Capacity investment under uncertainty:
The effect of volume flexibility

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Abstract

Real option theory is a central tool in today's investment theory as it integrates uncertainty and managerial flexibility in the analysis and valuation of investment projects. This paper studies the optimal time and size of investment for a monopolistic firm under demand uncertainty and volume flexibility. In our modeling framework, demand is random and the firm first decides the optimal time and size of the production process. After entry, the firm adjusts continuously production volume to match the observed demand. Volume flexibility comes at a cost which depends on both the current output and the established capacity. We study two different models of volume flexibility: *Downside* volume flexibility allows the firms to produce any quantity below the installed capacity. *Upside* volume flexibility also allows to expand production above the firm's capacity size. In both cases, the option to temporary suspend production is not given a priori, but it is part of the firm's optimal choice. With this feature, the model provides conclusions that contrast some of the most recent theoretical findings on the same subject. We find that an increase of the degree of downside volume flexibility makes the firm willing to invest earlier in a larger plant. We also show that downside volume flexibility reduces the utilization rates, especially in highly uncertain markets. Upside volume flexibility has the joint effect of reducing the size of the investment and the investment threshold at which the firm installs capacity. The utilization rates are significantly higher compared to the case of downside volume flexibility only, and there is an increasing relationship between increased upside flexibility and utilization rates.

1 Introduction

In today's business, changes in market and economic conditions have a tremendous impact on firms' performance. The high variability of demand in many markets has become the nightmare of managers, since it is the cause of the potential mismatch between supply and demand. An example is the breakdown of the dotcom industry described in Raturi and Jack (2004). They report a 50 percent of demand felt down between 2000 and 2002 in the technological and networking sector. This breakdown forced firms to perform a drastic revision of their production process. For instance, in 2001 Cisco Systems announced a suspension of 8,500 workers and a canceling of unused inventory for a total value of \$2,5 billion, due to a drastic change in market demand (in front of a forecast of a 70 percent increase in sales, they experienced a 30 percent decline), (see Raturi and Jack, 2004). Another example is the dramatic market's breakdown in the automotive industry during the recent financial crisis (2008-2011) also mentioned in Hagspiel et al. (2016). Bengtsson and Olhager (2002), among others, advocate the use of manufacturing flexibility to mitigate the risk exposure of firms' profit flow due to drastic changes in demand. In particualr, volume flexibility, that is the ability to adapt production to current demand level, is a central feature of today's investment planning. For instance Fleischmann et al. (2006), in describing BMW's model of strategic planning, argues that *"For BMW, flexibility of production capacities with regard to future unknown demand is a central issue"*.

Traditional capital budgeting techniques suggest that the valuation of a given investment project should be accomplished by computing the expected value of the project's future cash flow, the so-called project's Net Present Value (NPV), discounted with an opportunely risk-adjusted interest rate. In this context, Trigeorgis (1996), at page 25 states "In the absence of managerial flexibility, net present value (NPV) is the only currently available valuation measure consistent with a firms objective of maximizing its shareholders wealth". However, NPV analysis typically ignores situations where a project posses some degree of managerial flexibility, such as the ability to decide the starting date of the project, the possibility to abandon the project at some future date or the flexibility to decide about the scale of the project. In such cases the valuation of an investment project requires more sophisticated tools, such as modern real option analysis that recognizes an investment project as a portfolio of complex real options (that is, options on real assets).

This paper utilizes a real option framework to analyze the effect of volume flexibility on both the optimal time to invest and the optimal size of the production plant to install under uncertain market conditions. More specifically, the firm's decision problem consists in determining: *i)* when it is optimal to enter in the market; *ii)* what capacity maximizes the discounted sum of future profit flow and *iii)* the current production. In this context, we analyze how volume flexibility affects the firm's optimal choice.

Previous research utilizes different definitions of volume flexibility. Sethi and Sethi (1990) define it as the ability of a production process to profitably operate at different output levels. For Gerwin (1993), volume flexibility allows a firm to increase or decrease the aggregate production level. More recently, Hagspiel et al. (2016) uses the concept for which volume flexibility is the ability to adjust production costlessly over time, identified in Bengtsson (2001) as operational or production flexibility. We utilize the concept of volume flexibility elaborated in Goyal and Netessine (2011), for which volume flexibility is the ability to profitably produce at volumes different from installed capacity, so as to adapt the production output to the current level of demand. This is done by defining the firm's cost structure to depend linearly on the current output and by adding a quadratic term which measures and penalizes volumes of production different from installed capacity, see also Vives (1989). Our model accounts explicitly for two different variations. Downside volume flexibility is the ability to profitably downscale production when demand is low, while upside volume flexibility is the ability to increase production above the established capacity to face periods of high levels of demand.

In our model, the option to suspend production is neither given nor discarded a priori, but it is an endogenous choice. As it turns out, in fact, when the ability to adapt production at the current demand level comes at cost, the choice of the optimal capacity of the production plant impacts on the possibility to temporary stop production in the future. With a relatively low capacity, the option to suspend production in periods of market's crisis is part of the optimal solution. However, if the firm chooses a large capacity, stopping the production for a period of time might become so expensive to be never contemplated in the optimal solution.

When planning an investment in a production plant, especially in markets characterized by highly volatile levels of sale, firms face the following dilemma. On the one hand, the possibility of increased future levels of demand makes desiderable for the firm to invest in a large production plant. On the other hand, the high risk of market's breakdown makes more desiderable for the firm to invest in a small production plant. This two contrasting forces are the keys to understand how volume flexibility alters the firm's choice. In this paper we analyze separately the effects of upside and downside flexibility.

We start by considering a production process that only allows for downside flexibility. We show that the choice about whether to include or not the option to suspend production in the future depends on the level of uncertainty in the market and the degree of (downside) volume flexibility of the production process. When uncertainty is relatively low, the option is contemplated in the optimal strategy if the firm is sufficiently flexible and it is discarded if the firm is sufficiently inflexible. However, in markets characterized by high uncertainty, the incentive to increase the capacity of the plant in order to be able to face high levels of demand (Bar-Ilan and Strange, 1999; Dangl, 1999; Hagspiel et al., 2016) prevails over the need of the firm to hedge the profit flow by the risk of a market's crash. The result is that the firm always discards the option to suspend production from the optimal strategy. Looking at the effect of downside flexibility on the size of the optimal capacity, we note that the more downside-flexible the firm, the larger the production plant installed. Since downwards output adjustments are cheaper at high degrees of flexibility, the firm has an additional incentive to rise capacity. While this pattern is in line with previous findings Hagspiel et al. (2016), we note a huge difference in quantitative terms between the case in which flexibility is for free and the case in which it is costly. In the latter case, indeed, the firm does not have the option to suspend production and, to balance the risk of negative profits due to low levels of demand, requires a considerably lower capacity than that required by a fully flexible firm. Analyzing the impact of downside volume flexibility on the optimal investment timing, our model predicts that, as far as flexibility is costly, the more flexible firm has an incentive to invest earlier than the less flexible firm. This phenomenon is in contrast with the recent findings of Hagspiel et al. (2016), where the inverse relation is found in highly uncertain markets. Finally, looking at the impact of downside volume flexibility on the utilization rates at the moment of entry, our model with costly flexibility predicts percentages of capacity utilization significantly higher than the benchmark case of full flexibility. Moreover, as uncertainty increases, the utilization rates display a decreasing pattern.

We then analyze the impact of upside volume flexibility by introducing it into a setup where downside flexibility is already present. This results in the reduction of the optimal capacity installed by the firm. At higher degrees of upside volume flexibility correspond lower optimal capacity sizes. Here, economic intuition suggests that the firm, being now able to profitably increase output volume also above the established capacity, seeks for more protection of the downwards part of the production process. This effect is particularly pronounced at high levels of the uncertainty parameters, since in those cases the risk of a market's crash is higher. As the first consequence of this capacity-reducing effect, the option to suspend production is restored, also in highly uncertain markets. The rationale for this increased interest in the option to suspend production is that upside volume flexibility provides the firm with good chances to exploit high levels of demand also when capacity is sufficiently low. Thus the firm has the possibility to better hedge the profit flow from the risk of markets' crash. However, this happens only for sufficiently high degrees of upside flexibility. As the firm becomes less flexible in the upwards part of the production process the option to suspend production looses importance until is not contemplate in the optimal strategy. We also find that upside volume flexibility further reduces the optimal investment threshold. This is again a consequence of the capacity-reducing effect of upside flexibility, since to install the desired capacity the firm has to reach a lower level of demand. Finally, the model with upside flexibility predict higher utilization rates, and an increasing relationship between the degree of upside volume flexibility and the utilization rate. This phenomenon is persistent over different levels of uncertainty and different degrees of downside volume flexibility.

The remainder of the paper is organized as follows. In subsection 1.1 we provide a brief survey of related literature. Section 2 presents the firm's decision problem and describe the concept of volume flexibility we utilize throughout the paper. In section 3 we present our numerical results. In particular, in section 3.1 we present the results of the model with downside flexibility only, while in section 3.2 we analyze the model with upside flexibility. Section 4 concludes.

1.1 Related literature

This paper is related to the stream of literature studying volume flexibility as a tool to face long-term uncertainty. Previous research in this context investigates the problem of selecting the optimal technology (that is, level of flexibility). Considering a monopoly framework, this is done, for instance, in Vives (1989) in a one-product setup and Goyal and Netessine (2011) in a two-product setup. Our model of volume flexibility is borrowed from this literature. The research questions, however, are quite different, since we investigate how volume flexibility affects the optimal investment timing and the optimal capacity sizing of a production plant. Other research investigates the strategic aspects of volume flexibility. In a three-stage (capacity choice, pricing, and production) game, Anupindi and Jiang (2008) show that investments in volume flexibility are influenced by the nature of the random shock affecting the demand. Our paper does not consider competition for tractability reasons, but considers in addition the question of the optimal time of the investment, allowing us to draw testable hypotheses on the utilization rate at the moment of entry.

In order to mitigate the risk associated with highly fluctuating levels of demand, a number of relevant papers introduces real option contracts into the supply chain management. Started with Barnes-Schuster et al. (2002), this stream of literature has received considerable attention in recent years from both academics and practitioners. For instance, Chen and Shen (2012) study models of supply chain management that include service requirements as well as option contracts, showing how real options affect the supply chain performance. Chen et al. (2014) investigate the effects of the introduction of call option contracts into the supply chain when the retailer is loss-averse. Chen et al. (2017) extend Chen and Shen (2012) by introducing bidirectional option contracts into the supply chain model, showing that bidirectional contracts help members of the supply chain to enhance their profit flows. We view our paper related to this stream of literature, since they share the overall goal of facing the risks due to unanticipated demand.

This paper is also related to the literature studying investments under uncertainty started with Dixit and Pindyck (1994); McDonald and Siegel (1986), and more specifically with the previous research dealing with the problem of optimal capacity sizing under uncertainty. One of the first contribution in this vein is Bar-Ilan and Strange (1999), where the authors study the intensity of investment of a non flexible firm. In this context, Bengtsson (2001) review early papers that study manufacturing flexibility and real options from an industrial engineering/production management perspective. Among the most recent papers in this stream of literature, it is worth mentioning Savolainen et al. (2017) who analyze how the financing structure affects the value of a large investment project such as that metal mining. Our paper also studies investment timing and capacity, but the production process of our firm is volume-flexible.

This paper is also strictly related to Dangl (1999). Our model of downside flexibility collapses to the model of (Dangl, 1999) when the parameter of downside volume flexibility is set to zero. In that paper, however, the author focuses on the impact of demand uncertainty on the optimal strategy, while our focus is on the impact of volume flexibility. In this direction, even more related to our work is the recent analysis in Hagspiel et al. (2016), where the authors compare the case in which (downside) volume flexibility comes for free with the non flexible case. They conclude that, in highly uncertain markets, increased flexibility in production delays the optimal timing of the investment, since in those cases the incentive of the firm to install a large capacity is stronger than the incentive to invest earlier due to the increased flexibility. Although our model of downside flexibility cannot be seen as a direct generalization of Hagspiel et al. (2016), the key difference of our analysis compared with Hagspiel et al. (2016) is in the introduction of the cost associated to volume flexibility. This simple ingredient makes us conclude that, when (downside) volume flexibility comes at some cost, though small, increased flexibility reduces the optimal investment threshold, thus making the firm willing to invest earlier. This feature that cannot be seen by comparing the two extreme cases of full (costless) flexibility and inflexibility. Moreover, we also analyze the impact of upside volume flexibility, a factor that Dangl (1999); Hagspiel et al. (2016) do not mention in their work.

2 Model, project value, and optimal investment

Consider a risk-neutral firm that has the possibility to undertake an investment in a production facility. The decision problem involves both the timing of the investment and the capacity of the production plant, which we denote by K. The sequence of events is depicted in figure 1. Starting from time $t = 0$,

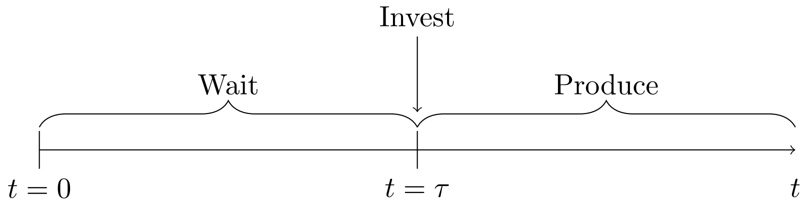


Figure 1: Model's sequence of events.

the firm posses a perpetual option to invest in a production plant. At time τ the firm exercises the option and installs the desired capacity. Once capacity is installed, the firm is able to produce the product. This involves observing, from time to time, the realized level of demand and adjusting production accordingly.

Denote by q_t the quantity produced at each time $t \geq 0$. The price at time t of the product is given by

$$p(q_t) = \theta_t - \gamma q_t, \tag{1}$$

where the positive parameter γ is the slope of the inverse demand, and the exogenous process $\{\theta_t\}$ models random fluctuations in the market and it is assumed to follow a geometric Brownian motion

$$d\theta_t = \mu\theta_t dt + \sigma\theta_t dW_t \tag{2}$$

where μ is the instantaneous growth rate and the positive parameter σ represents the volatility of the random process.

To simplify notation, from now on we omit the time subscript whenever it does not create misunderstanding. The function $C(q; K)$ describes the firm's cost of producing output q when installed capacity is K. It follows that the instantaneous profit of the firm is

$$\pi(q; \theta, K) = qp(q) - C(q; K) \tag{3}$$

We assume that the risk neutral firm is endowed with a discount rate r which satisfies $r > \mu$ and $r > 2\mu + \sigma^2$ to guarantee that the integrals below are well defined. The firm's decision problem is formalized as follows:

$$F_D(\theta) = \max_{T\geq 0, K\geq 0} E\left[\int_T^{\infty} \mathrm{e}^{-rt}\pi^*(\theta_t, K)dt - \mathrm{e}^{-rT}I(K) \mid \theta_0 = \theta\right] \tag{4}$$

where $\pi^*(\theta, K) = \max_{q\geq 0}\pi(q; \theta, K)$. Following Dangl (1999), the investment costs is assumed to have the the functional form $I(K) = \delta K^{\lambda}$, where $\delta > 0$ is a proportional parameter while $\lambda > 0$ measures the concavity of the investment costs' function. Any $\lambda < 1$ represents situations where the installation of capacity benefits of economies of scale. At $\lambda = 1$ the cost function is linear, so that doubling the size of the investment will double the investment cost. Any $\lambda > 1$ models diseconomies of scale.

The functional forms of the cost function captures the flexibility characteristics of the production process. We assume that the firm posses a volume-flexible technology, which allows production at levels different from installed capacity. Our model of volume flexibility mirrors (Goyal and Netessine, 2011), where volume flexibility is defined as the possibility for a firm to profitably produce levels of output different from installed capacity. We distinguish between two different forms of volume flexibility. When only downside flexibility is allowed, the firm can produce up to the established capacity. Sources of downside flexibility are shutting down production lines, reducing working hours, negotiating on volume with suppliers (Hagspiel et al., 2016; Jack and Raturi, 2002). With upside volume flexibility, the firm is able to adjust the current volume of production to levels of demand higher than installed capacity. Jack and Raturi (2002) list several sources of flexibility that firms use to satisfy high levels of demand, such as labor flexibility (hiring temporary workers, using overtime or part-time labor resources), inventory buffers, outsourcing arrangements, etc.

To model both sides of volume flexibility, we assume the cost of producing output q to explicitly depend on the deviation from the output and installed capacity. More precisely:[2]

$$C(q; K) = cq + (q - K)^2 \left(b_D \mathbb{1}_{\{q<K\}}(q) + b_U \mathbb{1}_{\{q>K\}}(q)\right). \tag{5}$$

In equation (5), in addition to the linear part of the cost function, the quadratic component reflects the additional cost the firm incurs for producing at levels different from installed capacity. The parameter $b_D \geq 0$, which is active only when the firm produces below the established capacity, may be interpreted as the degree of downside volume flexibility as it drives the steepness of the average cost curve around the minimum. The greater the value of b_D, the steeper the average cost curves around its minimum, and the less flexible the production process. This definition of degree of volume flexibility is in accordance with Stigler (1939) (see also Goyal and Netessine, 2011; Vives, 1989). Similarly, the parameter $b_U \geq 0$, which is active only when production

[2]The function $\mathbb{1}_{\mathcal{A}}(x)$ denotes the indicator function of x over the set $\mathcal{A}$, which is equal to 1 if $x \in \mathcal{A}$ and zero otherwise.

exceeds installed capacity, drives the degree of upside volume flexibility. We keep the distinction between the upside and downside volume flexibility by allowing the parameters b_D and b_U to be different.[3] This reflects the fact that two types of flexibility come from different sources and abilities. Moreover, different industries have different attitudes towards upside and downside flexibility. For instance, hydropower producers are characterized by a high degree of downside flexibility and a very low (if not completely absent) degree of upside flexibility, while in many sectors, service providers are usually very flexible on both sides of the production process, Kesavan et al. (2014).

The framework proposed in this paper also allows to study the case in which the production process is flexible only in the downside direction. Indeed, by letting b_U grow to infinity, we retrieve the case in which only downside flexibility is present. With this feature we are able to: *i)* make direct comparisons between the scenario in which downside volume flexibility is for free (Dangl (1999); Hagspiel et al. (2016)) and the scenario in which the possibility to scale down production is costly, and *ii)* separate the effects of both forms of volume flexibility in the optimal time of investment and capacity choice.

Having described the framework, we now turn to the solution of the decision problem. Suppose the firm has made the investment with capacity K and assume that the current level of demand is θ. The optimal current output $q^*(\theta, K)$ is computed by straightforward maximization:

$$q^*(\theta, K) = \begin{cases} 0 & \text{if } \theta \leq c - 2b_D K \\ \frac{2b_D K - c + \theta}{2(b_D + \gamma)} & \text{if } c - 2b_D K \leq \theta < c + 2\gamma K \\ \frac{2K b_U - c + \theta}{2(b_U + \gamma)} & \text{if } \theta \geq c + 2\gamma K. \end{cases} \tag{6}$$

Since the level of demand θ can never be negative, inspection of (6) reveals that when volume flexibility comes at cost the possibility to temporary stop the production depends on the (endogenous) capacity of the production process. The value $\frac{c}{2b_D}$ acts as a threshold: the firm finds optimal to suspend production only when the capacity is below the threshold. Thus, the firm might choose a capacity size either below the threshold $\frac{c}{2b_D}$ and get the profit flow

$$\pi^*(\theta, K) = \begin{cases} -b_D K^2 & \text{if } \theta \leq c - 2b_D K \\ \frac{(2b_D K - c + \theta)^2}{4(b_D + \gamma)} - b_D K^2 & \text{if } c - 2b_D K \leq \theta < c + 2\gamma K \\ \frac{(2b_U K - c + \theta)^2}{4(b_U + \gamma)} - b_U K^2 & \text{if } \theta \geq c + 2\gamma K, \end{cases} \tag{7}$$

or above $\frac{c}{2b_D}$ and get the profit flow

$$\pi^*(\theta, K) = \begin{cases} \frac{(2b_D K - c + \theta)^2}{4(b_D + \gamma)} - b_D K^2 & \text{if } \theta < c + 2\gamma K \\ \frac{(2b_U K - c + \theta)^2}{4(b_U + \gamma)} - b_U K^2 & \text{if } \theta \geq c + 2\gamma K. \end{cases} \tag{8}$$

[3]Observe that Dangl (1999)'s model is recovered when full downside flexibility is allowed ($b_D = 0$) and no upside flexibility is allowed (b_U is infinitely large).

In choosing the optimal capacity, the firm thus implicitly chooses whether to contemplate or not in its future activities the possibility to temporary shut down production. Here, intuition is clear. Since producing at levels different from installed capacities is costly, a low capacity will give the firm more freedom to adjust output volume when demand is low, including the possibility to stop the production. On the other hand, with a large capacity the firm is able to increase production at a cheaper cost, but scaling down production is more costly. Also observe that this dilemma is present whether or not the parameter driving the degree of upside volume flexibility is infinite. This is not surprising, as upside flexibility only modifies that part of the production process which exceeds capacity.

We observe that, in both cases, the profit flow can be negative for low levels of demand. For this reason we assume that the firm has the possibility to exit the market at the cost EC. This might happen provided that EC is lower than discounted value of future losses.

Next proposition characterizes the value of the investment project at the moment of entry for a fixed capacity, which is defined as the expect discounted sum of future profit flows. We relegate additional details and cumbersome expressions in A.

Proposition 1 *Define the functions $\bar{V}_1$, $\bar{V}_2$ and $\bar{V}_3$ as follows*

- $\bar{V}_1(K) = \frac{b_D K^2}{r}$;
- $\bar{V}_2(\theta, K) = \frac{-4b_D cK - 4b_D\gamma K^2 + c^2}{4r(b_D+\gamma)} + \frac{\theta(4b_D K - 2c)}{4(b_D+\gamma)(r-\mu)} + \frac{\theta^2}{4(b_D+\gamma)(-2\mu+r-\sigma^2)}$;
- $\bar{V}_3(\theta, K) = \frac{-4b_U cK - 4b_U\gamma K^2 + c^2}{4r(b_U+\gamma)} + \frac{\theta(4b_U K - 2c)}{4(b_U+\gamma)(r-\mu)} + \frac{\theta^2}{4(b_U+\gamma)(-2\mu+r-\sigma^2)}$;.

Define also β_1 and β_2 respectively as the positive and negative root of the equation $\frac{\sigma^2}{2}\beta^2 + (\mu - \frac{\sigma^2}{2})\beta = r$ and set $\theta_1 = c - 2b_D K$, $\theta_2 = c + 2\gamma K$.

1. *If the firm chooses its capacity such that $K < \frac{c}{2b_D}$, then the suspension option is part of the optimal strategy and the project has value $V^{Inc}(K,\theta)$ given by:*

$$V^{Inc}(K,\theta) = \begin{cases} -EC & \text{if } \theta < \theta_E^{Inc}(K) \\ A_1(K)\theta^{\beta_1} + A_2(K)\theta^{\beta_2} - \bar{V}_1(K) & \text{if } \theta_E^{Inc}(K) \leq \theta < \theta_1 \\ B_1(K)\theta^{\beta_1} + B_2(K)\theta^{\beta_2} + \bar{V}_2(\theta,K) & \text{if } \theta_1 \leq \theta < \theta_2 \\ C_2(K)\theta^{\beta_2} + \bar{V}_3(\theta,K) & \text{if } \theta \geq \theta_2. \end{cases} \tag{9}$$

2. *If the firm chooses its capacity such that $K \geq \frac{c}{2b_D}$, then the firm will not have the possibility to suspend production in its optimal strategy,*

and the value of the project is $V^{Exc}(K,\theta)$ given by:

$$V^{Exc}(K,\theta) = \begin{cases} -EC & \text{if } \theta < \theta_E^{Exc}(K) \\ D_1(K)\theta^{\beta_1} + D_2(K)\theta^{\beta_2} + \bar{V}_2(\theta,K) & \text{if } \theta_E^{Exc}(K) \leq \theta < \theta_2 \\ E_2(K)\theta^{\beta_2} + \bar{V}_3(\theta,K) & \text{if } \theta \geq \theta_2 \end{cases} \tag{10}$$

The expressions of functions $A_1(\cdot), A_2(\cdot), B_1(\cdot), B_2(\cdot), C_2(\cdot), D_1(\cdot), D_2(\cdot)$ and $E_2(\cdot)$ and exit thresholds $\theta_E^{Inc}(\cdot), \theta_E^{Exc}(\cdot)$ are relegated in A.

The relationship between installed capacity and downside volume flexibility has interesting implications in terms of managerial insights. From proposition 1, a fixed degree of downside volume flexibility, b_D, defines the maximum level of installed capacity, $\frac{c}{2b_D}$, which guarantees the future possibility to temporary shut down production. This implies that the degree of freedom the firm has when deciding about the size of the production plant is not sufficient to provide the firm with protection.

The firm's decision problem can be solved by backward induction. First, for any fixed level of demand θ, the firm chooses its optimal capacity $K^*(\theta)$. Then it determines the optimal demand level, which we denote θ^*, at which it is optimal to make the investment. When deciding about the size of the capacity, the firm faces implicitly the dichotomy between contemplating and excluding the suspension option. This implies that the firm compares, for each level of demand, the maximum value attainable by including the option to suspend production with the maximum value attainable by excluding the suspension option. We summarize the procedure in the next proposition.

Proposition 2 *Define*

$$K^{*,Inc}(\theta) = \underset{0\leq K<\frac{c}{2b_D}}{\arg\max} \left[V^{Inc}(\theta,K) - I(K)\right]$$

$$K^{*,Exc}(\theta) = \underset{K\geq\frac{c}{2b_D}}{\arg\max} \left[V^{Exc}(\theta,K) - I(K)\right]$$

If $V^{Inc}(\theta,K^{*,Inc}(\theta)) > V^{Exc}(\theta,K^{*,Exc}(\theta))$ then the firm chooses to include the suspension option in the optimal strategy. The optimal capacity size is $K^*(\theta) = K^{*,Inc}(\theta)$ and the value of the investment, right after the firm's entry, is $V(\theta) = V^{Inc}(\theta,K^*(\theta)) - I(K^*(\theta))$. Otherwise, the firm chooses to exclude the suspension option in the optimal strategy. The optimal capacity size is $K^*(\theta) = K^{*,Exc}(\theta)$ and the value of the investment, right after the firm's entry, is $V(\theta) = V^{Exc}(\theta,K^*(\theta)) - I(K^*(\theta))$.

In Proposition 2 the maximization of functions $V^{Inc}(\cdot,\theta)$ and $V^{Exc}(\cdot,\theta)$ must be performed numerically, since no closed form solution is available. This can be done by first deriving explicitly and then solve numerically the first and second order optimality conditions for a maximum of a one-dimensional function like in Dangl (1999); Hagspiel et al. (2016). However, given the cumbersome expressions of the first and second derivatives of the value functions with respect to capacity, we find more convenient the use of a derivative-free optimization routine (see, for instance, Judd (1998); Miranda and Fackler (2004)).

The last step of the decision problem is to find the threshold level, θ^*, at which the firm finds optimal to make the investment. As seen at time 0, that is before the firm makes the investment, the project has value $F(\theta)$ which, by the log-normality assumption of the state variable θ, can be expressed as (see, for instance, Karatzas and Shreve (1998)).

$$F(\theta) = \left(\frac{\theta}{\theta^*}\right)^{\beta_1} V(\theta^*). \tag{11}$$

The optimal threshold is the value θ^* that maximizes (11).

3 Results

In this section we perform a series of numerical experiments to show the main insights of the model. We use, as base case, the parameter values in Table 1. Note that this base case is also studied in Dangl (1999); Hagspiel et al. (2016). In what follows, we first analyze the effect of downside flexibility in

Table 1: Parameter values used in the analysis

μ	r	γ	c	δ	λ	EC
0.02	0.1	1	200	1000	0.7	0

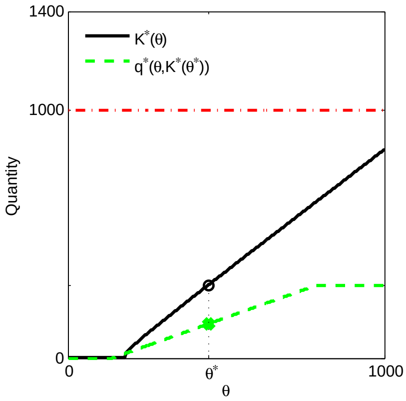
the optimal investment strategy. This is done by letting b_U grow to infinity. Then, we turn to the impact of upside volume flexibility.

3.1 Downside volume flexibility

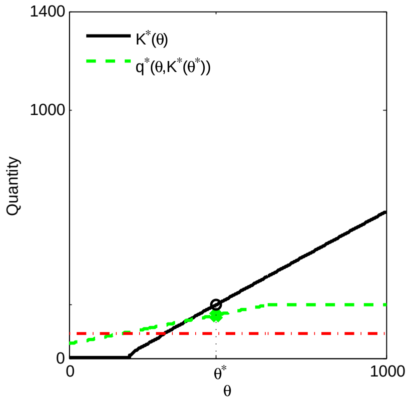
We recall that the degree of downside volume flexibility is driven by the parameter b_D. High values of b_D correspond to low degrees of flexibility and vice versa.

The option to suspend production

In our model the option to suspend production is endogenous. The firm can choose a capacity size for which future production levels either contemplate or neglect the possibility to temporary suspend production. It turns out that, depending on demand uncertainty and flexibility degree, both situations can occur. We first illustrate the optimal investment strategies when the uncertainty is relatively low. An example is depicted in Figure 2. In this case, the optimal strategy suggests to keep the option to suspend production for high degrees of downside flexibility in the production process. This is shown, for instance, in panel 2(a), which corresponds to a value of $b_D = 0.1$. Here, the optimal strategy tells the firm to enter the market whenever $\theta \geq \theta^* = 445$ and to choose a capacity size of $K^*(\theta^*) = 295.04$. After entry, the firm will



(a) $b_D = 0.1$



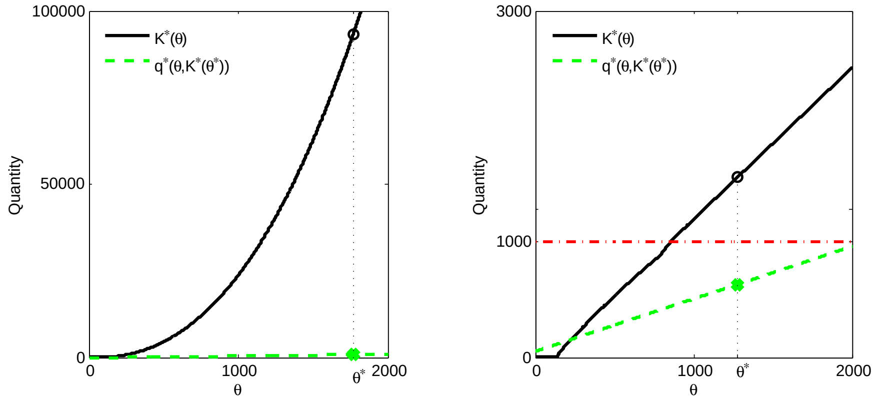
(b) $b_D = 1$

Figure 2: Investment strategy with downside volume flexibility only. Scenario with low uncertainty ($\sigma = 0.1$). The black (solid) line depicts the function $K^*(\cdot)$, while the black (empty) circle indicates the optimal capacity when entry, $K^*(\theta^*)$. The green (dashed) line depicts the production output function $q^*(\cdot, K^*(\theta^*))$ and the green (full) circle indicates the production when entry, $q^*(\theta^*, K^*(\theta^*))$. The red horizontal (dash-dotted) line marks the threshold $\frac{c}{2b}$. Capacities below (above) such value contemplate (neglect) the option to suspend production. Panel 2(a) shows a highly flexible scenario ($b_D = 0.1$). In this case the optimal values are: $\theta^* = 445$, $K^*(\theta^*) = 295.04$ and $q^*(\theta^*, K^*(\theta^*)) = 138.19$. Panel 2(b) shows a moderately flexible scenario ($b_D = 1$). In this case the optimal values are: $\theta^* = 465$, $K^*(\theta^*) = 215.11$ and $q^*(\theta^*, K^*(\theta^*)) = 173.8$. In both panels the remaining parameter values are those in Table 1.

suspend production whenever $\theta < c - 2bK^*(\theta^*) = 140.99$. The opposite situation occurs in panel 2(b), which considers a production process which is only moderately flexible ($b_D = 1$). In this case, the option to temporary suspend the production is not included in the optimal strategy: The firm will entry the market whenever $\theta \geq \theta^* = 465$ with the chosen capacity $K^*(\theta^*) = 215.11$ and start producing always positive quantities. To summarize, with low uncertainty, the firm will contemplate the suspension options only if the production process is sufficiently (downside) flexible. This is quite intuitive, since low degrees of downside volume flexibility imply high costs associated to the suspension of production. When demand is highly uncertain, however, we find that, as far as the parameter b_D is strictly positive (though small), it is never optimal for the firm to contemplate the possibility to temporary stop the production. An example of this effect is shown in Figure 3, where we set the volatility parameter to $\sigma = 0.2$ and compare the case $b_D = 0$ (full downside flexibility, Dangl (1999)) in panel 3(a) with the case $b_D = 0.1$ in panel 3(b). This result is a consequence of the firm's incentive to increase the size of the installed capacity in markets with high uncertainty ((Bar-Ilan and Strange, 1999; Dangl, 1999; Hagspiel et al., 2016)). Consider for instance panel 3(a), a situation in which the production process posses a high degree of downside flexibility. Under high uncertainty the firm foresees not only high risk of market's crash, but also a high probability of market's boom. Due to the high degree of downside flexibility, the firm is able to hedge the risk of breakdown. On the other hand, the firm is not able to adjust production beyond the established capacity, and the only way to get prepared to exploit future high demand levels is to increase the capacity size at the moment of the investment. This effect pushes the installed capacity well above the threshold $\frac{c}{2b}$ and the firm would find optimal to never suspend production.

The impact of downside flexibility

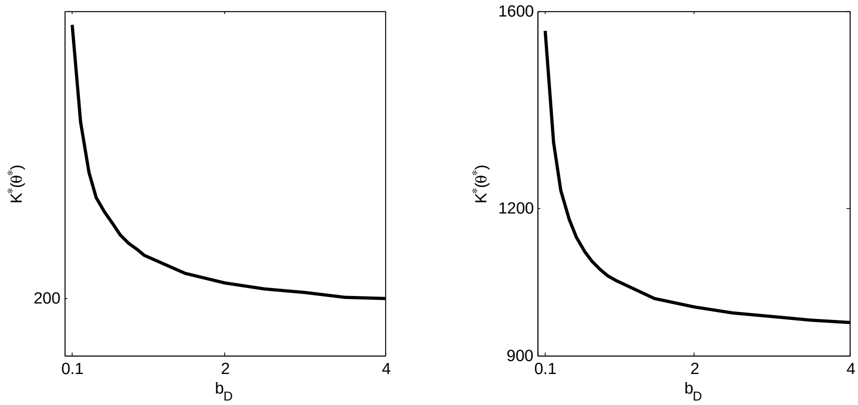
We now analyze how increased downside volume flexibility affects the investment strategy. Figure 4 shows the optimal capacity choice and Figure 5 the optimal investment threshold, both as functions of b_D. The two panels of Figure 4 (one with high uncertainty and one with low uncertainty) displays a monotonically increasing pattern between increased flexibility and capacity sizing. The more flexible the production process, the cheaper the adjustments of future production levels below from the established capacity. This pushes the firm to establish a greater capacity to exploit the possibility of future high levels of demand. Moreover, when uncertainty is high, we observe a huge quantitative difference between the established capacity of the full flexible case ($b_D = 0$) with that of any other intermediate case. For instance, with $\sigma = 0.2$, at level $b_D = 0.1$ we find $K^*(\theta^*) = 1558.7$, while the full flexible case gives $K^*(\theta^*) = 93232$. This is only in part due to the possibility, in the full flexible case, to adjust production output without additional cost. Indeed, we recall that in market with high uncertainty the optimal strategy suggests to exclude the option to suspend production. This means that the



(a) $b_D = 0.0$ (b) $b_D = 0.1$

Figure 3: Investment strategy with downside flexibility only. Scenario with high uncertainty ($\sigma = 0.2$). The black (solid) line depicts the function $K^*(\cdot)$, while the black (empty) circle indicates the optimal capacity when entry, $K^*(\theta^*)$. The green (dashed) line depicts the production output function $q^*(\cdot, K^*(\theta^*))$ and the green (full) circle indicates the production when entry, $q^*(\theta^*, K^*(\theta^*))$. The red (dash-dotted) horizontal line marks the threshold $\frac{c}{2b}$. Capacities below (above) such value contemplate (neglect) the option to suspend production. Panel 3(a) shows the full-flexible scenario ($b_D = 0$). The optimal values are: $\theta^* = 1770$, $K^*(\theta^*) = 93232$ and $q^*(\theta^*, K^*(\theta^*)) = 785$. Panel 3(b) shows a high flexible scenario ($b_D = 0.1$). The optimal values are: $\theta^* = 1279$, $K^*(\theta^*) = 1558.7$ and $q^*(\theta^*, K^*(\theta^*)) = 632.15$. In both panels the remaining parameter values are those in Table 1.

full flexible firm, having for free the possibility of not loosing money if the demand drastically drops down, requires a very large capacity to exploit the possible high levels of demand. On the other hand, although the firm with flexibility coefficient $b_D = 0.1$ can adjust production volume at a relatively low cost, at large capacity size it incurs in the risk of producing output with negative profits when demand is sufficiently low.

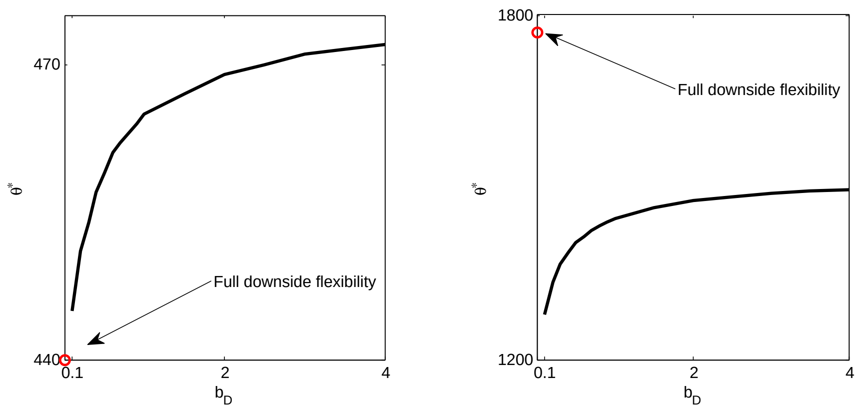


(a) $\sigma = 0.1$

(b) $\sigma = 0.2$

Figure 4: Impact of downside volume flexibility on capacity sizing. The line depicts the optimal capacity at the moment of entry $K^*(\theta^*)$ as a function b_D. Panel 4(a) shows a case with low uncertainty ($\sigma = 0.1$). The optimal capacity for the benchmark case of full downside flexibility is $K^*(\theta^*) = 589$. Panel 4(b) shows a case with high uncertainty ($\sigma = 0.2$). The benchmark optimal capacity is $K^*(\theta^*) = 93232$. In both panels the remaining parameter values are those in Table 1.

We now turn to the effect of downside flexibility on the optimal timing of investment. Figure 5 shows the optimal investment threshold as a function of b_D. Two cases are compared: panel 5(a) shows a low uncertainty scenario and panel 5(b) a high uncertainty scenario. In both cases, as far as the parameter governing flexibility is strictly positive, increased flexibility monotonically reduces the optimal investment threshold. This insight is only partially in line with the results in Hagspiel et al. (2016), where the authors find the same pattern only for low volatility levels. Instead, when demand is highly uncertain they conclude that increased (downside) flexibility further delays the optimal investment time. They correctly claim that in highly uncertain markets the firm's willingness to wait in order to establish a much higher capacity is stronger than the incentive of the firm to invest earlier because its ability to vary production over time increases the value of the investment. However, this is true only when adjustments in production levels are for free. As long as the ability to vary production comes at (possibly also small) cost, the value of increasing capacity is reduced. The firm looses the possibility to suspend



(a) $\sigma = 0.1$ (b) $\sigma = 0.2$

Figure 5: Impact of downside volume flexibility on investment timing. The line depicts the optimal investment threshold θ^* as a function of b_D, while the red circle indicates the optimal investment threshold in the benchmark case of full flexibility ($b_D = 0$). Panel 5(a) shows a case with low uncertainty ($\sigma = 0.1$). Panel 5(b) shows a case with high uncertainty ($\sigma = 0.2$). In both panels the remaining parameter values are those in Table 1.

production and requires a much lower capacity size to balance the trade-off between the protection from the risk of market's crash and the exploitation of future market's boom. This in turn implies that, when downside flexibility is present but costly, the incentive to invest earlier is stronger.

Downside volume flexibility and utilization rates

We continue the analysis of the model with downside volume flexibility only by analyzing the capacity utilization at the time of investment, defined in (Hagspiel et al., 2016) as the ratio $UR = \frac{q^*(\theta^*, K^*(\theta^*))}{K^*(\theta^*)}$.

Table 2: Utilization rates for different levels of volatility and degrees of downside flexibility. The remaining parameter values are those used in Table 1.

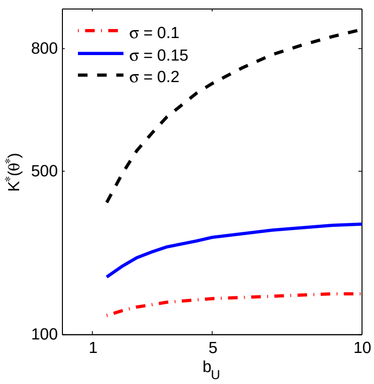
	σ		
b_D	0.1	0.15	0.2
0	0.2028	0.0795	0.0084
0.1	0.4684	0.4346	0.4056
0.5	0.7084	0.6970	0.6852
1	0.8080	0.8029	0.7962
2.5	0.9041	0.9030	0.9002
4	0.9360	0.9356	0.9339

Table 2 displays the utilization rates at the entry level, for different values of the uncertainty parameter and degrees of downside volume flexibility. The first row indicates the benchmark case where downside volume flexibility is for free (Dangl, 1999; Hagspiel et al., 2016). The capacity-reducing effect caused by the costly volume flexibility has the immediate consequence of rising the percentage of utilized capacity. The effect, particularly pronounced in highly uncertain markets, is impressive when one compares the benchmark case with a case of very high, though not full, downside flexibility. For instance, when $\sigma = 0.2$ the capacity utilization jumps from 0.8% when flexibility is for free to about 42% when $b_D = 0.1$, a scenario of very high downside volume flexibility. This is due to the already-mentioned effect for which as flexibility becomes costly the reduction of the optimal capacity is significant.

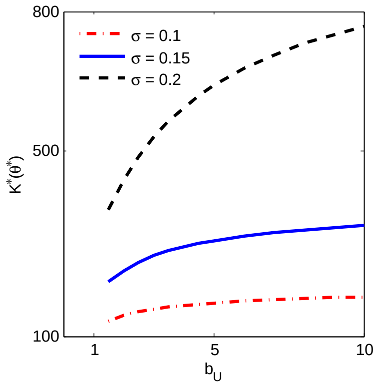
For what concerns the effect of uncertainty in the utilization rates, we also observe that the capacity utilization is decreasing with uncertainty, thus confirming the results Hagspiel et al. (2016). Intuitively, as uncertainty increases so does the willingness of the firm to invest in a large production plant so as to exploit a large capacity when the demand will rise. In other words, having enough protection for the risk of a market's crash, the firm gets prepared for a possible drastic increase in the level of sales. This results in a decreasing utilization rate as uncertainty increases.

3.2 The effects of upside flexibility

To analyze the effect of upside volume flexibility, we set to a finite value the parameter b_U. The greater the b_U, the more expensive the production above the established capacity and thus the less flexible the firm in the upwards part of the production process.



(a) $b_D = 1$



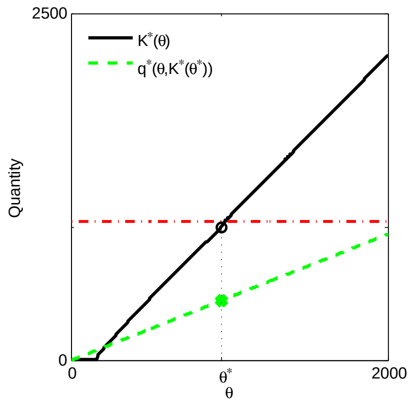
(b) $b_D = 5$

Figure 6: Effect of upside volume flexibility on optimal capacity size for different levels of volatility. The curves display the optimal capacity at the entry level, $K^*(\theta^*)$. In panel 6(a) and 6(b) the degree of downside flexibility is set to $b_D = 1$ (moderate) and $b_D = 5$ (low flexibility), respectively. The remaining parameter values are those in Table 1.

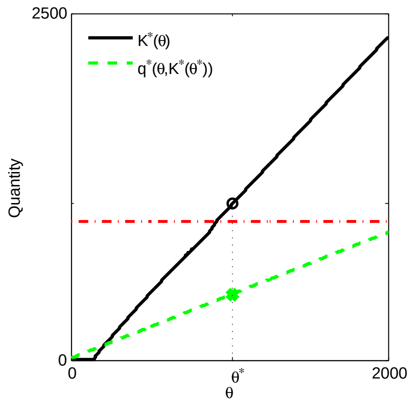
The introduction of some degree of upside volume flexibility into the production process has the effect of reducing the size of the installed capacity. Moreover, as the cost to adapt production at higher levels of demand decreases, the optimal strategy requires a lower capacity. Figure 6 displays several examples of this effect for different levels of uncertainty and fixed degrees of downside volume flexibility, showing that the phenomenon is persistent over different market configurations and production processes. However, the higher steepness of the curves at higher values of the volatility indicates that the phenomenon is much more pronounced in highly uncertain markets. The economic intuition is rather clear. If the firm can produce only up to capacity, the firm has an incentive to install a capacity size sufficient to exploit a possible future boom of the market. This incentive is stronger in highly uncertain markets since in those cases the probability of high levels of demand (as well as the probability of low levels of demand) is much higher. However, as soon as the firm can adapt (at some cost) its production process also toward the upside part of the production process, this effect vanishes. The firm, by installing a lower capacity, keeps the possibility to exploit future high demand levels, while simultaneously assuring a better protection for possible market's breakdowns.

The first immediate consequence of this effect consists in the increased importance of the option to suspend production. In fact, with the possibility to adapt the volume of production also above the established capacity, the firm endogenously chooses the size of its production plant so as to contemplate the possibility to temporary stop production also in highly uncertain markets. To show this, in Figure 7 we plot the investment strategy in presence of upside volume flexibility. For direct comparison, we use the same parameter values of Figure 3(b), so that the two panels represent the same scenario of Figure 3(b) (a case of high uncertainty and high degree of downside volume flexibility) where, in addition, we allow for some degree of upside volume flexibility. In panel 7(a), we consider a scenario where the upside flexibility parameter is set to $b_U = 2.5$. In this case, the introduction of upside volume flexibility changes the qualitative structure of the investment strategy compared with the optimal behavior of Figure 3(b), as it is now convenient for the firm to keep the possibility to temporary stop production in the future. The rationale for this strategy follows from the considerations below. Upside flexibility alleviates to some extent the risk of being unable to satisfy potential high demand levels, so that the firm's primary concern lies in protecting its profit flow from the risk of potential losses due to market's breakdowns. However, in panel 7(b) the situation is reversed. The increased cost associated to production above the established capacity reduces the value suspension option, relative to the value the firm would loose at the same installed capacity due to its reduced ability to adapt production at high demand levels. This pushes the firm to establish a lager capacity until the suspension option is not part of the optimal behavior anymore.

The introduction of upside flexibility has also the effect of further reducing



(a) $b_U = 2.5$

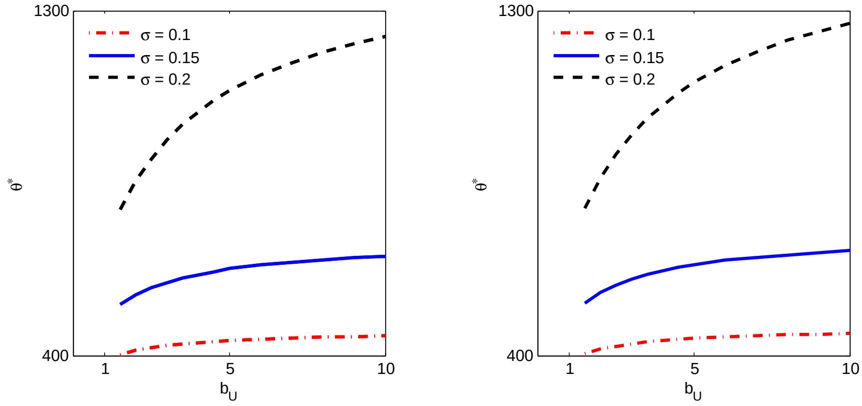


(b) $b_U = 5$

Figure 7: Investment strategy with high uncertainty in demand ($\sigma = 0.2$) and upside volume flexibility. The black (solid) line depicts the function $K^*(\cdot)$, while the black (empty) circle indicates the optimal capacity when entry, $K^*(\theta^*)$. The green (dashed) line depicts the production output function $q^*(\cdot, K^*(\theta^*))$ and the green (full) circle indicates the production when entry, $q^*(\theta^*, K^*(\theta^*))$. The red (dash-dotted) horizontal line marks the threshold $\frac{c}{2b}$. Capacities below (above) such value contemplate (neglect) the option to suspend production. Panel 7(a) shows the case in which b_U is set to 2.5. In this case the optimal values are: $\theta^* = 947$, $K^*(\theta^*) = 958.21$ and $q^*(\theta^*, K^*(\theta^*)) = 426.66$. Panel 7(b) shows the case in which $b_U = 5$. In this case the optimal values are: $\theta^* = 1021$, $K^*(\theta^*) = 1125.4$ and $q^*(\theta^*, K^*(\theta^*)) = 475.49$. For direct comparison with Figure 3(b), in both panels the parameter of downside flexibility is set to $b_D = 0.1$. The remaining parameter values are those in Table 1.

Table 3: Utilization rates for different levels of uncertainty, degrees of downside flexibility, and degrees of upside flexibility. The remaining parameter values are those displayed in Table 1.

		σ		
b_D	b_U	0.1	0.15	0.2
1	1	0.8685	0.8640	0.8628
1	3	0.8287	0.8243	0.8208
1	5	0.8204	0.8160	0.8115
1	10	0.8143	0.8096	0.8041
5	1	0.9687	0.9700	0.9720
5	3	0.9548	0.9553	0.9556
5	5	0.9519	0.9522	0.9520
5	10	0.9498	0.9498	0.9491



(a) $b_D = 1$ (b) $b_D = 5$

Figure 8: Effect of upside volume flexibility on optimal investment threshold for different levels of volatility. The curves display the optimal capacity at the entry level, θ^*. In panel 6(a) and 6(b) the degree of downside flexibility is set to $b_D = 1$ (moderate) and $b_D = 5$ (low flexibility), respectively. The remaining parameter values are those in Table 1.

the optimal investment threshold. This, too, is an immediate consequence of the capacity-reducing effect of upside flexibility, since the firm needs a lower level of market demand to reach the desired capacity size. This is illustrated in Figure 8 where, again, the effect is particularly pronounced at high uncertainty. Moreover, Table 3 displays the utilization rates for different configurations of market's uncertainty and degrees of downside and upside flexibility. The presences of upside flexibility increases the utilization rates. The capacity utilization is increasing as upside volume flexibility is less expensive. This effect is persistent over different volatility levels and degree of downside volume flexibility.

4 Conclusion

In this paper, we investigate the effect of volume flexibility for investment in a production plant under uncertainty. The firm's decision problem consists in determining both the investment threshold and the capacity of the production plant that maximize the investment value. We use the concept of volume flexibility proposed by (Goyal and Netessine, 2011), for which volume flexibility is the ability to profitably adapt the production volume to fluctuations of demand. We study separately downside volume flexibility (the ability to downscale production below installed capacity) and upside volume flexibility (the ability to produce above installed capacity). In both cases, the cost associated to volume flexibility is modeled by the introduction of a quadratic component which measures the distance between installed capacity and actual production. The degree of (upside and downside) volume flexibility is measured as the steepness of the cost curve around its minimum, a classic concept of volume flexibility due to Stigler (1939).

A distinctive feature of our model is that the possibility to temporary suspend production is not always part of the optimal decision, but it is itself an endogenous choice. The firm faces the following dilemma: choosing a small capacity allows the firm to contemplate the possibility to optimally suspend production in the future but makes more expensive adjustments of volume of production at higher levels of demand. On the other hand, a large capacity implies lower costs of production adjustment in periods of market's booms, but makes the firm unable to suspend production if the market crashes. Thus, the firm makes three explicit decisions, namely the investment time, the capacity, and the current volume of production, while the forth decision, namely contemplating or disregarding the option to suspend production, is implicitly included in the choice about capacity.

We first analyze the case in which the firm is volume-flexible only in the downside part of the production process. This allows to make direct comparisons with the results of Dangl (1999) and the more recent Hagspiel et al. (2016). For what concerns the option to suspend production, we find two distinct patterns. In market characterized by low uncertainty, the degree of (downside) flexibility drives the results. Highly (downside)-flexible firms choose to keep the option to suspend production, option that is discarded by firms with low degree of (downside) flexibility. In markets characterized by high uncertainty, instead, the incentive of the firm to invest in a large capacity in order to get prepared for market's boom is so strong that the (downside)-flexible firm always discards the option to suspend production. For what concerns the optimal investment in capacity, we find that at increased degrees of (downside) volume flexibility correspond increases of the optimal capacity. The firm, being able to hedge from the risk of market's boom at a cheaper cost, seeks to get prepared to exploit future high levels of demand. This incentive is stronger in highly uncertain markets, since the probability of an increase of the level of sales is higher. While the same qualitative effect is found in Hagspiel et al. (2016), we note however a huge quantitative difference between the case in which downside volume flexibility is for free with our case, where flexibility is costly. As capacity increases the firm looses the option to suspend production. This in turn modifies the risk exposure of the firm that is not able anymore to fully protect the profit flow from the risk of a market's breakdown, thus requiring a lower capacity. This capacity-reducing effect has dramatic implications on the optimal time of the investment. In this respect, Hagspiel et al. (2016) conclude that an increase of flexibility creates an incentive for the firm to invest earlier if uncertainty is sufficiently low, and delays the optimal time of investment if uncertainty is high. Our analysis reveals that the delaying effect is true only if flexibility comes for free. As far as the firm finds costly to reduce production below capacity, even for small costs, we find that increased downside flexibility has the effect of reducing the optimal investment threshold, thus making the firm willing to invest earlier. In analyzing the capacity utilization at the time of investment, we find that the introduction of the cost of flexibility sensibly rises the utilization rates. Moreover, analyzing the relationship between the percentage of capacity utilized at the moment of entry and uncertainty, we report a decreasing pattern, also found in Hagspiel et al. (2016).

We then analyze the effect of the introduction of some degree of upside volume flexibility in a production process where downside volume flexibility is present. The introduction of upside volume flexibility alters the strength of the incentives. Analyzing the optimal investment in capacity, we find that upside volume flexibility reduces the size of the investment. The firm, being able to profitably increases the volume of production above capacity, seeks better protection in the downside part of the production process. This effect is more pronounced in highly uncertain markets, since higher is the risk of market's breakdown. This effect impacts on the optimal strategy as follows. First, there is an increased incentive for the firm to contemplate the option to suspend production. Indeed, when upside flexibility is present, the option to suspend can be part of the optimal strategy also in markets characterized by high uncertainty. Second, the level of sales at which the firm invests is further lowered. Third, the utilization rates at the moment of entry are higher compared to the case of downside flexibility only. Moreover, the utilization rate is an increasing function of the degree of upside volume flexibility. This pattern is persistent over different degrees of downside volume flexibility and levels of uncertainty.

This paper leaves unexplored several aspects of the design of a manufacturing process that certainly deserve more attention. In our model, the levels of both downside and upside volume flexibility are given endogenously. Nevertheless, in many real world situations managers can choose to some extent the degree of volume flexibility. Analyzing a model where the degree of volume flexibility is chosen endogenously represents a promising direction for future research. Also, in this paper we concentrate volume flexibility. Further exploration is needed to understand the interaction between volume flexibility and product flexibility, following the lines of Goyal and Netessine (2011). Moreover, in this paper we completely ignored competition between two firms. In this context: *i)* extending our model to a duopoly setup by following the lines of Huisman and Kort (2015) seems to be a promising but daunting task; *ii)* Analyzing an Incumbent-Entrant model with varying degree of volume flexibility appears to be an interesting problem that is currently under investigation.

A Details on the value of the investment

For a fixed value of demand level and capacity size, this appendix determines the expected discounted value of future cash flows right after the investment has been made, that is

$$V^f(K,\theta) = \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-rt}\pi^*(\theta_t, K)dt|\theta_0 = \theta\right] \tag{12}$$

for $f = Inc, Exc$ and $\pi^*(\cdot,\cdot)$ is given in (7) is $f = Inc$ or in (8) is $f = Exc$. We use the standard machinery to determine V^f. By dynamic programming, V^f must satisfy the following Bellman equation

$$rV^f dt = \mathbb{E}\left[dV^f\right]. \tag{13}$$

A straightforward application of Ito's Lemma gives the non-homogeneous second order linear differential equation that V^f satisfies:

$$\frac{\sigma^2\theta^2}{2}\frac{\partial^2 V^f}{\partial\theta^2} + \mu\theta\frac{\partial V^f}{\partial\theta} + \pi^* = rV^f. \tag{14}$$

In case $f = Inc$, the solution of (14) can be written as

$$V^{Inc}(K,\theta) = \begin{cases} -EC & \text{if } \theta < \theta_E^{Inc} \\ A_1\theta^{\beta_1} + A_2\theta^{\beta_2} - \bar{V}_1(K) & \text{if } \theta_E^{Inc} \leq \theta \leq \theta_1 \\ B_1\theta^{\beta_1} + B_2\theta^{\beta_2} + \bar{V}_2(\theta,K) & \text{if } \theta_1 < \theta < \theta_2 \\ C_1\theta^{\beta_1} + C_2\theta^{\beta_2} + \bar{V}_3(\theta,K) & \text{if } \theta \geq \theta_2. \end{cases} \tag{15}$$

where the parameters $A_1, A_2, B_1, B_2, C_1, C_2$ and θ_E^{Inc} are determined as follows. Since $\beta_1 > 0$, the condition $\lim\limits_{\theta\to\infty} V^{Inc}(K,\theta) = \bar{V}_3(\theta,K)$ implies $C_1 = 0$. The parameters A_1, B_1, B_2, C_2 are determined by solving the linear system which imposes continuity and differentiability restrictions of $V^{Inc}(K,\cdot)$ at the two thresholds θ_1 and θ_2. This gives:

$$\begin{aligned}
A_1(K) =& \frac{\theta_1^{-\beta_1}\theta_2^{-\beta_1}}{\beta_1-\beta_2}\left(\theta_2^{\beta_1}\left(\theta_1\frac{\partial\bar{V}_2(\theta_1,K)}{\partial\theta} - \beta_2\left(\bar{V}_1(K)+\bar{V}_2(\theta_1,K)\right)\right)+\right. \\
&\left.\theta_1^{\beta_1}\left(\theta_2\left(\frac{\partial\bar{V}_3(\theta_2,K)}{\partial\theta} - \frac{\partial\bar{V}_2(\theta_2,K)}{\partial\theta}\right) + \beta_2\left(\bar{V}_2(\theta_2,K)-\bar{V}_3(\theta_2,K)\right)\right)\right) \\
B_1(K) =& \frac{\theta_2^{-\beta_1}\left(\theta_2\left(\frac{\partial\bar{V}_3(\theta_2,K)}{\partial\theta} - \frac{\partial\bar{V}_2(\theta_2,K)}{\partial\theta}\right) + \beta_2\left(\bar{V}_2(\theta_2,K)-\bar{V}_3(\theta_2,K)\right)\right)}{\beta_1-\beta_2} \\
B_2(K) =& A_2(K) + \frac{\theta_1^{-\beta_2}\left(\theta_1\frac{\partial\bar{V}_2(\theta_1,K)}{\partial\theta} - \beta_1\left(\bar{V}_1(K)+\bar{V}_2(\theta_1,K)\right)\right)}{\beta_1-\beta_2} \\
C_2(K) =& \frac{\theta_1^{-\beta_2}\theta_2^{-\beta_2}}{\beta_1-\beta_2}\left(\theta_2^{\beta_2}\left(\theta_1\frac{\partial\bar{V}_2(\theta_1,K)}{\partial\theta} - \beta_1\left(\bar{V}_1(K)+\bar{V}_2(\theta_1,K)\right)\right) - A_2(K)\beta_2\theta_1^{\beta_2}\theta_2^{\beta_2}+\right. \\
&\left.\theta_1^{\beta_2}\left(\beta_1\left(A_2(K)\theta_2^{\beta_2}+\bar{V}_2(\theta_2,K)-\bar{V}_3(\theta_2,K)\right)+\theta_2\left(\frac{\partial\bar{V}_3(\theta_2,K)}{\partial\theta}-\frac{\partial\bar{V}_2(\theta_1,K)}{\partial\theta}\right)\right)\right).
\end{aligned}$$

The remaining parameters, $\theta_E^{Inc}(K), A_2(K)$ are determined by imposing the value matching and smooth-pasting conditions at the exit threshold θ_E^{Inc}. This gives:

$$\theta_E^{Inc}(K) = \left(\frac{\beta_2\left(\bar{V}_1(K)-EC\right)}{(\beta_2-\beta_1)A_1(K)}\right)^{\frac{1}{\beta_1}}$$

$$A_2(K) = \theta_E^{Inc}(K)^{-\beta_2}\left(\frac{\beta_1\left(\bar{V}_1(K)-EC\right)}{(\beta_1-\beta_2)}\right)$$

In case $f = Exc$, the procedure follows the same steps as before opportunely adapted to the profit flow (8). The parameters $D_1(K), E_2(K)$ are computed by solving the value matching and smooth-pasting condition at the threshold θ_2. This gives:

$$\begin{aligned} D_1(K) =& \frac{\theta_2^{-\beta_1}}{\beta_1-\beta_2} \\ & \beta_2(\bar{V}_2(\theta_2,K)-\bar{V}_3(\theta_2,K)) - \theta_2\left(\frac{\partial\bar{V}_2(\theta_2,K)}{\partial\theta} - \frac{\partial\bar{V}_3(\theta_2,K)}{\partial\theta}\right) \\ E_2(K) =& D_2(K) + \frac{\theta_2^{-\beta_2}}{\beta_2-\beta_1} \\ & \left(\theta_2\left(\frac{\partial\bar{V}_2(\theta_2,K)}{\partial\theta} - \frac{\partial\bar{V}_3(\theta_2,K)}{\partial\theta}\right) + \beta_1(\bar{V}_3(\theta_2,K)-\bar{V}_2(\theta_2,K))\right). \end{aligned}$$

The value matching and smooth-pasting conditions at the exit threshold do not admit closed form solution. By straightforward manipulation of the differentiability condition, we express the exit threshold as the solution of the following implicit equation:

$$\begin{aligned} D_1(K)\theta_E^{Exc}(K)^{\beta_1}\left(1-\frac{\beta_1}{\beta_2}\right) + F\theta_E^{Exc}(K)\left(1-\frac{1}{\beta_2}\right) + \\ G\theta_E^{Exc}(K)^2\left(1-\frac{2}{\beta_2}\right) + H + EC = 0 \end{aligned} \tag{16}$$

where H, F, G are such that $\bar{V}_2(\theta,K) = H + F\theta + G\theta^2$. Equation (16) must be solved numerically. The parameter $D_2(K)$ can be computed as follows:

$$D_2(K) = \theta_E^{Exc}(K)^{-\beta_2}\left(-\frac{\beta_1 D_1(K)\theta_E^{Exc}(K)^{\beta_1} + F\theta_E^{Exc}(K) - 2G\theta_E^{Exc}(K)^2}{\beta_2}\right).$$

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