Disruptive Innovation in a Declining Market*

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Abstract

The paper considers the problem of a firm operating in a declining market. The firm has an option to innovate and has to derive the right time to do so, if at all. We find that it can be optimal for the firm to innovate because of two reasons. The first reason is that a new technology is available with which the firm can achieve higher profits. The second reason is that, due to demand saturation, profits of the established product have become so low that the firm will adopt a new technology even if the

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newest available innovation has not improved for some time. We obtain that the firm essentially has two candidate policies for optimality: innovate early and keep on producing both the established and the new product, or innovate late and replace the established product by the new one.

1 Introduction

In an evolving economy demand for existing products decreases over time due to the arrival of more exciting alternatives.[1] This induces that firms need to change their product portfolio over time, and thus have to innovate in order to keep on making profits. This paper has the aim to study optimal firm behavior in such a setting. To do so, we study a problem of an existing incumbent producing an established product of which demand declines over time. The firm has an option to innovate, where, due to technological progress, a newer technology can produce better products. The resulting higher demand leads to higher profits. As time passes the best available new technology that can be adopted by the firm improves. So, the longer the firm waits with investing, the better the technology is that the firm can acquire and the better the products are the firm can produce.

In such a scenario the firm has the necessity to innovate, because otherwise the declining demand of the existing product diminishes its revenue over time.[2] In evaluating its innovation option the firm faces the following tradeoff. Adopting soon means the firm soon gets rid of the existing technology with reducing revenues, while it attracts a newer technology with higher profits. Adopting late means that, on the one hand, the firm suffers for a long time from declining profits due to the demand decrease of the established product. On the other hand, later adoption implies that, due to technological progress, the firm can attract

[1] An example, among many others (like, e.g., the arrival of LCD television sets that influenced demand of CRT television sets and the replacement in the semiconductor industry of 200mm wafer plants by 300mm wafer plants (see Cho and McCardle (2009))) is the introduction of solid state drives as an alternative for hard disk drives for data storage in computers. Before the current transition to solid state drives, the computer storage market has in the past decades gone through significant innovations from 14-inch, via 8-inch and 5.25-inch to 3.5-inch drives (see Kwon (2010)).

[2] In fact, in the computer data storage industry (see footnote 1), Western Digital (producer of hard disk drives) announced in October 2015 that it plans to acquire SanDisk (producer of solid state drives) (https://www.sandisk.com/about/media-center/press-releases/2015/western-digital-announces-acquisition-of-sandisk).

a still better new technology with which the firm can obtain higher profits than when it adopted a new technology sooner.[3]

Existing works like Balcer and Lippman (1984), Farzin *et al.* (1998), Huisman (2001), and Hagspiel *et al.* (2015) consider similar innovation problems, but they do not consider the for us important characteristic of declining demand for the existing product. As a result we obtain that the time to innovate can be governed by two different causes. First, like in Farzin *et al.* (1998), a firm innovates right at the moment of arrival of a far better technology, the use of which enables the firm to produce products with much higher demand, leading to a considerable profit increase. Second, the fact that demand for the existing product declines over time implies that the firm's revenue gets lower and lower as long as it does not innovate. For this reason it could be optimal for the firm to adopt a new technology a time lag after its introduction.

The latter result is as such not new in the literature, but what is new is that it is caused by declining demand for the existing product. To exemplify, first consider Balcer and Lippman (1984) that also shows that as time passes without new technological advantages, it may become profitable to purchase an existing technology that is superior to the one in place at the firm even though it was not profitable to do so in the past. However, in that paper this is caused by the fact that the discovery time was not memoryless. Hagspiel *et al.* (2015) show that changing arrival rates over time of new technologies can result in firms adopting a new technology at a later point in time than when it was available for the first time. McCardle (1985) argues that such a time lag can be explained by the uncertainty regarding the profit potential of a new technology.

Unlike the just mentioned contributions, Kwon (2010) has in common with our paper that it also considers a firm with a declining profit stream over time. However, Kwon (2010), and also Hagspiel *et al.* (2016) that extends Kwon by considering capacity optimization, does not consider a sequence of new technologies arriving over time. Instead, it analyzes whether to exercise a single innovation opportunity. In addition the firm also has an option to exit the industry, which exists before and after the investment. Matomaki (2013) generalizes the work of Kwon (2010) by considering different stochastic processes representing profit uncertainty. Strategic interactions in a declining industry are studied by Fine and Li (1986) and Murto

[3] In the computer storage industry of footnote 1, the 8-inch drives were eventually superseded by 5.25-inch drives, which are currently replaced by solid state drives (Kwon (2010)).

(2004).

The described product innovation problem is attacked as follows in this paper. We start out by formulating a benchmark model. As in Farzin *et al.* (1998) technological progress is modeled as a Poisson process, where the level of the frontier technology jumps up at unknown points in time. At the moment the firm adopts the new technology, the revenue obtained from selling the new products is deterministic and linearly dependent on the level of the adopted technology. Demand for the existing product decreases over time, resulting in a reduction of the associated profit with a fixed rate. We obtain a threshold level for the technology that needs to be reached in order for the firm to invest optimally. The threshold level is increasing in the profit level of the established product, i.e. the firm delays the product innovation if the established product market is more profitable, which makes sense.

Next, we extend the model one by one to check for robustness of our obtained results. We show that, despite its simplicity, the the benchmark model has results that are generally valid. In particular we get that the threshold level does not change if the profit on the established product market is uncertain in a way that it behaves according to a geometric Brownian motion (GBM) process. Assuming uncertainty, again by introducing a GBM, for the revenue on the new product market after adoption leads to the result that again the threshold level does not change if the trend of the GBM process is zero. A positive trend on the new market accelerates the adoption process while a negative trend delays it. Remarkable is that the amount of uncertainty does not influence the threshold level. If the revenue of the new product is increasing in the technology level in a nonlinear way we can only obtain the threshold level implicitly. However, comparative statics results of the benchmark model carry over.

A more involved extension is the introduction of a cannibalization effect in the sense that introduction of the new product reduces demand and thus also revenue for the old product. Also, if the quantity of the old product is larger, revenue on the new product market is lower. This we call our main model. Where in the benchmark model the introduction of the new product at the same time stops production of the established product, here the firm also has the option to keep on producing the old product after the product innovation. Our analysis shows that essentially the firm will choose between innovating early and producing both the established and the new product simultaneously, or innovating late and abolish the established product. In the last case the firm specializes by just being active on the new product market.

The paper is organized as follows. In Section 2 the benchmark model is introduced, analyzed, and robustness is checked by considering the stochastic extensions regarding revenue on the established and the new product market, and by introducing a nonlinearity with respect to the dependence of the revenue of the new product on the technology level. Section 3 considers the main model, whereas Section 4 concludes.

2 Benchmark model

We consider an incumbent firm currently producing an established product. Over time demand for this product declines, because as time passes consumers get access to all kinds of alternative products in an evolving economy. For this reason profits earned on the established product market decrease over time. To capture this, we assume that the profit flow of the firm at time t is equal to $\pi_0 x(t)$, where x is a decreasing process, i.e.

$$dx(t) = \alpha x(t)\, dt, \tag{1}$$

with $\alpha < 0$.

Due to this falling profit, the firm wants to change its product portfolio. Therefore, the firm has an incentive to perform a product innovation. To do so it has to adopt a new technology with which it can produce the new products. We denote the level of technology at time t by $\theta(t)$. The higher θ is, the more advanced the technology will be and the better the products are that this technology will produce. Better products imply that there will be more demand for these products, which will result in a higher revenue.

The development of θ over time is governed by technological progress, which is exogenous to the firm. In particular we impose that new technologies arrive according to a Poisson process with rate $\lambda > 0$, where each arrival increases the technology level by $u > 0$. This gives

$$d\theta(t) = \begin{cases} u & \text{with probability } \lambda dt, \\ 0 & \text{with probability } 1 - \lambda dt, \end{cases} \tag{2}$$

$$\theta(0) = \theta_0 > 0. \tag{3}$$

The profit flow that results from applying the new technology after the investment does not depend on time and is equal to $\pi_1(\theta) = z\theta$, where θ is the level of the technology at which the firm has made the investment. So, once the new technology is adopted, from that moment on the firm's instantaneous profit is linearly dependent on the level of θ associated with the adopted technology. Innovating implies that the firm has to incur a cost of investment being equal to $I > 0$. We relax the assumption of constant profit in Section 2.2, where we consider other extensions as well.

2.1 Results

Essentially, the firm has two reasons to innovate. The first reason is that the established product market profit has reduced too much so that keeping on to produce this established product is not economically viable for the firm. The second reason is that over time alternative technologies have been invented that are much more profitable than staying in the established product market. Translated to our model, the first reason is equivalent to a low value of $x(t)$, whereas the second reason implies a high value of $\theta(t)$.

We conclude that innovating is optimal for low values of x and high values of θ, while the firm should keep on being active on the established product market when x is high and θ is low. It follows that in the $(x, \theta)-$plane an upward-sloping curve $\theta^*(x)$ exists on which the firm is indifferent between innovating and staying on the established product market.

The firm knows how to behave optimally once the curve $\theta^*(x)$ is determined: it should wait until θ has increased so much that $\theta \geq \theta^*(x)$, before it should innovate. Therefore, our aim in this section is to derive the threshold curve $\theta^*(x)$. Following Huisman (2001) we apply dynamic programming and in particular optimal stopping to achieve this. To do so, as a first step we define a value function $F(x, \theta)$ that represents the firm's value of the project. Proposition 1 presents the functional form for this value function.

Proposition 1 *Let $n = \lceil \frac{\theta^* - \theta}{u} \rceil$, then the value of the firm $F(\theta, x)$ is equal to*

$$F(\theta, x) = \begin{cases} \left(\frac{\lambda}{r+\lambda}\right)^n \left(\frac{z(\theta+nu)}{r} - I\right) & \\ + \left(1 - \left(\frac{\lambda}{r+\lambda-\alpha}\right)^n\right) \frac{\pi_0 x}{r-\alpha} & \text{for } \theta \in [0, \theta^* - u), \\ \frac{\pi_0 x}{r+\lambda-\alpha} + \frac{\lambda}{r+\lambda}\left(\frac{z(\theta+u)}{r} - I\right) & \text{for } \theta \in [\theta^* - u, \theta^*), \\ \frac{z\theta}{r} - I & \text{for } \theta \in [\theta^*, \infty). \end{cases} \tag{4}$$

Proposition 1 shows that the value function consists of three parts. Whenever $\theta \geq \theta^*(x)$ the firm will innovate, after which it earns a profit flow of $z\theta$. Adding it up and discounting gives a total discounted profit stream $\frac{z\theta}{r}$. Since innovating requires an investment outlay of I, this results in a value of the firm being equal to $\frac{z\theta}{r} - I$.

If the current technology level is such that $\theta^*(x) - u \leq \theta < \theta^*(x)$, it is optimal for the firm to innovate right after the next new technology arrival. The fraction $\frac{\lambda}{r+\lambda}$ stands for the stochastic discount factor under a Poisson process (Huisman (2001, p.46)), i.e. it represents the expected value of one unit of money right after the next jump. This is exactly the time that the firm innovates, so we get the amount $\frac{\lambda}{r+\lambda}\left(\frac{z(\theta+u)}{r} - I\right)$. The first term represents what the firm earns on sales of the established product until it innovates. The denominator $r+\lambda-\alpha$ makes sure that the resulting expected revenue stream is discounted (r), it is corrected for the fact that the revenue stream lasts up until the next technology jump (λ), and for the fact that revenue decreases over time with rate $-\alpha$ due to the declining demand of the established product.

If the current technology level, θ, falls below $\theta^*(x) - u$, the number of arrivals of new technologies until it is optimal to innovate, equals $n = \lceil \frac{\theta^*-\theta}{u} \rceil$. Consequently, the expected discounted value of the future new technology adoption amounts to $\left(\frac{\lambda}{r+\lambda}\right)^n \left(\frac{z\theta^*}{r} - I\right)$. Up until the innovation time the firm sells the established product.The term $\frac{\pi_0 x}{r-\alpha}$ stands for the discounted revenue stream if the firm were active on the established product market forever. However, after the firm innovates, it discontinues this activity, so we need to subtract the amount $\left(\frac{\lambda}{r+\lambda-\alpha}\right)^n \frac{\pi_0 x}{r-\alpha}$.

The value of the firm $F(\theta, x)$ is continuous in both arguments. It is straightforward to check that F is continuous for $\theta = \theta^*(x) - u$. Continuity of F for $\theta = \theta^*(x)$ gives the threshold function $\theta^*(x)$, as exemplified in the next proposition.

Proposition 2 *The threshold curve $\theta^*(x)$ is given by*

$$\theta^*(x) = \frac{r+\lambda}{r+\lambda-\alpha} \frac{\pi_0 x}{z} + \frac{\lambda u}{r} + \frac{rI}{z}. \tag{5}$$

Expression (5) is the result of equating the second and third row of the value function expression (4):

$$\frac{\pi_0 x}{r+\lambda-\alpha} + \frac{\lambda}{r+\lambda}\left(\frac{z(\theta+u)}{r} - I\right) = \frac{z\theta}{r} - I. \tag{6}$$

It reflects that when the technology level equals $\theta^*(x)$, the firm is indifferent between innovating, which gives the payoff $\frac{z\theta}{r} - I$, and waiting for the next technology arrival, which results in extra expected revenue on the established product market, $\frac{\pi_0 x}{r+\lambda-\alpha}$, and the discounted payoff of innovating after the next jump, which equals $\frac{\lambda}{r+\lambda}\left(\frac{z(\theta+u)}{r} - I\right)$.

From (5) we conclude that indeed $\theta^*(x)$ is an upward sloping curve in the $(x,\theta)-$plane. It is important to realize that the firm need not innovate at the moment a new technology arrival takes place, which corresponds to a vertical jump in the $(x,\theta)-$plane, but that it also can happen that the existing revenue for the established product becomes so low that innovating is optimal. This is reflected by the decrease in x over time, so this corresponds to a horizontal movement in the $(x,\theta)-$plane, such that innovating takes place at the moment the $\theta^*(x)-$curve is hit from the right. These two possibilities are graphically illustrated in Figure 1.

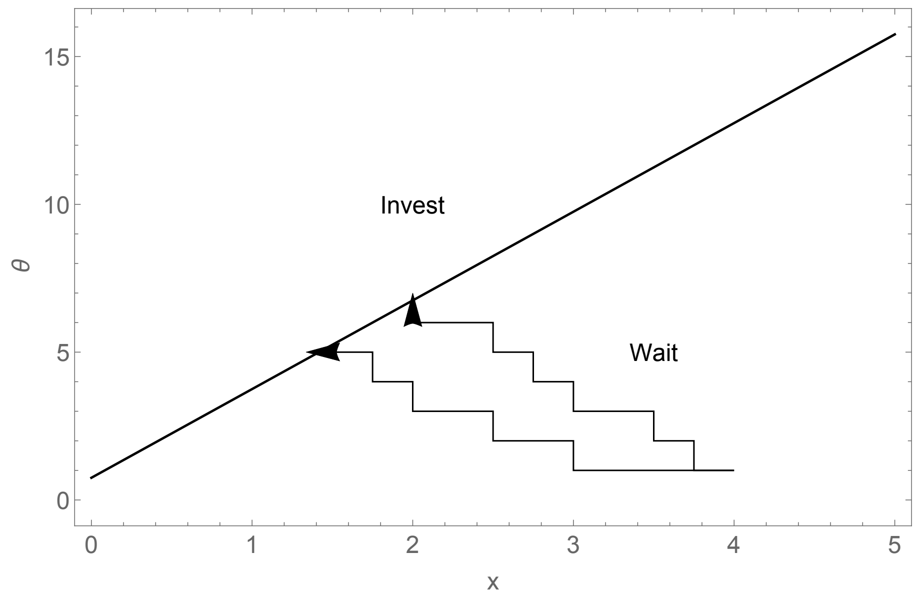

Figure 1: Illustration of the two possible ways of adopting: at the arrival of a new technology (vertical crossing of threshold curve) or after sufficient decrease of the current market (horizontal crossing). Parameter values used: $r = 0.1$, $\lambda = 0.05$, $u = 0.5$, $I = 50$, $\pi_0 = 50$, $\alpha = -0.1$, and $z = 10$.

Proposition 3 *The threshold curve $\theta^*(x)$ is decreasing in z and increasing in α, λ, π_0, I and u for a given value x*

The firm will innovate later if profits on the established product market are higher, which implies that $\theta^*(x)$ is increasing in π_0 and α. On the other hand, the firm will innovate sooner if the revenue from innovating is higher, so that $\theta^*(x)$ will be decreasing in z. Waiting for the next technology arrival is more worthwhile if it is expected to occur sooner or when the technology arrival results in a higher increase of the technology level. Therefore, it holds that $\theta^*(x)$ is increasing in λ and u. If innovating is more expensive it will happen later, implying that $\theta^*(x)$ is increasing in I.

Concerning the interest rate r there are conflicting effects. On the one hand, waiting for the next jump and innovating then gives a lower payoff because of more heavy discounting, but on the other hand the same holds for the revenue stream after innovating immediately. Therefore, we have to conclude that upfront it is not clear how the threshold level $\theta^*(x)$ is influenced by r. This is illustrated in Figure 2.

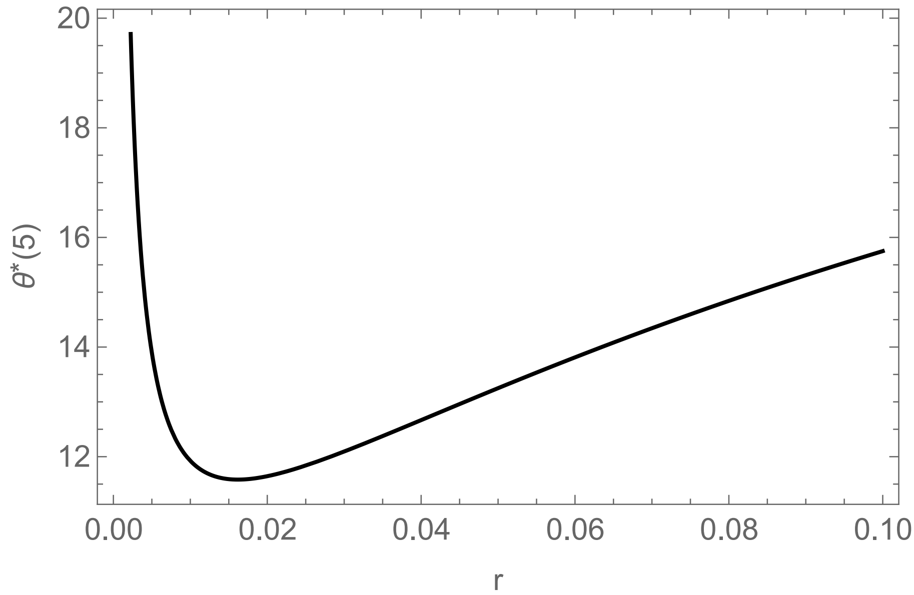

Figure 2: Example showing that the threshold curve θ^* for a given $x = 5$ is first decreasing in the discount rate r and then increasing. Parameter values used: $\lambda = 0.05$, $u = 0.5$, $I = 50$, $\pi_0 = 50$, $\alpha = -0.1$, and $z = 10$.

2.2 Robustness

In this subsection we explore the robustness of our results by extending the model in three ways. First, it is assumed that the profit flow before the investment follows a geometric Brownian motion. Second, we impose that the profit flow after the investment follows a geometric Brownian motion. Finally, we assume that the profit flow after the investment is non-linear in the adopted technology level.

2.2.1 Geometric Brownian Motion before the investment

To make te profit flow before the investment uncertain we let the development of $x(t)$ be governed by a Geometric Brownian motion process. To do so, we add a stochastic term to expression (1) so that we get

$$dx(t) = \alpha x(t)\,dt + \sigma x(t)\,dz(t), \tag{7}$$

where, as before, $\alpha < 0$. Furthermore, it holds that $\sigma > 0$, and $dz(t)$ is the increment of a Wiener process. The following proposition states that the results of the previous subsection still hold, implying that the level of uncertainty, σ, does not influence the investment decision.

Proposition 4 *Making the decrease of the profit flow of the existing product follow a geometric Brownian motion does not change the value function of the firm and the optimal investment threshold boundary.*

2.2.2 Geometric Brownian Motion after Investment

At the moment the firm innovates, the instantaneous profit obtained by selling the new product equals, as before, $z\theta_\tau$, with τ being the moment of investment. Where in our benchmark model the instantaneous profit keeps on being fixed at this level, we now consider the situation that future profits are uncertain and satisfy a geometric Brownian motion process. So, the profit flow of the new product now equals $zy(t)$, where

$$dy(t) = \mu y(t)\,dt + \sigma y(t)\,dz(t), \tag{8}$$

with $y(0) = \theta_\tau$.

The next proposition establishes the value of the firm.

Proposition 5 *Let* $n = \lceil \frac{\theta^* - \theta}{u} \rceil$*, then the value of the firm* $F(\theta, x)$ *is equal to*

$$F(\theta, x) = \begin{cases} \left(\frac{\lambda}{r+\lambda}\right)^n \left(\frac{z(\theta+nu)}{r-\mu} - I\right) & \\ + \left(1 - \left(\frac{\lambda}{r+\lambda-\alpha}\right)^n\right) \frac{\pi_0 x}{r-\alpha} & \textit{for } \theta \in [0, \theta^* - u), \\ \frac{\pi_0 x}{r+\lambda-\alpha} + \frac{\lambda}{r+\lambda}\left(\frac{z(\theta+u)}{r-\mu} - I\right) & \textit{for } \theta \in [\theta^* - u, \theta^*), \\ \frac{z\theta}{r-\mu} - I & \textit{for } \theta \in [\theta^*, \infty). \end{cases} \tag{9}$$

If we compare with the value function resulting from the benchmark model given in Proposition 1, we conclude that it is the same, except that the total discounted profit stream on the new market has been corrected for the trend parameter μ.

We next derive the threshold function $\theta^*(x)$.

Proposition 6 *The threshold curve* $\theta^*(x)$ *is given by*

$$\theta^*(x) = \left(\frac{r+\lambda}{r+\lambda-\alpha}\right)\left(\frac{r-\mu}{r}\right)\frac{\pi_0 x}{z} + \frac{\lambda u}{r} + \frac{I(r-\mu)}{z}. \tag{10}$$

Compared to our benchmark model, not much changes, except that the trend parameter μ enters the expression.

The expression for the threshold function gives rise to the following comparative statics results.

Proposition 7 *The threshold curve* $\theta^*(x)$ *is decreasing in* z *and* μ *and increasing in* α*,* λ*,* π_0*,* I *and* u *for a given value* x*.*

So the conclusions from the benchmark model are not affected by this model extension. In addition we find that, again, the uncertainty parameter σ does not influence the investment decision. Further, we see that the firm invests earlier if the profit flow on the new market is expected to increase.

2.2.3 Non-linear new-market profit flow

A substantial simplification in the benchmark model is that the profit flow on the new market is proportional to the technology level θ. Here we relax this assumption. In particular, Let us consider a profit flow expressed by $\pi(\theta) = z\theta^b$ with b being a positive constant (note that for $b = 1$ we are in the benchmark case).

The following proposition derives the value of the firm.

Proposition 8 *Let $n = \lceil \frac{\theta^* - \theta}{u} \rceil$, then the value function is given by*

$$F(\theta, x) = \begin{cases} \left(\frac{\lambda}{r+\lambda}\right)^n \left(\frac{z(\theta+nu)^b}{r} - I\right) & \\ + \left(1 - \left(\frac{\lambda}{r+\lambda-\alpha}\right)^n\right) \frac{\pi_0 x}{r-\alpha} & \text{for } \theta \in [0, \theta^* - u), \\ \frac{\pi_0 x}{r+\lambda-\alpha} + \frac{\lambda}{r+\lambda}\left(\frac{z(\theta+u)^b}{r} - I\right) & \text{for } \theta \in [\theta^* - u, \theta^*), \\ \frac{z\theta^b}{r} - I & \text{for } \theta \in [\theta^*, \infty). \end{cases} \tag{11}$$

We conclude that the value of the firm in the benchmark case, as given in Proposition 1, essentially stays except for the obvious correction of the new market profit flow now influenced by the parameter b.

We next obtain the threshold function $\theta^*(x)$.

Proposition 9 *The threshold curve $\theta^*(x)$ is implicitly given by the following equation*

$$\left(\frac{\lambda}{r+\lambda}\right) \frac{z(\theta^* + u)^b}{r} - \frac{z(\theta^*)^b}{r} + \frac{\pi_0 x}{r+\lambda-\alpha} + \left(\frac{r}{r+\lambda}\right) I = 0. \tag{12}$$

Now the expression for the threshold curve is just implicit. Note however that $b = 1$ results in expression (6), i.e. this expression is quite similar to the result of the benchmark model. We just have a correction for nonlinearity of the new market profit flow governed by the parameter b.

From the implicit expression of the threshold curve (11) the following comparative statics results are derived.

Proposition 10 *The threshold curve $\theta^*(x)$ is decreasing in z, increasing in α, λ, π_0, I, u, and b (for $b > 1$), for a given value x.*

So again the conclusions from the benchmark model do not change. If the profit flow on the new market is convex in θ, the value of waiting for a larger θ goes up, which explains why $\theta^*(x)$ is increasing in b for $b > 1$.

3 Main model: Keep old product alive or replace

In our main model we assume that the firm can keep producing the old product after investing in the innovative product. The price in the existing market is assumed to be equal to

$$p_0(t) = x(t)(1 - \gamma q_0). \tag{13}$$

After introduction of the innovative product in the market the market structure changes and the following demand system arises in case the firm chooses to keep the old product alive:

$$p_0\left(t\right) = x\left(t\right)\left(1-\gamma q_0-\eta q_1\theta\right), \tag{14}$$

$$p_1\left(t\right) = \theta\left(1-\gamma q_1-\eta q_0 x(t)\right). \tag{15}$$

In this case the profit flow after the investment is given by $\hat{\pi}_0\left(\theta\right)x+\hat{z}^A\left(x\right)\theta$ in case the old product stays alive, where $\hat{\pi}_0\left(\theta\right)=\left(1-\gamma q_0-\eta q_1\theta\right)q_0$ and $\hat{z}^A\left(x\right)=\left(1-\gamma q_1-\eta q_0 x\right)q_1$.

If it is optimal to replace the old product by the innovative one, the profit flow in the new market is equal to $\hat{z}^R\theta$, where $\hat{z}^R=\left(1-\gamma q_1\right)q_1$.

The optimal stopping problem is given by

$$\begin{aligned} F(\theta,x) &= \sup_{\tau_1}\mathbb{E}\left[\int_0^{\tau_1}\pi_0 x(s)e^{-rs}ds-e^{-r\tau_1}I\right. \\ &+ \left.\sup_{\tau_2}\left[\int_{\tau_1}^{\tau_2}\left(\hat{\pi}_0(\theta_{\tau_1})x(s)+\hat{z}^A(x(s))\theta\right)e^{-rs}ds+\int_{\tau_2}^{\infty}\hat{z}^R\theta e^{-rs}ds\right]\right]. \end{aligned} \tag{16}$$

Proposition 11 *The firm will keep the old product if it adopts a technology with level θ less than $\hat{\theta}$, where*

$$\hat{\theta}=\frac{1-\gamma q_0}{2\eta q_1}, \tag{17}$$

and will replace the olde product otherwise.

From Proposition 11 we conclude that the choice of whether the firm adds the innovative product to the product portfolio or replaces the old product depends solely on the technology level θ adopted. If the firm decides to adopt the technology level $\theta>\hat{\theta}$, then it is optimal to replace the old product with the new one, otherwise, it is optimal to add the new product to the product portfolio. Therefore, τ_2 is always equal to either τ_1 or ∞, and we can simplify the optimal stopping problem to

$$\begin{aligned} F(\theta,x) &= \sup_{\tau_1}\mathbb{E}\left[\int_0^{\tau_1}\pi_0 x(s)e^{-rs}ds-e^{-r\tau_1}I\right. \\ &+ \left.\max\left[\int_{\tau_1}^{\infty}\left(\hat{\pi}_0(\theta_{\tau_1})x(s)+\hat{z}^A(x(s))\theta\right)e^{-rs}ds,\int_{\tau_2}^{\infty}\hat{z}^R\theta e^{-rs}ds\right]\right]. \end{aligned} \tag{18}$$

Figure 3 shows the different regions that should be considered to derive the threshold boundary. Regions 1 and 5 are defined as the region where even after the next jump the firm will not invest, where in region

5 the technology level is already that high that we know that the firm will never choose add, but will always go for replace. Mathematically, we have that $R1 = \left\{(\theta, x) \middle| \theta < \min\left(\theta^*(x) - u, \hat{\theta}\right)\right\}$ and $R5 = \left\{(\theta, x) \middle| \hat{\theta} \leq \theta < \theta^*(x) - u\right\}$. In region 2, defined as $R2 = \left\{(\theta, x) \middle| \theta^*(x) - u \leq \theta < \min\left(\theta^*(x), \hat{\theta} - u\right)\right\}$, the firm will invest after the next jump or after sufficient decrease of the first market in region 4, which is defined as $R4 = \left\{(\theta, x) \middle| \theta^*(x) \leq \theta < \hat{\theta}\right\}$. In region 4 the firm will add the new product to its portfolio. After a technology arrival in regions 3 or 6 the firm will invest in region 7, where it will replace the current technology. Region 3 is defined as $R3 = \left\{(\theta, x) \middle| \max\left(\theta^*(x) - u, \hat{\theta} - u\right) \leq \theta < \min\left(\theta^*(x), \hat{\theta}\right)\right\}$. A sufficient decrease of the current market will bring the firm from region 3 to region 4. Region 6 is defined such that we know that the technology level is high enough to ensure that investment will result in replacing the current technology, mathematically $R6 = \left\{(\theta, x) \middle| \max\left(\hat{\theta}, \theta^*(x) - u\right) \leq \theta < \theta^*(x)\right\}$. Region 7 is the investment region for replace, i.e. $R7 = \left\{(\theta, x) \middle| \max\left(\hat{\theta}, \theta^*(x)\right) \leq \theta\right\}$.

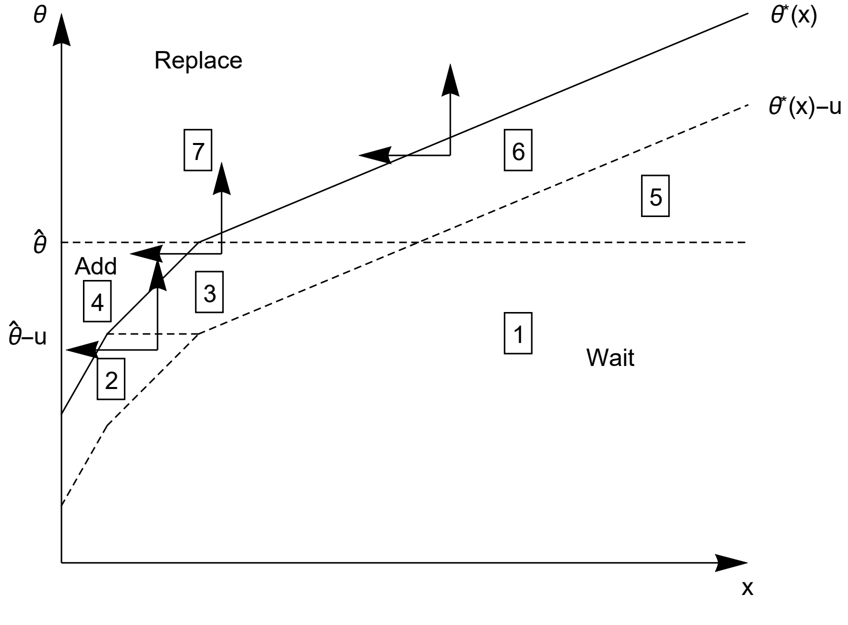

Figure 3: The regions that need to be considered to derive the threshold boundary and the possible ways to end up in the investment regions 4 and 7.

To derive the threshold curve we need the value functions for regions 2, 3, 4, 6, and 7.[4] The following

[4]We do not specify the value function for regions 1 and 5, the reason is that it will just be a mathematically exercise with good bookkeeping of the number of jumps needed till the other regions are reached and the value function is not needed to

proposition specifies the value function for these regions.

Proposition 12 *The value function in the investment regions (4 and 7) and the regions where investment will take place after the next technology jump or sufficient decrease of the profitability of the first market (2, 3, and 6) is given by*

$$F(\theta, x) = \begin{cases} \frac{\pi_0 x}{r+\lambda-\alpha} + \frac{\lambda}{r+\lambda}\left(\frac{\hat{\pi}_0(\theta+u)x}{r-\alpha} + q_1\left(\frac{1-\gamma q_1}{r} - \frac{\eta q_0 x}{r-\alpha}\right)(\theta+u) - I\right) & \textit{for region 2,} \\ \frac{\pi_0 x}{r+\lambda-\alpha} + \frac{\lambda}{r+\lambda}\left(\frac{\hat{z}^R(\theta+u)}{r} - I\right) & \textit{for region 3 and 6,} \\ \frac{\hat{\pi}_0(\theta)x}{r-\alpha} + q_1\left(\frac{1-\gamma q_1}{r} - \frac{\eta q_0 x}{r-\alpha}\right)\theta - I & \textit{for region 4,} \\ \frac{\hat{z}^R\theta}{r} - I & \textit{for region 7.} \end{cases} \tag{19}$$

Using value matching at the threshold boundary and the value function of Proposition 13 we can derive the threshold boundary which is specified in Proposition 13.

Proposition 13 *Let us define $\hat{x}_1$ and $\hat{x}_2$ as follows $\hat{x}_1 = \inf\left(x \,\middle|\, \theta^*(x) = \hat{\theta} - u\right)$ and $\hat{x}_2 = \inf\left(x \,\middle|\, \theta^*(x) = \hat{\theta}\right)$, which leads to*

$$\hat{x}_1 = \frac{\frac{1-\gamma q_0}{2\eta q_1} - \left(\frac{Ir}{q_1(1-\gamma q_1)} + \frac{\lambda u}{r}\right)}{\frac{(q_0(1-\gamma q_0))(\lambda+r)}{(q_1(1-\gamma q_1))(-\alpha+\lambda+r)}}, \tag{20}$$

$$\hat{x}_2 = \frac{(r-\alpha)(-\alpha+\lambda+r)\left(2\eta I r^2 - r(\gamma q_1 - 1)(\gamma q_0 + 2\eta q_1 u - 1) - 2\eta\lambda q_1 u(\gamma q_1 - 1)\right)}{2\eta q_0 r(\lambda+r)((r-\alpha)(\gamma q_0 + 2\eta q_1 u - 1) + 2\eta\lambda q_1 u)}. \tag{21}$$

The theshold curve $\theta^*(x)$ is given by

$$\theta^*(x) = \begin{cases} \frac{\lambda u}{r} + \frac{I - \frac{\pi_0 x}{r(r-\alpha)}\left(\frac{\alpha\lambda}{r+\lambda-\alpha}\right)}{\frac{\hat{z}^R}{r} - \frac{2\eta q_0 q_1 x}{r-\alpha}} & \textit{for } x \in [0, \hat{x}_1), \\ \frac{\frac{r}{r+\lambda}I + \frac{\lambda}{r+\lambda}\frac{\hat{z}^R u}{r} - \frac{\lambda}{r+\lambda-\alpha}\frac{\pi_0 x}{r-\alpha}}{\frac{\hat{z}^R}{r+\lambda} - \frac{2\eta q_0 q_1 x}{r-\alpha}} & \textit{for } x \in [\hat{x}_1, \hat{x}_2), \\ \frac{r+\lambda}{r+\lambda-\alpha}\frac{\pi_0 x}{\hat{z}^R} + \frac{\lambda u}{r} + \frac{rI}{\hat{z}^R} & \textit{for } x \in [\hat{x}_2, \infty). \end{cases} \tag{22}$$

It turns out that it is not always the case that $\hat{x}_1 < \hat{x}_2$ and as a result the threshold curve between regions 3 and 4 can be decreasing, see Figure 4 for an illustration.

Following the arc in Figure 4 different θ regions are passed. Consider the x-interval $(\hat{x}_2, \hat{x}_1)$ and let θ gradually increase from 0 on. If we do so, we pass the following regions. First, $0 < \theta < \theta^*(x) - u$, which is region 1, where firm is active on the old market and also does so after the next jump. Second, $\theta^*(x) - u < \theta < \hat{\theta} - u$, which is region 2, where firm is active on the old market but will enter the add

derive our economic results.

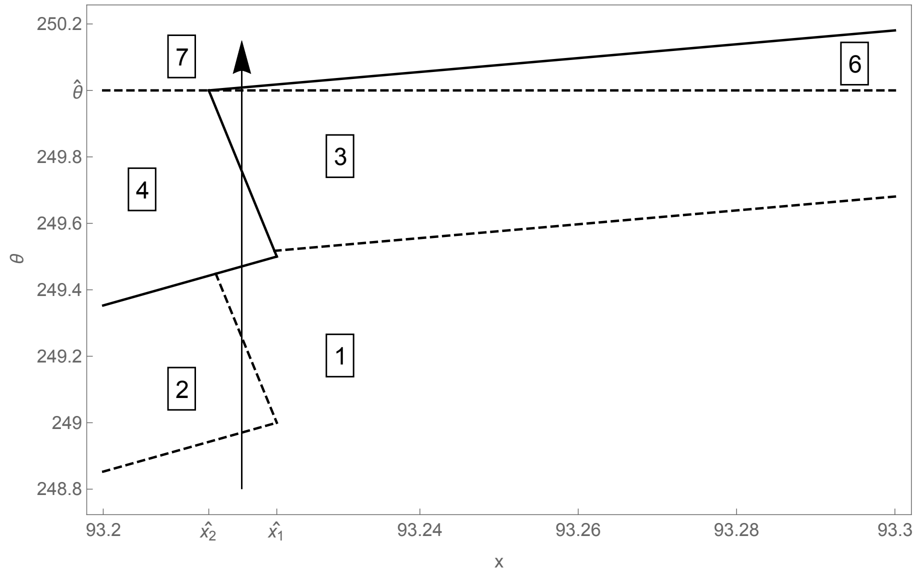

Figure 4: Parameter values $r = 0.1$, $\lambda = 0.05$, $u = 0.5$, $I = 50$, $q_0 = 0.5$, $q_1 = 0.1$, $\alpha = -0.05$, and $z = 10$ lead to a decreasing threshold curve between regions 3 and 4.

investment region after the next jump. Third, $\hat{\theta} - u < \theta < \theta^* (x)$, region 1 where firm is active on old market and also does so after the next jump. Fourth, $\theta^* (x) < \theta < \theta^* (x)$ (note this interval is not empty (mathematicians do not ?faint?, notation is just sloppy), but when θ goes up we intersect $\theta^* (x)$ three times): the firm innovates in region 4 being the add region $\theta^* (x) < \theta < \hat{\theta}$: region 3 where the firm is active on the old market but will enter replace investment region after the next jump and enters add investment region as soon as x horizontally hits the $\theta^* (x)$ curve. Fifth, $\hat{\theta} < \theta < \theta^* (x)$, region 6, which is the same as region 3 except that now firm also enters the replace investment region as soon as x horizontally hits the $\theta^* (x)$ curve. Sixth, $\theta^* (x) < \theta$: the firm innovates in region 7 being the replace investment region.

So we have an inaction region, region 3 (and also region 6) in between the innovation regions 4 and 7. This is caused by the fact that the boundary between regions 3 and 4 has decreasing slope. This implies that, given that no jump occurs, the firm innovates later for larger theta. To find out why this can be optimal, distinguish the following two effects:

EFFECT 1: if a firm moves from region 3 into region 4, then instantaneous profit changes as follows:

$$\begin{aligned}\pi_{add} - \pi_{oldmarket} &= x(1-\gamma q_0 - \eta q_1 \theta)q_0 + \theta(1-\gamma q_1 - \eta q_0 x)q_1 - x(1-\gamma q_0)q_0 \\ &= \theta(1-\gamma q_1 - 2\eta q_0 x)q_1.\end{aligned}$$

The part between brackets is positive, otherwise there would not have been an incentive to innovate. So we conclude due to this effect innovation is more profitable for larger θ.

EFFECT 2: if a firm moves from region 3 into region 4 it gives up the option to replace instead of add. Hence, as long as the firm stays in region 3, it keeps the option to replace. The value of this option is larger when the difference between the instantaneous profit in case of replace and in case of add is larger, where replace takes place after the next jump. Hence the value of the option to replace goes up with $\pi_{replace}(\theta+u) - \pi_{add}(\theta)$, where

$$\begin{aligned}\pi_{replace}(\theta+u) - \pi_{add}(\theta) &= (\theta+u)(1-\gamma q_1)q_1 - (x(1-\gamma q_0 - \eta q_1\theta)q_0 + \theta(1-\gamma q_1 - \eta q_0 x)q_1) \\ &= u(1-\gamma q_1)q_1 - (x(1-\gamma q_0 - 2\eta q_1 \theta)q_0).\end{aligned}$$

Hence, keeping the option to replace alive goes up with θ, which explains why innovating immediately is less profitable for larger θ. Therefore one stays in region 3 for a longer time when θ is larger so that the boundary has decreasing slope.

So, if the boundary between regions 3 and 4 increases, as in Figure 4, EFFECT 1 dominates. In case the boundary decreases, EFFECT 2 dominates.

It would be interesting to see under which circumstances EFFECT 2 dominates. Probably when η is relatively large. And also when the next jump occurs soon (λ large) and the jump is large (u large). Also when the expression for discounting after the next jump, $\frac{\lambda}{r+\lambda}$ is close to one requiring that r is small. I think some numerical experiments where we verify this, would be fruitful.

We found that for small η that the threshold curve 2b5 is decreasing in x. This has to be checked further. In particular we have to check the slope of $\theta^*2b5 = \frac{ax-b}{cx-d}$. It follows that $\frac{\partial\theta^*2b5}{\partial x} = \frac{ad-bc}{(cx-d)^2}$. Furthermore we should check the same for the threshold curve 2a5.

The derivative of θ^*_{2b5} is given by $\frac{\partial\theta^*2b5}{\partial x} = \frac{ad-cb}{(cx-d)^2}$. The sign of the derivative depends on the sign of the numerator $ad - cb$ which is equal to

$$\frac{(q_0 q_1)\left(2\eta I r^2(-\alpha+\lambda+r)+\lambda(\gamma q_1-1)(r(\gamma(-q_0)-2\eta q_1 u+1)+2\eta q_1 u(\alpha-\lambda))\right)}{r(r-\alpha)(\lambda+r)(-\alpha+\lambda+r)} \tag{23}$$

Therewith we can conclude that $\frac{\partial\theta^*_{2b5}}{\partial x} > 0$ if $\eta > \frac{r(1-\gamma q_0)}{2(r+\lambda-\alpha)q_1 u} - \frac{r^2 I}{2\lambda q_0 q_1^2(1-\gamma q_1)}$

The derivative of θ^*_{2a5} is given by $\frac{\partial\theta^* 2a5}{\partial x} = \frac{ad-cb}{(cx-d)^2}$. The sign of the derivative depends on the sign of the numerator $ad - cb$ which is equal to

$$\frac{q_0 q_1\left(2\eta I r^2(-\alpha+\lambda+r)-\alpha\lambda(\gamma q_0-1)(\gamma q_1-1)\right)}{r^2(r-\alpha)(-\alpha+\lambda+r)} \tag{24}$$

$$\frac{q_0 q_1}{r^2(r-\alpha)(r+\lambda-\alpha)}\left[2\eta I r^2(r+\lambda-\alpha)-\alpha\lambda(1-\gamma q_0)(1-\gamma q_1)\right] \tag{25}$$

Therewith, we conclude that $\frac{\partial\theta^*_{2a5}}{\partial x} > 0$ if $2\eta I r^2(r+\lambda-\alpha) - \alpha\lambda(1-\gamma q_0)(1-\gamma q_1) > 0$ which always holds since $\alpha > 0$.

Add proposition for result on η and state it holds for η small, I small, α very negative, q_0 small and u small. Look at price equation, make sure price is positive and use that for the denominator of the threshold between 3 and 4

4 Conclusion

Further research: capacity choice, game, learning.

A Proofs of proposition

Needs adjustments to reflect propositions in main text.

Proof of Proposition 1 We have two states variables θ, the state of the technology, and x, the level of the current profit. The optimal investment threshold θ^* will then be a function of x. In the continuation region the firm chooses not to undertake the investment and in the stopping region the firm does invest. The continuation region can be split up in two parts, in the first the firm will not invest after the next jump $\{(\theta, x)\,|\theta < \theta^*(x) - u\}$ and in the second part the firm will invest after the next technology arrival $\{(\theta, x)\,|\theta^*(x) - u \leq \theta < \theta^*(x)\}$.

The value of the firm in the stopping region is independent of the level of x and denoted by $V(\theta)$ and is equal to

$$V(\theta) = \int_{t=0}^{\infty} \pi_1(\theta) \exp(-rt)\, dt - I = \frac{\pi_1(\theta)}{r} - I = \frac{z\theta}{r} - I. \tag{26}$$

In the second part of the continuation the following Bellman equation should hold

$$rF(\theta, x) = \pi_0 x + \lim_{dt \downarrow 0} \frac{1}{dt} E\left[dF(\theta, x)\right]. \tag{27}$$

Using Itô's lemma we find

$$E\left[dF(\theta, x)\right] = \alpha x \frac{\partial F(\theta, x)}{\partial x} dt + \lambda dt \left(V(\theta + u) - F(\theta, x)\right) + o(dt). \tag{28}$$

Substitution of (71) and (78) into equation (77) gives

$$(r + \lambda) F(\theta, x) = \pi_0 x + \lambda \left(\frac{z(\theta + u)}{r} - I\right) + \alpha x \frac{\partial F(\theta, x)}{\partial x}. \tag{29}$$

To find the solution of equation (79) we first search for the solution to the homogeneous differential equation that is implied by equation (79):

$$(r + \lambda) F(\theta, x) = \alpha x \frac{\partial F(\theta, x)}{\partial x}. \tag{30}$$

We guess the solution

$$F(\theta, x) = A_0 x^{\beta_0}. \tag{31}$$

Substitution of this guess in equation (30) gives

$$(r + \lambda) A_0 x^{\beta_0} = \alpha \beta_0 A_0 x^{\beta_0}, \tag{32}$$

which leads to

$$\beta_0 = \frac{r + \lambda}{\alpha}. \tag{33}$$

We conclude that the solution to the homogeneous equation is equal to

$$F(\theta, x) = A_0 x^{\frac{r+\lambda}{\alpha}}. \tag{34}$$

A guess for the particular solution of differential equation (79) is

$$F(\theta, x) = \gamma_0 x + \gamma_1 \theta + \gamma_2, \tag{35}$$

which leads to

$$\gamma_0 = \frac{\pi_0}{r+\lambda-\alpha}, \tag{36}$$

$$\gamma_1 = \frac{\lambda}{r+\lambda}\frac{z}{r}, \tag{37}$$

$$\gamma_2 = \frac{\lambda}{r+\lambda}\left(\frac{zu}{r} - I\right). \tag{38}$$

We conclude that the solution of the equation (79) is given by

$$F(\theta, x) = A_0 x^{\frac{r+\lambda}{\alpha}} + \frac{\pi_0 x}{r+\lambda-\alpha} + \frac{\lambda}{r+\lambda}\left(\frac{z(\theta+u)}{r} - I\right). \tag{39}$$

In the first part of the continuation the following Bellman equation should hold

$$rF(\theta, x) = \pi_0 x + \lim_{dt\downarrow 0}\frac{1}{dt}E[dF(\theta, x)]. \tag{40}$$

Using Itô's lemma we find

$$E[dF(\theta, x)] = \alpha x \frac{\partial F(\theta, x)}{\partial x}dt + \lambda dt\,(F(\theta+u, x) - F(\theta, x)) + o(dt). \tag{41}$$

Substitution of (41) into equation (40) gives

$$rF(\theta, x) = \pi_0 x + \alpha x \frac{\partial F(\theta, x)}{\partial x} + \lambda\,(F(\theta+u, x) - F(\theta, x)). \tag{42}$$

To find the solution of equation (42) we first search for the solution to the homogeneous differential equation that is implied by this equation:

$$(r+\lambda)F(\theta, x) = \alpha x \frac{\partial F(\theta, x)}{\partial x} + \lambda F(\theta+u, x). \tag{43}$$

Considering this differential equation, we expect that the solution will have separate terms in x, θ, and a cross term both containing x and θ. Following Huisman (2001, p. 38) and Øksendal (2009, p. 55), we guess the solution

$$F(\theta, x) = A_1 x^{\beta_1} + A_2 \beta_2^{\theta} + A_3 x \beta_3^{\theta}. \tag{44}$$

Substitution of this guess in equation (43) gives

$$(r+\lambda)\left(A_1 x^{\beta_1} + A_2\beta_2^{\theta} + A_3 x\beta_3^{\theta}\right) = \alpha x\left(\beta_1 A_1 x^{\beta_1 - 1} + A_3\beta_3^{\theta}\right) + \lambda\left(A_1 x^{\beta_1} + A_2\beta_2^{\theta+u} + A_3 x\beta_3^{\theta+u}\right), \tag{45}$$

which leads to

$$\beta_1 = \frac{r}{\alpha}, \tag{46}$$

$$\beta_2 = \left(\frac{\lambda}{r+\lambda}\right)^{-\frac{1}{u}}, \tag{47}$$

$$\beta_3 = \left(\frac{\lambda}{r+\lambda-\alpha}\right)^{-\frac{1}{u}}. \tag{48}$$

We conclude that the solution to the homogeneous equation is equal to

$$F(\theta,x) = A_1 x^{\frac{r}{\alpha}} + A_2\left(\frac{\lambda}{r+\lambda}\right)^{-\frac{\theta}{u}} + A_3 x\left(\frac{\lambda}{r+\lambda-\alpha}\right)^{-\frac{\theta}{u}}. \tag{49}$$

A guess for the particular solution of differential equation (42) is

$$F(\theta,x) = \gamma_3 x, \tag{50}$$

which leads to

$$\gamma_3 = \frac{\pi_0}{r-\alpha}. \tag{51}$$

We conclude that the solution of the equation (42) is given by

$$F(\theta,x) = A_1 x^{\frac{r}{\alpha}} + A_2\left(\frac{\lambda}{r+\lambda}\right)^{-\frac{\theta}{u}} + A_3 x\left(\frac{\lambda}{r+\lambda-\alpha}\right)^{-\frac{\theta}{u}} + \frac{\pi_0 x}{r-\alpha}. \tag{52}$$

The previous derivations lead to the following value function for the firm

$$F(\theta,x) = \begin{cases} A_1 x^{\frac{r}{\alpha}} + A_2\left(\frac{\lambda}{r+\lambda}\right)^{-\frac{\theta}{u}} + A_3 x\left(\frac{\lambda}{r+\lambda-\alpha}\right)^{-\frac{\theta}{u}} + \frac{\pi_0 x}{r-\alpha} & \text{for } \theta < \theta^*(x) - u, \\ A_0 x^{\frac{r+\lambda}{\alpha}} + \frac{\pi_0 x}{r+\lambda-\alpha} + \frac{\lambda}{r+\lambda}\left(\frac{z(\theta+u)}{r} - I\right) & \text{for } \theta^*(x) - u \le \theta < \theta^*(x), \\ \frac{z\theta}{r} - I & \text{for } \theta \ge \theta^*(x). \end{cases} \tag{53}$$

There are five unknowns left A_0, A_1, A_2, A_3, and the boundary threshold $\theta^*(x)$. There are two value matching conditions to be satisfied, between the first and second part of the continuatuon region and between the continuation and stopping region. Furthermore we know that for $x = 0$ the value of the firm is finite, therefore since $\alpha < 0$ we know that $A_0 = 0$ and $A_1 = 0$.

Value matching at $\theta = \theta^* - u$ gives

$$A_2\left(\frac{\lambda}{r+\lambda}\right)^{-\frac{\theta^*-u}{u}} + A_3 x\left(\frac{\lambda}{r+\lambda-\alpha}\right)^{-\frac{\theta^*-u}{u}} + \frac{\pi_0 x}{r-\alpha} = \frac{\pi_0 x}{r+\lambda-\alpha} + \frac{\lambda}{r+\lambda}\left(\frac{z\theta^*}{r} - I\right). \tag{54}$$

Taking together terms with x gives

$$A_3 x \left(\frac{\lambda}{r+\lambda-\alpha}\right)^{-\frac{\theta^*-u}{u}} + \frac{\pi_0 x}{r-\alpha} = \frac{\pi_0 x}{r+\lambda-\alpha}, \tag{55}$$

which leads to

$$A_3 = -\frac{\pi_0}{r-\alpha}\left(\frac{\lambda}{r+\lambda-\alpha}\right)^{\frac{\theta^*}{u}}. \tag{56}$$

The remainder terms are

$$A_2 \left(\frac{\lambda}{r+\lambda}\right)^{-\frac{\theta^*-u}{u}} = \frac{\lambda}{r+\lambda}\left(\frac{z\theta^*}{r} - I\right), \tag{57}$$

which gives

$$A_2 = \left(\frac{\lambda}{r+\lambda}\right)^{\frac{\theta^*}{u}}\left(\frac{z\theta^*}{r} - I\right). \tag{58}$$

Concluding we have that

$$F(\theta,x) = \begin{cases} \left(\frac{\lambda}{r+\lambda}\right)^{\frac{\theta^*-\theta}{u}}\left(\frac{z\theta^*}{r} - I\right) + \left(1 - \left(\frac{\lambda}{r+\lambda-\alpha}\right)^{\frac{\theta^*-\theta}{u}}\right)\frac{\pi_0 x}{r-\alpha} & \text{for } \theta < \theta^*(x) - u, \\ \frac{\pi_0 x}{r+\lambda-\alpha} + \frac{\lambda}{r+\lambda}\left(\frac{z(\theta+u)}{r} - I\right) & \text{for } \theta^*(x) - u \leq \theta < \theta^*(x), \\ \frac{z\theta}{r} - I & \text{for } \theta \geq \theta^*(x). \end{cases} \tag{59}$$

Proof of Proposition 2 The threshold boundary can be found by solving the value matching condition between the second part of the continuation region and the stopping region:

$$\frac{\pi_0 x}{r+\lambda-\alpha} + \frac{\lambda}{r+\lambda}\left(\frac{z(\theta+u)}{r} - I\right) = \frac{z\theta}{r} - I.; \tag{60}$$

Rewriting gives

$$\theta^*(x) = \frac{r+\lambda}{r+\lambda-\alpha}\frac{\pi_0 x}{z} + \frac{\lambda u}{r} + \frac{rI}{z}. \tag{61}$$

Proof of Proposition 3 Assume that equation (1) is replaced by

$$dx(t) = \alpha x dt + \sigma x d\omega(t), \tag{62}$$

where $\alpha < 0$, $\sigma > 0$, and $d\omega(t)$ is the increment of a Wiener process.

We can follow the same steps as in the proof of Proposition 1. The value of the firm in the stopping region will still be given by equation (71). Equation (79) will change into

$$(r+\lambda)F(\theta,x) = \pi_0 x + \lambda\left(\frac{z(\theta+u)}{r} - I\right) + \alpha x\frac{\partial F(\theta,x)}{\partial x} + \frac{1}{2}\sigma^2 x^2\frac{\partial^2 F(\theta,x)}{\partial x^2}. \tag{63}$$

It turns out that the solution to the homogeneous equation that is implied by this last equation is given by

$$F(\theta, x) = B_1 x^{\epsilon_1} + B_2 x^{\epsilon_2}, \tag{64}$$

where $\epsilon_1 > 1$ and $\epsilon_2 < 0$ are the solutions of the following equation

$$\frac{1}{2}\sigma^2 \epsilon (\epsilon - 1) + \alpha\epsilon - (r + \lambda) = 0. \tag{65}$$

The same particular solution is still valid, so that the solution to (63) is given by

$$F(\theta, x) = B_1 x^{\epsilon_1} + B_2 x^{\epsilon_2} + \frac{\pi_0 x}{r + \lambda - \alpha} + \frac{\lambda}{r + \lambda}\left(\frac{z(\theta + u)}{r} - I\right). \tag{66}$$

The differential equation for the first part of the continuation region now becomes

$$(r + \lambda) F(\theta, x) = \pi_0 x + \lambda F(\theta + u, x) + \alpha x \frac{\partial F(\theta, x)}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 F(\theta, x)}{\partial x^2}. \tag{67}$$

The solution to the implied homogeneous equation is given by

$$F(\theta, t) = B_3 x^{\epsilon_3} + B_4 x^{\epsilon_4} + B_5 \left(\frac{\lambda}{r + \lambda}\right)^{-\frac{\theta}{u}} + B_6 x \left(\frac{\lambda}{r + \lambda - \alpha}\right)^{-\frac{\theta}{u}}, \tag{68}$$

with $\epsilon_3 > 1$ and $\epsilon_4 < 0$ the solutions of the equation

$$\frac{1}{2}\sigma^2 \epsilon (\epsilon - 1) + \alpha\epsilon - r = 0. \tag{69}$$

Concluding we have that

$$F(\theta, x) = \begin{cases} B_3 x^{\epsilon_3} + B_4 x^{\epsilon_4} + B_5 \left(\frac{\lambda}{r+\lambda}\right)^{-\frac{\theta}{u}} + B_6 x \left(\frac{\lambda}{r+\lambda-\alpha}\right)^{-\frac{\theta}{u}} & \text{for } \theta < \theta^*(x) - u, \\ B_1 x^{\epsilon_1} + B_2 x^{\epsilon_2} + \frac{\pi_0 x}{r+\lambda-\alpha} + \frac{\lambda}{r+\lambda}\left(\frac{z(\theta+u)}{r} - I\right) & \text{for } \theta^*(x) - u \le \theta < \theta^*(x), \\ \frac{z\theta}{r} - I & \text{for } \theta \ge \theta^*(x). \end{cases} \tag{70}$$

There are seven unknowns left B_1, B_2, B_3, B_4, B_5, B_6, and the boundary threshold $\theta^*(x)$. There are two value matching conditions to be satisfied, between the first and second part of the continuatuon region and between the continuation and stopping region. Furthermore we know that for $x = 0$ the value of the firm is finite, therefore since $\epsilon_2 < 0$ and $\epsilon_4 < 0$ we know that $B_2 = 0$ and $B_4 = 0$. Moreover, by ruling out speculative bubbles (see Dixit and Pindyck (1994, p. 181) we have $B_1 = 0$ and $B_3 = 0$. After substituting these values for B_1, B_2, B_3, and B_4 we conclude that equation (70) is equivalent to equation (86) after substituting $A_1 = 0$ and $A_0 = 0$. Therefore the solution for the problem with a geometric Brownian motion before the technology adoption is equal to the solution of the base model.

A.1 Proofs and Derivations for Robustness Sections 4.2, 4.3, 4.4

A.1.1 Appendix Geometric Brownian Motion after Investment

The value of the firm in the stopping region is denoted by $V(\theta)$ and is equal to

$$V(\theta) = E_\theta \left[\int_{t=0}^{\infty} zY(t)(\theta) \exp(-rt)\, dt - I \,\middle|\, Y(0) = \theta \right] = \frac{z\theta}{r-\mu} - I. \tag{71}$$

The functional form of the value function in the first part of the continuation region is the same as for the main model. Straightforward derivations show that the value function in the second part of the continuation region is equal to

$$F(\theta, x) = A_0 x^{\frac{r+\lambda}{\alpha}} + \frac{\pi_0 x}{r+\lambda-\alpha} + \frac{\lambda}{r+\lambda}\left(\frac{z(\theta+u)}{r-\mu} - I\right) \tag{72}$$

$$F(\theta, x) = \begin{cases} A_1 x^{\frac{r}{\alpha}} + A_2 \left(\frac{\lambda}{r+\lambda}\right)^{-\frac{\theta}{u}} + A_3 x \left(\frac{\lambda}{r+\lambda-\alpha}\right)^{-\frac{\theta}{u}} + \frac{\pi_0 x}{r-\alpha} & \text{for } \theta < \theta^*(x) - u, \\ A_0 x^{\frac{r+\lambda}{\alpha}} + \frac{\pi_0 x}{r+\lambda-\alpha} + \frac{\lambda}{r+\lambda}\left(\frac{z(\theta+u)}{r-\mu} - I\right) & \text{for } \theta^*(x) - u \leq \theta < \theta^*(x), \\ \frac{z\theta}{r-\mu} - I & \text{for } \theta \geq \theta^*(x). \end{cases} \tag{73}$$

Include derivations and arguments for the constants.

A.1.2 Appendix - Keep old product alive

The value of the firm in the stopping region is now not only depending on θ but also on x. The value function, denoted by $V(\theta, x)$, is equal to

$$\begin{aligned} V(\theta, x) &= \int_{t=0}^{\infty} \pi_1(\theta, x) \exp(-rt)\, dt - I & (74)\\ &= \frac{\tilde{\pi}_0 x}{r-\alpha} + \frac{\tilde{z}\theta}{r} - I & (75)\\ &= \frac{\theta(1-\gamma q_1 - \eta q_0) q_1}{r} + \frac{x(t)(1-\gamma q_0 - \eta q_1) q_0}{r-\alpha} - I & (76) \end{aligned}$$

In the second part of the continuation the following Bellman equation should hold

$$rF(\theta, x) = \pi_0 x + \lim_{dt \downarrow 0} \frac{1}{dt} E[dF(\theta, x)]. \tag{77}$$

Using Itô's lemma we find

$$E[dF(\theta, x)] = \alpha x \frac{\partial F(\theta, x)}{\partial x} dt + \lambda dt \left(V(\theta+u, x) - F(\theta, x)\right) + o(dt). \tag{78}$$

Substitution of (71) and (78) into equation (77) gives

$$(r+\lambda)\,F(\theta,x)=\pi_0 x+\lambda\left(\frac{\tilde{\pi}_0 x}{r-\alpha}+\frac{\tilde{z}\,(\theta+u)}{r}-I\right)+\alpha x\frac{\partial F(\theta,x)}{\partial x}. \tag{79}$$

As before the solution to the homogeneous differential equation implied by equation (79) is equal to

$$F(\theta,t)=A_0 x^{\frac{r+\lambda}{\alpha}}. \tag{80}$$

A guess for the particular solution of differential equation (79) is

$$F(\theta,t)=\gamma_0 x+\gamma_1\theta+\gamma_2, \tag{81}$$

which leads to

$$\gamma_0 = \frac{1}{r+\lambda-\alpha}\left[\pi_0+\frac{\lambda\tilde{\pi}_0}{r-\alpha}\right], \tag{82}$$

$$\gamma_1 = \frac{\lambda}{r+\lambda}\frac{\tilde{z}}{r}, \tag{83}$$

$$\gamma_2 = \frac{\lambda}{r+\lambda}\left(\frac{\tilde{z}u}{r}-I\right). \tag{84}$$

We conclude that the solution of the equation (79) is given by

$$F(\theta,x)=A_0 x^{\frac{r+\lambda}{\alpha}}+\frac{x}{r+\lambda-\alpha}\left[\pi_0+\frac{\lambda\tilde{\pi}_0}{r-\alpha}\right]+\frac{\lambda}{r+\lambda}\left(\frac{\tilde{z}\,(\theta+u)}{r}-I\right). \tag{85}$$

Same as in main case? Check derivations

This leads to the following value function for the firm

$$F(\theta,x)=\begin{cases} A_1 x^{\frac{r}{\alpha}}+A_2\left(\frac{\lambda}{r+\lambda}\right)^{-\frac{\theta}{u}}+A_3 x\left(\frac{\lambda}{r+\lambda-\alpha}\right)^{-\frac{\theta}{u}}+\frac{\pi_0 x}{r-\alpha} & \text{for } \theta<\theta^*(x)-u, \\ A_0 x^{\frac{r+\lambda}{\alpha}}+\frac{\pi_0 x}{r+\lambda-\alpha}+\frac{\lambda}{r+\lambda-\alpha}\left[\frac{\tilde{\pi}_0 x}{r-\alpha}\right]+\frac{\lambda}{r+\lambda}\left(\frac{\tilde{z}(\theta+u)}{r}-I\right), & \\ \frac{\tilde{\pi}_0 x}{r-\alpha}+\frac{\tilde{z}\theta}{r}-I & \text{for } \theta\geq\theta^*(x). \end{cases} \tag{86}$$

Derive and argue five unknowns

The threshold boundary can be found by solving the value matching condition between the second part of the continuation region and the stopping region:

$$\frac{x}{r+\lambda-\alpha}\left[\pi_0+\frac{\lambda\tilde{\pi}_0}{r-\alpha}\right]+\frac{\lambda}{r+\lambda}\left(\frac{\tilde{z}\,(\theta+u)}{r}-I\right)=\frac{\tilde{\pi}_0 x}{r-\alpha}+\frac{\tilde{z}\theta}{r}-I.; \tag{87}$$

Rewriting gives

$$\theta^*(x)=\left(\frac{r+\lambda}{r+\lambda-\alpha}\right)\frac{(\pi_0-\tilde{\pi}_0)\,x}{\tilde{z}}+\frac{\lambda u}{r}+\frac{rI}{\tilde{z}}. \tag{88}$$

If we insert π_0, $\tilde{\pi}_0$ and $\tilde{z}$ into the threshold expression we get:

$$\theta^*(x) = \left(\frac{r+\lambda}{r+\lambda-\alpha}\right)\frac{(\eta q_0 q_1)\,x}{(1-\gamma q_1-\eta q_0)q_1} + \frac{\lambda u}{r} + \frac{rI}{(1-\gamma q_1-\eta q_0)q_1}. \tag{89}$$

A.1.3 Appendix - Non-linear in θ

Do the comparative statics for the threshold curve.

First comparative statics results:

Define $g(.) = \left(\frac{\lambda}{r+\lambda}\right)\frac{z(\theta^*+u)^b}{r} - \frac{z\theta^{*,b}}{r} + \frac{\pi_0 x}{r+\lambda-\alpha} + \left(\frac{r}{r+\lambda}\right)I$

$$\frac{\partial g}{\partial \alpha} = \pi_0 x\left[\frac{1+\alpha(r+\lambda-\alpha)}{(r+\lambda-\alpha)^2}\right] \tag{90}$$

It holds that $\frac{\partial g}{\partial \alpha} < 0$ if $\alpha < \alpha_2$ and $\frac{\partial g}{\partial \alpha} < 0$ if $\alpha_2 \leq \alpha \leq 0$, where $\alpha_2 = \frac{1}{2}\left(r+\lambda-\sqrt{(\lambda+r)^2+4}\right)$.

Proof: The sign of $\frac{\partial g}{\partial \alpha}$ depends on the sign of $1+\alpha(r+\lambda-\alpha)$ as all other terms are positive. We define $Q(\alpha) = 1+\alpha(r+\lambda-\alpha)$, which is a downward pointing parabola for which it holds that $Q(0) = 1$, $Q'(\alpha) > 0$ for $\alpha \leq 0$ and $Q(\alpha) = 0$ for $\alpha_{1,2} = \frac{1}{2}\left(r+\lambda+-\sqrt{4+(r+\lambda)^2}\right)$.

It can easily be concluded that $\frac{\partial g}{\partial I} > 0$, $\frac{\partial g}{\partial \pi_0} > 0$,

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