Production cap and the lumpy-flexible investment choice

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Abstract

An upper production limit is incorporated as a cap in a real-option formulation for investigating the conditions discriminating between a lumpy single-stage investment and a flexible consecutive two-stage variant. Although having no effect on the investment threshold and timing decision unless the cap is breached, the production cap deflates the investment option value evaluated in the absence of any cap and consequently influences the lumpy-flexible choice. Also, the cap disengages the accepted positive link between volatility and option value, since for volatilities above a certain level, the option value declines with increasing volatility in the presence of a cap. We demonstrate that a flexible as opposed to a lumpy investment policy can be more attractive and sometimes significantly so.

1 Introduction

In formulating analytical real-option models, it is customary to select geometric Brownian motion to represent the dynamics of the stochastic factor. This assumes the factor can adopt any value along $\mathbb{R}^+$. In practice, the factor may be naturally bounded as illustrated by the physical output produced by most plant and equipment that is limited by a capacity constraint even when supplemented with overtime. However, this capacity constraint is like a cap that by having option-like payoff features can be formulated within a real-option model. This representation, while having no effect on the investment threshold and timing decision provided the cap remains unbreached at implementation, does deflate the option value of the opportunity because of the consequential loss of attractiveness. An effective comparison of alternative policy opportunities often relies on evaluating their investment option values so any possible distortion caused by ignoring the capacity limit may lead to erroneous decisions. We investigate the impact of a cap on the investment option value and the resulting policy choice between a lumpy strategy that is implemented in a single stage and a flexible strategy that is implemented in two consecutive stages, where the flexible strategy is disadvantaged by a greater overall capital cost..

Uncertainty in real-option models plays a crucial role in understanding the scope for managerial discretion in making decisions on investment timing and capacity choice. Originally, demand volatility was considered to favour more frequent investments in small capacity increments when as needed, Dixit and Pindyck (1994). Since then, this view has been re-assessed by several authors. Dangl (1999) examines the claim by expressing the project value in terms of the maximum profit level when idle, active and active but exceeding the capacity constraint to show that a firm invests later in a larger capacity as market volume volatility increases. Similarly, Hagspiel et al. (2016) consider firms with and without the flexibility to adjust production capacity in line with output volume changes to examine their consequences on timing and size. De Giovanni and Massabò (2018) extend their analysis by including the costs associated with volume flexibility. Chronopoulos et al. (2017) show also that the lumpy strategy is preferred under high price uncertainty for fixed capacity projects but not when the firm has freedom over the capacity choice. Huberts et al. (2015) provides a survey on capacity choice. In work closest to our own, Kort et al. (2010) evaluates the relative merits of a lumpy and flexible strategy to show that greater uncertainty favours the former despite the latter's intrinsic flexibility. The authors make two telling points that flexibility has several connotations and that increased volatility engenders further inertia that defers the second stage investment.

In our formulation, we impose a production cap, modelled as a pair of call and put options, to represent the limit on output volume for both the lumpy and flexible investments. Call and put option features have been adopted elsewhere to represent a factor limit due to a government subsidy policy. Takashima et al. (2010) and Armada et al. (2012) evaluate a project with a floor representing a government support mechanism, while Adkins and Paxson (2019) and Adkins et al. (2019) assess the effect of a collar involving both a floor and cap on the project value. Here, the capacity limit is formulated as a cap. This means that when we are considering an additional investment for the flexible strategy, the increment can only be installed once the capacity limit has been attained, since it is uneconomic to expand capacity when under-utilized capacity exists. Representing capacity as a cap has three advantages: (1) intrinsically, a production cap reflects reality, as observed by Dangl (1999) and Hagspiel et al. (2016); (2) it enables the upside investment potential to be modelled, and (3) it is unnecessary to impose conditions on cost and revenue structure to guarantee the timing of the consecutive stages of the flexible strategy to be appropriately ordered.

Revisiting the lumpy-flexible debate with a cap yields some interesting results because in certain instances, the cap significantly modifies the solution. If the cap for the lumpy investment is not breached, then the with-cap and without-cap investment thresholds are identical but the with-cap investment option value is lower, since the threshold decides project viability while the lost upside potential reduces the option value. On the other hand, if the without-cap threshold breaches the cap, then the with-cap threshold increases and the option value decreases because of the lost latent value when market demand exceeds the cap. The volatility option value relationship is not monotonic increasing as expected, but concave with a maximum, since volatility increases lead eventually to the cap being breached and the consequential deterioration in value. The findings concerning the thresholds and option values for the flexible strategy are very similar to those for the lumpy strategy. Further, we find that flexible strategy is preferred to the lumpy strategy for a low capital cost disadvantage, but as this increases, the superiority of the flexible strategy wanes until a point of indifference is attained. While the flexible strategy is superior, the firm has the flexibility to select from a range of first stage investment levels, but this range narrows as the capital cost disadvantage worsens. Volatility also affects the relative merits of the two strategies. As expected, a flexible strategy is strongly favoured for low volatility projects while a lumpy strategy is favoured for very high volatility projects. The superiority of the flexible strategy is negatively affected by increases in both volatility and the capital cost disadvantage.

2 Analysis

2.1 Fundamental Model

A firm set in a monopolistic situation is considering a project opportunity that is subject to a single source of uncertainty due to demand volume variability, which is described by the geometric Brownian motion process:

$$\mathrm{d}\,\mathrm{q} = \alpha q\,\mathrm{d}t + \sigma q\,\mathrm{d}W\ , \tag{1}$$

where q denotes the periodic demand volume, α the expected drift, σ the volatility, and $\mathrm{d}W$ an increment of the standard Wiener process. Project output is constrained at the upper end by a capacity cap, q_U, so demand is only fully met provided $q \le q_U$. For a known unit price p and periodic operating cost f, the idle project value V dependent on demand volume q is described by:

$$\tfrac{1}{2}\sigma^2 q^2 \frac{\partial^2 V}{\partial q^2} + (r-\delta) q \frac{\partial V}{\partial q} - rV = 0 \tag{2}$$

where $r > \alpha$ denotes the risk-free rate and $\delta = r - \alpha$ the rate of return shortfall. Since $V(0) = 0$, the solution to (2) is $V(q) = A\,q^{\beta_1}$, where $A \ge 0$ is a to-be-determined coefficient and β_1, β_2 are the respective positive and negative roots of the characteristic equation:

$$\tfrac{1}{2}\sigma^2\ \beta(\beta - 1) + (r-\delta)\beta - r = 0\,, \tag{3}$$

with solution values:

$$\beta_1, \beta_2 = \left(\tfrac{1}{2} - \frac{r-\delta}{\sigma^2}\right) \pm \sqrt{\left(\tfrac{1}{2} - \frac{r-\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}\ , \tag{4}$$

where $\beta_1 > 1, \beta_2 < 0, \beta_1 - \beta_2 > 0.$ For an idle project, the firm holds a positive call option to invest in the project while the market demand is no more than the exercise threshold.

The value for an active project is given by:

$$\tfrac{1}{2}\sigma^2 q^2 \frac{\partial^2 V}{\partial q^2} + \left(r - \delta\right) q \frac{\partial V}{\partial q} + pq - f - rV = 0 . \tag{5}$$

with generic solution:

$$V\left(q\right) = \frac{pq}{\delta} - \frac{f}{r} + A_1 q^{\beta_1} + A_2 q^{\beta_2} , \tag{6}$$

where A_1, A_2 are to-be-determined generic coefficients. In (6), $A_1 q^{\beta_1}, A_2 q^{\beta_2}$ respectively represent the American perpetuity call, Samuelson (1965), and put options, Merton (1973), Merton (1990). The call and put options may be held or written. The coefficient for a held option is positive and that for a written option negative. For an active project, these options represent the opportunity value of switching to an alternative state. If the switch is value-creating, held by the firm and economically advantageous, then the option coefficient is positive. In the present context, this arises when the firm owns an investment opportunity to install or expand a productive asset. But, if the switch is value-destroying, written by the firm and economically disadvantageous, then the option coefficient is negative. Again, this arises when the firm's output volume attains the productive capacity limit, either from below or from above. If the volume attains the cap from below, then the loss in value is due to the cap forcing the firm to fail in meeting market demand. If the volume attains the cap from above, then the loss in value is due to market demand falling short of the cap.

If demand is not satisfied and exceeds the productive cap, $q > q_U$, then the maximum attainable project volume is $\min\left\{q, q_U\right\}$, and (6) is amended to:

$$V\left(q\right) = \frac{pq_U}{r} - \frac{f}{r} + A_1 q^{\beta_1} + A_2 q^{\beta_2} . \tag{7}$$

2.2 Model I: Lumpy Investment with a Cap

A firm has an opportunity to invest in a project in its entirety, whose output is constrained by an upper capacity limit. Installing this lumpy project requires a one-time capital investment cost K, and once installed incurs a known periodic operating cost f. In the presumed absence of any abandonment optionality, the active project may operate at a loss. The capacity of the installed project is capped by the upper limit q_U. The optimal demand threshold for an investing in the project is denoted by $\hat{q}_{(I)01}$, given that the threshold does not exceed the cap, $\hat{q}_{(I)01} \le q_U$, although this condition is subsequently relaxed. The firm managing the project can be said to exist in one of three possible distinct states, see Figure 1a. In the idle pre-investment state-0, the firm is waiting for more favourable information to emerge before making the investment, so $0 < q < \hat{q}_{(I)01}$. In the active post-investment state-1, the project is already installed and actively producing output q but the production cap is not breached, so $q \le q_U$. When the cap is breached, the firm is the active state-11, $q > q_U$, so demand is unmet and production output is restricted to q_U. The values for the three distinct states, denoted by $V_{(I)0}(q), V_{(I)1}(q), V_{(I)11}(q)$, respectively, are:

$$V_{(I)0}(q) = A_{(I)01} q^{\beta_1} \quad \text{for } q < \hat{q}_{(I)01}, \tag{8}$$

$$V_{(I)1}(q) = \frac{pq}{\delta} - \frac{f}{r} + A_{(I)11} q^{\beta_1} \quad \text{for } q \le q_U, \tag{9}$$

$$V_{(I)11}(q) = \frac{pq_U}{r} - \frac{f}{r} + A_{(I)112} q^{\beta_2} \quad \text{for } q > q_U. \tag{10}$$

In (8)-(10), $A_{(I)01} \ge 0$ denotes the investment option coefficient, while $A_{(I)11}$, $A_{(I)112}$ denote the switch option coefficients when migrating between state-1 and -11. Subsequently, we show both these coefficients to be negative because of the value sacrificed when switching. When in state-1, the firm is confronted with the possibility of being switched to state-11 and obliged to accept a less attractive output volume imposed by the cap q_U instead of meeting the market demand level q. The accompanying loss in value is represented by selling the written call option $A_{(I)11} q^{\beta_1}$. Similarly, when in state-11, the firm is has to face the possibility of being switched from state-1 and obliged to accept a less attractive market demand level q instead of operating at full capacity q_U. The accompanying loss in value is represented by selling the written put option $A_{(I)112}q^{\beta_2}$.

The values $A_{(I)11}, A_{(I)112}$ are obtained from the value-matching relationship and smooth-pasting condition ruling at the boundary between state-1 and -11, and given by[1]:

$$A_{(I)11} = \frac{pq_U^{1-\beta_1}\left(-r + r\beta_2 - \delta\beta_2\right)}{r\left(\beta_1 - \beta_2\right)\delta},\ A_{(I)112} = \frac{pq_U^{1-\beta_2}\left(-r + r\beta_1 - \delta\beta_1\right)}{r\left(\beta_1 - \beta_2\right)\delta}. \tag{11}$$

In (11), $A_{(I)11} \le 0, A_{(I)112} \le 0$. $A_{(I)11} < 0$ because $r - \delta > 0$ so $\beta_2\left(r - \delta\right) - r < 0$, and $A_{(I)112} < 0$ because $\frac{1}{2}\sigma^2\ \beta_1\left(\beta_1 - 1\right) > 0$ so $\left(r - \delta\right)\beta_1 - r < 0$ due to (3). Any relaxation in the capacity limit, as reflected by an increasing cap level q_U, causes $A_{(I)11}$ to increase towards zero to make the switch from state-1 to -11 less unlikely, but also causes $A_{(I)112}$ to decrease towards minus infinity to make a switch from state-11 to -1 more likely.

Provided the productive cap is not breached, the threshold $\hat{q}_{(I)1}$ and coefficient $A_{(I)01}$ are obtained from the value-matching relationship and associated smooth-pasting condition ruling at the boundary of state-0 and -1, and are given by:

$$\hat{q}_{(I)01} = \frac{\beta_1}{\beta_1 - 1}\left(f + rK\right)\frac{\delta}{rp}, \tag{12}$$

$$A_{(I)01} = \frac{p\hat{q}_{(I)01}^{1-\beta_1}}{\beta_1\delta} + A_{(I)11}. \tag{13}$$

From (12), the cap has no effect on the threshold and the investment timing, which contrasts directly with the effect of a floor acting as a subsidy policy instrument, Takashima et al. (2010), Armada et al. (2012), and Adkins and Paxson (2019), Adkins et al. (2019). Under certain circumstances, a floor guarantee positively influences both the threshold and investment option value, since the former decreases and the latter increases for increasing floor levels. In the absence of a cap or floor, the investment threshold level reflects the underlying uncertainty and represents the minimum exercise price ensuring project viability against the downside risk. But,

[1] The derivation for Model I is relegated to Appendix A.

the floor mitigates the downside risk and accordingly the threshold is reduced relative to that without a floor. In contrast, the cap provides no protection against the downside risk and consequently the threshold is unaffected.

From (13), the cap makes the investment option less attractive, since the coefficient $A_{(I)01}$ is the sum of that without a cap $p\hat{q}_{(I)1}^{1-\beta_1} / (\beta_1 \delta)$ and $A_{(I)11} < 0$, the sacrifice levied by switching from state-1 to -11. The cap offers no downside protection, and because it limits upside potential, the opportunity is less attractive, a floor, on the other hand, enhances the option value by limiting adverse outcomes.

2.2.1 Breached Capacity

When the opportunity threshold $\hat{q}_{(I)01}$ exceeds the cap, exercising the investment option entails switching from state-0 to -11, see Figure 1b. This necessitates amending the value-matching and smooth pasting expressions accordingly, from which the amended threshold $\hat{q}_{(I)011}$ and coefficient $A_{(I)011}$ are given by:

$$\hat{q}_{(I)011} = q_U \left(\frac{(pq_U - f - rK)\beta_1 \delta}{pq_U (r - r\beta_1 + \delta\beta_1)} \right)^{\frac{1}{\beta_2}}, \tag{14}$$

$$A_{(I)011} = \frac{-(pq_U - f - rK)\beta_2}{(\beta_1 - \beta_2) r} \hat{q}_{(I)01}^{-\beta_1}. \tag{15}$$

When $\hat{q}_{(I)01} > q_U$, the amended threshold $\hat{q}_{(I)011}$ is always at least equal to the cap. If we set $q_U = \hat{q}_{(I)01}$, then (14) simplifies to (12) and (15) to (13). Further, if viability is assured because $pq_U > f + rK$, then $\partial \hat{q}_{(I)011} / \partial q_U < 0$ and $\partial A_{(I)011} / \partial q_U > 0$. As q_U falls in value and the bite of the cap intensifies, the resulting increase in the threshold is due to the need to compensate the downside risk and mitigate the possibility of the project making a loss, while the decline in the investment option value reflects the project's decreasing attractiveness.

2.3 Model II: Flexible Investment with a Cap

Model I represents a lumpy opportunity exercisable as a single project. An alternative strategy is Model II, which splits the project into two constituent elements, stage-1 and -2, both of which require completion for the project to be complete. The two stages are managed consecutively, such that the second can only be activated provided the first is operational and operating at full capacity. This inherent flexibility has the merit of initially exposing less capital to the misfortunes of downside risk while offering the opportunity to gain from upside potential by upscaling. But this flexibility is attended with additional capital costs, otherwise all divisible capital projects would always be established in a piecewise fashion. We impose the condition $K_1 + K_2 > K$ to ensure completing the project in a single stage is less expensive than completing it in two consecutive stages, where K_1 and K_2 denote the capital costs for the two stages, respectively. Despite this, the two stage framework is not further disadvantaged. When stage-2 is attained, both Model I and II share identical caps and periodic operating costs. Model II stage-1 cap is denoted by $q_u < q_U$ and stage-2 by q_U. The known additional periodic operating costs for stage-1 and -2 are denoted by f_1 and f_2, respectively, where $f_1 + f_2 = f$. Consequently, Model I and II are directly comparable. The distinction between two models is attributable to only capital costs and not to any operating cost differences, although this constraint can be subsequently relaxed. Again, we exclude the possibility of abandonment optionality.

The capacity expansion option is presumed to become available only if the installed capacity is fully utilized. Expanding an under-utilized installed capacity is seen as uneconomic because of the accompanying additional capital and operating costs rendered by the expansion. When the prevailing output volume is below the stage-1 capacity, $q < q_u$, a small output increase accompanied with no capacity change incurs no additional costs, but a similar increase with a capacity change entails both a capital expense and an increase in operating costs. We conceptualize the expansion option to be triggered only if full capacity at stage-1 is attained. This does not necessarily entail that the stage-2 investment threshold occurs at the stage-1 cap q_u, but rather that the market volume trigger is at least equal to q_u.

The firm managing the project exists in one of five possible distinct states, see Figure 2a. In the idle state-0, the firm is waiting for more propitious conditions to emerge before making the investment, so $0 < q < \hat{q}_{(II)01}$ where $\hat{q}_{(II)01}$ denotes the optimal threshold for investing K_1 to establish stage-1. In the active state-1, the firm has already invested in the project and is actively producing output q provided the cap is not breached, so $q \le q_u$. In active state-11, the market demand exceeds the stage-1 cap so output is constrained to q_u, but the firm now owns the expansion option and is deliberating on raising the output cap to q_U, so $q_u < q < \hat{q}_{(II)12}$ where $\hat{q}_{(II)12}$ denotes the optimal threshold for investing K_2 to establish stage-2. In the active state-2, the firm has already invested in additional capacity and is producing an output q provided the stage-2 cap q_U is not breached. $q \le q_U$. In the active state-21, market demand exceeds the cap and the firm is producing at full capacity q_U. Again, we initially assume the two thresholds, investment and expansion, do not exceed the caps for the respective stage, $\hat{q}_{(II)01} \le q_u$, $\hat{q}_{(II)12} \le q_U$, although this condition is subsequently relaxed.

The respective project values for the five distinct states, $V_{(II)0}(q), V_{(II)1}(q), V_{(II)11}(q), V_{(II)2}(q)$, $V_{(II)21}$ can now be specified:

$$V_{(II)0}(q) = A_{(II)01} q^{\beta_1} \quad \text{for } q < \hat{q}_{(II)01}, \tag{16}$$

$$V_{(II)1}(q) = \frac{pq}{\delta} - \frac{f_1}{r} + A_{(II)11} q^{\beta_1} \quad \text{for } q \le q_u, \tag{17}$$

$$V_{(II)11}(q) = \frac{pq_u}{r} - \frac{f_1}{r} + A_{(II)111} q^{\beta_1} + A_{(II)112} q^{\beta_2} \quad \text{for } q > q_u, \tag{18}$$

$$V_{(II)2}(q) = \frac{pq}{\delta} - \frac{f_1}{r} - \frac{f_2}{r} + A_{(II)21} q^{\beta_1} \quad \text{for } q \le q_U, \tag{19}$$

$$V_{(II)21}(q) = \frac{pq_U}{r} - \frac{f_1}{r} - \frac{f_2}{r} + A_{(II)212} q^{\beta_2} \quad \text{for } q > q_U. \tag{20}$$

In (16)-(20), the option coefficients, $A_{(II)01} \ge 0$, $A_{(II)111} \ge 0$, are associated with the stage-1 investment opportunity at state-0, which incurs a cost K_1 to produce a volume subject to the cap

q_u, and the stage-2 expansion opportunity at state-11, which incurs a cost K_2 to produce a volume subject to the cap q_U, respectively. The option coefficients, $A_{(II)11} \leq 0$, $A_{(II)112} \leq 0$, and $A_{(II)21} \leq 0$ $A_{(II)212} \leq 0$, reflect the sacrificial value inflicted by the two caps, q_u, q_U, when switching between state-1 and -11, state-2 and -21, respectively.

The coefficients, $A_{(II)21}$, $A_{(II)212}$, are obtainable from the value matching relationship and smooth pasting condition ruling at the boundary between stage-2 and -21, and are given by:[2]

$$A_{(II)21} = \frac{pq_U^{1-\beta_1}\left(-r + r\beta_2 - \delta\beta_2\right)}{r\left(\beta_1 - \beta_2\right)\delta} < 0, \; A_{(II)212} = \frac{pq_U^{1-\beta_2}\left(-r + r\beta_1 - \delta\beta_1\right)}{r\left(\beta_1 - \beta_2\right)\delta} < 0 \tag{21}$$

By comparing (11) and (21), $A_{(II)21} = A_{(I)11}, A_{(II)212} = A_{(I)112}$. These are representative of the value sacrificed when switching between state-1 and -11 for Model I, and between state-2 and -21 for Model II. They are identical because their respective coefficients depend on only the cap level q_U, which is common to the two models.

The coefficients $A_{(II)111}, A_{(II)112}$ are obtainable from the value matching relationship and smooth pasting condition ruling at the boundary between stage-11 and -2, and given by:

$$A_{(II)111} = A_{(II)21} + \frac{\hat{q}_{(II)12}^{-\beta_1}}{\beta_1 - \beta_2}\left[\frac{\left(1-\beta_2\right)p\hat{q}_{(II)12}}{\delta} + \beta_2\left(K_2 + \frac{f_2}{r} + \frac{pq_u}{r}\right)\right], \tag{22}$$

$$A_{(II)112} = \frac{\hat{q}_{(II)12}^{-\beta_2}}{\beta_1 - \beta_2}\left[\frac{\left(\beta_1 - 1\right)p\hat{q}_{(II)12}}{\delta} - \beta_1\left(K_2 + \frac{f_2}{r} + \frac{pq_u}{r}\right)\right], \tag{23}$$

where $\hat{q}_{(II)12}$ denotes the to-be-determined stage-2 expansion threshold. This is obtained from the value matching relationship and smooth pasting condition ruling at the boundary between stage-1 and -11, which yields the non-linear relationship:

$$\hat{q}_{(II)12}^{-\beta_2}\left[r\left(\beta_1 - 1\right)p\hat{q}_{(II)12} - \delta\beta_1\left(rK_2 + f_2 + pq_u\right)\right] = q_u^{-\beta_2}\left(r\left(\beta_1 - 1\right) - \delta\beta_1\right)pq_u. \tag{24}$$

[2] The derivation for Model II is relegated to Appendix B.

No feasible closed-form solution exists unless $\beta_2 = -1$, so $\hat{q}_{(II)12}$ has to be solved numerically. The boundary conditions enacted between stage-1 and -11 also yields the coefficient $A_{(II)11}$:

$$A_{(II)11} = \frac{\beta_2(r-\delta)-r}{r(\beta_1-\beta_2)\delta}\left(pq_U^{1-\beta_1} + pq_u^{1-\beta_1}\right) + \frac{\hat{q}_{(II)12}^{-\beta_1}}{r(\beta_1-\beta_2)\delta}\left[p\hat{q}_{(II)12}(1-\beta_2)r + (pq_u + f_2 + K_2)\delta\beta_2\right]. \tag{25}$$

Finally, $\hat{q}_{(II)01}$ and $A_{(II)01}$ are obtainable from the value matching relationship and smooth pasting condition ruling at the boundary between stage-0 and -1. This yields:

$$\hat{q}_{(II)01} = \frac{\beta_1(f_1 + rK_1)\delta}{p(\beta_1-1)r}, \tag{26}$$

$$A_{(II)01} = \frac{p\hat{q}_{(II)01}}{\beta_1\delta\hat{q}_{(II)01}^{\beta_1-1}} + A_{(II)11}. \tag{27}$$

The investment threshold (26) and option coefficient are similar in form to those obtained for Model I, (12) and (13), respectively. Provided the stage-1 cap is not breached by the threshold, $\hat{q}_{(II)01} \le q_u$, the cap does not affect the threshold or investment timing. Also, the cap makes the stage-1 investment option less attractive since $A_{(II)11} < 0$.

2.3.1 Stage-2 Breached Capacity

If the threshold $\hat{q}_{(II)12}$ obtained from (24) exceeds the stage-2 cap q_U, then exercising the expansion option entails switching from state-11 to -21. This change necessitates amending the value-matching and smooth pasting expressions accordingly, from which we obtain the amended solutions for the threshold $\hat{q}_{(II)121}$ and coefficients $A_{(II)111}$, $A_{(II)112}$, $A_{(II)11}$:

$$\hat{q}_{(II)121}^{-\beta_2} = \frac{\left(pq_U^{1-\beta_2} - pq_u^{1-\beta_2}\right)\left(\delta\beta_1 - r(\beta_1-1)\right)}{\left(pq_U - pq_u - f_2 - rK_2\right)\beta_1\delta}, \tag{28}$$

$$A_{(II)111} = \frac{-\hat{q}_{(II)121}^{-\beta_1}}{r(\beta_1-\beta_2)}\left(pq_U - pq_u - f_2 - rK_2\right)\beta_2 \tag{29}$$

$$A_{(II)112}=\frac{q_U^{-\beta_2}\left(-r+r\beta_1-\delta\beta_1\right)pq_U}{r\left(\beta_1-\beta_2\right)\delta}+\frac{\hat{q}_{(II)121}^{-\beta_2}}{r\left(\beta_1-\beta_2\right)\delta}\left(pq_U-pq_u-f_2-rK_2\right)\beta_1\delta \tag{30}$$

$$A_{(II)11}=\frac{-\hat{q}_{(II)121}^{-\beta_1}\left(pq_U-pq_u-f_2-rK_2\right)\beta_2\delta}{r\left(\beta_1-\beta_2\right)\delta}+\frac{pq_u^{1-\beta_1}\left(r\left(\beta_2-1\right)-\beta_2\delta\right)}{r\left(\beta_1-\beta_2\right)\delta}. \tag{31}$$

The amended solutions (28)-(31) affect $A_{(II)01}$ through (27), but the solutions for $A_{(II)21}, A_{(II)212}$ and $\hat{q}_{(II)01}$ remain unaffected. Consequently, an expansion threshold exceeding the stage-2 cap has a bearing on the investment option, but unlike the breached capacity solution for Model I, §2.2.1, the investment threshold is unaffected.

3 Numerical Illustrations

Further insights on the behaviour of the models presented in the previous section are obtained through numerical illustrations. Unless otherwise specified in the subsequent text, the values used to investigate the comparative behaviours of the models are initially as follows. The parameter values for the stochastic process are $r=0.04$, $\delta=0.04$ and $\sigma=0.2$, which implies that $\beta_1=2$ and $\beta_2=-1$. A constant product price at $p=1$ is kept throughout analysis. The values used to initially illustrate Model I, the lumpy strategy, are a capital expenditure cost set at $K=100$ and a periodic operating cost at $f=1.6$. The capacity limit capping the productive output level is set at $q_U=15$. The flexible alternative, represented by Model II, echoes the lumpy investment but splits the entire project into two constituent, consecutive stages, stage-1 and -2. For convenience, we conceptualize the two distinct stages as behaving in accordance to the proportions, $\phi, 1-\phi\left(0<\phi<1\right)$ of the entire project, This means that the stage-1 and -2 operating costs are $f_1=\phi f$ and $f_1=\left(1-\phi\right)f$, respectively. Further, the productive capacity limits are obtained in a similar way, but since stage-2 is an addition to stage-1, the stage-2 cap is $q_U=15$, the same as for the lumpy investment. The cap for stage-1 is given by $q_u=\phi\, q_U$. In contrast, because of a disproportionally greater capital cost, the flexible alternative is disadvantaged compared with the lumpy investment. If K_1, K_2 denote the respective capital costs for stage-1 and -2, then

$$K_1 + K_2 = K + \kappa,\, K_1 = \phi K + \kappa,\, K_2 = (1-\phi)K,\, \kappa > 0.$$

The additional capital charge κ is assigned to only stage-1 because of the extra cost of initially establishing a smaller scale production plant. Initially, we set $\kappa = 5$.

Clearly, whenever $\phi = 0$ or $\phi = 1$ the lumpy investment project is always more attractive than the flexible alternative, since the two project forms are identical by having an equal productive cap but the flexible alternative incurs a greater capital cost. Yet, the flexible alternative with its lower stage-1 capital investment cost and operating cost is more likely to be installed earlier than the lumpy investment because of having a lower investment threshold. The critical question is then for those values of ϕ in between the extremes, which of the two alternatives is more attractive, and how is this finding influenced by parameter value changes.

3.1 Model I

In the absence of a productive cap, the investment threshold is obtained as $\hat{q}_{(I)0} = 11.2$ and the option coefficient as $A_{(I)0} = 1.1161$. The threshold and coefficient values are expected to change, sometimes significantly, when a cap is present. The effect of introducing a productive cap on the project solution depends on whether the upper limit is breached by market demand or not. This is illustrated in Figure 3a for varying cap levels q_U where the threshold limit is defined by $\hat{q}_{(I)1}$. When the cap is not breached, the investment threshold adopts the identical level as if the cap is absent, but the option coefficient declines in value as the cap becomes increasingly more stringent. The decline in option value reflects the increasing loss of project value as the cap increasingly restricts the upper limit of the project's realised cash-flow. Projects with a production cap are always less attractive. When the production cap is breached, there is a difference in outcome. As the cap q_U falls below the investment threshold without a cap, the resulting investment threshold increases for two dependent reasons. First, there is a sacrifice of the potential project value generated whenever demand exceeds the production cap, and secondly, the investment threshold has to be sufficiently high to guarantee the project's viability. These two factors not only raise the investment threshold but also render the project less attractive, and as a consequence the option value as reflected by its coefficient declines. For a cap of twice the threshold level of 11.2 without a cap, the option coefficient value is one-third less than that without the cap. As the restrictedness of the production cap intensifies, the investment threshold tends to infinity while the option coefficient tends to zero. Also, while the coefficients $A_{(I)11}$, $A_{(I)112}$ are both negative, since value is lost in migrating in between state-1 and -11, a cap reduction produces, in absolute terms, an $A_{(I)11}$ increase and a $A_{(I)112}$ decrease, and consequently a fall in the investment option coefficient $A_{(I)01}$.

Figure 3a

Model I Investment Threshold and Option Coefficient for Cap Variations

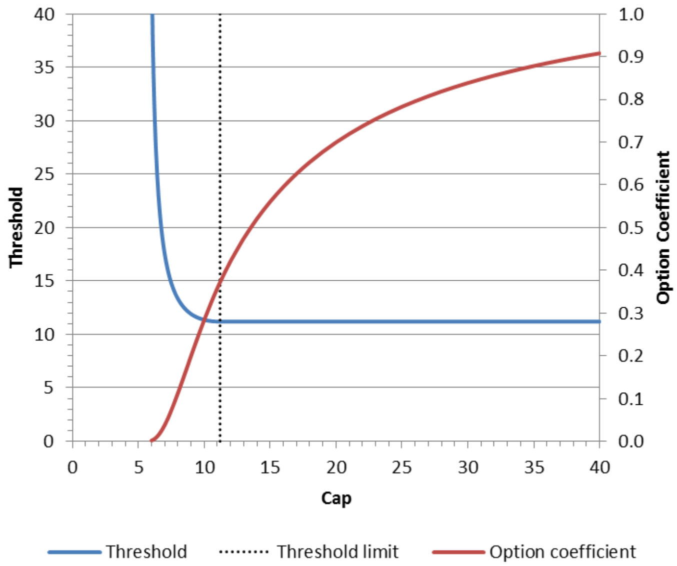
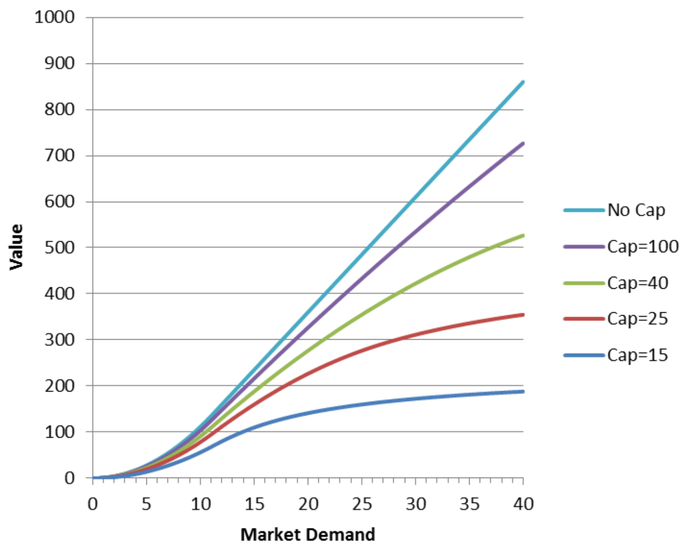


Figure 3b

Model I Investment Option Value for Cap Variations



The impact of the cap on the project value is illustrated in Figure 3b for specified cap levels including "no-cap" versus market demand variations. None of the displayed cap levels is breached and the project value is defined as $V_{(I)0}(q)$ for $q < \hat{q}_{(I)1}$, $V_{(I)1}(q) - K$ for $\hat{q}_{(I)1} \le q \le q_U$, $V_{(I)11}(q) - K$ for $q > q_U$. Figure 3b reveals that decreases in the cap escalate the gap between the without-cap and with-cap project values, and that both the relative and absolute differences increase with market demand. Since a cap in some form or other plausibly exists for most practical projects, analyses ignoring the cap can yield project values that are significantly inflated and overstate their attractiveness. Figure 3b also shows all the profiles to be monotonic increasing with curvatures depending on whether $q < \hat{q}_{(I)1}$, $\hat{q}_{(I)1} \le q \le q_U$, or $q > q_U$.

Although volatility is generally recognised to exert a positive influence on the investment threshold and investment option value, this finding appears to hold only partially for the option value in the presence of a cap. In Figure 4, the threshold and option coefficient profiles are illustrated for volatility variations up to 40% with $q_U = 15$. While the threshold continuously increases with volatility over the illustrated range, the option value increases until the volatility attains a sufficiently high level for the threshold to equal the cap, which occurs for $\sigma = 0.3$. For greater volatility levels the cap is breached, and then the option value begins and continues to decline. Without a cap, the option value and the project attractiveness increase with volatility owing to the expectation that a higher volatility entails a greater likelihood of higher values for the underlying. However, when the cap is present and the production level is constrained to an upper limit, the beneficial consequences of a high volatility fail to materialise owing to the cap, which results in a declining project value. Volatility has a positive impact on the option value unless the cap is breached, in which case the impact is negative.

Figure 4a

Lumpy Investment Threshold and Option Coefficient for Volatility Variations

with $q_U = 15$

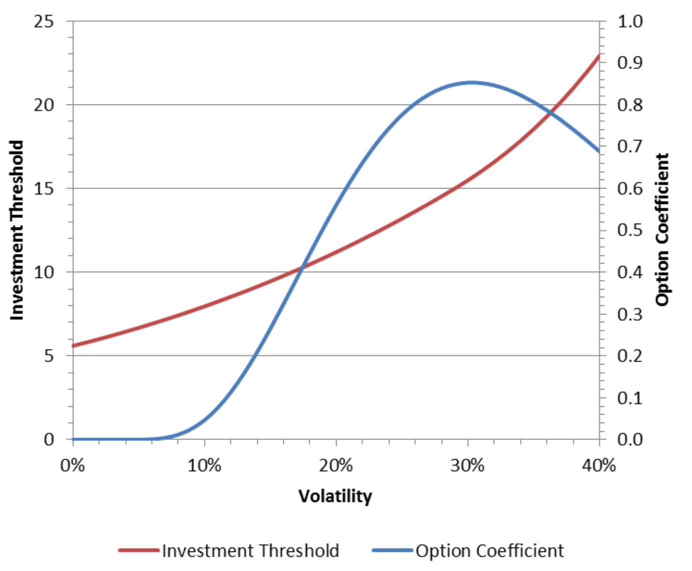


Figure 4b

Lumpy Investment Option Coefficient for Volatility Variations and Cap Levels

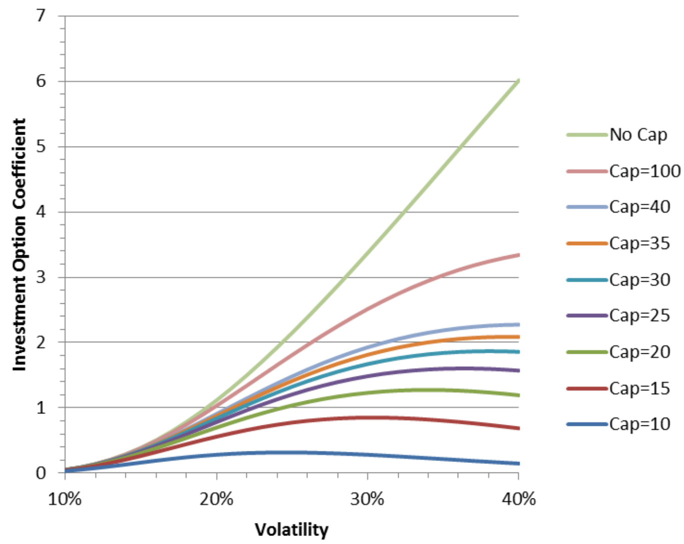


Figure 4b illustrates the joint effects of volatility and cap level on the investment option coefficient. This reveals the functional relationship between the option coefficient and volatility to be concave for finite cap levels, but the curvature at the maximum varies. The cap level positively influences the maximum option coefficient and its corresponding volatility value. For cap levels close to the investment threshold, the volatility maximising the option coefficient is relatively low, while for those tending to no-cap, the corresponding volatility is very high, in excess of levels commonly met in practice. Further, the curvature for these two extremes is relatively low. This suggests first, that volatility has an almost insignificant effect on the option coefficient for relatively low caps, and second, that the relationship can be treated as monotonic increasing for relatively high caps. When the cap level lies between the two extremes, the relationship is concave. Figure 4b demonstrates again the significance of the cap in modifying the option coefficient due to the presence and absence of the cap. Further, the difference between the option coefficients for the cap and no-cap cases intensifies as the volatility increases.

3.2 Model II

While the lumpy investment is expected to be more attractive at extreme ϕ values, the flexible alternative is often judged as being more valuable because of its earlier exercise and lower capital injection, but its value is expected to decline as ϕ approaches 1. We illustrate in Figure 5a and b the stage-1 and -2 investment thresholds and investment option coefficients for Model II with $q_U = 15, 40$, respectively, for variations in ϕ values over the range [0.05, 0.95]. The respective stage-1 and -2 caps are breached for $\phi \leq 0.10$ and $\phi \geq 0.26$ when $q_U = 15$, and for $\phi \geq 0.72$ when $q_U = 40$. As the stage-1 fraction ϕ increases, the stage-1 option coefficient initially increases, peaks and then declines, a feature which is primarily due to the additional capital charge $\kappa = 5.0$ added to the stage-1 investment cost. For stage-2, there is no additional capital charge and its option coefficient is monotonically decreasing since its investment cost declines as ϕ increases. The effect of the additional capital charge is also reflected in the investment thresholds. As ϕ increases, the stage-1 threshold initially dips and then increases[3]. Despite ϕ increasing and the stage-2 capital investment cost declining, the stage-2 threshold is increasing continuously over its range because of the increasing stage-11 value foregone when exercising the investment option.

[3] This feature occurs for a cap of 40 but is not visible due to the range displayed.

Figure 5a

Flexible Investment Threshold and Investment Option Coefficient for Variations in ϕ

with $q_U = 15$

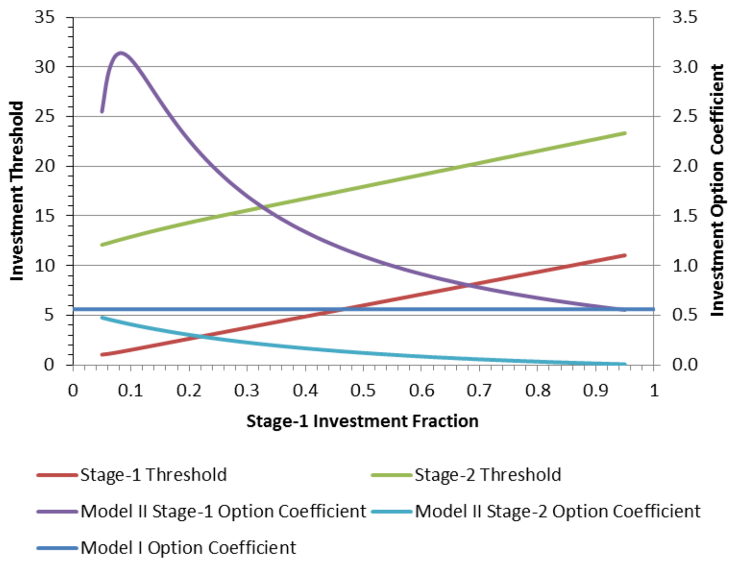
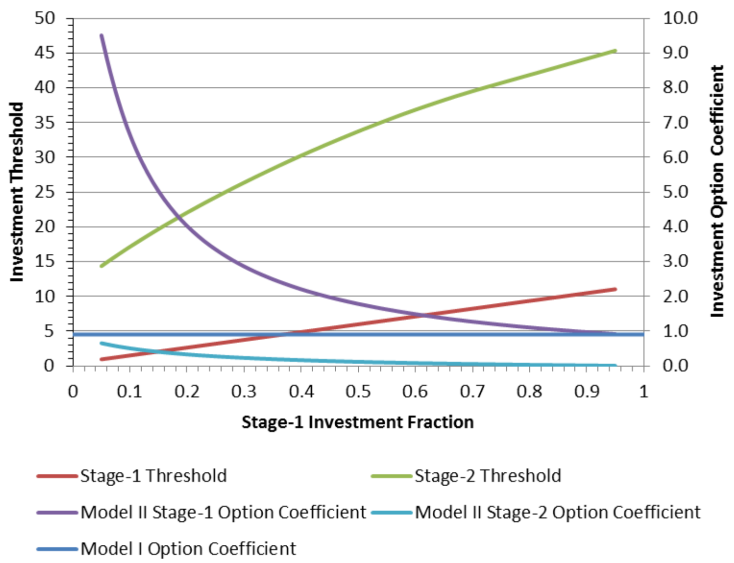


Figure 5b

Flexible Investment Threshold and Investment Option Coefficient for Variations in ϕ

with $q_U = 40$



The effects of variations in ϕ and q_U on the relative performance of the lumpy and flexible investment strategies are illustrated in Figure 6a, which shows the ratio of the investment option coefficients for variations in ϕ values over the range [0.05, 0.95] with $\kappa = 5.0$ and $q_U = 15, 40$. In Figure 6, a ratio in excess of 1.0 indicates the flexible strategy as having a greater stage-1 investment option coefficient than the option coefficient for the lumpy strategy, both strategies having the identical cap level. For the reported range, a less than 1.0 ratio occurs only for the $q_U = 15$ profile when $\phi \geq 0.95$. Further, since the stage-1 investment threshold for the flexible strategy is always less than the lumpy strategy threshold, this suggests that for many plausible ϕ values, the flexible strategy is the more attractive. The actual values of the investment option coefficients for the two flexible strategies are reported in Figure 6b. This reveals that relaxing the cap raises the option value for all ϕ values over the range [0.05, 0.95]. On the basis of a greater option value and earlier exercise, a more relaxed flexible strategy is more valuable than a less relaxed one, and a flexible strategy is the more attractive than a lumpy strategy for many plausible ϕ values given $\kappa = 5.0$. However, as the additional capital charge is allowed to increase and to additionally disadvantage the flexible strategy, this latter finding may become less secure.

Figure 6a

Ratio of the Flexible to Lumpy Investment Option Coefficients for Variations in ϕ

with $q_U = 15, 40$ and $\kappa = 5.0$

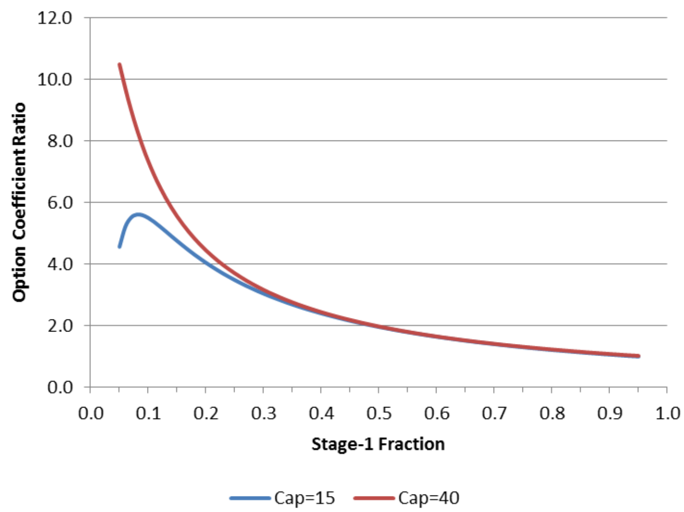
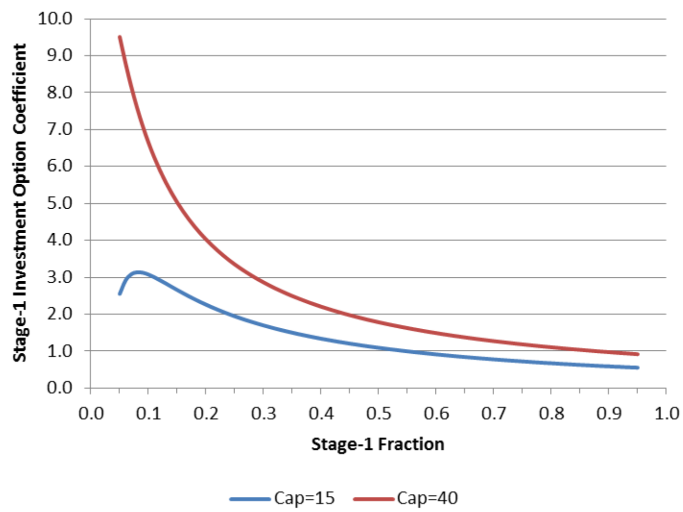


Figure 6b

Actual Flexible Stage-1 Investment Option Coefficients for Variations in ϕ

with $q_U = 15, 40$ and $\kappa = 5.0$



3.2.1 Volatility Changes

The effects of volatility changes on the Model I and II results share a very similar pattern. We present in Figure 7a and 7b, the stage-1 and -2 investment thresholds and option coefficients profiles with $q_U = 15$ for the investment fractions $\phi = 0.2, 0.4$, respectively. This reveals the expected result of an increasing relationship between the threshold and volatility. However, the stage-1 and -2 option coefficients profiles for the flexible strategy have a similar shape as that for the lumpy strategy, are concave and exhibit a maximum. The explanation for this unexpected finding is probably due to the breach in the productive cap. When $\phi = 0.2$, the stage-1 and -2 breaches occur at $\sigma \geq 0.25, \geq 0.22$, respectively; when $\phi = 0.4$, the stage-1 and -2 breaches occur at $\sigma \geq 0.27, \geq 0.17$, respectively. Whenever the investment threshold exceeds the cap, there is a commensurate decline in the option value reflecting the latent value lost due to the capacity failing to satisfy market demand.

Figure 7a

Flexible Investment Threshold and Option Coefficient for Volatility Variations

with $\phi = 0.2$ and $\kappa = 5.0$, $q_U = 15.0$

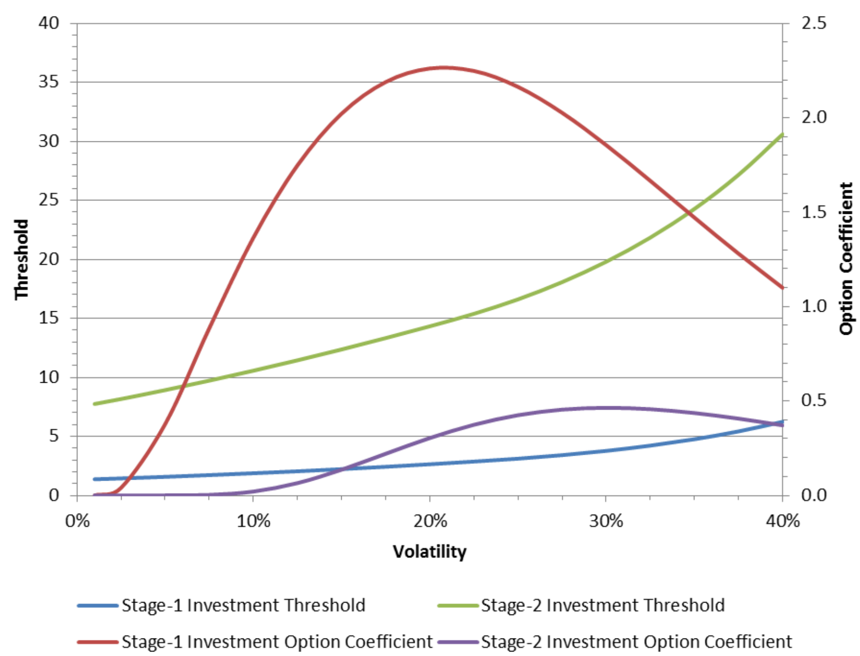
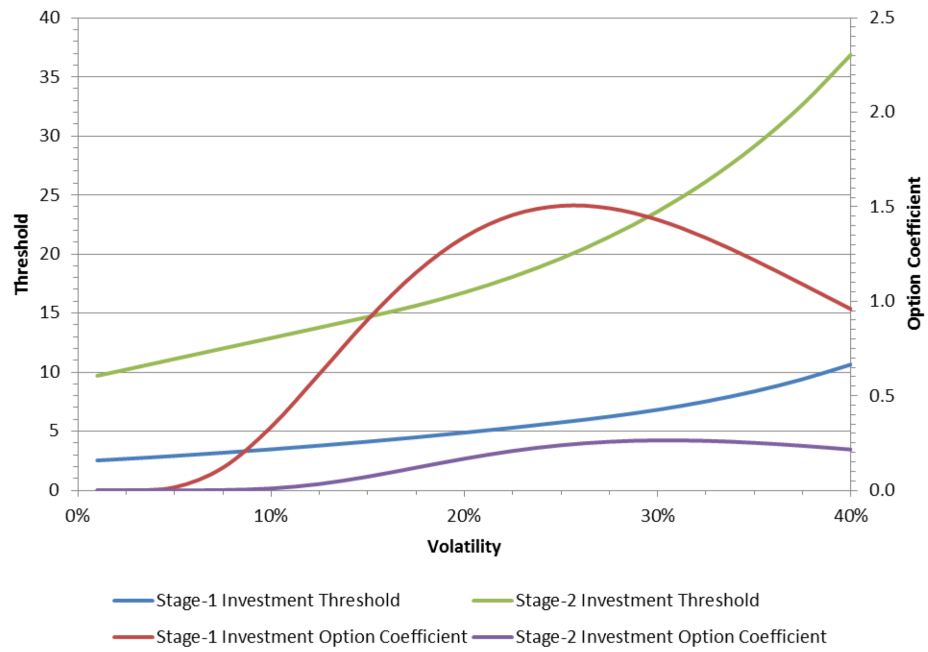


Figure 7b

Flexible Investment Threshold and Option Coefficient for Volatility Variations

with $\phi = 0.4$ and $\kappa = 5.0$, $q_U = 15.0$



A comparison of Figure 7a and b reveals that the case having a lower investment ratio $\phi = 0.2$ is more attractive, since it possesses the greater investment option value and is exercised earlier for all represented volatility values. The comparative performance for the two flexible alternative cases relative to the lumpy investment is presented in Figure 8a and b, which shows respectively the ratio of the investment option coefficient and investment threshold with $\phi = 0.2, 0.4$ relative to that for the lumpy investment for various volatility levels. In Figure 8a, a ratio exceeding 1.0 implies the flexible strategy as having a greater investment option coefficient, while in Figure 8b, a ratio less than 1.0 implies the flexible strategy as having a lower investment threshold and consequently an earlier exercise. Justified by only having a greater investment option coefficient, the virtual superiority of the flexible strategy is apparent in Figure 8a. However, the ratios continuously decline with volatility, falling below 1.0 at $\sigma \geq 70\%, \geq 72\%$ for $\phi = 0.2, 0.4$, respectively. We can conclude, for our data set, that almost certain projects definitely warrant a flexible strategy for their execution, while extremely risky projects should be executed using a lumpy strategy, except that the lumpy strategy appears to engender only a modest additional value. In Figure 8b, the threshold ratios are clearly less than 1.0 for all reported volatilities, which suggest the flexible strategies with $\phi = 0.2, 0.4$ having earlier exercise times than the lumpy strategy. Low volatility projects should be installed in two stages by adopting a flexible strategy in preference to a lumpy strategy because of having a greater investment option value and earlier exercise time, but as the volatility increases, the flexible strategy loses its superiority. A sufficiently high volatility may justify adopting a lumpy strategy, but its comparative advantage is modest.

Figure 8a

Option Coefficient Ratio with $\phi = 0.2, 0.4$ Relative to the Lumpy Investment

for Volatility Variations with $q_U = 15.0$

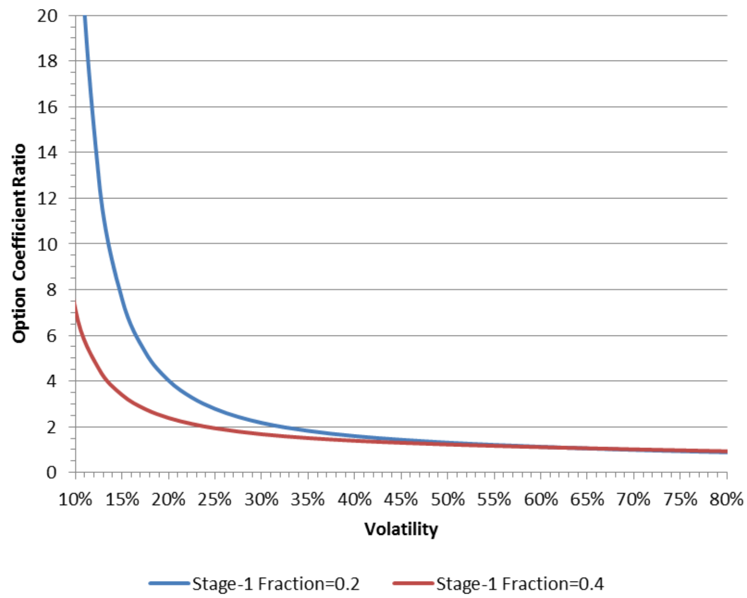
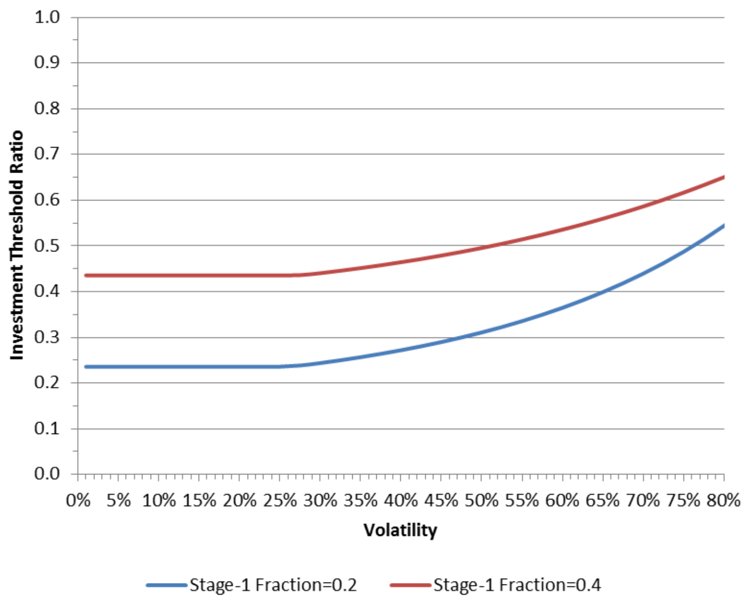


Figure 8b

Threshold Ratio with $\phi = 0.2, 0.4$ Relative to the Lumpy Investment

for Volatility Variations with $q_U = 15.0$



3.2.2 *Additional Capital Charge Changes*

Since it may be argued that the flexible strategy is favoured simply because of a relatively low additional capital charge, we reconsider our findings in light of a higher charge set at $\kappa = 20.0$. Figure 9a illustrates the stage-1 and -2 investment thresholds and the investment option coefficients for Model II with $q_U = 15$ $\kappa = 20.0$ and for varying values of ϕ between 10%-90%. This reveals that the stage-1 threshold both falls and rises as the stage-1 fraction ϕ increases. For $\phi \le 0.17$, the threshold falls since the additional capital charge represents a significant proportion of the stage-1 investment cost, but as that proportion diminishes for $\phi > 0.17$ the threshold rises in line with the stage-1 investment cost. Even so, it is not until $\phi \ge 0.17$ that the cap is not breached by the investment threshold. Similar to Figure 5 where $\kappa = 5.0$, the stage-2 investment threshold rises as ϕ rises, but the stage-2 cap is breached for $\phi \ge 0.26$. The more interesting finding from Figure 9a derives from comparing the investment option coefficient profiles for the lumpy and flexible strategies with $q_U = 15.0$. With $\kappa = 20.0$, the flexible strategy is less valuable than the lumpy strategy provided $\phi \le 0.13$ or $\phi > 0.73$. Over this range, the flexible strategy will never be selected despite having a lower investment threshold. For $0.13 < \phi \le 0.73$, the flexible strategy has the greater investment option value and attains a maximum for $\phi = 0.28$. The effect of raising the additional capital charge from $\kappa = 5.0$ to $\kappa = 20.0$ is revealed by comparing Figure 5 with Figure 9a. Although the stage-2 investment threshold remains unchanged, assuming the cap is not breached, the result of increasing the additional capital charge is a rise in the stage-1 threshold and a fall in the stage-1 option coefficient. An increase in the additional capital charge κ makes the flexible opportunity less attractive and to raise the ϕ value maximising the option coefficient. This effect is illustrated in Figure 9b for $\kappa = 30$. The result of increasing κ from 20 to 30 is to raise the ϕ value maximising the option coefficient from 0.28 to 0.38 and to lower the maximum option coefficient from 0.8990 to 0.5995. For an additional κ increase, $\kappa = 32$ as illustrated in Figure 9c, the firm is indifferent between the lumpy and flexible strategy with $\phi = 0.44$, but for $\phi \ne 0.44$, the lumpy strategy with its higher investment option value is superior. Provided that the additional capital charge to the stage-1 investment cost is no more than 32% of the lumpy strategy's investment cost, then for certain ϕ values, the flexible strategy is the more attractive of the two. And, the more the flexible opportunity is disadvantaged by a greater additional capital charge, the less likely it is of being selected.

Figure 9a

Flexible Investment Threshold and Investment Option Coefficient for Variations in ϕ with $\kappa = 20.0$

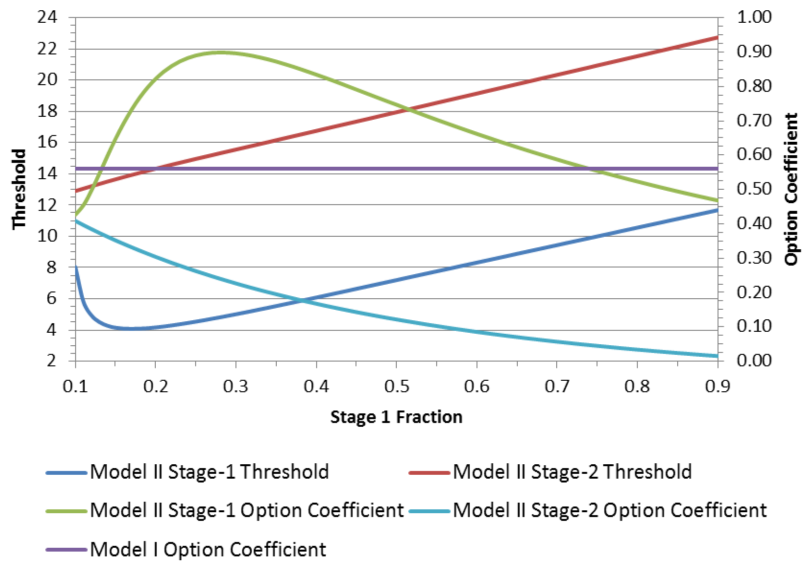


Figure 9b

Flexible Investment Threshold and Investment Option Coefficient for Variations in ϕ

with $\kappa = 30.0$

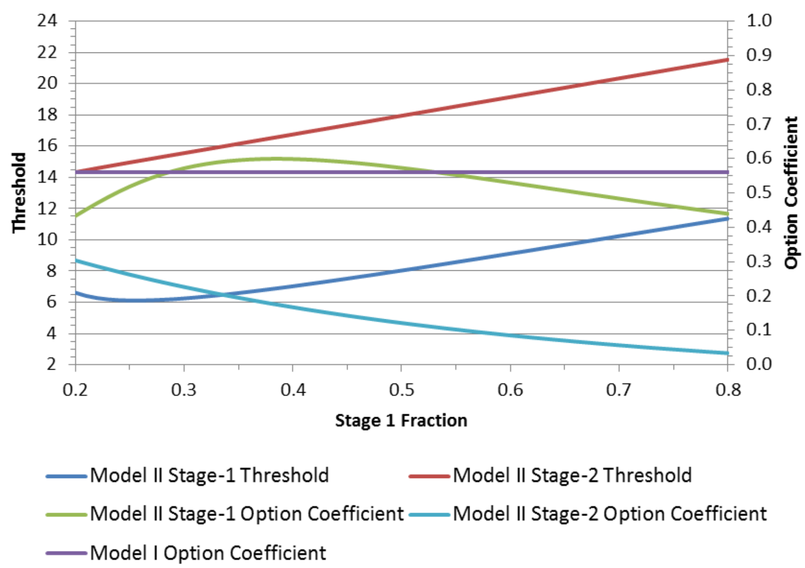
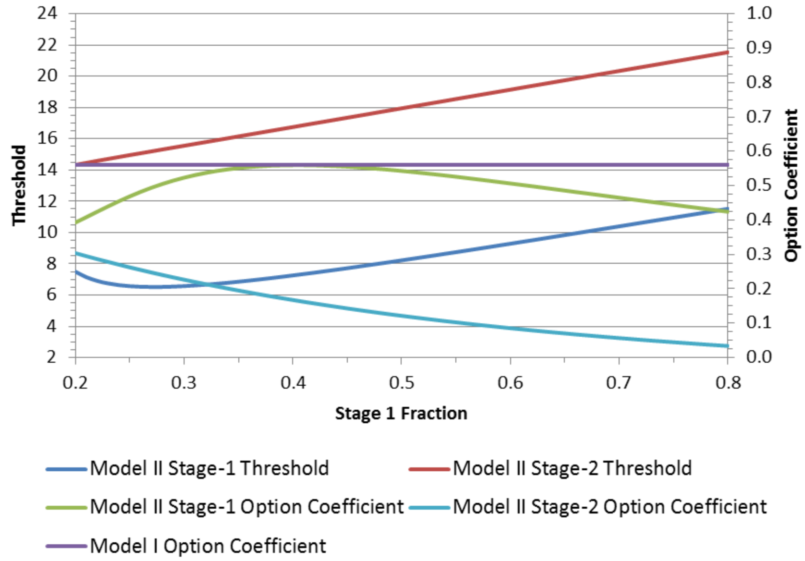


Figure 9c

Flexible Investment Threshold and Investment Option Coefficient for Variations in ϕ

with $\kappa = 32.0$



4 Conclusion

A cap representing a capacity limit on the production output is formulated as a pair of written call and put options. Intrinsically, since any limit on an underlying factor has the attributes of an option payoff, this description of a productive cap is plausible with the merit of yielding some interesting tractable analytical results. The options are written because of the lost latent value as the market demand volume increases to exceed the cap and decreases to fall below the cap. We investigate the role of the cap in making the decision between a lumpy investment strategy, represented by Model I, and a flexible strategy, represent by Model II.

The presence of a cap complicates the analysis because of its effect on the solution depending on whether the cap is breached or not. For Model I, if the cap is not breached by the investment threshold, then the resulting with-cap threshold is identical to the without-cap solution, since the threshold level is determined optimally to be sufficiently high to ensure project viability. However, the cap limits the upside potential and impacts negatively on the investment option value, thereby resulting in a with-cap option value less than that without a cap. A without-cap investment threshold exceeding the cap produces a greater with-cap threshold to compensate the value lost due to limited output enforced by the cap while at the same time ensuring project viability. The investment option value is then further reduced by the resulting threshold increase owing to the lost latent value. The effect of relaxing the cap by increasing the limit is to raise the investment option value for all reported market demand volumes. As expected, volatility increases are reflected in a corresponding increase in the investment threshold. However, the relationship between volatility and the investment option value is not monotonic increasing as for many real-option formulations but concave with the option value experiencing a maximum for a plausible volatility. This feature is replicated for cap level increases, but the effect becomes increasingly less pronounced as the cap increases.

The flexible strategy is composed of two consecutive stages such that only until the attainment of stage-1 can stage-2 be implemented. It is relatively disadvantaged compared with the lumpy strategy and is designed to have a greater overall capital cost with the additional loading at stage-1. Our findings are similar in content to those for Model I. If the stage-1 cap is not breached, then the with- and without-cap investment thresholds are identical and the with-cap investment option value is less than the without-cap option value. Compared with Model I, Model II has a lower investment threshold and consequently, if exercised is exercised earlier. The numerical investigation demonstrates that for an additional capital charge equalling 5.0, the flexible strategy is more attractive than the lumpy strategy for most stage-1 and -2 capital distributions, both in terms of greater investment option value and lesser investment threshold. Volatility variations have a similar effect on the results for Model II as for Model I. An increase in volatility produces a corresponding increase in the stage-1 and -2 investment thresholds. Also, there is a similar effect on the stage-1 and -2 investment option values, which are both concave in shape with a maximum, with the former being the more pronounced. Volatility variations change the relative merits of the flexible versus the lumpy strategy. Low risk projects, having low volatilities, are best executed using a flexible strategy because of having a greater option value and lower threshold. However, the attractiveness of the flexible strategy diminishes as volatility increases, until the lumpy strategy becomes eventually superior, except that its relative advantage is modest. This endorses the conclusion of Kort et al. (2010) that increased uncertainty favours the lumpy to the flexible strategy.

Compared with the lumpy strategy, preference for the flexible alternative depends on the level of disadvantage imposed by the additional capital charge. The flexible strategy is preferred for low additional capital charges, which allows management to select from a relative wide range of stage-1 investment levels despite the existence of an optimal level. However, the choice range of investment levels narrows for increases in the additional capital charge until the point of indifference between the two strategies is reached. The flexibility in selecting the stage-1 investment proportion is limited by the additional capital charge and the volatility.

Our findings are subject to reservations. The only forms of optionality represented in the model are investment opportunities and the productive cap. This could be enhanced by including abandonment, but preliminary results suggest that its inclusion engenders only a slight change in solution values. More insightful results may be obtained by making the productive cap less severe by allowing overtime and introducing downside volume flexibility and suspension.

Appendix A: Derivations for Model I

The coefficients $A_{(I)11}, A_{(I)112}$ are obtained from the value-matching relationship and associated smooth-pasting condition ruling at the boundary between state-1 and -11, namely:

$$V_{(I)1}(q)\Big|_{q=q_U} = V_{(I)11}(q)\Big|_{q=q_U}, \frac{\partial V_{(I)1}(q)}{\partial q}\bigg|_{q=q_U} = \frac{\partial V_{(I)11}(q)}{\partial q}\bigg|_{q=q_U}, \tag{A.1}$$

or respectively as:

$$A_{(I)11} q_U^{\beta_1} + p q_U/\delta - f/r = A_{(I)112} q_U^{\beta_2} + p q_U/\delta - f/r, \tag{A.2}$$

$$\beta_1 A_{(I)11} q_U^{\beta_1 - 1} + p/\delta = \beta_2 A_{(I)112} q_U^{\beta_2 - 1}. \tag{A.3}$$

Substituting $A_{(I)112} = q_U^{-\beta_2}\left(p q_U/\delta + \beta_1 A_{(I)11} q_U^{\beta_1}\right)\Big/\beta_2$ from (A.3) into (A.2) and simplifying yields:

$$A_{(I)11} = -\frac{p q_U^{1-\beta_1}\left(r - r\beta_2 + \delta\beta_2\right)}{r\left(\beta_1 - \beta_2\right)\delta}, \tag{A.4}$$

$$A_{(I)112} = \frac{p q_U^{1-\beta_2}\left(r - r\beta_1 + \delta\beta_1\right)}{r\left(\beta_1 - \beta_2\right)\delta}. \tag{A.5}$$

The investment threshold $\hat{q}_{(I)}$ and option coefficient $A_{(I)01}$ are obtained from the value-matching relationship and associated smooth-pasting condition ruling at the boundary between state-0 and -1:

$$V_{(I)0}(q)\Big|_{q=\hat{q}_{(I)1}} = V_{(I)1}(q) - K\Big|_{q=\hat{q}_{(I)1}}, \frac{\partial V_{(I)0}(q)}{\partial q}\bigg|_{q=\hat{q}_{(I)1}} = \frac{\partial V_{(I)1}(q)}{\partial q}\bigg|_{q=\hat{q}_{(I)1}} \tag{A.6}$$

or respectively as:

$$A_{(I)01} \hat{q}_{(I)1}^{\beta_1} = A_{(I)11} \hat{q}_{(I)1}^{\beta_1} + p\hat{q}_{(I)1}\Big/\delta - f/r - K, \tag{A.7}$$

$$\beta_1 A_{(I)01} \hat{q}_{(I)1}^{\beta_1 - 1} = \beta_1 A_{(I)11} \hat{q}_{(I)1}^{\beta_1 - 1} + p/\delta. \tag{A.8}$$

From (A.8):

$$A_{(I)01} = \frac{p \hat{q}_{(I)1}^{1-\beta_1}}{\beta_1 \delta} + A_{(I)11}, \tag{A.9}$$

which when substituted in (A.7) yields:

$$\hat{q}_{(I)1} = \frac{\beta_1}{\beta_1 - 1}\left(f + rK \right)\frac{\delta}{rp}. \tag{A.10}$$

If $f/r + K > \left(p\,q_U/\delta \right)\left(\beta_1 - 1 \right)/\beta_1$, then $\hat{q}_{(I)01} > q_U$ and the value-matching relationship and smooth-pasting conditions require amending accordingly. From (A.6), those ruling at the boundary between state-0 and -11 are:

$$V_{(I)0}\left(q \right)\Big|_{q=\hat{q}_{(I)11}} = V_{(I)11}\left(q \right) - K\Big|_{q=\hat{q}_{(I)11}}, \frac{\partial V_{(I)0}\left(q \right)}{\partial q}\Bigg|_{q=\hat{q}_{(I)11}} = \frac{\partial V_{(I)11}\left(q \right)}{\partial q}\Bigg|_{q=\hat{q}_{(I)11}}. \tag{A.11}$$

The respective value-matching relationship and smooth-pasting condition are:

$$A_{(I)011}\hat{q}_{(I)011}^{\beta_1} = A_{(I)112}\hat{q}_{(I)011}^{\beta_2} + pq_U/r - f/r - K, \tag{A.12}$$

$$\beta_1 A_{(I)011}\hat{q}_{(I)011}^{\beta_1 - 1} = \beta_2 A_{(I)112}\hat{q}_{(I)011}^{\beta_2 - 1} \tag{A.13}$$

From (A.13):

$$A_{(I)011} = \frac{\beta_2}{\beta_1} A_{(I)112}\hat{q}_{(I)011}^{\beta_2 - \beta_1} = \frac{-\left(pq_U - f - rK \right)\beta_2}{\left(\beta_1 - \beta_2 \right)r}\hat{q}_{(I)11}^{-\beta_1} \tag{A.14}$$

which when substituted in (A.13) yields:

$$\hat{q}_{(I)11} = \left(\frac{\beta_1\left(pq_U - f - rK \right)}{\left(\beta_1 - \beta_2 \right)r\left(-A_{(I)22} \right)} \right)^{\frac{1}{\beta_2}} = q_U\left(\frac{\left(pq_U - f - rK \right)\beta_1\delta}{pq_U\left(r - r\beta_1 + \delta\beta_1 \right)} \right)^{\frac{1}{\beta_2}}. \tag{A.15}$$

From (A.15):

$$\frac{\partial\hat{q}_{(I)11}}{\hat{q}_{(I)11}\partial q_U} = \frac{\beta_2 p + f + rK}{\beta_2\left(pq_U - f - rK \right)} < 0, \tag{A.16}$$

provided $\beta_2 p + f + rK > 0$. From (A.14):

$$\frac{\partial A_{(I)01}}{A_{(I)01}\partial q_U} = \frac{p}{pq_U - f - rK} - \frac{\beta_1}{\hat{q}_{(I)11}}\frac{\partial\hat{q}_{(I)11}}{\partial q_U} > 0. \tag{A.17}$$

Appendix B: Derivations for Model II

The coefficients $A_{(II)21}, A_{(II)212}$ are obtained from the value-matching relationship and associated smooth-pasting condition ruling at the boundary between state-2 and -21, namely:

$$V_{(II)2}(q)\Big|_{q=q_U} = V_{(II)21}(q)\Big|_{q=q_U}, \frac{\partial V_{(II)2}(q)}{\partial q}\Bigg|_{q=q_U} = \frac{\partial V_{(II)21}(q)}{\partial q}\Bigg|_{q=q_U} \tag{B.1}$$

or, respectively, as:

$$A_{(II)21} q_U^{\beta_1} + \frac{pq_U}{\delta} - \frac{f_1 + f_2}{r} = A_{(II)212} q_U^{\beta_2} + \frac{pq_U}{r} - \frac{f_1 + f_2}{r}, \tag{B.2}$$

$$\beta_1 A_{(II)21} q_U^{\beta_1 - 1} + \frac{q_U}{\delta} = \beta_2 A_{(II)212} q_U^{\beta_2 - 1}. \tag{B.3}$$

Substituting $A_{(II)212} = q_U^{-\beta_2}\left(pq_U / \delta + \beta_1 A_{(II)21} q_U^{\beta_1 - 1} \right) \Big/ \beta_2$ from (B.3) into (B.4) yields:

$$A_{(II)21} = \frac{\beta_2 (r - \delta) - r}{r(\beta_1 - \beta_2)\delta} pq_U^{1-\beta_1}, \tag{B.5}$$

so:

$$A_{(II)212} = \frac{\beta_1 (r - \delta) - r}{r(\beta_1 - \beta_2)\delta} pq_U^{1-\beta_2}. \tag{B.6}$$

The coefficients $A_{(II)111}, A_{(II)112}$ are obtained from the value matching relationship and smooth pasting condition ruling at the boundary between state-11 and -2:

$$V_{(II)11}(q)\Big|_{q=\hat{q}_{(II)12}} = V_{(II)2}(q)\Big|_{q=\hat{q}_{(II)12}} - K_2, \frac{\partial V_{(II)11}(q)}{\partial q}\Bigg|_{q=\hat{q}_{(II)12}} = \frac{\partial V_{(II)2}(q)}{\partial q}\Bigg|_{q=\hat{q}_{(II)12}}, \tag{B.7}$$

where $\hat{q}_{(II)12}$ denotes the stage-2 investment opportunity threshold. The value matching relationship and smooth pasting condition are, respectively:

$$A_{(II)111} \hat{q}_{(II)12}^{\beta_1} + A_{(II)112} \hat{q}_{(II)12}^{\beta_2} + \frac{pq_u}{r} - \frac{f_1}{r} = A_{(II)21} \hat{q}_{(II)12}^{\beta_1} + \frac{p\hat{q}_{(II)12}}{\delta} - \frac{f_1 + f_2}{r} - K_2, \tag{B.8}$$

$$\beta_1 A_{(II)111} \hat{q}_{(II)12}^{\beta_1 - 1} + \beta_2 A_{(II)112} \hat{q}_{(II)12}^{\beta_2 - 1} = \beta_1 A_{(II)21} \hat{q}_{(II)12}^{\beta_1 - 1} + \frac{p}{\delta}. \tag{B.9}$$

Substituting $A_{(II)112} = \hat{q}_{(II)12}^{-\beta_2}\left(\beta_1 A_{(II)21} \hat{q}_{(II)12}^{\beta_1} - \beta_1 A_{(II)111} \hat{q}_{(II)12}^{\beta_1} + p\hat{q}_{(II)12} / \delta \right) \Big/ \beta_2$ from (B.9) into (B.8) yields:

$$A_{(II)111} = A_{(II)21} + \frac{\hat{q}_{(II)12}^{-\beta_1}}{\beta_1 - \beta_2}\left[\frac{(1-\beta_2)\,p\hat{q}_{(II)12}}{\delta} + \beta_2\left(K_2 + \frac{f_2}{r} + \frac{pq_u}{r}\right)\right], \tag{B.10}$$

so:

$$A_{(II)112} = \frac{\hat{q}_{(II)12}^{-\beta_2}}{\beta_1 - \beta_2}\left[\frac{(\beta_1 - 1)\,p\hat{q}_{(II)12}}{\delta} - \beta_1\left(K_2 + \frac{f_2}{r} + \frac{pq_u}{r}\right)\right]. \tag{B.11}$$

The coefficient $A_{(II)11}$ and threshold $\hat{q}_{(II)12}$ are obtained from the value matching relationship and smooth pasting condition ruling at the boundary between state-1 and -11:

$$V_{(II)1}(q)\Big|_{q=q_u} = V_{(II)11}(q)\Big|_{q=q_u}, \frac{\partial V_{(II)1}(q)}{\partial q}\bigg|_{q=q_u} = \frac{\partial V_{(II)11}(q)}{\partial q}\bigg|_{q=q_u}. \tag{B.12}$$

or, respectively, as:

$$A_{(II)11}q_u^{\beta_1} + \frac{pq_u}{\delta} - \frac{f_1}{r} = A_{(II)111}q_u^{\beta_1} + A_{(II)112}q_u^{\beta_2} + \frac{pq_u}{r} - \frac{f_1}{r}, \tag{B.13}$$

$$\beta_1 A_{(II)11}q_u^{\beta_1 - 1} + \frac{p}{\delta} = \beta_1 A_{(II)111}q_u^{\beta_1 - 1} + \beta_2 A_{(II)112}q_u^{\beta_2 - 1}. \tag{B.14}$$

Substituting $A_{(II)11} = q_u^{-\beta_1}\left(\beta_1 A_{(II)111}q_u^{\beta_1} + \beta_2 A_{(II)112}q_u^{\beta_2} - pq_u/\delta\right)\big/\beta_1$ from (B.14) into (B.13), and simplifying yields:

$$A_{(II)112} = \frac{\hat{q}_u^{-\beta_2}\left(r(\beta_1 - 1) - \delta\beta_1\right)pq_u}{r(\beta_1 - \beta_2)\delta}. \tag{B.15}$$

Eliminating $A_{(II)112}$ from (B.11) and (B.15) yields:

$$\hat{q}_{(II)12}^{-\beta_2}\left[r(\beta_1 - 1)\,p\hat{q}_{(II)12} - \delta\beta_1(rK_2 + f_2 + pq_u)\right] = q_u^{-\beta_2}\left(r(\beta_1 - 1) - \delta\beta_1\right)pq_u. \tag{B.16}$$

The solution for $\hat{q}_{(II)12}$ is obtained from (B.16) by using numerical methods with

$$\frac{\delta\beta_1(rK_2 + f_2 + pq_u)}{r(\beta_1 - 1)\,p}$$

as a possible initial estimate. Also from (B.14), and substituting (B.5), (B.10) and (B.11) to yield after simplification:

$$A_{(II)11} = \frac{\beta_2(r-\delta)-r}{r(\beta_1-\beta_2)\delta}\left(pq_U^{1-\beta_1} + pq_u^{1-\beta_1}\right) + \frac{\hat{q}_{(II)12}^{-\beta_1}}{r(\beta_1-\beta_2)\delta}\left[p\hat{q}_{(II)12}(1-\beta_2)r + (pq_u + f_2 + K_2)\delta\beta_2\right] \tag{B.17}$$

The coefficient $A_{(II)01}$ and the stage-1 investment threshold $\hat{q}_{(II)01}$ are obtained from the value matching relationship and smooth pasting condition ruling at the boundary between state-0 and -1:

$$V_{(II)0}(q)\Big|_{q=\hat{q}_{(II)01}} = V_{(II)1}(q)\Big|_{q=\hat{q}_{(II)01}} - K_1, \ \frac{\partial V_{(II)0}(q)}{\partial q}\bigg|_{q=\hat{q}_{(II)01}} = \frac{\partial V_{(II)1}(q)}{\partial q}\bigg|_{q=\hat{q}_{(II)01}}, \tag{B.18}$$

The respective value matching relationship and smooth pasting condition are:

$$A_{(II)01}\hat{q}_{(II)01}^{\beta_1} = A_{(II)11}\hat{q}_{(II)01}^{\beta_1} + \frac{p\hat{q}_{(II)01}}{\delta} - \frac{f_1}{r} - K_1, \tag{B.19}$$

$$\beta_1 A_{(II)01}\hat{q}_{(II)01}^{\beta_1-1} = \beta_1 A_{(II)11}\hat{q}_{(II)01}^{\beta_1-1} + \frac{p}{\delta} \tag{B.20}$$

Substituting $A_{(II)01} = A_{(II)11} + \hat{q}_{(II)01}^{-\beta_1}\, p\hat{q}_{(II)01}\big/\beta_1\delta$ from (B.20) into (B.19), and simplifying yields:

$$\hat{q}_{(II)01} = \frac{\beta_1(f_1 + rK_1)\delta}{p(\beta_1-1)r}. \tag{B.21}$$

Also from (B.20) and substituting (B.21) yields:

$$A_{(II)01} = \frac{p\hat{q}_{(II)01}}{\beta_1\delta\hat{q}_{(II)01}^{\beta_1-1}} + A_{(II)11}. \tag{B.22}$$

If $\hat{q}_{(II)12} > q_U$, then the value matching relationship and smooth pasting condition ruling at the boundary between state-11 and -2, (B.7), is replaced by those between state-11 and -21, while the others remain intact. The revised expressions are:

$$V_{(II)11}(q)\Big|_{q=\hat{q}_{(II)121}} = V_{(II)21}(q)\Big|_{q=\hat{q}_{(II)121}} - K_2, \ \frac{\partial V_{(II)11}(q)}{\partial q}\bigg|_{q=\hat{q}_{(II)121}} = \frac{\partial V_{(II)21}(q)}{\partial q}\bigg|_{q=\hat{q}_{(II)121}}, \tag{B.23}$$

or respectively as:

$$A_{(II)111}\hat{q}_{(II)121}^{\beta_1}+A_{(II)112}\hat{q}_{(II)121}^{\beta_2}+\frac{pq_u}{r}-\frac{f_1}{r}=A_{(II)212}\hat{q}_{(II)121}^{\beta_2}+\frac{p\hat{q}_U}{r}-\frac{f_1+f_2}{r}-K_2\,, \tag{B.24}$$

$$\beta_1 A_{(II)111}\hat{q}_{(II)121}^{\beta_1-1}+\beta_2 A_{(II)112}\hat{q}_{(II)121}^{\beta_2-1}=\beta_2 A_{(II)212}\hat{q}_{(II)121}^{\beta_2-1}\,. \tag{B.25}$$

From (B.25), $A_{(II)112}=A_{(II)212}-\beta_1 A_{(II)111}\hat{q}_{(II)121}^{\beta_1-\beta_2}\big/\beta_2$, which when substituted in (B.24) yields:

$$A_{(II)111}=\frac{-\hat{q}_{(II)121}^{-\beta_1}}{r\left(\beta_1-\beta_2\right)}\left(pq_U-pq_u-f_2-rK_2\right)\beta_2\,, \tag{B.26}$$

so:

$$A_{(II)112}=\frac{q_U^{-\beta_2}\left(-r+r\beta_1-\delta\beta_1\right)pq_U}{r\left(\beta_1-\beta_2\right)\delta}+\frac{\hat{q}_{(II)121}^{-\beta_2}}{r\left(\beta_1-\beta_2\right)\delta}\left(pq_U-pq_u-f_2-rK_2\right)\beta_1\delta\,. \tag{B.27}$$

Since the expressions pertaining to stage-11 are unaffected, then by combining (B.27) with (B.15) yields:

$$\left(pq_U^{1-\beta_2}-pq_u^{1-\beta_2}\right)\left(r\left(\beta_1-1\right)-\delta\beta_1\right)+\hat{q}_{(II)121}^{-\beta_2}\left(pq_U-pq_u-f_2-rK_2\right)\beta_1\delta=0\,, \tag{B.28}$$

so:

$$\hat{q}_{(II)121}^{-\beta_2}=\frac{\left(pq_U^{1-\beta_2}-pq_u^{1-\beta_2}\right)\left(\delta\beta_1-r\left(\beta_1-1\right)\right)}{\left(pq_U-pq_u-f_2-rK_2\right)\beta_1\delta}\,. \tag{B.29}$$

Combining (B.27) and (B.28) with (B.14) and simplifying yields:

$$A_{(II)11}=\frac{-\hat{q}_{(II)121}^{-\beta_1}\left(pq_U-pq_u-f_2-rK_2\right)\beta_2\delta}{r\left(\beta_1-\beta_2\right)\delta}+\frac{pq_u^{1-\beta_1}\left(r\left(\beta_2-1\right)-\beta_2\delta\right)}{r\left(\beta_1-\beta_2\right)\delta}\,. \tag{B.30}$$

Since the expressions pertaining to stage-1 are unaffected, then the investment opportunity threshold $\hat{q}_{(II)01}$ is found from (B.21) and the option coefficient $A_{(II)01}$ from:

$$A_{(II)01}=A_{(II)11}+\hat{q}_{(II)01}^{-\beta_1}\,p\hat{q}_{(II)01}\big/\beta_1\delta\,, \tag{B.31}$$

where $A_{(II)11}$ is given by (B.30).

It is straightforward to demonstrate that $\hat{q}_{(II)12}=\hat{q}_{(II)121}$ when $\hat{q}_{(II)12}$ is determined from (B.16) given $\hat{q}_{(II)12}=q_U$ and $\hat{q}_{(II)121}$ is determined from (B.29) given $\hat{q}_{(II)121}=q_U$.

If $\hat{q}_{(II)01} > q_u$, then the value matching relationship and smooth pasting condition ruling at the boundary between state-0 and -1, (B.18), are replaced by those between state-0 and -11. The revised expressions are:

$$V_{(II)0}(q)\Big|_{q=\hat{q}_{(II)011}} = V_{(II)11}(q)\Big|_{q=\hat{q}_{(II)011}} - K_1 \, , \; \frac{\partial V_{(II)0}(q)}{\partial q}\Bigg|_{q=\hat{q}_{(II)011}} = \frac{\partial V_{(II)11}(q)}{\partial q}\Bigg|_{q=\hat{q}_{(II)011}} . \qquad \text{(B.32)}$$

The respective value matching relationship and smooth pasting condition are:

$$A_{(II)011}\hat{q}^{\beta_1}_{(II)011} = A_{(II)111}\hat{q}^{\beta_1}_{(II)011} + A_{(II)112}\hat{q}^{\beta_2}_{(II)011} + \frac{pq_u}{r} - \frac{f_1}{r} - K_1 , \qquad \text{(B.33)}$$

$$\beta_1 A_{(II)011}\hat{q}^{\beta_1-1}_{(II)011} = \beta_1 A_{(II)111}\hat{q}^{\beta_1-1}_{(II)011} + \beta_2 A_{(II)112}\hat{q}^{\beta_2-1}_{(II)011} . \qquad \text{(B.34)}$$

From (B.34), $A_{(II)011} = A_{(II)111} + \beta_2 A_{(II)112}\, \hat{q}^{\beta_2-\beta_1}_{(II)011} \big/ \beta_1$, which when substituted in (B.33) yields:

$$\hat{q}^{\beta_2}_{(II)011} = \frac{-\beta_1\left(pq_u - f_1 - rK_1\right)}{\left(\beta_1 - \beta_2\right) r A_{(II)112}} , \qquad \text{(B.35)}$$

where $A_{(II)112}$ is determined from (B.15) if the stage-2 cap is not breached, or from (B.27) if otherwise. Also, from (B.34):

$$A_{(II)011} = A_{(II)111} + \frac{-\beta_2\left(pq_u - f_1 - rK_1\right)}{\left(\beta_1 - \beta_2\right) r}\hat{q}^{-\beta_1}_{(II)011} \qquad \text{(B.36)}$$

where $A_{(II)111}$ is determined from (B.10) if the stage-2 cap is not breached, or from (B.26) if otherwise.

Figure 1a

Model I: Unbreached Cap

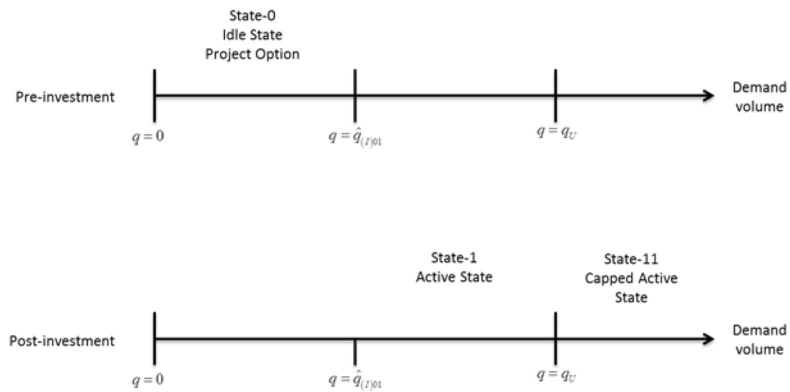


Figure 1b

Model I: Breached Cap

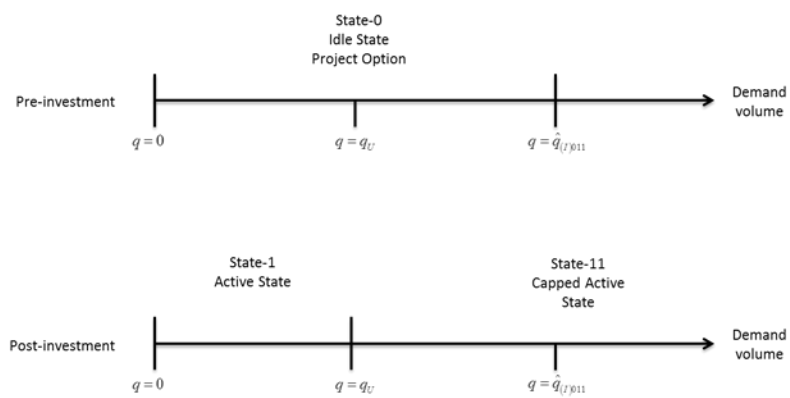
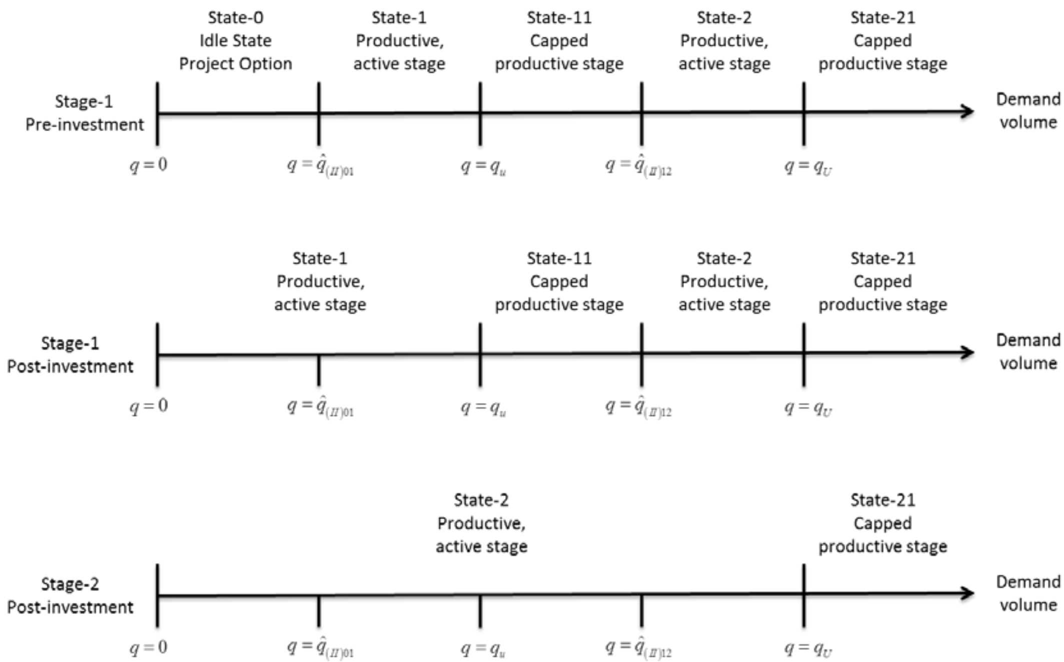


Figure 2

Model II



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