standard issue in preemption games pertaining to their representation in continuous time. There generally exist payoff configurations where each firm prefers to invest first but simultaneous investments are jointly suboptimal (*i.e.* $L_i^\xi(t) \geq F_i^\xi(t) > M_i^\xi(t)$, $i \in \{1, 2\}$) and the timing of investments would in effect be coordinated in an equilibrium of the discrete time game that the continuous time formulation approximates. This issue is generally resolved either by expanding the strategy space to allow a continuous time representation of such strategy coordination (Fudenberg and Tirole [8]) or by positing a random assignment of leader and follower roles (Dutta and Rustichini [7]). The former approach is parsimonious in its assumptions but involves notationally costly strategies, whereas like the latter approach, the simultaneous investment profile in the text restricts the economic environment but induces a unique equilibrium in pure strategies. So long as firm 1 can realize its first-mover advantage if a given stage is reached sufficiently early, the choice of approach to simultaneous investments does not alter outcomes along the equilibrium path and is consequently secondary to the present analysis.

4.2 Equilibrium

Let H denote the set of histories of the investment game. A strategy for firm i is a map $\Gamma_i : H \to \bigcup_{\xi \in \{0,...,N-1\}} A_i^{\xi}$. Let $\Gamma_i^{h^\xi}$ denote the restriction of Γ_i to histories that include h^ξ. A subgame of the investment game is defined by firm 1's capital stock $\overline{K}_1 = \xi$ and a starting time t_ξ.[11] A strategy profile (Γ_1, Γ_2) is a subgame perfect Nash equilibrium of the investment game if and only if, for all $h^\xi \in H$, $\left(\Gamma_1^{h^\xi}, \Gamma_2^{h^\xi}\right)$ is a Nash equilibrium of the subgame starting at (ξ, t_ξ). Many strategy profiles can constitute a subgame perfect Nash equilibrium, but the main result of this section establishes that such strategy profiles identify a unique path of equilibrium capital accumulation in the industry. Along this path, firm 1 invests incrementally at a succession of preemptive times $T^P(\xi/N)$, $\xi = 0, 1, ..., \lfloor N\widehat{\kappa} \rfloor$, up until it reaches a pivotal stage and enters the product market at time $\widehat{T}^N = \max\left\{T^L\left(\left(\lfloor N\widehat{\kappa}\rfloor + 1\right)/N\right), T^P\left(\lfloor N\widehat{\kappa}\rfloor/N\right)\right\}$ whereas firm 2 enters the product market subsequently at T^F.

Proposition 3 *Along a subgame perfect equilibrium path investment times satisfy* $\left(\widehat{T}_1^\xi, \widehat{T}_2^\xi\right) = \left(T^P(\xi/N), T^P(\xi/N)\right)$ *for* $\xi \in \{0, 1, ..., \lfloor N\widehat{\kappa}\rfloor\}$. *The magnitude of firm 1's investment is* $\eta^* = 1$ *for* $\xi < \lfloor N\widehat{\kappa}\rfloor$, *and for* $\xi = \lfloor N\widehat{\kappa}\rfloor$ *either* $\eta^* = N - \lfloor N\widehat{\kappa}\rfloor$ *if* $T^P(\xi/N) = \widehat{T}^N$ *or* $\eta^* = 1$ *if* $T^P(\xi/N) < \widehat{T}^N$. *In the latter case firm 1 subsequently invests at* $\widehat{T}_1^{\lfloor N\widehat{\kappa}\rfloor+1} = T^L\left(\left(\lfloor N\widehat{\kappa}\rfloor + 1\right)/N\right)$, *with* $\eta^* = N - 1 - \lfloor N\widehat{\kappa}\rfloor$.

According to Proposition 3, firm 1's capital accumulation follows the path

$$k_1^N(t) = \sum_{j=0}^{\lfloor N\widehat{\kappa}\rfloor} \frac{1}{N}\mathbf{1}_{t \geq T^P\left(\frac{j}{N}\right)} + \frac{N-1-\lfloor N\widehat{\kappa}\rfloor}{N}\mathbf{1}_{t \geq \widehat{T}^N},$$

whereas firm 2's follows $k_2^N(t) = 1_{t \geq T^F}$.

[11] As actions are planned investment times which are not revised unless rival investment occurs, for given (ξ, t_ξ) $\xi' > \xi$ and $t_{\xi'} \geq t_\xi$ define a subgame but $\xi' = \xi$ and $t_{\xi'} > t_\xi$ do not.

Having stated the main result, it is useful to return for a moment to the role of the tie-breaking rule described further above in the section. Observe that in any proper subgame, firm 1 could behave as if its capital were no longer divisible and obtain an equilibrium asymmetric preemption payoff (with $\kappa = \xi/N$) by just setting $T_1^{\xi} = \max\left\{t_{\xi}, \min\left\{T^{L}(\xi/N), T^{P}(\xi/N)\right\}\right\}$ with $\eta = N - \xi$. Firm 1's leader payoff in any stage ξ therefore satisfies $L_1^{\xi}(t) \geq L_1(t; \xi/N)$ by revealed preference, and by the same token in subsequent stages firm 1 holds firm 2 to a follower payoff in equilibrium so that $F_2^{\xi}(t) = F(t)$. Using a standard property of asymmetric preemption for the second inequality (see footnote 5) it follows that

$$\begin{aligned} L_1^{\xi}(t) - F_1\left(t; \frac{\xi}{N}\right) &\geq L_1\left(t; \frac{\xi}{N}\right) - F_1\left(t; \frac{\xi}{N}\right) \\ &\geq L_2\left(t; \frac{\xi}{N}\right) - F(t) = L_2\left(t; \frac{\xi}{N}\right) - F_2^{\xi}(t). \end{aligned}$$

The tie-breaking rule therefore ensures that firm 1 can realize its first-mover advantage for $\xi \geq 1$. The reasoning for $\xi = 0$ is slightly more involved, but the proof of Proposition 3 establishes $L_1^{0}(t) > L_1(t)$ for any $t_0 < \widehat{T}^{N}$ which allows firm 1 to realize its first-mover advantage in this case too. Alternative approaches to simultaneous investments (see footnote 10) induce similar firm 1 behavior and therefore also support the equilibrium outcome in the proposition.

Proof

The argument involves identifying a profile of subgame perfect equilibrium strategies that support the outcome described in the proposition and then verifying that no other equilibrium outcomes arise. Let $\left(\widehat{\Gamma}_1, \widehat{\Gamma}_2\right)$ denote the strategy profile

$$\widehat{T}_1^{\xi} = \begin{cases} \max\left\{t_{\xi}, T^{P}\left(\frac{\xi}{N}\right)\right\}, & \xi \in \{0, 1, ..., \lfloor N\widehat{\kappa}\rfloor\} \\ \max\left\{t_{\xi}, T^{L}\left(\frac{\xi}{N}\right)\right\}, & \xi \in \{\lfloor N\widehat{\kappa}\rfloor + 1, ..., N-1\} \end{cases}$$

and

$$\widehat{T}_2^{\xi}=\begin{cases}\max\left\{t_\xi,T^P\left(\frac{\xi}{N}\right)\right\}, & \xi\leq\lfloor N\overline{\kappa}\rfloor \text{ and } t_\xi\leq\bar{t}_{\xi/N}\\ \max\left\{t_\xi,T^F\right\}, & \text{otherwise}\end{cases}$$

for $\xi\in\{0,...,N-1\}$.[12] It is necessary to verify that these strategies are mutual best replies in every subgame. Observe first of all that if $t_\xi\geq T^F$ equilibrium decisions are straightforward, as they consist of immediate investments for both firms that result in terminal payoffs $(F_1(t_\xi;\xi/N),F(t_\xi))$. The analysis therefore focuses on starting times $t_\xi<T^F$.

Consider first subgames $\xi>N\widehat{\kappa}$. The argument proceeds by induction on firm 1's possible capital stock levels.

For $\xi=N-1$, the subgame starting at any t_{N-1} is an asymmetric preemption game (Section 4.1.1) of which $\left(\widehat{T}_1^{N-1},\widehat{T}_2^{N-1}\right)$ is an equilibrium (Appendix A.2).

Next, let ξ be given with $N\widehat{\kappa}<\xi<N-1$ and $t_\xi\geq0$. For the induction argument assume that for all $\xi'>\xi$ and any $t_{\xi'}\geq t_\xi$ the restricted strategy profiles $\left(\widehat{\Gamma}_1^{h_{\xi'}},\widehat{\Gamma}_2^{h_{\xi'}}\right)$ are mutual best replies.

Consider firm 2 first. Either $t_\xi<T^P(\xi/N)$ in which case $\widehat{T}_1^{\xi}<T^P(\xi/N)$ as $T^P(\xi/N)>T^L(\xi/N)$, so setting $T_2<\widehat{T}_1^{\xi}$ (leading) is unprofitable for firm 2 and it is indifferent between all $T_2\geq\widehat{T}_1^{\xi}$ that yield it a follower payoff (for $T_2=\widehat{T}_1^{\xi}$ this is due to the tie-breaking rule). Otherwise $t_\xi\geq T^P(\xi/N)$ and $\widehat{T}_1^{\xi}=t_\xi$ so firm 2 cannot lead and is similarly indifferent between all $T_2\geq\widehat{T}_1^{\xi}$. In either case $\widehat{T}_2^{\xi}$ in particular constitutes a (weak) best-response to $\widehat{T}_1^{\xi}$.

Consider firm 1 next. There are two subcases to examine. If $\xi>N\overline{\kappa}$, $\widehat{T}_2^{\xi}=T^F$ and firm 1 effectively faces a single-firm decision problem. Its continuation payoff $L_1^{\xi}(t)$ is determined as follows. If $t\geq T^L((\xi+1)/N)$ then an investment of size η with $1\leq\eta<N-1$ is followed by immediate entry in stage $\xi+\eta$ by

[12] This strategy profile, which has firm 2 investing immediately whenever the preemption range is reached, is chosen for consistency with an equilibrium in the extended mixed strategies discussed in footnote 10.

the induction hypothesis ($\widehat{T}_1^{\xi+\eta} = t$ as $T^L((\xi+\eta)/N) < T^L(\xi/N)$) so $L_1^{\xi}(t) = L_1(t;\xi/N)$, which is maximized at $t = T^L(\xi/N)$. If $t < T^L((\xi+1)/N)$ then $\eta = N-1$ is not an optimal investment size as $L_1(t;\xi/N)$ is increasing, and for $1 \leq \eta < N-1$, by the induction hypothesis, either $\widehat{T}_1^{\xi+\eta} = t$ resulting in a suboptimal payoff $L_1(t;\xi/N)$ or $\widehat{T}_1^{\xi+\eta} = T^L((\xi+\eta)/N) > t$ (firm 1's investments are staggered) in which case it obtains

$$
\begin{aligned}
& L_1\left(T^L\left(\frac{\xi+\eta}{N}\right);\frac{\xi+\eta}{N}\right) - \frac{\eta}{N}X(t)\,e^{-rt} \\
= {} & L_1\left(T^L\left(\frac{\xi+\eta}{N}\right);\frac{\xi}{N}\right) - \frac{\eta}{N}\left(X(t)\,e^{-rt} - X\left(T^L\left(\frac{\xi+\eta}{N}\right)\right)e^{-rT^L\left(\frac{\xi+\eta}{N}\right)}\right).
\end{aligned}
$$

If investing at t firm 1 would therefore set $\eta^* = 1$ and obtain $L_1^{\xi}(t) = L_1\left(T^L((\xi+1)/N);(\xi+1)/N\right)$ $-(1/N)X(t)\,e^{-rt} = L_1\left(T^L((\xi+1)/N);\xi/N\right) -(1/N)\left(X(t)\,e^{-rt} - X\left(T^L((\xi+1)/N)\right)e^{-rT^L((\xi+1)/N)}\right)$ $< L_1\left(T^L(\xi/N);\xi/N\right)$, as $T^L(\xi/N)$ is a global maximum. In stage ξ firm 1's optimal choice ($\arg\max_{t\geq t_\xi} L_1^{\xi}(t)$) is therefore $\widehat{T}_1^{\xi}$. The second subcase to examine is $N\widehat{\kappa} < \xi \leq N\overline{\kappa}$. The main difference is that $\widehat{T}_2^{\xi} = \max\left\{t_\xi, T^P(\xi/N)\right\}$ if $t_\xi \leq \bar{t}_{\xi/N}$, in which case firm 1 does not face a single-firm decision problem. If $t_\xi < T^P(\xi/N)$ then $\widehat{T}_2^{\xi} = T^P(\xi/N)$ does not constrain firm 1's optimal choice $\widehat{T}_1^{\xi} = \max\left\{t_\xi, T^L(\xi/N)\right\}$. If $T^P(\xi/N) \leq t_\xi \leq \bar{t}_{\xi/N}$, then $\widehat{T}_2^{\xi} = t_\xi$ and firm 1 earns $L_1^{\xi}(t_\xi) - \left(L_2^{\xi}(t_\xi) - F(t_\xi)\right)$ by the tie-breaking rule if it sets $T_1 = \widehat{T}_2^{\xi} = t_\xi$ and a lower follower payoff $F_1(t_\xi;\xi/N)$ otherwise. If $t_\xi > \bar{t}_{\xi/N}$ then the argument is as in the previous subcase. For any given t_ξ therefore, firm 1's optimal choice is $\widehat{T}_1^{\xi}$. Hence, in any subgame with $\xi > N\widehat{\kappa}$ investing at the times $\widehat{T}_1^{\xi}$ and $\widehat{T}_2^{\xi}$ specified above is an equilibrium.

Consider next subgames with $\xi \leq N\widehat{\kappa}$. Take firm 2 first. Either $t_\xi < T^P(\xi/N)$ in which case $\widehat{T}_1^{\xi} = T^P(\xi/N)$ so leading is unprofitable for firm 2 which is indifferent between all $T_2 \geq T^P(\xi/N)$, or $\widehat{T}_1^{\xi} = t_\xi \geq T^P(\xi/N)$ and firm 2 is similarly indifferent between simultaneous investment and following. In either case $\widehat{T}_2^{\xi}$ in particular constitutes a (weak) best-response to $\widehat{T}_1^{\xi}$. Next take firm 1. If $t_\xi > \bar{t}_{\xi/N}$, $\widehat{T}_2^{\xi} = T^F$ and by the same argument as for $\xi > N\widehat{\kappa}$, $\widehat{T}_1^{\xi} = t_\xi$. If $T^P(\xi/N) \leq t_\xi \leq \bar{t}_{\xi/N}$ then as $\widehat{T}_2^{\xi} = t_\xi$ firm 1 obtains $S_1^{\xi}(t_\xi) > F_1^{\xi}(t_\xi)$ by setting $\widehat{T}_1^{\xi} = t_\xi$. Finally if $t_\xi < T^P(\xi/N) = \widehat{T}_2^{\xi}$, firm 1's continuation payoff from leading is determined by an induction argument as for $\xi > N\widehat{\kappa}$, so that

$$L_1^{\xi}(t) = L_1\left(\widehat{T}^N; \frac{\lfloor N\widehat{\kappa}\rfloor + 1}{N}\right) - \mathbf{1}_{\xi < \lfloor N\widehat{\kappa}\rfloor - 1} \sum_{i=\xi+1}^{\lfloor N\widehat{\kappa}\rfloor} \frac{1}{N} X\left(T^P\left(\frac{i}{N}\right)\right) e^{-rT^P\left(\frac{i}{N}\right)} - \frac{1}{N} X(t)\, e^{-rt} \tag{4}$$

which is maximized by setting $\widehat{T}_1^{\xi} = T^P(\xi/N)$. Hence in any subgame with $\xi \leq N\widehat{\kappa}$ investing at the times $\widehat{T}_1^{\xi}$ and $\widehat{T}_2^{\xi}$ specified above is an equilibrium.

To verify that there are no equilibrium outcomes than that described in the proposition, observe that firm 1 can assure itself of leading arbitrarily near $\widehat{T}^N$ by accumulating capital along the path $k_1^{\delta}(t) = k_1^N(t+\delta)$ for small δ, as $k_1^{\delta}(t) > k^P(t)$ for $t < \widehat{T}^N$ and investment at any given time t is not individually rational for firm 2 if $\lim_{s \to t^-} k_1(s) > k^P(t)$. By setting investment times according to the policy $k_1^{\delta}(t)$ firm 1 obtains a payoff arbitrarily close to $L_1^0(t)$ (see (4) above). Letting investment increments become arbitrarily small and picking $\delta(N)$ accordingly, $\liminf_{N\to\infty} \widehat{V}_1^0 = L_1^0(0) = V^P\left(\widehat{T}\right)$ (see (1) above), as $\lim_{N\to\infty} \widehat{T}^N = \widehat{T}$ and $\lim_{N\to\infty} k_1^N(t) = k_1^*(t)$. Firm 1 cannot obtain more than $V^P\left(\widehat{T}\right)$ either, since that would imply violating the no-preemption constraint. □

To relate Proposition 3 to the analysis in the rest of the article, suppose that the divisibility N of firm 1's capital is arbitrarily large. For larger N the size of firm 1's investment steps becomes arbitrarily small and its entry time $\widehat{T}^N$ approaches $\widehat{T}$. Therefore $\lim_{N\to\infty} k_1^N(t) = k_1^*(t)$ pointwise, and the equilibrium outcome of the N-stage preemption game converges to the optimal policy in a Stackelberg choice of capital accumulation policies discussed in Section 3.

5 Leadership dynamics under input price uncertainty

The dynamics of industry investments are generally governed by market uncertainty, giving rise to a value of waiting for a more opportune moment to invest. Firms then have to trade two motives off against one another, as on the one hand the threat of competition creates a strategic incentive to act quickly and commit, but on the other hand option value due to uncertainty calls for flexibility and delay (Chevalier-Roignant and Trigeorgis [3]). A natural extension of the analysis of the above sections is to inquire into the effect of uncertainty on the capital accumulation policy of the firm whose capital is relatively divisible.

Suppose that the capital price process includes a noise term, so as to follow a geometric Brownian motion $dX(t) = \mu X(t)dt + \sigma X(t)dW(t)$ where $W(t)$ is a standard Wiener process, and with $X(0) = x > \Pi^M$.[13] Firm 1 is assumed to have the capability of implementing threshold policies that allow it to invest incrementally. Suppose that its capital is perfectly divisible, so firm 1's feasible accumulation policies are represented by nondecreasing and right-continuous functions $k_1(x) : R_+ \to [0, 1]$ that define its capital stock process absent rival investment as a function of the input price path, $K_1(t) = \sup_{s \in [0,t]} \{k_1(X(s))\}$. Assume that firm 2 stands ready to enter whenever profitable so that as in Section 3 firm 1's problem consists in regulating its rival's entry threat up until its own desired entry threshold is reached. It accordingly determines an optimal investment threshold under the constraint that firm 2's entry incentive is maintained at zero. Despite the more complex structure underlying the payoff functions, the main difference that the stochastic term in the input price adds to the model if the firms follow threshold policies is that choices and hence payoffs

[13] The model in Section 2 corresponds to $\mu = -\lambda$ and $\sigma = 0$ whereas if $\sigma > 0$ firms have a motive to wait even if $\mu > 0$.

are defined over price levels which determine stochastic investment times, but the broad economic intuitions are otherwise similar.

Let X^P, X^L and X^F denote the respective input price thresholds in symmetric preemption equilibrium, for monopoly and for duopoly. These thresholds are analogs to T^P, T^L and T^F in the deterministic case where firms set investment times directly. Like T^P, the preemption time X^P does not have a closed form. On the other hand $X^L = (\gamma/(\gamma+1))\,\Pi^M$ and $X^F = (\gamma/(\gamma+1))\,\Pi^D$, where γ is a parameter function that reflects expected discounting in the stochastic case.[14] The preemptive capital accumulation policy is defined in the same manner as in Section 2.3 over $(X^F, x]$ and has the specific form

$$k^P(x) = \begin{cases} \overline{\kappa}_X, & X^F < x \leq X^L \\ 1 - \left(\dfrac{\frac{\Pi^M}{\Pi^D} - 1}{\left(\frac{\Pi^M}{X^F} - \frac{x}{X^F}\right)\left(\frac{x}{X^F}\right)^{\gamma} - \frac{1}{\gamma+1}} \right)^{\frac{1}{\gamma}}, & X^L < x \leq X^P \\ 0, & x > X^P \end{cases}$$

where $\overline{\kappa}_X$ is a constant (See Appendix A.3).

A similar argument to Section 3 establishes that firm 1's optimal capital accumulation policy is $k_1^*(x) = k^P(x)\mathbf{1}_{x > \widehat{X}} + \mathbf{1}_{x \leq \widehat{X}}$. This policy is preemptive up until a critical threshold is reached and the firm's capital stock jumps to the level required for market entry. This critical threshold $\widehat{X}$ is implicitly defined by firm 1's first-order condition

$$\widehat{X} = \frac{\gamma}{\gamma+1} \frac{\Pi^M}{1 - k^P\left(\widehat{X}\right)}.$$

Figure 2 illustrates the optimal policy in (x, K_1)-space. $\widehat{X}$ is determined by

[14] Specifically

$$\gamma(\mu, r, \sigma) = -\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right) + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}$$

satisfies $\gamma > 0$ and $\partial\gamma/\partial\sigma < 0$. Letting $\tau(X) = \inf\{t \in R_+ |\, X(t) \leq X\}$ denote the first hitting time for threshold $X \leq x$, $E_x e^{-r\tau(X)} = (X/x)^{\gamma}$.

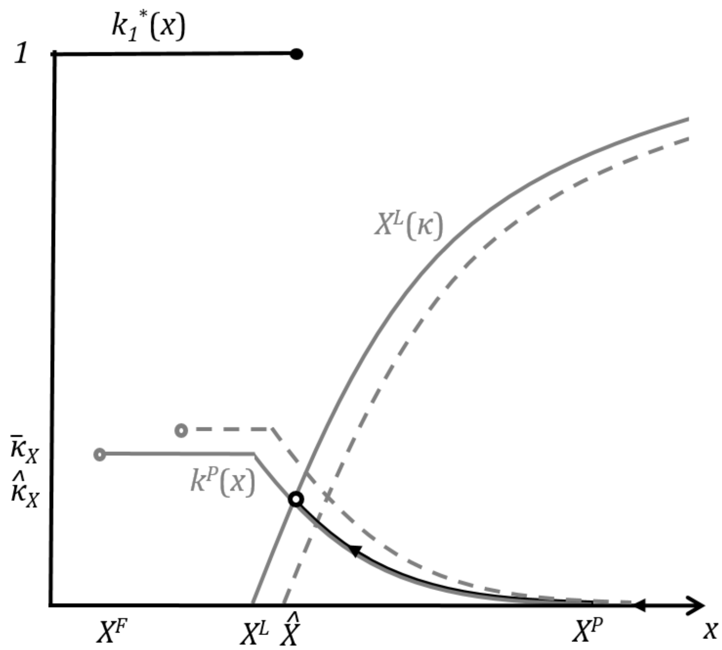

Figure 2: In the stochastic case greater volatility shifts $X^L(\kappa)$ and $k^P(x)$ leftward, lowering the threshold $\widehat{X}$ and delaying the jump to completion.

the intersection of $k^P(x)$ and the locus of monopoly investment thresholds $X_1^L\left(\overline{K}_1\right) = (\gamma/(\gamma+1))\left(\Pi^M/\left(1-\overline{K}_1\right)\right)$. Because the input price has positive volatility, the time at which the threshold $\widehat{X}$ is first hit and firm 1 enters is stochastic. Up until then, were it to have the follower role, firm 1's probability of completing its investment in a given future time interval is an increasing function of its capital stock. Its optimal capital accumulation policy thus involves a form of brinkmanship insofar as along the path of its capital accumulation, its follower threat presents firm 2 with a gradually increasing risk of losing any monopoly position it might seek to secure.

Incorporating a stochastic input price allows the effect of uncertainty on investment decisions to be studied by varying the value of the volatility parameter. An increase in σ, which reduces γ, has several effects. First it raises option value, which leads firms to delay product market entry. For a given capital stock level firm 1 thus lowers its monopoly threshold $X_1^L(\kappa)$. The effect of greater volatility on preemption is complex, but with geometric Brownian motion greater volatility raises the value of the follower's duopoly option more than it decreases the rents of moving early, attenuating preemption and thus shifting the preemptive capital accumulation policy $k^P(x)$ down and to the left. Therefore $\partial\widehat{X}/\partial\sigma < 0$, as can be checked by direct calculation (see Appendix A.3.2). Because the limited and asymmetric form of competition in the present model allows the leader to invest at a (myopically) optimal threshold despite the presence of a rival, the effect of greater volatility thus turns out to mirror the role this parameter plays in investment decisions in the absence of a competitive threat. Finally, the effect of greater volatility on the size of firm 1's last investment step depends generally on the relative magnitude of the effects on monopoly investment and preemption.

6 Conclusion

This article has studied dynamic competition in an industry in which one firm subdivides its investment in a capital input to make a gradual strategic commitment. Provided that the price of this input is initially high enough for firms to prefer delaying, such a firm follows a policy that involves accumulating capital preemptively and leading product market entry at a threshold that is both an instantaneous monopoly optimum and a preemptive equilibrium. Moreover if the input price follows a stochastic process, the timing of this firm's entry is positively related to a measure of market uncertainty.

The emblematic example of strategic commitment, with capital having been literally sunk in order to gain a military rather than a business advantage, is the scuttling of the conquistador Hernando Cortés' ships before marching on the city of Mexico. Dixit and Nalebuff [5] explain that this move both signalled the Spaniards' determination to their adversaries and compelled Cortés' men to fight by physically barring them from retreat. The perspective sketched in the present article suggests that a full economic account of this historical episode should also emphasize the more incremental moves made during the first months of the conquistador's expedition.

According to Prescott [13] the destruction of the fleet occurred nearly six months after Cortés and his men first reached Mexican shores. The Spaniards had already had several exchanges with local populations and with the Aztec emperor Montezuma's emissaries over the course of the spring and early summer. They had begun to build a colony near the present day city of Veracruz, and several key new figures such as Cortés' charismatic mistress, La Malinche, now numbered among their party. Finally, a few of the men had just conspired to escape back to Cuba. Reinstated into its context, the sinking of their ships is better understood therefore as the culmination of a lengthier process during which the conquistadors learned about their surroundings and made various forms of commitments before the subsequent leap in their military engagement.

The piece of evidence that best supports viewing the destruction of the fleet as a part of a broader plan is that the Spaniards did not actually sink every single one of their ships strategically (but one to be exact), or at least not literally so. In fact, the first ship to be sunk in a game-theoretic sense was sent not to lie on the Caribbean seabed but rather across the Atlantic, for a motive complementary to the one that governed the subsequent destruction of the fleet. Upon setting off from Cuba, Cortés had defied the orders of the island's governor Velasquez, and he needed to make a provision for the continuation game that would arise in the event that the conquest of Mexico succeeded. Cortés therefore cemented his *pronunciamento* by sending several men away on his best vessel as emissaries bearing the expedition's accumulated treasure, to pledge allegiance directly to Spain's monarch.[15] It is after this, when a few of his men had nearly commandeered one of the remaining ships in order to return to Cuba, that Cortés forged and carried out the more fabled scheme to wreck the remainder of his fleet.

Interpreting this last step as a link in a chain rather than an isolated move is consistent with a view that strategic investment involves exercising a compound option (or "moving in small steps") to which this article has sought to contribute. The formal analysis then predicts that whereas higher monopoly rents accelerate capital accumulation, greater uncertainty delays the jump to completion all else equal. When interpreting Cortés' conquest through an economic lens therefore, one should understand the aforementioned incremental steps over the first months of the expedition as together playing as considerable a role, even if less striking, as the nine ships the conquistadors eventually scuttled in the optimal conduct of their highly lucrative but also most improbable undertaking.

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[15] See Prescott [13], p. 362:

> "Cortés ... knew that all the late acts of the colony, as well as his own authority, would fall to the ground without royal sanction. He knew, too, that the interest of Velasquez, which was great at court, would, so soon as he was acquainted with his secession, be wholly employed to circumvent and crush him. He resolved to anticipate his movements, and to send a vessel to Spain, with despatches addressed to the emperor himself, announcing the nature and extent of his discoveries, and to obtain, if possible, the confirmation of his proceedings."

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A Appendix

A.1 Preemption time $T^P(\kappa)$ and upper bound $\overline{\kappa}$

The time $T^P(\kappa) = \underline{t}_\kappa$ and the bound $\overline{\kappa}$ are both obtained from the condition $L_2(t,\kappa) - F(t) = 0$ which has the specific form

$$\left(\Pi^M - X(0)e^{-\lambda t}\right)e^{-rt} - \left(\Pi^M - \Pi^D\right)\left(\frac{r}{\lambda + r}\frac{\Pi^D}{(1-\kappa)X(0)}\right)^{\frac{r}{\lambda}} - \frac{\lambda r^{\frac{r}{\lambda}}}{(\lambda + r)^{\frac{\lambda + r}{\lambda}}}\frac{\left[\Pi^D\right]^{\frac{\lambda + r}{\lambda}}}{[X(0)]^{\frac{r}{\lambda}}} = 0 \tag{5}$$

for $t < T^F$. $L_2(t;\kappa) - F(t)$ is a strictly quasiconcave function of t with a maximum at T^L, which shifts downward as κ increases. From the analysis of the symmetric case we know $L_2(t;0) - F(t)$ has two roots, $T^P < T^F$. Up to $\overline{\kappa}$, an increase in κ shifts $L_2(t;\kappa) - F(t)$ downward, increasing the lower root $\underline{t}_\kappa$ and decreasing the upper root $\bar{t}_\kappa$, so that $d\underline{t}_\kappa/d\kappa > 0$ whereas $d\bar{t}_\kappa/d\kappa < 0$. For $\kappa \in (0, \overline{\kappa})$ therefore $\underline{t}_\kappa$ is well-defined and lies in $\left(T^P, T^L\right)$. As $\overline{\kappa}$ is the capital stock for which firm 2's investment incentive vanishes at the maximizer T^L, it satisfies $L_2\left(T^L;\overline{\kappa}\right) = F\left(T^L\right)$. Substituting for $t = T^L$ in (5) (note that $e^{-\lambda T^L} = r\Pi^M/(\lambda+r)X(0)$ and $e^{-rT^L} = \left(r\Pi^M/(\lambda+r)X(0)\right)^{r/\lambda}$) and dividing by $(r/((\lambda+r)X(0)))^{r/\lambda}$ gives

$$\frac{\lambda}{\lambda+r}\left[\Pi^M\right]^{\frac{\lambda+r}{\lambda}} - \left(\Pi^M - \Pi^D\right)\left(\frac{\Pi^D}{1-\overline{\kappa}}\right)^{\frac{r}{\lambda}} - \frac{\lambda}{\lambda+r}\left[\Pi^D\right]^{\frac{\lambda+r}{\lambda}} = 0$$

which after rearrangement yields the expression in footnote 5.

A.2 Asymmetric preemption equilibrium

In the text, the investment game is discussed in terms of static choices of investment plans whose outcomes unfold over time and Proposition 1 gives the equilibrium outcome assuming the initial input price satisfies $X(0) \geq \Pi^M$. In order to develop a comprehensive view of the investment game, Proposition A1 below allows for an arbitrary initial input price. Let $t_0 \geq 0$ therefore denote the initial time at which both firms choose when to invest absent interim rival entry by selecting planned investment times $T_i \geq t_0$, $i \in \{1,2\}$, which they revise if they observe their rival has entered. If the firms set $T_1 = T_2 = T$ efficient tie-breaking is assumed and they obtain continuation payoffs $S_1(T;\kappa) = L_1(T;\kappa) - \max\{L_2(T;\kappa) - F(T), 0\}$ and $S_2(T;\kappa) = F(T)$ respectively.[16] The payoff functions in this game are therefore

$$V_1(T_1,T_2) = \begin{cases} L_1(T_1;\kappa), & T_1 < T_2 \\ S_1(T;\kappa), & T_1 = T_2 = T \\ F_1(T_2;\kappa), & T_1 > T_2 \end{cases}$$

[16] An economic motivation for this simultaneous investment profile is if there is an instantaneous capacity constraint so that firm 1, which has a higher valuation as $L_1(T;\kappa) - F_1(T;\kappa) > L_2(T;\kappa) - F(T)$, would bid more for the input (see Section 4.1). Alternative approaches to simultaneous investments are discussed at the end of this section.

and

$$V_2(T_1,T_2) = \begin{cases} L_2(T_2;\kappa), & T_2 < T_1 \\ S_2(T;\kappa), & T_2 = T_1 = T \\ F(T_1), & T_2 > T_1. \end{cases}$$

An equilibrium consists of a pair of planned investment times $\left(\widehat{T}_1, \widehat{T}_2\right)$ such that $\widehat{T}_1 \in \arg\max_{T_1 \geq t_0} V_1\left(T_1, \widehat{T}_2\right)$ and $\widehat{T}_2 \in \arg\max_{T_2 \geq t_0} V_1\left(\widehat{T}_1, T_2\right)$.

Proposition A1 *In an equilibrium of the investment game starting at* t_0*,* $\widehat{T}_1 = \max\{t_0, \min\{T^L(\kappa), T^P(\kappa)\}\}$ *and* $\widehat{T}_2 \in \{\max\{t_0, T^P(\kappa)\}\}$ *if* $\max\{t_0, T^P(\kappa)\} < T^L(\kappa)$ *or* $[\max\{t_0, \min\{T^L(\kappa), T^P(\kappa)\}\}, \infty]$ *otherwise.*

Proof If $t_0 \geq T^F$ immediate investment is weakly dominant for both firms as $L_i(t;\kappa) = F_i(t;\kappa)$ with $F_i(t;\kappa)$ decreasing over (T^F, ∞) for $i \in \{1,2\}$. Any $\left(\widehat{T}_1, \widehat{T}_2\right) \in [t_0, \infty)^2$ satisfying $\min\left\{\widehat{T}_1, \widehat{T}_2\right\} = t_0$ is an equilibrium in this case. The remainder of the argument therefore focuses on starting times $t_0 < T^F$. Moreover as $L_i(t;\kappa)$ is decreasing over (T^F, ∞), leading after T^F is dominated and both firms can be constrained to choosing times $T_i \in [t_0, T^F]$.

The equilibrium characterization involves three cases that depend on the level of κ, which determines whether the high cost firm's preemption threshold is defined and, when this threshold is defined, whether it is greater or smaller than the low cost firm's monopoly threshold.

If $\kappa > \overline{\kappa}$, then $\Theta_2(\kappa) = \{T^F\}$. As $L_2(t;\kappa) - F(t) < 0$ for all $t < T^F$, leading before T^F is not individually rational for firm 2. The game therefore reduces to an individual decision problem for firm 1. Its optimum is $\widehat{T}_1 = \max\{t_0, T^L(\kappa)\}$ and firm 2 indifferently chooses any $\widehat{T}_2$ in $\left[\widehat{T}_1, T^F\right]$, which leaves firm 1's equilibrium profit unaffected given the tie-breaking assumption.

If $\kappa \in [\widehat{\kappa}, \overline{\kappa}]$, then $T^L(\kappa) \leq T^P(\kappa) \leq \bar{t}_\kappa < T^F$ (see Figure 1) so $\Theta_2(\kappa) = [T^P(\kappa), \bar{t}_\kappa] \cup \{T^F\}$. If $\kappa < \overline{\kappa}$ then the preemption range is non-empty whereas $\kappa = \overline{\kappa}$ is the limiting case $\Theta_2(\kappa) = \{T^L, T^F\}$. Consider first starting times

$t_0 < T^P(\kappa)$. Firm 1's myopic optimum is $\max\{t_0, T^L(\kappa)\}$, and as $T^P(\kappa) \geq T^L(\kappa)$ so that $L_2(t;\kappa) - F(t) < 0$ for all $t < T^L(\kappa)$ no individually rational choice of firm 2 can constrain firm 1. Firm 1 therefore sets its optimal time $\widehat{T}_1 = \max\{t_0, T^L(\kappa)\}$ whereas firm 2 indifferently chooses any in $\widehat{T}_2 \in \left[\widehat{T}_1, T^F\right]$, which leaves firm 1's equilibrium profit unaffected given the tie-breaking assumption. Consider next starting times t_0 with $T^P(\kappa) \leq t_0 \leq \bar{t}_\kappa$. As $t_0 \geq T^L(\kappa)$, $L_1(t;\kappa)$ is decreasing over $\left(t_0, T^F\right)$ and moreover $L_1(t;\kappa) \geq S_1(t;\kappa) > F_1(t;\kappa)$. Immediate investment is therefore strictly dominant for firm 1 so that $\widehat{T}_1 = t_0$. Because $S_2(t_0;\kappa) = F(t_0)$, firm 2 is indifferent between simultaneous investment and following and hence between all $\widehat{T}_2 \in \left[t_0, T^F\right]$. Note that if $t_0 \in \left(T^P(\kappa), \bar{t}_\kappa\right)$ (with $\kappa < \overline{\kappa}$) firm 1's payoff is a correspondence whose value is either $L_1(t_0;\kappa) - (L_2(t_0;\kappa) - F(t_0))$ (if $\widehat{T}_2 = \widehat{T}_1$) or $L_1(t_0;\kappa)$ (if $\widehat{T}_2 > \widehat{T}_1$). Consider finally starting times $t_0 > \bar{t}_\kappa$. As $L_1(t;\kappa) = S_1(t;\kappa)$ is decreasing and $L_1(t;\kappa) > F_1(t;\kappa)$, immediate investment is dominant for firm 1 which sets $\widehat{T}_1 = t_0$ whereas firm 2 indifferently chooses any in $\widehat{T}_2 \in \left[\widehat{T}_1, T^F\right]$, which leaves firm 1's equilibrium profit unaffected given the tie-breaking rule. To summarize therefore, in this case $\widehat{T}_1 = \max\{t_0, T^L(\kappa)\}$ and $\widehat{T}_2 \in \left[\max\{t_0, T^L(\kappa)\}, T^F\right]$.

If $\kappa \in (0, \widehat{\kappa})$ then $T^P(\kappa) < T^L(\kappa) < \bar{t}_\kappa$ (see Figure 1) so $\Theta_2(\kappa) = \left[T^P(\kappa), \bar{t}_\kappa\right] \cup \left\{T^F\right\}$ and the preemption range is non-empty. Consider first starting times $t_0 < T^L(\kappa)$. If moreover $t_0 < T^P(\kappa)$, individual rationality rules out any choice $T_2 \in \left[t_0, T^P(\kappa)\right)$ by firm 2. This implies that $T_1 \geq T^P(\kappa)$, as $L_1(t;\kappa)$ is increasing over $\left[t_0, T^L(\kappa)\right)$ and $S_1(T_2;\kappa) = L_1(T_2;\kappa)$ if $T_2 = T^P(\kappa)$. For given $t_0 < T^L(\kappa)$ therefore, an equilibrium must satisfy $\widehat{T}_1, \widehat{T}_2 \geq \max\{t_0, T^P(\kappa)\}$. Moreover $T_j > T_i \geq \max\{t_0, T^P(\kappa)\}$, $i, j \in \{1, 2\}$, $i \neq j$ cannot constitute an equilibrium. If $T_i = \max\{t_0, T^P(\kappa)\}$ then firm i finds if profitable to delay by choosing some T_i' with $\max\{t_0, T^P(\kappa)\} < T_i' < T_j$, whereas if $T_i > \max\{t_0, T^P(\kappa)\}$ then as leader payoffs are continuous with $L_1(t;\kappa) -$

$F_1(t;\kappa), L_2(t;\kappa) - F(t) > 0$, all $t \in \left\{\max\left\{t_0, T^P(\kappa)\right\}, T_i\right\}$, preemption by firm j is profitable and it chooses some $T_j^{\prime}$ with $\max\left\{t_0, T^P(\kappa)\right\} < T_j^{\prime} < T_i$. $T_1 = T_2 > \max\left\{t_0, T^P(\kappa)\right\}$ cannot constitute an equilibrium, as firms similarly have an incentive to set any T_i with $\max\left\{t_0, T^P(\kappa)\right\} < T_i < T_j$. Only $\widehat{T}_1 = \widehat{T}_2 = \max\left\{t_0, T^P(\kappa)\right\}$ remains as a candidate equilibrium. $\widehat{T}_i$ is a (possibly weak) best response to $\widehat{T}_j = \max\left\{t_0, T^P(\kappa)\right\}$ as $L_i(t;\kappa)$ is increasing over $\left(t_0, \widehat{T}_j\right)$ (if $t_0 < T^P(\kappa)$) and either $S_1\left(\widehat{T}_2;\kappa\right) > F_1(t;\kappa)$, all $t > \widehat{T}_2$ (firm 1), or $S_2\left(\widehat{T}_1;\kappa\right) = F(t)$, all $t > \widehat{T}_1$ (firm 2). Note that by the tie-breaking rule the equilibrium payoffs are $L_1\left(\max\left\{t_0, T^P(\kappa)\right\};\kappa\right) - \left[L_2\left(\max\left\{t_0, T^P(\kappa)\right\};\kappa\right) - F\left(\max\left\{t_0, T^P(\kappa)\right\}\right)\right]$ for firm 1 and $F(t_0)$ for firm 2. Consider next $t_0 \geq T^L(\kappa)$. A similar argument to the previous case establishes that firm 1 invests immediately whereas firm 2 is indifferent between simultaneous investment and following. To summarize therefore, in this case $\widehat{T}_1 = \max\left\{t_0, T^P(\kappa)\right\}$ and $\widehat{T}_2 \in \left\{\max\left\{t_0, T^P(\kappa)\right\}\right\}$ if $t_0 < T^L(\kappa)$ or $\left[t_0, T^F\right]$ otherwise. $\square$

For any κ and any t_0, firm 1 leads in equilibrium either directly because it chooses $\widehat{T}_1 < \widehat{T}_2$ or due to efficient rationing. Firm 2 invariably obtains a follower payoff $\widehat{V}_2(t_0) = F(t_0)$, but when the game starts in the interior of the preemption range firm 1's payoff is a correspondence whose value is the sum of its follower payoff and either a differential rent or the full positional rent depending upon whether firm 2 invests simultaneously or as a follower:

$$\widehat{V}_1(t_0) =$$
$$\begin{cases} L_1\left(\max\left\{t_0, T^P(\kappa)\right\};\kappa\right) - \left[L_2\left(\max\left\{t_0, T^P(\kappa)\right\};\kappa\right) - F(t_0)\right], & t_0 < T^L(\kappa) \\ L_1(t_0;\kappa) - \mathbf{1}_{\widehat{T}_1=\widehat{T}_2}\left[L_2(t_0;\kappa) - F(t_0)\right], & T^L(\kappa) \leq t_0 \leq \bar{t}_\kappa \\ L_1(t_0;\kappa), & t_0 > \bar{t}_\kappa \end{cases} \quad (\text{if } \kappa < \widehat{\kappa}),$$

or

$$\begin{cases} L_1\left(\max\left\{t_0, T^L\left(\kappa\right)\right\}; \kappa\right), & t_0 < T^P\left(\kappa\right) \\ L_1\left(t_0; \kappa\right) - \mathbf{1}_{\widehat{T}_1 = \widehat{T}_2}\left[L_2\left(t_0; \kappa\right) - F(t_0)\right], & T^P\left(\kappa\right) \leq t_0 \leq \bar{t}_\kappa \quad (\text{if } \widehat{\kappa} \leq \kappa \leq \overline{\kappa}), \\ L_1\left(t_0; \kappa\right), & t_0 > \bar{t}_\kappa \end{cases}$$

or

$$L_1\left(\max\left\{t_0, T^L\left(\kappa\right)\right\}; \kappa\right) \text{ (if } \kappa > \overline{\kappa}).$$

In order to have a full dynamic view of the game, consider what would happen if firms decided at any point in time to either wait or invest. In this perspective any $t_0' > t_0$ defines a subgame starting at t_0' of the investment game which is reached if no firm invests in the interval $[t_0, t_0')$. The equilibrium timing choices in Proposition 1A can thus be associated with closed-loop strategies which prescribe the actions *{wait}* if $t_0' < \widehat{T}_i$ and *{invest}* otherwise, $i \in \{1, 2\}$. Almost all of the firm 2 timing choices given in Proposition 1A are suboptimal if some $t_0' \in \left(T^P\left(\kappa\right), \bar{t}_\kappa\right)$ is reached and firm 1 does not invest, whereas only $\widehat{T}_2 = \left\{\max\left\{t_0, T^P\left(\kappa\right)\right\}\right\}$ (if $\kappa \leq \overline{\kappa}$ and $t_0 \leq \bar{t}_\kappa$) or $\left\{\max t_0, T^F\right\}$ (otherwise) remains, and can accordingly be viewed as a perfect equilibrium.

The main novelty is the simultaneous move profile $\left(S_1\left(T; \kappa\right), S_2\left(T; \kappa\right)\right)$, which is discussed in more detail in Section 4.1. Simultaneous moves raise well-known and somewhat technical issues in preemption games. A common alternative tie-breaking assumption in the literature is to posit that roles are determined by a coin toss so that $S_i = 0.5 L_i + 0.5 F_i, i \in \{1, 2\}$. If the continuous time game is understood to be the limit of a series of discrete time games over a grid that becomes arbitrarily fine and provided that $t_0 < T^P\left(\kappa\right)$, the same equilibrium outcome as described above results as firm 1 prefers to lead just before $T^P\left(\kappa\right)$ rather than to invest simultaneously right after $T^P\left(\kappa\right)$ is reached. Fudenberg and Tirole [8] have a different approach which does not impose an instantaneous capacity constraint but instead augments the strategy space to allow the continuous time approximation to represent mixed strategy discrete time equilibria. Multiple equilibria arise if $t_0 \in \left[T^P(\kappa), \bar{t}_\kappa\right]$ including a mixed strategy equilibrium involving rent dissipation. Fudenberg and Tirole also identify a simultaneous investment equilibrium in their model, but as here $M_2(t) = F(t)$ over (T^F, ∞) is decreasing, $M_2(t)$ is maximized at T^F in the present model so the equilibrium that they identify cannot arise.

A.3 Stochastic case

Fudenberg and Tirole [8]'s analysis extends to the investment under uncertainty framework described by Dixit and Pindyck [6]. This supplemental section gives the main steps involved in defining continuation payoffs, the loci $X_1^L(\kappa)$ and $k^P(x)$, and obtaining the comparative static $\partial k^P\left(\widehat{X}\right)/\partial\sigma$ discussed in the text.

A.3.1 Continuation payoffs

For a given level of the current price $X(t) = x$ of the capital good, let $V^F(x)$ denote the value of an option on the stationary duopoly profit stream with capitalized value Π^D. In the continuation region, this option value satisfies

$$rV^F(x)\,dt = EdV^F(x).$$

Developing the right hand side using Itô's lemma and taking the expectation yields

$$\frac{\sigma^2}{2}x^2\left[V^F(x)\right]'' + \mu x\left[V^F(x)\right]' - rV^F(x) = 0$$

after rearranging. The boundary conditions are $\lim_{x\to\infty} V^F(x) = 0$ and $V^F\left(X^F\right) = \Pi^D - X^F$, and the smooth pasting condition is $\left[V^F\right]'\left(X^F\right) = -1$. For $\sigma > 0$ the fundamental quadratic $(\sigma^2/2)b\,(b-1)+b\mu-r=0$ has two distinct roots of which only the negative root $b' = -\left(\left(\mu/\sigma^2\right)-(1/2)\right)-\sqrt{\left((1/2)-(\mu/\sigma^2)\right)^2+(2r/\sigma^2)}$ is consistent with the first boundary condition. The option value is therefore of the form $V^F(x) = Ax^{b'}$. Setting $\gamma=-b'$, the option's exercise threshold is $X^F = (\gamma/(\gamma+1))\,\Pi^D$, the multiplicative constant is $A = \left[X^F\right]^{\gamma+1}/\gamma$ and

$$V^F(x) = \begin{cases} \Pi^D - x, & x \le X^F \\ Ax^{\gamma}, & x > X^F. \end{cases}$$

The value in initial currency units of obtaining the follower option at the stochastic time $\tau(X) = \inf\{t \ge 0 |\, X(t) \le X\}$ at which the input price first hits a threshold $X \le x$ is therefore

$$F(X) = E_x\left[V^F(X)\,e^{-r\tau(X)}\right] = V^F(X)\left(\frac{X}{x}\right)^{\gamma}.$$

The leader payoffs are

$$\begin{aligned} L_1\left(X;\overline{K}_1\right) &= E_x\left[\int_{\tau(X)}^{\tau\left(\min\{X,X^F\}\right)} \pi^M e^{-rs}ds + \int_{\tau(\min\{X,X^F\})}^{\infty} \pi^D e^{-rs}ds - \left(1-\overline{K}_1\right)X(\tau(X))e^{-r\tau(X)}\right] \\ &= \left(\Pi^M - \left(1-\overline{K}_1\right)X\right)\left(\frac{X}{x}\right)^{\gamma} - \left(\Pi^M - \Pi^D\right)\left(\frac{\min\{X,X^F\}}{x}\right)^{\gamma} \end{aligned}$$

for firm 1 and

$$\begin{aligned} L_2\left(X;\overline{K}_1\right) &= E_x\left[\int_{\tau(X)}^{\tau\left(\min\{X,X_1^F(\overline{K}_1)\}\right)} \pi^M e^{-rs}ds + \int_{\tau\left(\min\{X,X_1^F(\overline{K}_1)\}\right)}^{\infty} \pi^D e^{-rs}ds - X(\tau(X))e^{-r\tau(X)}\right] \\ &= \left(\Pi^M - X\right)\left(\frac{X}{x}\right)^{\beta} - \left(\Pi^M - \Pi^D\right)\left(\frac{\min\{X,X_1^F\left(\overline{K}_1\right)\}}{x}\right)^{\beta} \end{aligned}$$

for firm 2, where $X_1^F\left(\overline{K}_1\right) = (\gamma/(\gamma+1))\left(\Pi^D/\left(1-\overline{K}_1\right)\right)$ denotes firm 1's duopoly investment threshold. Maximization of the strictly quasiconcave function $L_1\left(X;\overline{K}_1\right)$ with respect to X results in the monopoly threshold $X_1^L\left(\overline{K}_1\right)$ given in the text.

A.3.2 Preemptive capital accumulation policy

Firm 2's preemption threshold $X^P(\kappa)$ is the upper root of the condition $L_2(X;\kappa) = F\left(X^F\right)$, $X \in \left(X^F, x\right)$, which is well-defined provided that κ does not exceed the upper bound $\overline{\kappa}_X$ derived below. After normalization by $x^{-\gamma}$ this condition is

$$\left(\Pi^M - X\right) X^\gamma - \left(\Pi^M - \Pi^D\right)\left(\frac{\gamma}{\gamma+1}\frac{\Pi^D}{1-\kappa}\right)^\gamma - \frac{\gamma^\gamma\left[\Pi^D\right]^{\gamma+1}}{(\gamma+1)^{\gamma+1}} = 0.$$

Similarly to Appendix A.1, setting $X = X^L = (\gamma/(\gamma+1))\,\Pi^M$ yields the upper bound beyond firm 2's preemption range vanishes,

$$\overline{\kappa}_X = 1 - \left(\frac{\frac{\Pi^M}{\Pi^D} - 1}{\left(\frac{\Pi^M}{\Pi^D}\right)^{\gamma+1} - \frac{1}{\gamma+1}}\right)^{\frac{1}{\gamma}}.$$

For $\kappa \in (0, \overline{\kappa}_X)$, the upper root $X^P(\kappa) \in \left(X^L, X^P\right)$ is well-defined and rearranging yields the expression of $k^P(x)$ given in the text.

A.3.3 Equilibrium and sensitivity analysis

In (X, K_1)-space, the locus $X_1^L\left(\overline{K}_1\right) = (\gamma/(\gamma+1))\left(\Pi^M/\left(1-\overline{K}_1\right)\right)$ is an increasing and convex function of $\overline{K}_1 \in [0,1)$, with $X_1^L(0) = X^L$ and $\lim_{\overline{K}_1\to 1} X_1^L\left(\overline{K}_1\right) = \infty$. The locus $k^P(X)$ is decreasing for $X \in \left(X^L, X^P\right)$ with $k^P(X^L) = \overline{\kappa}_X$ and $k^P(X^P) = 0$. There is therefore a unique intersection of the two loci, which is $\left(\widehat{X}, k^P(\widehat{X})\right)$.

As $\partial X_1^L/\partial\gamma = X_1^L/(\gamma(\gamma+1)) > 0$ and $\partial\gamma/\partial\sigma < 0$, greater volatility shifts

$X_1^L\left(\overline{K}_1\right)$ leftward. The effect of greater volatility on $k^P(x)$ is more complex, but for geometric Brownian motion a standard result is that $\partial X^P(\kappa)/\partial\gamma > 0$ (Huisman [9]) so an increase in volatility shifts $k^P(x)$ down and to the left.

To study the sensitivity of $\widehat{X}$, substitute for $1-k^P(\widehat{X})$ in the equilibrium condition and raise to the power γ to get

$$\widehat{X}^\gamma = \frac{\left(\frac{\Pi^M}{\Pi^D}-\frac{\widehat{X}}{\Pi^D}\right)\left(\frac{\widehat{X}}{\Pi^D}\right)^\gamma - \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}}}{\frac{\Pi^M}{\Pi^D}-1}\left[\Pi^M\right]^\gamma .$$

Define $\widehat{x}=\widehat{X}/\Pi^D$ and $m=\Pi^M/\Pi^D$ to express this condition more compactly. Rearranging yields an implicit definition of $\widehat{x}$,

$$m^\beta \widehat{x}^{\beta+1} - \left(m^{\beta+1}-m+1\right)\widehat{x}^\beta + \frac{m^\beta\beta^\beta}{(\beta+1)^{\beta+1}} = 0.$$

Letting $\Gamma(\widehat{x},\gamma)$ denote the left-hand side, $d\widehat{x}/d\gamma = -\left(\partial\Gamma/\partial\gamma\right)/\left(\partial\Gamma/\partial\widehat{x}\right)$. First,

$$\frac{\partial\Gamma}{\partial\widehat{x}} = \left((\gamma+1)\,m^\gamma\widehat{x} - \gamma\left(m^{\gamma+1}-m+1\right)\right)\widehat{x}^{\gamma-1}.$$

As $\widehat{X}\geq X^L$, $\widehat{x}\geq(\gamma/(\gamma+1))\,m$ and hence $(\gamma+1)\,m^\gamma\widehat{x}\geq\gamma m^{\gamma+1}$ so $\partial\Gamma/\partial\widehat{x}\geq\gamma(m-1)\widehat{x}^{\gamma-1}>0$. Next,

$$\frac{\partial\Gamma}{\partial\gamma} = m^\gamma\widehat{x}^{\gamma+1}\ln(m\widehat{x}) - m^{\gamma+1}\widehat{x}^\gamma\ln m - \left(m^{\gamma+1}-m+1\right)\widehat{x}^\gamma\ln\widehat{x} + \frac{m^\gamma\gamma^\gamma}{(\gamma+1)^{\gamma+1}}\ln\left(\frac{m\gamma}{\gamma+1}\right).$$

As $m^\gamma\gamma^\gamma/(\gamma+1)^{\gamma+1} = \left(m^{\gamma+1}-m+1\right)\widehat{x}^\gamma - m^\gamma\widehat{x}^{\gamma+1}$ by definition,

$$\frac{\partial\Gamma}{\partial\gamma} = m^\gamma\widehat{x}^{\gamma+1}\ln\left(\frac{(\gamma+1)\,\widehat{x}}{\gamma}\right) - m^{\gamma+1}\widehat{x}^\gamma\ln m + \left(m^{\gamma+1}-m+1\right)\widehat{x}^\gamma\ln\left(\frac{\gamma m}{(\gamma+1)\,\widehat{x}}\right).$$

Because $\widehat{x}>\gamma/(\gamma+1)$ the first summand is positive whereas the other terms are negative. Moreover $m^{\gamma}\widehat{x}^{\gamma+1} < \left(m^{\gamma+1} - m + 1\right)\widehat{x}^{\gamma}$, so

$$\begin{aligned}
\frac{\partial \Gamma}{\partial \gamma} &< \left(m^{\gamma+1} - m + 1\right)\widehat{x}^{\gamma} \ln\left(\frac{(\gamma+1)\,\widehat{x}}{\gamma}\right) - m^{\gamma+1}\widehat{x}^{\gamma} \ln m + \left(m^{\gamma+1} - m + 1\right)\widehat{x}^{\gamma} \ln\left(\frac{\gamma m}{(\gamma+1)\,\widehat{x}}\right) \\
&= -(m-1)\,\widehat{x}^{\gamma} \ln m < 0.
\end{aligned}$$

Therefore $\partial\widehat{x}/\partial\gamma > 0$ and hence $\partial\widehat{X}/\partial\sigma < 0$.