Exercise Boundary Fitting for Real Option Valuation *

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Abstract

Real option analysis is recognized as a superior method to quantify the value of real-world investment opportunities where managerial flexibility can influence their worth, as compared to standard discounted cash-flow methods typically used in industry. However, realistic models that try to account for a number of risk factors can be mathematically complex, and in situations where many future outcomes are possible, many layers of analysis may be required. The focus of this research is the development of a real options valuation methodology geared towards practical use with mining valuation as a context. A key innovation of the methodology to be presented is the idea of fitting optimal decision making boundaries to optimize the expected value, based on Monte Carlo simulated stochastic processes that represent important uncertain factors. Our specific emphasis in this work will be to explore theoretical / numerical aspects associated with the simulation methodology as they pertain to 1) a Bermudan put option, 2) a Bermudan-like option with variable strike price, 3) an American put option and 4) a build / abandon real option example.

1 Introduction

Real option analysis (ROA) is recognized as a superior method to quantify the value of real-world investment opportunities where managerial flexibility can influence their worth, as compared to standard discounted cash-flow methods typically used in industry. ROA stems from the work of Black and Scholes (1973) on financial option valuation. Myers (1977) recognized that both financial options and project decisions are exercised after uncertainties are resolved. Early techniques therefore applied the Black-Scholes equation directly to value put and call options on tangible assets (see, for example, Brennan and Schwartz (1985)). Since then, ROA has gained significant attention in academic and business publications, as well as textbooks (Copeland and Tufano (2004), Trigeorgis (1996)).

The ability for managers to react to uncertainties at a future time adds value to projects, and since this value is not captured by standard DCF methods, erroneous decision making may result (Trigeorgis (1996)). An excellent empirical review of ex-post investment decisions made in copper mining showed that fewer than half of investment timing decisions were made at the right time and 36 of the 51 projects analyzed should have chosen an extraction capacity of 40% larger or

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smaller (Auger and Guzman (2010)). The authors were unaware of any mining firm basing all or part of their decision making on the systematic use of ROA and emphasize that the "failure to use ROA to assess investments runs against a basic assumption of neoclassical theory: under uncertainty, firms ought to maximize their expected profits". They make the case that irrational decision making exists within the industry due to a lack of real option tools available for better analysis. A number of surveys across industries have found that the use of ROA is in the range of 10-15% of companies, and the main reason for lack of adoption is model complexity (Hartmann and Hassan (2006), Block (2007), Truong, Partington, and Peat (2008), Bennouna, Meredith, and Marchant (2010), Dimitrakopoulos and Abdel Sabour (2007)). As mentioned, this work is focused on developing a practical Monte Carlo simulation-based real options methodology as Monte Carlo simulation can be easily understood by managers and allows for the modelling of multiple stochastic factors (Longstaff and Schwartz (2001)).

Realistic models that try to account for a number of risk factors can be mathematically complex, and in situations where many future outcomes are possible, many layers of analysis may be required. As a motivating example, consider the case of a greenfield mining site, where the life of the mine lease is 2 years, construction will take half a year, the ore price, S_t, follows geometric Brownian motion (GBM) and the per unit costs are K to construct and C_{ab} to abandon, and C_{op} is the operating cost rate. For a given set of parameters, the scenarios are depicted in Figure 1 in a binomial tree. The S_t process of the first panel is used to determine the operating cash-flow, calculated as $CF_t = S_t - C_{op}$. For this case, we assume that abandonment can occur at year 2 only, with cost C_{ab}. The real option can be valued in a recursive manner and the different scenarios are presented in Figure 2. Since it takes half a year for construction, the latest we would construct the mine is at year 1. In this case, only the cash-flows associated with the last period are of value and these are discounted twice to year 1 (relevant probabilities and discounting factor were used) to determine the expected value. At year 1, there are 3 possible values for S_t and thus three possible valuations for the cash-flows. Clearly, we would only invest if the total expected value of the cash-flows minus the investment cost, K, is greater than 0. As shown, only one of the three scenarios has a positive value, the others are set to 0. We continue to discount these expected values to reach a valuation of \$1.0 at year 0. Similar valuations are done for the case of building at years 0.5 and 0. Based on the analysis, we see that it is best to wait one period (half year) before constructing and then choosing to construct only if the price $S_{t=0.5} = \$10.7$ is realized. The overall project value at $t = 0$ is determined to be \$2.9. Note that even for this very simple problem, a separate binomial tree was required at each decision making time point. If we allowed for early abandonment, many more trees would be required. If we added a second stochastic factor, we would have another spatial dimension. Clearly, to value a complex real option the model's complexity increases substantially. This complexity leads us to the overall objective of developing a practical simulation based real options methodology that can model realistic decision-making scenarios encountered in industry.

The focus of this research is the development of a real options valuation methodology geared towards practical use with mining valuation as a context. A key innovation of the methodology to be presented is the idea of fitting optimal decision making boundaries to optimize the expected value, based on Monte Carlo simulated stochastic processes that represent important uncertain factors. Our specific emphasis in this work will be to explore theoretical / numerical aspects associated with the simulation methodology as they pertain to 1) a Bermudan put option, 2) a Bermudan-like option

Price Process (St)					Cash-Flow per Period				
0	**0.5**	**1**	**1.5**	**2**	**0**	**0.5**	**1**	**1.5**	**2**
				13.3					15.3
			12.4					11.8	
		11.5		11.5			7.6		6.6
	10.7		10.7			3.7		3.7	
10.0		10.0		10.0	0.0		0.0		-1.0
	9.3		9.3			-3.4		-3.4	
		8.7		8.7			-6.6		-7.6
			8.1					-9.6	
				7.5					-13.3

Figure 1: Price process and resulting cash-flow.

Build at Year 1					Build at Year 0.5					Build at Year 0				
0	**0.5**	**1**	**1.5**	**2**	**0**	**0.5**	**1**	**1.5**	**2**	**0**	**0.5**	**1**	**1.5**	**2**
				15.3					15.3					15.3
			11.6					23.4					23.4	
		$\boxed{3.1}$		6.6			16.4		6.6			24.0		6.6
	1.8		3.4			$\boxed{5.0}$		7.1			14.5		7.1	
1.0		~~0.0~~		-1.0	**2.9**		1.3		-1.0	$\boxed{0.6}$		1.3		-1.0
	0.0		-3.7			~~0.0~~		-7.1			-6.8		-7.1	
		~~0.0~~		-7.6			-11.9		-7.6			-18.5		-7.6
			-9.8					-19.4					-19.4	
				-13.3					-13.3					-13.3

Figure 2: Real option valuation based on different build options.

with variable strike price K, 3) an American put option and 4) a simple build / abandon real option example.

2 Relevant Literature

We make the argument that a majority of real-world real options are either American or Bermudan type options – i.e. managers typically make strategic decisions either when there is a noticeable shift in important state variables (American), or decisions are made at predefined intervals (Bermudan). With this in mind, below we provide a brief review of relevant frameworks to estimate American / Bermudan options. Then, we provide a review of valuation methods utilized in mining as we see mining valuation in the real option context the leading use case example for our methodology.

A somewhat recent review of the valuation of American options was provided by Barone-Adesi (2005) where the LSMC of Longstaff and Schwartz (2001) was highlighted as "the most innovative", but other similar Monte Carlo based approaches have been proposed (Barraquand and Martineau (2007)) and the literature is abundant on the utilization of simulation and dynamic programming to value American options. There are many articles providing numerical or analytical approximations to an American exercise boundary (e.g. Barone-Adesi and Whaley (1987), Ju (1998), Tung (2016), Del Moral, Remillard, and Rubenthaler (2012)), however very few articles utilize a "forward" Monte Carlo approach, where the valuation method does not rely on the end result, but rather, the problem is worked forwards in time. Miao and Lee (2013) propose the use of forward Monte Carlo valuation, however the exercise boundary was estimated using the analytical method of Barone-Adesi and Whaley (1987), which negates the ability to develop a general model. Nasakkala and Keppo (2008) do utilize forward Monte Carlo simulation with a parametric boundary fitting approach in a hydropower planning problem utilizing two stochastic factors, namely the electricity price forward curve and random water inflow. However, in their approach the parameters are optimized for each path, an approach that is similar to that provided by (Broadie and Glasserman 1997) for calculating the upper bound for an American put option called the perfect foresight solution. We emphasize that in our approach we are proposing a general simulation approach to solve American / Bermudan models by optimizing a parameterized exercise boundary. Aside from convergence issues, the accuracy of our approach will be based on the assumed parametric boundary equation. One reason why our proposed approach may not have been presented in the financial derivatives literature is that most works are focused on improving efficiency and accuracy of the pricing models. In the real options context, where many assumptions are required to estimate the cash-flows, accuracy is not as important – what is important is ease of implementation, comprehension by decision makers and a tool for better decision making.

The academic literature is very rich in the field of mining valuation. Mining projects are laced with uncertainty and many discounted cash-flow (DCF) methods have been proposed in the literature to try to account for the uncertainty (Bastante, Taboada, Alejano, and Alonso (2008), Dimitrakopoulos (2011), Everett (2013), Ugwuegbu (2013)). Several guidelines/codes have been developed to standardize mining valuation (CIMVAL (2003), VALMIN (2015)). The main mining valuation approaches are income (i.e. cash-flows), market or cost based and the focus of this paper is on income-based real option valuation, which resemble American (or Bermudan) type financial options. Earlier real option works focused on modelling price uncertainty only (Brennan and Schwartz (1985), Dixit and Pindyck (1994), Schwartz (1997)), however the complexity in mining is significant and there are numerous risk factors. Simpler models based on lattice and finite difference methods (FDM) are difficult to implement in a multi-factor setting (Longstaff and Schwartz (2001)) and, also, it is extremely difficult to account for time dependent costs with multiple decision making points (Dimitrakopoulos and Abdel Sabour (2007)). Nevertheless, the simpler models continue to merit attention (Haque, Topal, and Lilford (2014), Haque, Topal, and Lilford (2016)). Dimitrakopoulos and Abdel Sabour (2007) utilize a multi-factor least squares Monte Carlo (LSMC) approach to account for price, foreign exchange and ore body uncertainty under multiple pre-defined operating scenarios (states). However, the model only allows for operation and irreversible abandonment — aspects such as optimal build time, expansion and mothballing are not considered. Similarly, Mogi and Chen (2007) use ROA and the method developed by Barraquand and Martineau (2007) to account for multiple stochastic factors in a four-stage gas field project. Abdel Saboura and Poulin (2010) develop a multi-factor LSMC model for a single mine expansion.

A review of 92 academic works found that most real options research is focused on dealing with very specific situations where usually no more than two real options are considered (Savolainen (2016)). While the LSMC allows for a more realistic analysis, methods presented to date are applicable only for the case where changes from one state to another does not change the fundamental stochastic factors with time. For example, modular expansion would be difficult to implement in such a model if the cost to expand was a function of time and impacts extracted ore quality due to the changing rate of extraction – these issues were considered in Davison, Lawryshyn, and Zhang (2015) and Kobari, Jaimungal, and Lawryshyn (2014). Also, modeling of multiple layers is still complex and will not lead to a methodology that managers can readily utilize.

3 Theory

As mentioned above, the specific objective of this work is to explore the theoretical / numerical aspects associated with the proposed simulation methodology. Specifically, we first consider a Bermudan put option for which we can determine a pseudo-analytical value for the option and the optimal early exercise threshold value, which we can then compare to those determined utilizing the proposed simulation method. Then we introduce a Bermudan-like option with a variable strike price, which can be considered to be a simplification of an optimal plant build size of a real option project valuation. Again, we present a pseudo-analytical formulation which can be compared to the proposed simulation method. Next, we explore the use of our methodology to price an American put option, where we examine different boundary fitting strategies, convergence and accuracy. Finally, we develop a simple build / abandon real option formulation utilizing our proposed simulation method. All results are presented in the Results section.

3.1 Bermudan Put Option

For the Bermudan put option, we consider a GBM stock price process, S_t, as

$$dS_t = rS_t dt + \sigma S_t d\widehat{W}_t, \tag{1}$$

where r is the risk-free rate, σ is the volatility and $\widehat{W}_t$ is a Wiener process in the risk-neutral measure. We assume the payoff of the option to be $\max(K - S_t, 0)$ and can be exercised at times $t = \tau$ and

$t = T$ where $\tau < T$. The value of the put option can be written as

$$V_0 = e^{-r\tau} \int_0^\infty \max\left(K - x, P_{BS_{put}}(x, \tau, T, r, \sigma, K)\right) f_{S_\tau}(x|S_0) dx, \tag{2}$$

where $P_{BS_{put}}(x, \tau, T, r, \sigma, K)$ is the Black-Scholes formula for the value of a European put option with current stock price x, maturity $T - \tau$, risk-free rate r, volatility σ and strike K, and $f_{S_\tau}(x|S_0)$ is the density for S_τ given S_0. As can be seen in equation (2), the optimal exercise occurs when

$$K - \theta^* = P_{BS_{put}}(\theta^*, \tau, T, r, \sigma, K), \tag{3}$$

where θ^* is used to denote the exercise price at $t = \tau$. Equation (3) can be solved using numerical methods and thus the option value simplifies to

$$V_0 = e^{-r\tau} \left(\int_0^{\theta^*} (K - x) f_{S_\tau}(x|S_0) dx + \int_{\theta^*}^\infty P_{BS_{put}}(x, \tau, T, r, \sigma, K) f_{S_\tau}(x|S_0) dx \right), \tag{4}$$

which, too, can be solved using standard numerical methods.

To explore numerical issues regarding the proposed boundary fitting methodology in the context of the Bermudan put option, we simulate N risk-neutral paths for S_t. For a given exercise price θ at $t = \tau$, the value of the option for the i-th path is given by

$$V_0^{(i)}(\theta) = \mathbb{1}_{S_\tau^{(i)} \le \theta} \left(K - S_\tau^{(i)}\right) e^{-r\tau} + \mathbb{1}_{S_\tau^{(i)} > \theta} \max\left(K - S_T^{(i)}, 0\right) e^{-rT}. \tag{5}$$

where $S_t^{(i)}$ represnets the value of S_t of the i-th simulated path. The optimal exercise price can then be estimated as

$$\theta^* = \arg\max_\theta \frac{1}{N} \sum_{i=1}^N V_0^{(i)}(\theta), \tag{6}$$

and the option value estimate becomes

$$V_0^{sim} = \frac{1}{N} \sum_{i=1}^N V_0^{(i)}(\theta^*). \tag{7}$$

Note that $\lim_{N \to \infty} V_0^{sim} = V_0$, as required.

3.2 Bermudan Option with Variable Strike

Next, we consider a Bermudan-like option with a variable strike K. This scenario represents a simplification of the idea of the optimal plant build size of a real option project valuation. We utilize the same stock price process as above (equation (1)). In this scenario, the option holder has the opportunity to exercise the option at $\tau < T$ at a cost of

$$C_K = \mathbb{1}_{K>0}(aK + b), \tag{8}$$

where $a > 0$ and $b > 0$ are some constants, to receive a payoff of $\min(S_T, K)$ at time T.

The value of the option at $t = \tau$ if exercised is

$$V_\tau^+ = e^{-r(T-\tau)} \int_0^\infty \min(x, K) f_{S_T}(x|S_\tau) dx \tag{9}$$

$$= S_\tau \Phi(A) + e^{-r(T-\tau)} K \left(1 - \Phi(B)\right) \tag{10}$$

where $\Phi(\cdot)$ is the standard normal distribution and

$$A \equiv \frac{\ln \frac{K}{S_\tau} - \left(r + \frac{\sigma^2}{2}\right)(T-\tau)}{\sigma\sqrt{T-\tau}}, \;\; B \equiv \frac{\ln \frac{K}{S_\tau} - \left(r - \frac{\sigma^2}{2}\right)(T-\tau)}{\sigma\sqrt{T-\tau}}. \tag{11}$$

To find the optimal K we set $\frac{\partial\left(V_\tau^+ - C_K\right)}{\partial K} = 0$ and solve for K,

$$K_{opt}(S_\tau) = S_\tau e^{\left(r - \frac{\sigma^2}{2} - \sqrt{2}\sigma\,\mathrm{erf}^{-1}\left(2ae^{r(T-\tau)}-1\right)\right)(T-\tau)}. \tag{12}$$

Furthermore, if we assume a maximum capacity of K_{max} then we can define

$$K^*(S_\tau) \equiv \min(K_{opt}(S_\tau), K_{max}) \tag{13}$$

and substituting $K = K^*(S_\tau)$ in equation (10),

$$V_\tau^{+*}(S_\tau) \equiv S_\tau \Phi(A) + e^{-r(T-\tau)} K^*(S_\tau)\left(1 - \Phi(B)\right). \tag{14}$$

The option value at $t = 0$ is thus

$$V_0 = e^{-r\tau}\mathbb{E}\left[\max\left(V_\tau^{+*}(S_\tau) - C_{K^*}(S_\tau), 0\right)\right] \tag{15}$$

$$= e^{-r\tau}\int_0^\infty \max\left(V_\tau^{+*}(x) - C_{K^*}(x), 0\right) f_{S_\tau}(x|S_0)dx. \tag{16}$$

To utilize simulation to estimate the option value, we assume a parametric function for K^* as a function of S_τ for the form $g(S_\tau|\vec{\theta})$, where $\vec{\theta} = [\theta_1, \theta_2, ..., \theta_n]'$ is a vector of constants. For a given $\vec{\theta}$, the option value of the i-th path is given by

$$V_0^{(i)}(\vec{\theta}) = e^{-rT}\min\left(S_T^{(i)}, g(S_\tau^{(i)}|\vec{\theta})\right) - e^{-r\tau}C_K\left(g(S_\tau^{(i)}|\vec{\theta})\right) \tag{17}$$

and the optimal parameters can be determined by

$$\vec{\theta}^* = \arg\max_{\vec{\theta}} \frac{1}{N}\sum_{i=1}^N V_0^{(i)}(\vec{\theta}), \tag{18}$$

from which the option value can be estimated as

$$V_0^{sim} = \frac{1}{N}\sum_{i=1}^N V_0^{(i)}(\vec{\theta}^*). \tag{19}$$

3.3 American Put Option

While there is no analytical solution for the value of an American put option, we utilize a binomial tree to estimate the value and the exercise boundary. To utilize the proposed exercise boundary fitting simulation method, we again simulate N risk-neutral paths for S_t and assume the exercise boundary to be function of the form $h(t|\vec{\eta})$, where $t \in [0, T]$ and $\vec{\eta} = [\eta_1, \eta_2, ..., \eta_n]'$ is a vector of constant parameters. We define the first passage of time when the i-th simulation path, $S_t^{(i)}$ (equation 1), hits the boundary as

$$\tau^{(i)} \equiv \min\{t > 0,\, T \,:\, S_t^{(i)} \leq h(t;\vec{\eta})\}, \tag{20}$$

where T is the time to maturity of the option.

The option value of the i-th path is thus

$$V_0^{(i)}(\vec{\eta}) = \mathbb{1}_{\tau^{(i)}<T}\left(K - S_t^{(i)}\right)e^{-r\tau^{(i)}} + \mathbb{1}_{\tau^{(i)}=T}\max\left(K - S_T^{(i)}, 0\right)e^{-rT}, \tag{21}$$

where K is the strike price. The optimal parameters can be determined by using methods similar to those above, where

$$\vec{\eta}^* = \arg\max_{\vec{\eta}} \frac{1}{N}\sum_{i=1}^{N} V_0^{(i)}(\vec{\eta}), \tag{22}$$

from which the option value can be estimated as

$$V_0^{sim} = \frac{1}{N}\sum_{i=1}^{N} V_0^{(i)}(\vec{\eta}^*). \tag{23}$$

3.4 Build / Abandon Real Option Example

In this subsection we develop our boundary fitting methodology for a build / abandon real option example. As above, we simulate N risk-neutral paths for S_t. We assume parametric functions $f_B(t;\vec{\theta}_B)$ for the construction (build) boundary and $f_A(t;\vec{\theta}_A)$ for the abandon boundary. Defining $\lambda_t^{(i)} = \{0,1,2,3\}$ as the state variable of the i-th simulation such that $\lambda_0^{(i)} = 0$, where 0 denotes the state where no construction has taken place, 1 denotes state where the plant is under construction, 2 denotes the state where the plant is in operation and 3 denotes the state where the plant has been abandoned. We define the first passage of time when $S_t^{(i)}$ hits the build boundary,

$$\tau_B^{(i)} \equiv \min\{t > 0 \,:\, S_t^{(i)} \geq f_B(t;\vec{\theta}_B)\}. \tag{24}$$

Similarly, the first passage of time when $S_t^{(i)}$ hits the abandon boundary after construction has begun can be defined as

$$\tau_A^{(i)} \equiv \min\left\{t > 0 \,:\, S_t^{(i)} \leq f_A(t;\vec{\theta}_A),\, \lambda_t^{(i)} \in \{1,2\}\right\}. \tag{25}$$

Clearly, the state variable is set as follows,

$$\lambda_t^{(i)} = \begin{cases} 0, & \text{for } t < \tau_B^{(i)} \text{ or } \tau_B^{(i)} \in \varnothing, \\ 1 & \text{for } \tau_B^{(i)} \leq t < \tau_B^{(i)} + \tau_c, \\ 2 & \text{for } \left\{\tau_B^{(i)} + \tau_c \leq t < \tau_A^{(i)}\right\} \text{ or } \left\{\tau_B^{(i)} + \tau_c \leq t \text{ and } \tau_A^{(i)} \in \varnothing\right\}, \\ 3 & \text{for } t \geq \tau_A^{(i)}, \end{cases} \tag{26}$$

where τ_c is a constant representing the time required for construction.

The real option value of the i-th path can be written as

$$V_0^{(i)}(\vec{\theta}) = -\mathbb{1}_{\lambda_T^{(i)}\geq 1}\left(Ke^{-r\tau_B^{(i)}} + C_{ab}\, e^{-r\left(\mathbb{1}_{\lambda_T^{(i)}=3}\tau_A^{(i)} + \mathbb{1}_{\lambda_T^{(i)}\in\{1,2\}}T\right)}\right) + \int_{\tau_B^{(i)}+\tau_c}^{T} \mathbb{1}_{\lambda_t^{(i)}=2}\, e^{-rs}\gamma\left(S_s^{(i)} - C_{op}\right) ds \tag{27}$$

where, C_{ab} is the cost to abandon or close the plant, C_{op} is the per unit operating cost and γ is the rate of extraction of the mineral. Defining $\vec{\theta} \equiv [\vec{\theta}_B, \vec{\theta}_A]'$, the optimal parameters defining the build and abandon exercise boundaries can be determined as

$$\vec{\theta}^* = \arg\max_{\vec{\theta}} \frac{1}{N} \sum_{i=1}^{N} V_0^{(i)}(\vec{\theta}), \tag{28}$$

from which the option value can be estimated as

$$V_0^{sim} = \frac{1}{N} \sum_{i=1}^{N} V_0^{(i)}(\vec{\theta}^*). \tag{29}$$

4 Results

In the following subsections we present some results of the simulation experiments that were performed for 1) the Bermudan put option, 2) the Bermudan-like option with variable strike price K, 3) the American put option and 4) the build / abandon real option example.

4.1 Bermudan Put Option

For the Bermudan option, we assume the following parameters:

- $S_0 = 5$
- $K = 5$
- $\tau = 1$
- $T = 2$
- $r = 3\%$
- $\sigma = 10\%$.

For these parameters the pseudo-analytical results, using equations (4) and (3), respectively, are:

- $V_0 = 0.1688$
- $\theta^* = 4.7571$.

Histograms of V_0^{sim} of equation (7) resulting from the simulations are presented in Figure 3, where the number of simulation paths was varied from $N = 10^2$ to $N = 10^6$. In each case, 1000 simulations were performed. In Table 1 we present the mean and standard deviation of the 1000 simulation runs for increasing N for V_0 and θ^*. As expected, as N is increased, the values for V_0 and θ^* approach those of the pseudo-analytical solution.

Table 1: Bermudan put option convergence for 1000 simulation runs; mean value and (standard deviation).

	100 Paths		1000 Paths		10,000 Paths		100,000 Paths		1,000,000 Paths	
V_0	0.1744	(0.0257)	0.1705	(0.0080)	0.1692	(0.0025)	0.1689	(0.0008)	0.1689	(0.0003)
θ^*	4.7052	(0.2324)	4.7353	(0.0819)	4.7538	(0.0334)	4.7564	(0.0157)	4.7572	(0.0070)

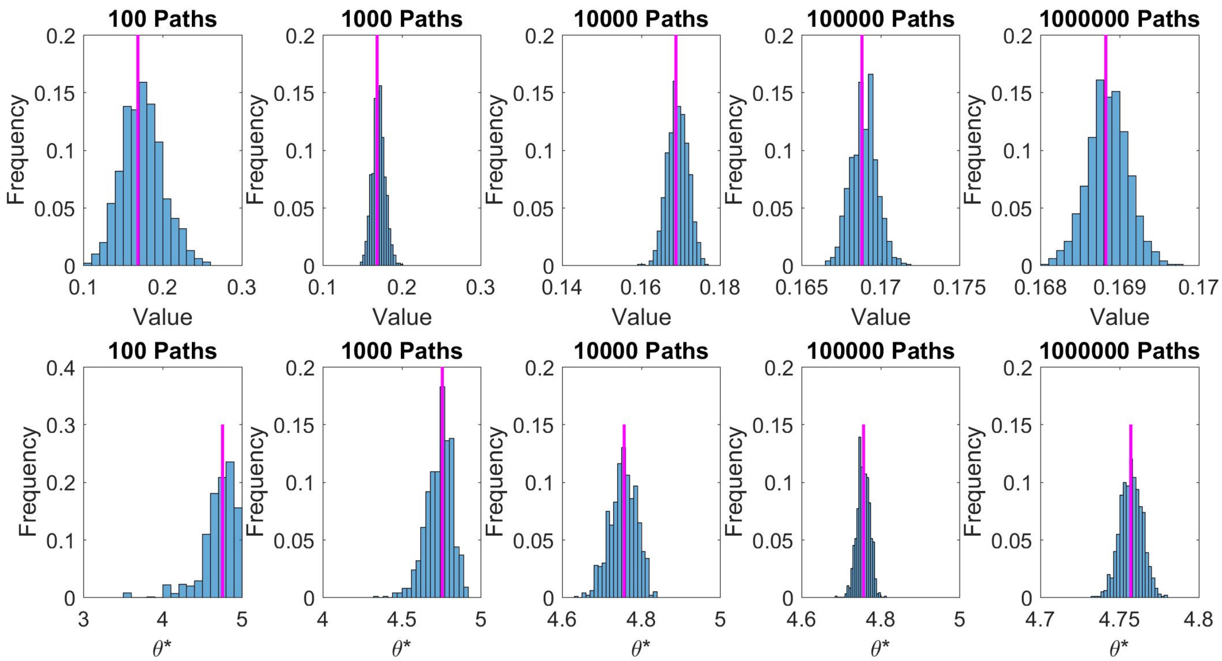

Figure 3: Histograms of V_0^{sim} for the Bermudan put option (note that each case was simulated 1000 times).

4.2 Bermudan Option with Variable Strike

In this subsection we present the results of the Bermudan-like option where the strike K is variable. We use the same parameter values as in Subsection 4.1 with $a = 0.5$ and $b = 1.0$ of equation (8). For the parametric function representing K^* we present the results assuming a second order polynomial,

$$g(x|\vec{\theta}) = \theta_1 x^2 + \theta_2 x + \theta_3. \tag{30}$$

In Figure 4 we plot $V_\tau^+ - C_K$ as a function of S_τ and K using equations (10) and (8), respectively. As can be seen in the figure, we expect K_{opt} to be quite linear, however, a practitioner would not necessarily know this fact in advance, and therefore, we utilized a more general, second order polynomial for $g(x|\vec{\theta})$ for presentation purposes. While not presented here, using a first order polynomial (i.e. line) produced even better results.

Setting $K_{max} = 10$, the resulting histograms of V_0^{sim} of equation (19) are plotted in Figure 5 and those of $\vec{\theta}^*$ of equation (18) in Figure 6. Note the bi-modal distribution for θ_3^* is likely due to the fact that we are using a second order polynomial where a line would likely suffice. In Figure 7 we present the standard deviation of the option value using 200 simulations as a function of N. As expected, the variance in the results reduces as N is increased and the converged values approach those of the analytical solution. A plot of the simulated and actual K^* as a function of S_τ in Figure 8 shows that as N is increased, the simulated results converge to the actual analytical ones.

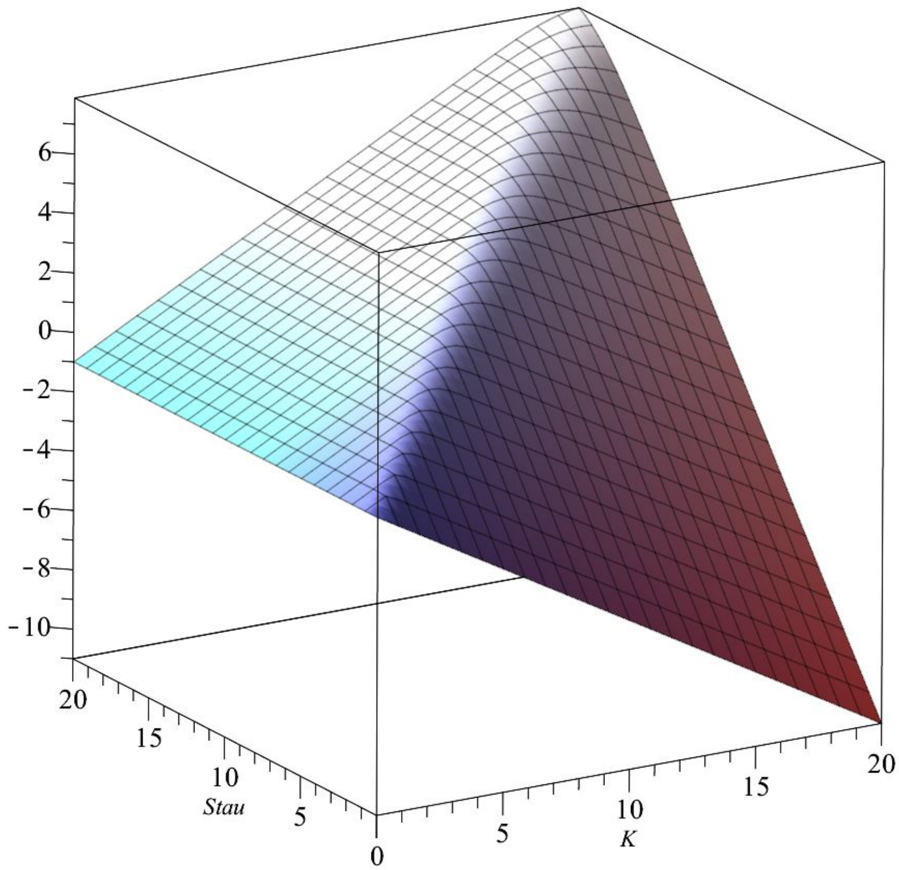

Figure 4: $V_{\tau+} - C_K$ as a function of S_τ and K.

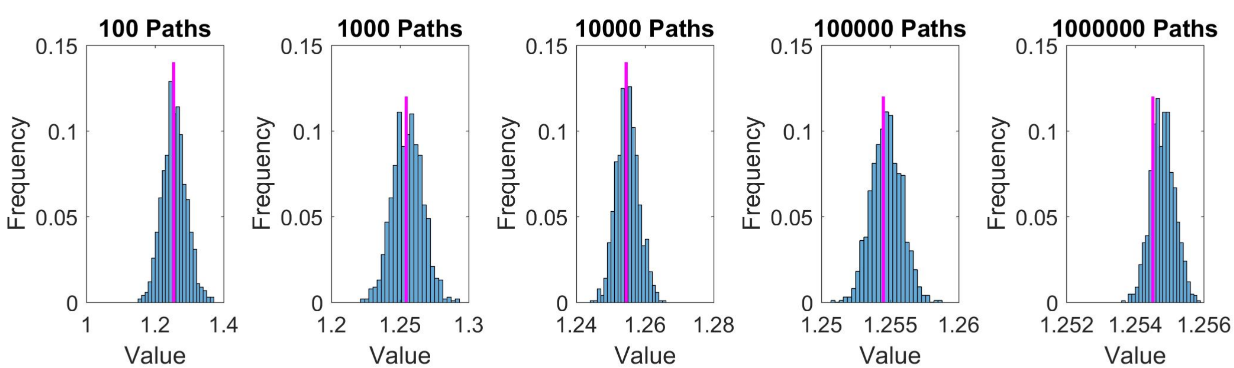

Figure 5: Histograms of V_0^{sim} for the variable strike Bermudan-like option (note that each case was simulated 200 times).

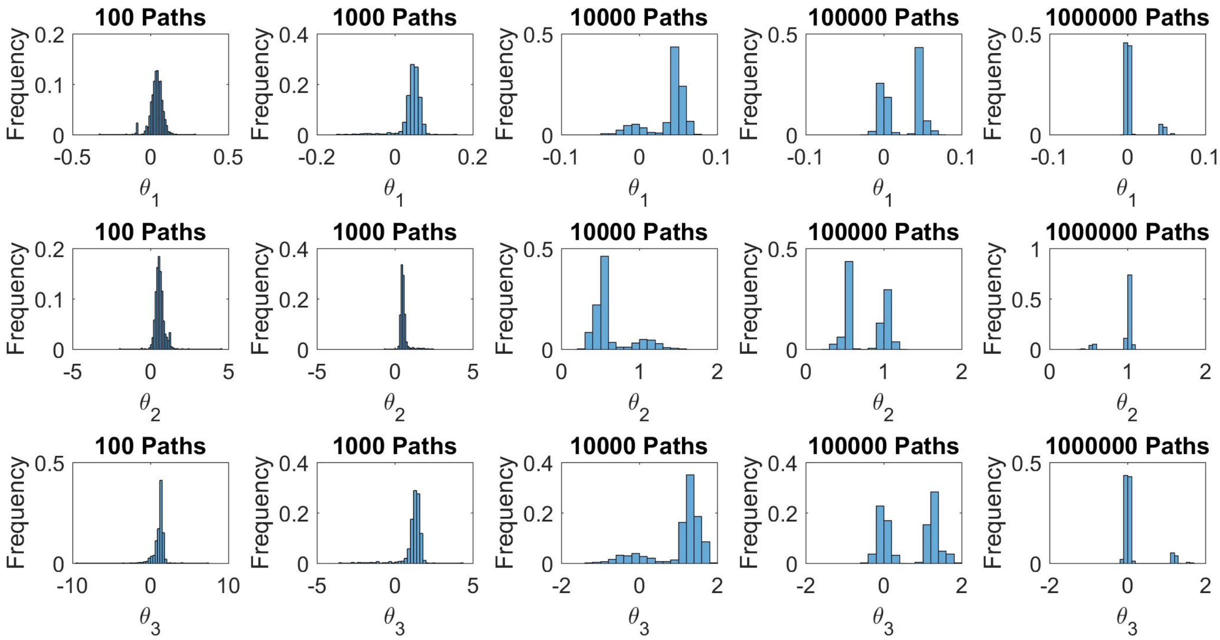

Figure 6: Histograms of $\vec{\theta}^*$.

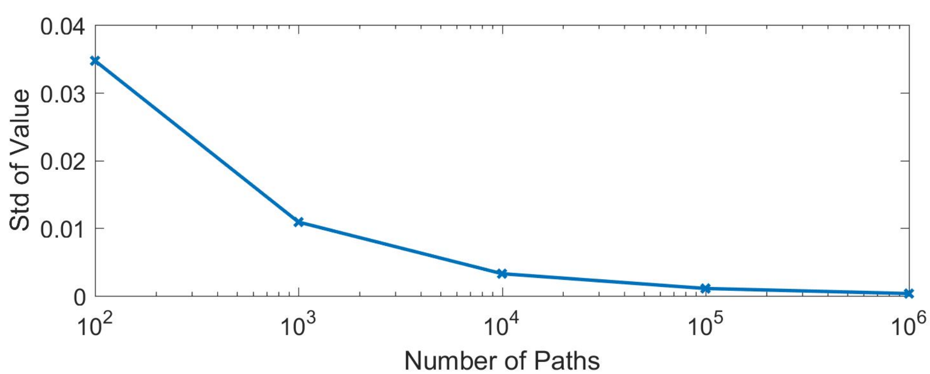

Figure 7: Simulation convergence for the variable strike Bermudan-like option.

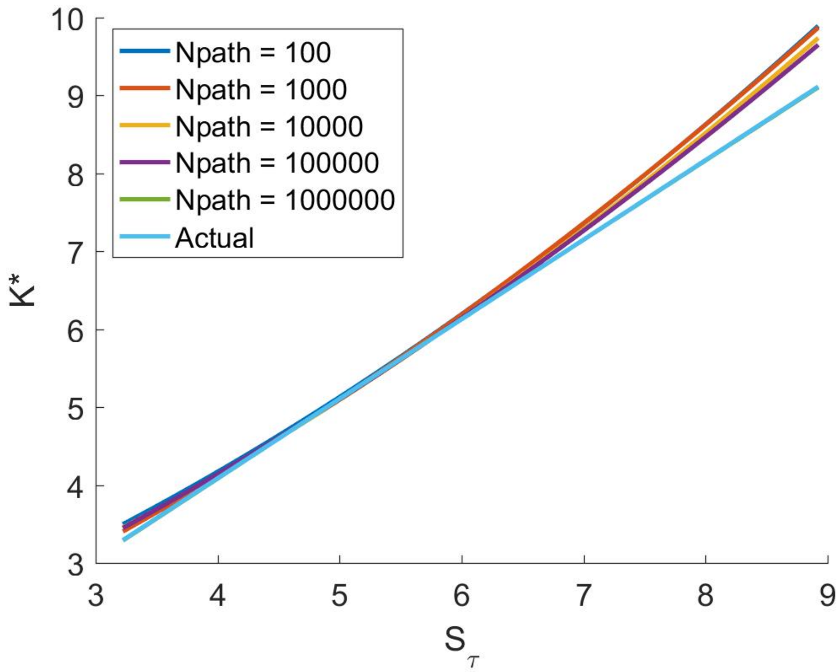

Figure 8: Simulated and actual K^* as a function of S_τ for the variable strike Bermudan-like option (note that the green line for the simulated case of $N = 10^6$ lies directly under the light blue (actual) line).

4.3 American Put Option

For the American put option we assume the same parameters for S_0, K, T, r and σ as in Subsections 4.1 and 4.2. Using a binomial tree, the value of the American put option was determined to be \$0.1835[1]. When simulating S_t we have the freedom to pick both the number of paths N and the number of simulated time steps for each path, N_{step}. To better gauge appropriate values for N and N_{step}, we utilized the exercise boundary determined from the binomial tree and ran 20 simulations for varying N and N_{step}. The mean and standard deviation of each set of 20 simulations is presented in Table 2. As expected, as N and N_{step} are increased, convergence to the true American put option value is achieved. Normally, one would not have the true exercise boundary a priori and clearly, a number of trial runs would need to be done. However, for the purpose of presenting the merits of the proposed methodology, we select $N = 100,000$ and $N_{step} = 500$ for all forthcoming simulations in this subsection. Using these values for N and N_{step} we create a simulation set $\{S_j^{(i)}\}$, where $i \in \{1, 2, 3, ..., N\}$ and $j \in \{1, 2, 3, ..., N_{step}\}$. For this particular simulation set the American put option value was 0.1836 using the true exercise boundary. Naturally, in all forthcoming results we would expect this value to be the upper bound of the American put option values calculated through exercise boundary fitting.

We first explore the use of cubic splines for $h(t; \vec{\eta})$. In Figure 9 we plot the results of the optimization using 4 to 8 node cubic splines. Also plotted in the figure are 20 randomly selected paths for S_t. The nodes are spaced evenly over time. As can be seen, while the error in option

[1]Note that convergence was assured by increasing the binomial tree size appropriately.

Table 2: Mean and (Standard Deviation) of American Put Option Values with Known Exercise Boundary (20 Simulations)

No. of Steps (N_{step})	No. of Paths (N) 1,000		10,000		100,000		500,000	
100	0.1853	(0.00659)	0.1821	(0.00205)	0.1828	(0.00078)	0.1827	(0.00027)
500	0.1836	(0.00654)	0.1831	(0.00260)	0.1833	(0.00066)	0.1834	(0.00026)
1,000	0.1818	(0.00670)	0.1830	(0.00250)	0.1836	(0.00064)	0.1835	(0.00034)

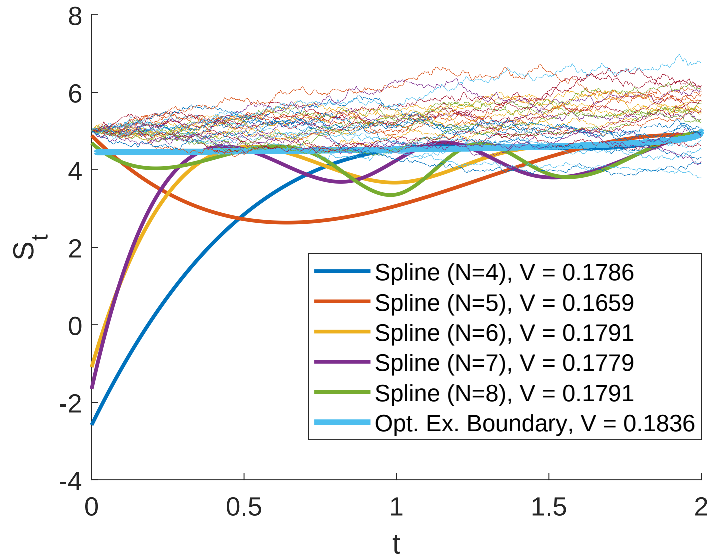

Figure 9: American put option exercise boundary simulation using cubic splines for $h(t;\vec{\eta})$.

calculation is less than 5% on average, the exercise boundaries are oscillatory and exhibit significant error. Next, we explore the use of second, third and fourth order polynomials for $h(t;\vec{\eta})$. The results are plotted in Figure 10. The average error is approximately 0.5% and the exercise boundaries are close to the optimal boundary. In Figure 11 we plot the results where we assume a piecewise linear function for $h(t;\vec{\eta})$. As can be seen, as the order increases, the optimal exercise boundary does not necessarily lead to better results. We also note that the exercise boundary was somewhat sensitive to the initial node values assumed for the piecewise function however, while not shown here, the results were further improved by forcing convexity on $h(t;\vec{\eta})$. In general, the results of these trials look promising.

4.4 Build / Abandon Real Option Example

For the build / abaondon real option example, we assume the following parameters:

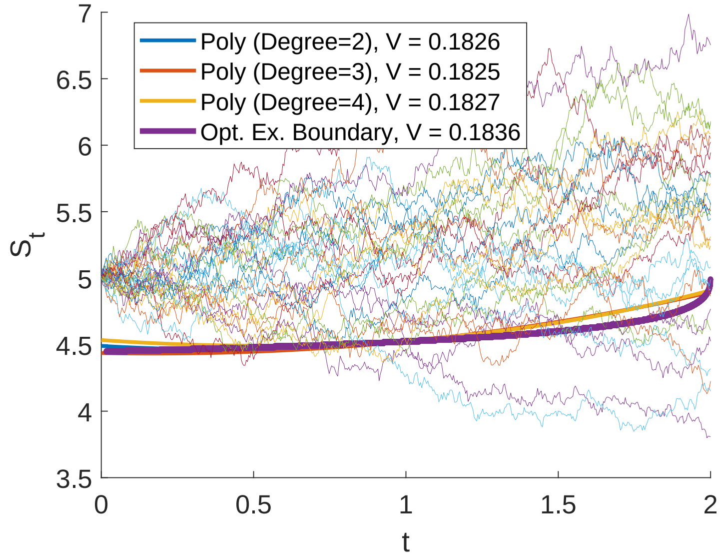

Figure 10: American put option exercise boundary simulation using polynomial functions for $h(t;\vec{\eta})$.

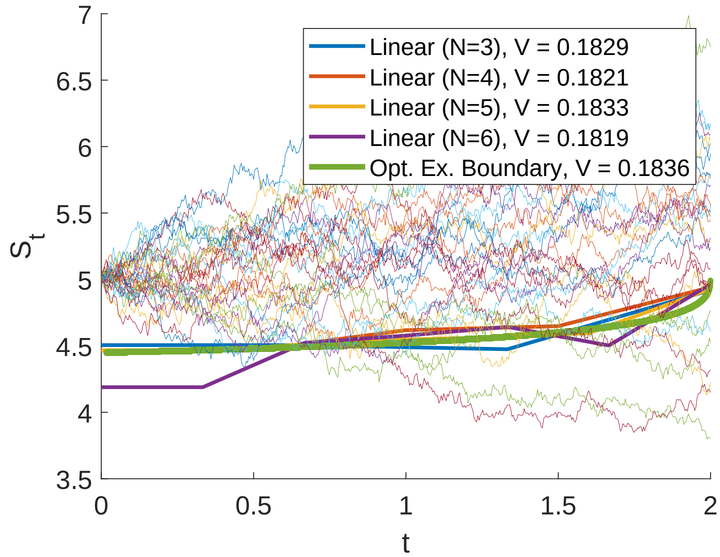

Figure 11: American put option exercise boundary simulation using piecewise linear functions for $h(t;\vec{\eta})$.

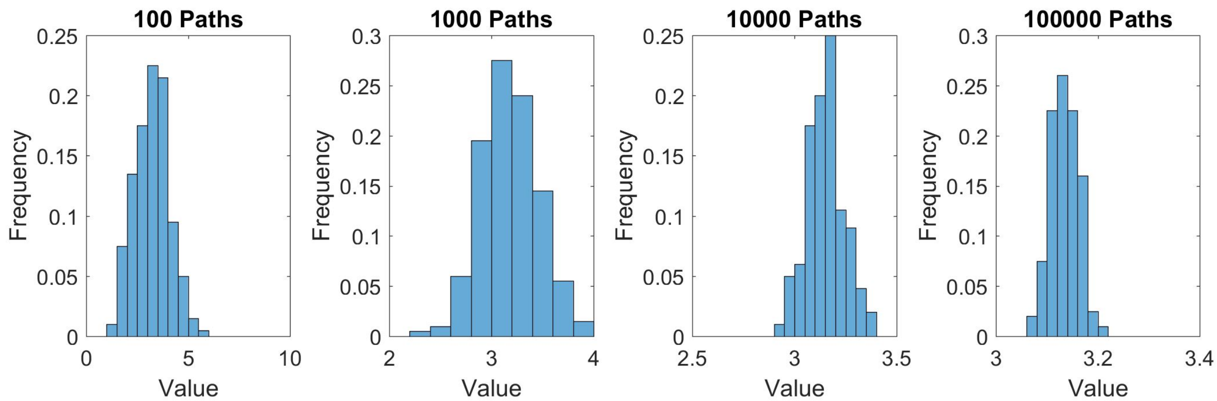

Figure 12: Histograms of V_0^{sim} for the build / abaondon real option (note that each case was simulated 200 times).

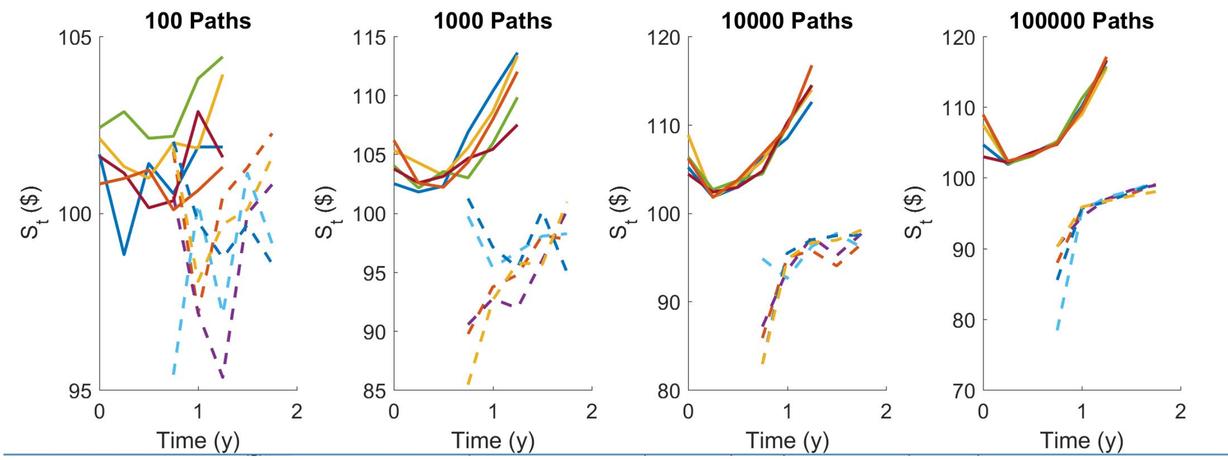

Figure 13: Example build / abaondon real option boundaries.

- $S_0 = 100$
- $K = 5$
- $C_{op} = 100$
- $C_{ab} = 1$
- $T = 2$ years
- $\tau_c = 0.5$ years
- $\gamma = 1.0$
- $r = 3\%$
- $\sigma = 10\%$.

In Figure 12 we plot the histograms for V_0^{sim} of equation (29) for varying N using 200 simulations. A few select build / abandon boundaries for varying N are plotted in Figure 13. Again, we see convergence is achieved. We note too, that exercise boundary values for small t will fluctuate as very few, if any, paths cross the exercise boundary at initiation.

5 Conclusions

The focus of this research was to present of a real options valuation methodology geared towards practical use. A key innovation of the methodology is the idea of fitting optimal decision making boundaries to optimize the expected value, based on Monte Carlo simulated stochastic processes that represent important uncertain factors. We showed how the methodology can be used to value a simple Bermudan put option. Then, we presented a Bermudan-like option where the strike was variable. This type of option is a simplification of the situation where managers have the option to build an optimal sized plant. For both the Bermudan and the Bermudan-like variable strike options convergence to the analytical values was achieved as the number of simulation paths were increased. We then considered an American put option where we showed convergence for exercise boundaries that were non-oscillatory. Under many circumstances, the practitioner may not know the true shape of an exercise boundary but may have intuition regarding its approximate location as well as convexity or concavity, reducing the potential of the oscillatory boundary solutions. Finally, we presented a simple build / abandon real option. As mentioned, to value a complex real option with multiple stochastic factors leads to model complexity that may make the analysis intractable. Our theoretical and numerical presentation of exercising boundary fitting shows how the complexity can be overcome through the use of Monte Carlo simulation. We emphasize that in a real options context, often many parameters can only be estimated. Errors associated with an approximate boundary fit may be significantly less than not modelling important complexities in the quest of a mathematically accurate solution. We feel that the methodology presented here is much more tractable in an industry setting for it is simple enough for managers to understand, yet can account for important real world factors that make the real options model suitable for valuation.

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