Investment under uncertainty in a market with network effects for consumers

Nick F.D. Huberts[1] and Jacco J.J. Thijssen[2]

[1] *CentER, Department of Econometrics and Operations Research, Tilburg University, Post Office Box 90153, 5000 LE Tilburg, The Netherlands*

[2] *The York Management School, University of York, Freboys Lane, Heslington, York YO10 5DD, United Kingdom*

February 6, 2017

Abstract

In this project we look at the investment behavior of firms in a market where consumers are exposed to network effects. In this model, the profitability of each product depends on the number of consumers. The investment timing is then related to the number of consumers. We show that, under various specifications of consumer incentives, that the resulting process of consumers is mean reverting. Despite standard real options models, this leads to a concave option value. Moreover, this model shows that more uncertainty does not necessarily lead to delaying the investment moment. Finally, for increasing populations we show that the value of waiting disappears.

Keywords: Network effects, Poisson Jumps, Investment under Uncertainty

1 Introduction

In a standard real options setting for investment under uncertainty problems, market uncertainty is assumed to follow a geometric Brownian motion (see, e.g., Dixit and Pindyck (1994) and McDonald and Siegel (1986)). Despite being reasonably appropriate in a good number of situations, this is not always the most accurate description of the market uncertainty behind economical problems. One of the advantages of a traditional set-up, is that the use of a geometric Brownian motion, generally, results in analytical expressions. This is an advantage that, e.g., mean-reverting processes like an Ornstein-Uhlenbeck process, generally, do not possess. A disadvantage is that, and this holds in general for Brownian motions, one is unable to vary with or discriminate between the upward trend and downward trend across state. These diffusion processes are assumed to be exogenously given. However, in this paper we propose a model where the underlying market uncertainty is implicitly determined as a continuous time Markov Chain. In our application, we study a market where the upward trend and downward trend of the underlying process are independently determined and vary across state. In this way one can obtain any type of underlying process. The methodology in this paper is able to solve each of these processes, i.e. this paper shows how to obtain the optimal investment rule for a given upward and downward trend. This makes it possible to solve optimal investment problems for, e.g., mean-reverting processes.

In particular, in our base model, we look at an application where the state process reflects the number of consumers, so that the profitability of the project is stochastic and depends on the number of consumers. We are studying a market with network effect for consumers. This means that the growth of consumers is positively related to the current number of consumers. At the same time, consumers can independently decide to stop purchasing (temporarily). As a result of these underlying incentives, the number of consumers is a mean-reverting process. Even though one can potentially choose any form of underlying incentives, this paper explicitely determines the optimal investment rule.

An example of such an application is a product that requires a subscription as, e.g., memberships for a fitness center, a sports club or for a video on demand (VOD) company (Netflix, HBO, Time Warner Cable and others). Here, a larger number of members has a positive effect on (the quality of) the supply. One can think of, e.g., better and more facilities when considering a fitness center or a better and broader supply of series and movies for the case of a VOD company. In this way there is a network effect for consumers. Even companies offering social media fit the model. More members increases the value of the social media, while the companies profit from advertisement and the sale of data.

Another example of such a market is the market for smart products, as e.g. navigation systems, where (technology) products become more valuable when they are more widely adopted so that the rate at which new consumers arive is increasing in the current number of consumers. A third example entails service systems where factors like word-of-mouth induce an acceleration of the number of consumers. Here, for instance, consumer loyalty plays an important role (see, e.g., Liu and Zhou (2008)). A fourth example where our model could be applied to, is the repeated purchase behavior of consumers under influence of a social network. From the studies on homophily[1] (see, e.g., McPherson *et al.* (2001)) it follows that purchase decisions are inter-dependent.

Another novel feature of our model is that jumps are assumed to be discrete. Although economically very intuitive, this is not a common assumption in the real options literature. This results in a discontinuous process. This paper shows that this does not interfere with our proposed solution concept.

> Section with contributions

This paper is organized as follows. After describing this paper's set-up in Section 2, Section 3 analyzes the process characteristics of the baseline model. Then, Section 4 studies the value of investment and Section 5 looks at the investment problem. The latter section also studies the feasibility of investment and does some robustness checks. This part of the paper is followed by Section 6, where the baseline model is generalized.

[1]Theory that people connect themselves or make relationships with other people with comparable characteristics.

Various other types of distributions are included here so that a general investment rule is obtained. Section 7 concludes the paper.

2 Model description

This model considers the investment decision of a rational, value maximizing firm on a market where the underlying market uncertainty is assumed to follow a continuous time Markov Chain with state dependent transition rates. More particularly, it is a birth-death process, where in our baseline model, we assume a finite number of states. The interarrival time of new consumers is assumed to be exponentially distributed with a state dependent rate, as is the time between two 'deaths'. At investment the firm starts up production to serve its consumers.

In our baseline model, we consider a market where each state of the chain represents the number of consumers. Then, the profitability of the considered product depends on consumer behavior. In this model $N \in \mathbb{N}$ agents have the option to purchase the product. This means that, at each moment of time, the group of N agents can be split into two groups: agents that are currently purchasing, i.e. consumers, and agents that represent potential consumers. We will refer to the former group as *active* costumers and to the latter as *inactive*. For inactive consumers, the willingness to start purchasing the product depends on the number of active consumers. For each active consumer the rate to switch between inactive to active goes up with η. Define by $n_t \in \mathcal{N} = \{1, \ldots, N\}$ the amount of consumers at time t buying the product. Then the time each of the $N - n_t$ inactive consumers switches to active is exponentially distributed with rate ηn_t. Notice that in this way state $n = 0$ becomes an absorbing state[2]. This implies that the market is dead when this state is reached. At the same time, active consumers are assumed to switch back to inactive with rate λ. This results in a path $n = (n_t)_{t \geq 0}$, denoting at each point in time, the number of active consumers, i.e. the number of consumers buying the product. The next section studies the process n_t in greater detail[3].

The firm receives net profits for each served active consumer. Analogue to standard models (see, e.g., Dixit and Pindyck (1994)), we assume that the firm is a price taker.[4] To serve consumers the firm needs to undertake an one-off investment at a sunk cost $I \in \mathbb{R}$.[5] After investment, the firm receives pn_t at each

[2]Section 5.5 studies the effect of this assumption and also studies this model without absorbing states.

[3]This work is closely related to and inspired by epidemiology models where a distinction is made between suspectible (S) and infected (I) agents. This paper's set-up is comparable to SIS models where infected agents become susceptible again after curing, as opposed to SIR models where a cured agent becomes immune and is called recovered (R). The main difference is that in these models, the development of the infected population is studied and one is interested in the process as such. Contrarily, our focus is on the investment of the firm where the underlying process only plays a role in the pursuit to find the optimal investment moment. This work is also related to queueing systems, where our baseline model could be classified as an $M_n/M/\infty/N$ queueing system. Here, as in the relation to epidemiology models, we do not only study the process itself, but aim to apply the process in an investment under uncertainty setting. Generally speaking, the process could be classified as a $G_n/H_n/\infty/N$ queueing system.

[4]FUTURE RESEARCH SECTION

[5]In Section 5.4 we consider alternative cost structures.

time t where p is the unit price per product. To evaluate the firm's option to undertake investment we make use of standard optimization techniques as applied in the real options literature. Here, the moment of investment relates to the value of the state process n. The firm then undertakes investment at the threshold level $n^* \in \mathcal{N}$ such that waiting for the process to go up to n^*+1 and then evaluate the option again does not yield more value than immediate investment. Here we apply Bellman's Principle of Optimality. Analogue to standard real option problems the optimization problem is solved by applying the Bellman equation and then solve the problem backwards.

3 Distribution of the process

This section aims to more deeply analyze the behavior and the distribution of the process n_t. By n_t the number of active consumers at time t is defined. Let $h > 0$ be small. Then the number of arrivals in the period $[t, t+h]$ is Poisson distributed, or, the time until the next arrival is exponentially distributed with rate $\eta n_t(N - n_t)$. As a result, at time $t+h$ there is one additional active consumer with probability

$$\eta n_t(N-n_t)he^{-\eta n_t(N-n_t)h} = \eta n_t(N-n_t)h + \mathcal{O}(h) =: p_u(n_t)h + \mathcal{O}(h),$$

where $\mathcal{O}(h)$ is defined such that $\frac{\mathcal{O}(h)}{h} \to 0$ as $h \to 0$. One can show that the probability of two jumps is of the magnitude of $\mathcal{O}(h)$ so that, for small h, this probability converges to zero. In a similar fashion, one consumers drops out with probability

$$\lambda n_t he^{-\lambda n_t h} = \lambda n_t h + \mathcal{O}(h) = p_d(n_t)h + \mathcal{O}(h).$$

This leads to the following characterization of the distribution of n_{t+h}, i.e. the process after a time period of h units, for a given value of n_t

$$n_{t+h} = \begin{cases} n_t + 1 & \text{with probability } p_u(n_t)h + \mathcal{O}(h) \\ n_t & \text{with probability } 1 - p_u(n_t)h - p_d(n_t)h + \mathcal{O}(h) \\ n_t - 1 & \text{with probability } p_d(n_t)h + \mathcal{O}(h) \\ otherwise & \text{with probability } \mathcal{O}(h). \end{cases}$$

This means that

$$\mathbb{E}[\Delta n_t | n_t] = (\eta(N-n_t)n_t - \lambda n_t)\Delta t + \mathbb{E}_t[\mathcal{O}(\Delta t)].$$

Taking limits leads to

$$\frac{\mathbb{E}_t[\mathrm{d}n]}{n} = -\eta\,(n - \bar{n})\,\mathrm{d}t,$$

with $\bar{n} = N - \frac{\lambda}{\eta}$. This shows that the process is mean reverting around $\bar{n}$ with mean reversion rate ηn. When $n > \bar{n}$ the process is expected to go down, which is caused by a higher downward rate relative to the upward rate. A larger value of λ decreased the mean as a result of a shorter expected time that consumers switch from active to inactive. Conversely, an increase in η would have a positive effect on the level of $\bar{n}$. This mean depends not only on the values of λ and η but also on the value of N, the total amount of consumers. Rather, the mean $\bar{n}$ is defined by the average number of consumers not purchasing the product $\theta = \frac{\lambda}{\eta}$. This comes as a result of the cap imposed on the number of consumers. This drives the parabolic nature of the upward rate balancing out the upward and downward movement of the process, the former being related to $n = N$.

More technically, one should write the diffusion process as

$$\mathrm{d}n_t = Y(n_t)\mathrm{d}\mathfrak{J}(n_t),$$

where

$$\mathrm{d}\mathfrak{J}(n_t) = \begin{cases} 1 & \text{with probability } (p_u(n_t) + p_d(n_t))\mathrm{d}t, \\ 0 & \text{with probability } 1 - (p_u(n_t) + p_d(n_t))\mathrm{d}t. \end{cases}$$

and

$$Y(n_t) = \begin{cases} 1 & \text{with probability } \frac{p_u(n_t)}{p_u(n_t)+p_d(n_t)}, \\ -1 & \text{with probability } \frac{p_d(n_t)}{p_u(n_t)+p_d(n_t)}. \end{cases}$$

One can rewrite this process $\mathrm{d}n(t)$ as a martingale $\mathrm{d}\tilde{n}(t)$, the so called compensated Poisson process,

$$\begin{aligned} \mathrm{d}\tilde{n}(t) &= -(p_u + p_d)\mathbb{E}[Y]\mathrm{d}t + Y(t)\mathrm{d}\mathfrak{J}(t) \\ &= -(p_u - p_d)\mathrm{d}t + Y(t)\mathrm{d}\mathfrak{J}(t) \\ &= -\eta n(N - \theta - n)\mathrm{d}t + Y(t)\mathrm{d}\mathfrak{J}(t). \end{aligned}$$

Again, this shows that the process $\mathrm{d}n$ is mean reverting with mean $\bar{n} = N - \theta$.

Standard real option analyses often assume a geometric Brownian motion,

$$\frac{\mathrm{d}x}{x} = \mu\mathrm{d}t + \sigma\mathrm{d}z,$$

where z_t is a Wiener process. This process differs in various ways from the process derived above. The most obvious difference between the two is the fact that this process has discrete jumps, whereas a geometric Brownian motion is a continuous function. Apart from that, more subtly, one can clearly see that the jump rates continuously change, depending on the state $n_t \in \mathcal{N}$. This is not the case for the Brownian motion, where $\log(x)$ is in no way dependent on the actual state x_t. Noise, and therefore uncertainty, in the standard model is exogenously inserted, where in this paper the fluctuations are endogenously determined.

> Make different section with comparison with Bm-s at the end of chapter

3.1 Additional analyses

The only absorbing state in this model is the state $n = 0$. Therefore not necessarily all states have been visited before the process dies out. The first proposition determines the probability a certain state will be visited in a finite amount amount of time. Since all states communicate it is implicit that with probability one, at some point, the process will hit zero. The second proposition formally proves this.

Proposition 1 *Let $P_i(n) = \mathbb{P}[$reaching state n at some $t < \infty$ starting with $i = n(0) < n$ active consumers$]$, then*

$$P_i(n) = \frac{A_i}{A_n} \ \forall \ i = 1, ..., n$$

where

$$A_i = \sum_{j=1}^{i} \left(\frac{\lambda}{\eta}\right)^{j-1} \frac{(N-j)!}{(N-1)!}.$$

Since $A_1 = 1$ we see that the probability of reaching state n starting with one consumers equals $1/A_n$. One can easily check that A_n is an increasing function. Interestingly, A_n increases concavely for $n - 1 < N - \theta$ and convexly for the opposite. This is in line with the finding that the process n_t is mean reverting around $N - \theta$.

Example 1 *Let there be $N = 4$ consumers, define $\theta = \frac{\lambda}{\eta}$. The probability that, at some point, all consumers buy the product equals*

$$P_1 = \left[\sum_{j=1}^{4} \left(\frac{\lambda}{\eta}\right)^{j-1} \frac{(N-j)!}{(N-1)!}\right]^{-1} = \left[1 + \frac{\lambda}{\eta}\frac{2!}{3!} + \left(\frac{\lambda}{\eta}\right)^2 \frac{1!}{3!} + \left(\frac{\lambda}{\eta}\right)^3 \frac{0!}{3!}\right]^{-1} = \frac{6}{6 + 2\theta + \theta^2 + \theta^3}$$

$$P_2 = \left[1 + \theta\frac{2!}{3!}\right] P_1 = P_1 + \frac{2\theta}{\eta + 4\lambda + 2\lambda^2 + 2\lambda^3} = \frac{6 + 2\theta}{6 + 2\theta + \theta^2 + \theta^3}$$

$$P_3 = \left[1 + \theta\frac{2!}{3!} + \theta^2\frac{1!}{3!}\right] P_1 = \frac{6 + 2\theta + \theta^2}{6 + 2\theta + \theta^2 + \theta^3}.$$

If $\lambda = \eta$ then $P_1 = \frac{3}{5}$, $P_2 = \frac{4}{5}$ and $P_3 = \frac{9}{10}$. However, for $\theta = 2$ these probabilities become $P_1 = \frac{3}{11}$, $P_2 = \frac{5}{11}$ and $P_3 = \frac{7}{11}$, while $P_1 = \frac{48}{59}$, $P_2 = \frac{56}{59}$ and $P_3 = \frac{58}{59}$ for $\theta = \frac{1}{2}$.

Notice that in this case the probability that one will ever reach one more consumer equals

$$P_{i,i+1} = \frac{A_i}{A_{i+1}} = 1 - \left(\frac{\lambda}{\eta}\right)^i \frac{(N-i-1)!}{(N-1)!}\frac{1}{A_{i+1}}.$$

Proposition 2 *Let $Q_i = \mathbb{P}[$reaching state 0 at some $t < \infty$ starting with $i = n(0)$ consumers$]$, then*

$$Q_i = 1 \ \forall \ i = 1, ..., N.$$

3.2 Noise of the process

After deriving Δn_t, one can, in a similar fashion, look at $\mathbb{E}[(\Delta n)^2]$. It follows that,

$$(\Delta n_t)^2 = \begin{cases} 1 & \text{with probability } p_u(n_t)\Delta t + \mathcal{O}(\Delta t) \\ 1 & \text{with probability } p_d(n_t)\Delta t + \mathcal{O}(\Delta t) \\ 0 & \text{with probability } 1 - p_u(n_t)\Delta t - p_d(n(t))\Delta t + \mathcal{O}(\Delta t) \\ \textit{otherwise} & \text{with probability } \mathcal{O}(\Delta t). \end{cases}$$

leading to, in expectation,

$$\mathbb{E}[(\Delta n_t)^2|n_t] = (\eta(N - n_t)n_t + \lambda n_t)\Delta t + \mathbb{E}[\mathcal{O}(\Delta t)].$$

Taking limits leads to

$$\frac{\text{Var}_t[\text{d}n]}{n} = \frac{\mathbb{E}_t[(\text{d}n)^2]}{n} = \eta(N + \theta - n)\text{d}t.$$

Here, since $N + \theta > n$ for all n, the noise increases for larger values of η, λ, and N. The rates at which consumers change their purchase decisions are going up with λ and η, indicating that the system faces jumps more rapidly when these rates go up. When η goes up, the process more quickly reaches the mean, if not having reached state $n = 0$ in the meanwhile. Ultimately, this leads to a higher probability that process will not die out anywhere soon. Conversely, λ has a comparable effect on $\text{Var}_t[\text{d}n]$ as η, while having a different effect on the investment problem and the evolution of n_t.

Notice that the concept of noise and uncertainty is different in this set-up than in the case of a Brownian motion. In the latter case, more noise implies more uncertainty and therefore risk, which is generally perceived as undesirable. Here, more noise does not (necessarily) mean more uncertainty. Instead, a larger value of η increases the mean and makes that the process more rapidly converges to its mean, which reduces the risk. A larger value of N not only increases variance but also increases the mean and the mean reversion rate. Therefore, since the process generally converges to the mean, a larger spread arises explaining the larger variance. Moreover, if N increases, p_u goes up, but p_d remains unaffected. A larger value of λ, though, is undesirable for three reasons. First, the mean of the process decreases, which decreases profits. Second, the time above the mean decreases since the downward rate is larger. Thirdly, the risk increases that the process dies out. Hence, the relation between risk and noise is different for this set-up.

Notice that at the mean $n = \bar{n}$ the noise becomes

$$\frac{\text{Var}_t[\text{d}n]}{\bar{n}} = 2\lambda\text{d}t.$$

4 Expected value at investment

In the previous section, the distribution of n_t is determined, allowing one to calculate the firm's value of investment. When the firm undertakes investment it receives a positive cash flow stream until the process dies out. With p denoting the net profit for each consumer, the firm gains, at each point in time, a value of pn_t. The firm obtains $V(n_t) - I$, at investment, denoting the expected discounted cash inflow stream from time t onwards corrected for the investment outlay, where, at time t, $V(n_t)$ reflects the expected accumulated future cash-inflow stream,

$$V(n_t) = \mathbb{E}\left[\int_{s=t}^{\infty} p\ n_s e^{-rs}\text{d}s \mid \mathcal{F}(t)\right]. \tag{1}$$

Here, r is the discount rate and $\mathcal{F}$ is the filtration with observations of the process n up till time t. Then the following proposition shows that the value $V(n_t)$ is recursively defined.

Proposition 3 *Let $V(n)$ be the value function as defined in (1) when the underlying state process is at state n exactly. Then,*

$$V(n) = \frac{np}{n\lambda + (N-n)n\eta + r} + \frac{(N-n)n\eta}{n\lambda + (N-n)n\eta + r} V(n+1) + \frac{n\lambda}{n\lambda + (N-n)n\eta + r} V(n-1). \tag{2}$$

Notice that the boundary conditions for $n = 0$ and $n = N$, i.e. $V(0) = 0$ and $p_u(N) = 0$, are incorporated. In this recurrence relation the value at n is defined as a function of $V(n+1)$ and $V(n-1)$, denoting the discounted expected future value of the chain after the next transition, plus a term reflecting the intermediate profit while being in state n.

The recurrence relation in (2) can be solved using the tridiagonal matrix algorithm, or sometimes called the Thomas algorithm after Thomas (1949)[6]. This leads to the following proposition.

Proposition 4 *Define*

$$\gamma_{n+1} = \frac{\eta n(N-n)}{r + \eta n(N-n) + \lambda n(1-\gamma_n)}, \tag{3}$$

$$\beta_{n+1} = \frac{pn + \lambda n \beta_n}{r + \eta n(N-n) + \lambda n(1-\gamma_n)}, \tag{4}$$

with $\gamma_1 = 0$ and $\beta_1 = 0$. Then

$$V(n) = \gamma_{n+1} V(n+1) + \beta_{n+1} = \frac{\eta n(N-n)V(n+1) + pn + \lambda n \beta_n}{r + \eta n(N-n) + \lambda n(1-\gamma_n)}$$

solves (2), with

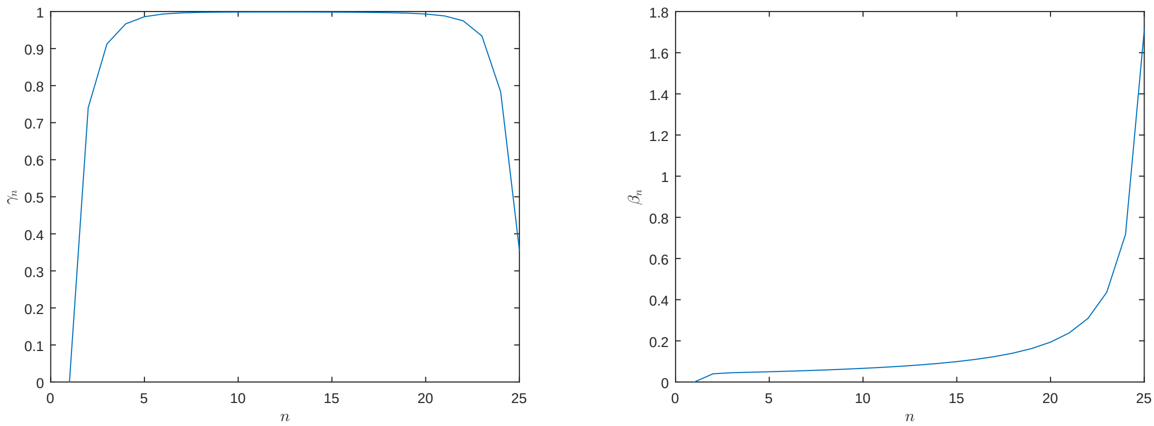
$$V(N) = \frac{pN + \lambda N \beta_N}{r + \lambda N(1-\gamma_N)}.$$

Here, the function γ_{n+1} can be interpreted as a stochastic discount factor for the period until the chain reaches the state with one additional consumer. From (3) and (4) one can conclude, by induction, that $0 \leq \gamma_{n+1} < 1$ and $\beta_{n+1} > 0$ for all $n \geq 1$.[7] The accumulated profits during the time it takes for the process to reach state $n+1$ from state n, i.e. the intermediate profit, is then captured by the function β_{n+1}. This is either the profit gained when only one jump is required to reach $n+1$ reflected by the pn term or when more jumps are required in case the process goes down first. As β_{n+1} reflects the profits in the time between the process goes from n to $n+1$, one can conclude that $\lambda n \beta_n$ accounts for the profits obtained when the process does not directly reach $n+1$, but first visits state $n-1$.

This explains the shape of the functions γ_{n+1} and β_{n+1}, as shown in Figure 1. Around the mean, for different values of n, the value function $V(n)$ stays relatively the same, for the process is expected to remain around its current value, which results in a discount factor close to 1 and small values of β_{n+1}. For large values of n the process is expected to go down so that, in expectation, the period until reaching one additional consumer is long. This results in a small value of γ_{n+1}. Since it is expected to take a while before one reaches one additional consumer, the additional profits are much larger for $n > \bar{n}$, as Figure 1b shows.

[6]See Appendix B for more about this algorithm.

[7]Although γ_{n+1} always seems to be strictly positive, is equals zero for $n = N$.



(a) The function γ_n.

(b) The function β_n.

Figure 1: Examples of the functions γ_n and β_n, for different values of $n = 1, \ldots, N$.

At the same time, for values close to $n = 0$, the probability that the process dies out is large so that reaching one additional consumer is not certain. Therefore, the discount factor is closer to zero[8]. The function β_n is linear in p and therefore, so is $V(n)$. The unit price p can therefore be seen as a scaling parameter.

Lemma 1 *Define*

$$\bar{\gamma}(n) = \frac{r + \eta n(N - n + \theta) - \sqrt{r^2 + 2r\eta n(N - n + \theta) + (\eta n)^2(\bar{n} - n)^2}}{2\lambda n}.$$

Then $\bar{\gamma}(n) \in [0, 1)$*, moreover,* γ_n *is an increasing function, i.e.* $\gamma_{n+1} > \gamma_n$*, if* $\gamma_n < \bar{\gamma}(n)$ *and decreasing, i.e.* $\gamma_{n+1} < \gamma_n$*, if* $\gamma_n > \bar{\gamma}(n)$*.*

One can check that $\bar{\gamma}(n)$ is continuous and $\bar{\gamma}(N) = 0$. It follows that $\gamma_n > \bar{\gamma}(n)$ for sufficiently large n. This implies that $\bar{\gamma} \to 0$ as $n \to N$ and so $\gamma \to 0$ as $n \to N$. At $n = \bar{n}$ this expression can be reduced to

$$\bar{\gamma}(n) = \frac{r}{2\lambda\bar{n}} + 1 - \sqrt{\frac{r}{2\lambda\bar{n}}\left(\frac{r}{2\lambda\bar{n}} + 1\right)}.$$

The function $V(n)$ looks as shown in Figure 2. From (1) it follows that $V(n)$ is increasing in n. Most remarkably, this is a concave function, as opposed to general real option models with a geometric Brownian motion. This is mostly explained by the concave nature of the drift term of the underlying stochastic process,

$$-\eta n(n - \bar{n}),$$

which is in turn the relative value of the upward and downward trend of the process. This means that the added value of one additional consumers decreases in n. For values below $\bar{n}$ this is explained by the fact that for larger n one is closer to the mean $\bar{n}$ so that the overall difference in function value is smaller. Afterall, it

[8]In Section 5.5 it is shown that this effect disappears when there is no absorbing state.

is expected that the process converges to the mean. For values above $\bar{n}$, the process is expected to go down so that one additional consumer only generates a temporary additional stream of profits. At the same time, since the variance goes up with n, it also means that the process moves quicker for larger values of n, so that the intermediate period between state is smaller[9]. This drives the concave nature of the function.

5 Investment problem

In this section the optimal investment rule is determined and studied. The firm faces the following optimization problem,

$$\sup_{\tau^* \geq 0} \left\{ V(n_{\tau^*}) - I \right\},$$

where τ^* is the investment moment. The investment moment τ^* is then the first hitting time of the process reaching n_{τ^*}. First, the value of waiting is determined with, consequently, the optimal trigger $n^* = n_{\tau^*}$. Then, the trigger is studied for different parameter values, the feasibility of investment is determined and the end of this section looks at the effect of the absorbing state $n = 0$.

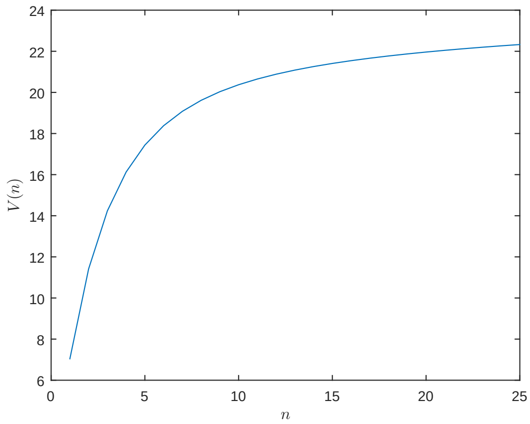


Figure 2: Value at investment $V(n)$ for different values of n.

[9]Technically, the variance is only increasing in n for $n < \frac{1}{2}(N + \theta)$. However, for values of n close to N, the downward trend is sufficiently larger relative to the upward trend to outweigh the decrease in variance.

5.1 Value of waiting

Denote by $F(n)$ the value of waiting, this is, the value when currently being at state n_t one invests at some later point in the future. The option value $F(n)$ satisfies the Bellman equation given by

$$\begin{cases} (\eta n(N-n)+\lambda n+r)F(n)=\eta n(N-n)F(n+1)+\lambda nF(n-1) & \text{for } n<n^*-1, \\ (\eta n(N-n)+\lambda n+r)F(n)=\eta n(N-n)(V(n^*)-I)+\lambda nF(n-1) & \text{for } n=n^*-1, \\ F(n)=V(n)-I & \text{for } n>n^*-1, \end{cases} \tag{5}$$

where $n^* \leq N$ reflects the minimum amount of active costumers required for investment. This means that for values of $n < n^*$ waiting yields a larger value than investment, equivalent to the *continuation region.* The *stopping region* is then defined for all values of n such that $n \geq n^*$.

Proposition 5 *Let $F(n)$ be defined as the value the firm faces before investment, $n < n^*$. Then (5) is solved by*

$$F(n) = \frac{\eta n(N-n)}{r+\eta n(N-n)+\lambda n(1-\gamma_n)}F(n+1).$$

with $F(n^*) = V(n^*) - I$.

Notice that, since $\gamma_n \in [0,1)$, $F(n)$ is an increasing function. By induction, one can conclude that for $n < n^*$, $F(n)$ can be written as

$$F(n) = (V(n^*)-I)\prod_{k=n}^{n^*-1}\frac{\eta k(N-k)}{r+\eta k(N-k)+\lambda k(1-\gamma_k)}.$$

This means that the product on the right-hand side should be seen as the stochastic discount factor,

$$\mathbb{E}\left[e^{-rT_n(n^*)}\right] = \prod_{k=n}^{n^*-1}\frac{\eta k(N-k)}{r+\eta k(N-k)+\lambda k(1-\gamma_k)},$$

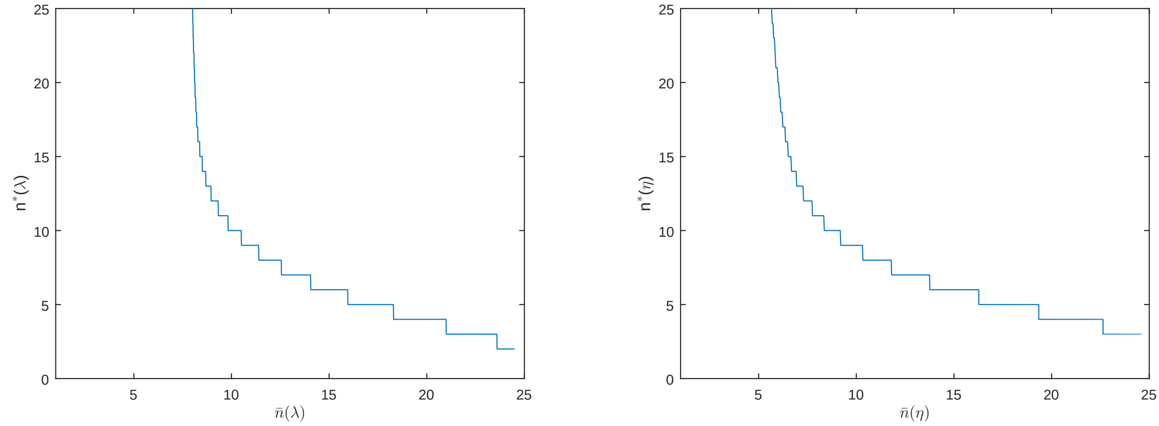
where $T_{n(0)}(k)$ is the first hitting time of state k at time $t = 0$. Here, each term

$$\frac{\eta k(N-k)}{r+\eta k(N-k)+\lambda k(1-\gamma_k)}$$

represents the discounting over the time between k active consumers and $k+1$ active consumers. The fraction takes the form of a discrete discount factor where in the numerator the rate of going up is present. The denominator consist of three terms: the discount rate r, the rate at which the process goes up and a term including the rate at which the process goes down. The stochastic discount factor entails the time until the process hits one more consumer. This either happens at the first next event, with an upward jump, or requires more steps when the process first goes down. In that respect, γ_k accounts for these events.

5.2 Investment threshold

The following proposition determines the optimal moment of investment. Here, we make use of the fact that at $n = n^* - 1$, firms find it optimal to delay their investment, while this is no longer true for $n = n^*$.



(a) Threshold for different values of $\lambda \in (0.5, 25)$.

(b) Threshold for different values of $\eta \in (0.4, 25)$.

Figure 3: Threshold n^* as a function of $\bar{n}$, with $K = 25$, $\lambda = 10$, $\eta = 1$ and $I = 10$.

Proposition 6 *Let n^* be the investment trigger, then*

$$n^* = \min\left\{ n \in \{1, \ldots, N\} \;\middle|\; I \leq \frac{pn + \lambda n \beta_n}{r + \lambda n(1 - \gamma_n)} \right\}.$$

Intuitively, investment is undertaken for the first time the fraction

$$\frac{pn + \lambda n \beta_n}{r + \lambda n(1 - \gamma_n)}$$

exceeds the level of I. In that situation, the expected future profit stream covers the investment outlay I required at the time of investment.

Figure 3 shows how the threshold looks as a function of the mean $\bar{n} = N - \frac{\lambda}{\eta}$. Figure 3a shows this for different values of λ and Figure 3b does this for different values of η. Here one can see that a change in $\bar{n}$ has different effects depending on whether the change was caused by a change in λ or η. Together shown in Figure 4 one can clearly see that a change in $\bar{n}$ as a result of a change in λ is much stronger. Clearly, a change from $\bar{n} = 15$ to $\bar{n} = 10$ requires a change in λ of $\Delta\lambda = +5$. However, to obtain a similar result η ought te be changed by merely $\Delta\eta = -\frac{1}{3}$. Clearly, such a relatively strong change in λ is much more noticeable. A larger λ decreases the expected path of $n(t)$ so that firms delays their investments. The opposite holds of course when the mean goes up so that firms accelerate more quickly their investment moment when this is caused by a change in λ compared to a change caused by η. The effect of N will be discussed later.

Alternative formulations of the threshold n^* include

$$n^* = \min\left\{ n \in \{1, \ldots, N\} \;\middle|\; I \leq \frac{\beta_{n+1}}{1 - \gamma_{n+1}} \right\}, \tag{6}$$

so that the stopping region includes all points n such that

$$I \leq \frac{\beta_{n+1}}{1 - \gamma_{n+1}}.$$

This implies that, under the assumption that $n(0) = 1$, investment is undertaken immediately if and only if[10]

$$I \leq \frac{p}{r+\lambda}.$$

> *equivalent to traditional models with* PP(λ)

However, the probability that the process dies immediately equals

$$\frac{\lambda}{(N-1)\eta+\lambda}.$$

It follows that, even though this probability can become very close to 1 by setting η sufficiently small, investment is still undertaken as long as I is small enough.

Example 2 *Let $N = 10$. Suppose $\lambda = 5$ and $\eta = 1$ so that n is mean reverting around $N - \theta = 5$. Furthermore, assume $p = 1$, $r = 0.05$ and $I = 9$. Table 1 shows all values. Recall that the stopping region includes all points n such that*

$$I \leq \frac{\beta_{n+1}}{1-\gamma_{n+1}}.$$

From this Table we see that investment is undertaken when the process hits $n = 7$, which would have been $n = 4$ in case of the NPV rule. Since the mean is quite low, the probability of the process dying out soon is quite substantial. Therefore, to cover the investment cost, a relatively long project durability should be expected in order for the project to be profitable. Therefore, one requires a threshold that is relatively large.

From the example one can infer that, for I being sufficiently large, it is possible that $n^* > \bar{n}$. In these cases, the investment cost is too large to be covered by the regular expected future profit stream. Instead,

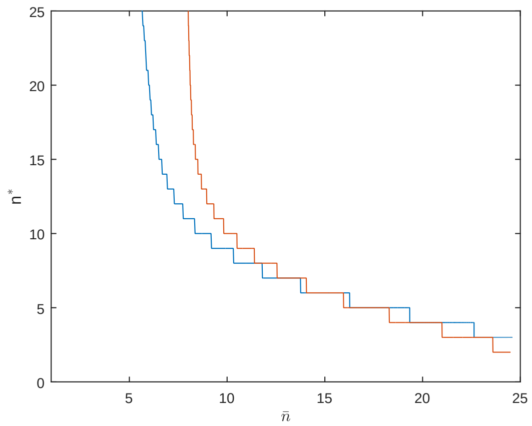


Figure 4: Threshold n^* as a function of $\bar{n}$, for different values of λ (orange) and η (blue), with $K = 25$, $\lambda = 10$, $\eta = 1$ and $I = 10$.

[10] As a result, when the inequality is satisfied, investment is immediately undertaken for any $n(0)$.

n	$V(n)$	$\frac{\beta(n+1)}{1-\gamma(n+1)}$
1	4.75	0.20
2	7.30	0.74
3	8.79	1.79
4	9.74	3.40
5	10.38	5.46
6	10.84	7.64
7	11.19	9.53
8	11.46	10.84
9	11.69	11.55
10	11.88	11.88

Table 1: Value function and decision fraction

one requires some additional profits, gained when the process takes values above the mean, to be able to pay for the investment.

To prove that n^* is uniquely determined, it is sufficient to show that the fraction $\frac{\beta_{n+1}}{1-\gamma_{n+1}}$ is increasing in n. Then, the firm waits in the continuation region

$$\mathcal{C} = \left\{ n \in \mathbb{N}_0,\ n \leq N \ :\ I > \frac{\beta_{n+1}}{1-\gamma_{n+1}} \right\}$$

and undertakes investment in the stopping region $\mathcal{S} := \mathbb{N}_0 \backslash \mathcal{C} \cap [0, N)$.

Proposition 7 *Let β_n and γ_n be defined as in , then,*

$$\frac{\beta_n}{1-\gamma_n} < \frac{\beta_{n+1}}{1-\gamma_{n+1}}$$

for all n.

The investment rule fraction $\frac{\beta_{n+1}}{1-\gamma_{n+1}}$ is shown in Figure 5. Characteristically the fraction follows $\frac{pn}{r}$ for values below $\bar{n}$, but then changes for values beyond $\bar{n}$. Lemma 5 in Appendix A shows that the fraction is always below $\frac{pn}{r}$. For values below $\bar{n}$, higher investment cost can be covered by additional consumers at the moment of investment. However, when the process reaches values above $\bar{n}$, the process is expected to return to its mean so that the additional value of one more consumer is only very small and only results in temporary short term revenues. This is not the case for $n < \bar{n}$ since then all future payoff streams are expected to be relatively higher. This explains the shape of the investment rule fraction.

Figure 6 shows how the option value $F(n)$ typically looks. Similar to the value function $V(n)$ this function takes a concave shape. The intuition behind this result is of a similar nature. Assume $n^* \leq \bar{n}$. Then, since the process has the tendency to go up, being closer to the threshold $n = n^*$ implies that one additional consumer does not yield relatively more value. Since the process is expected to converge to the mean, one

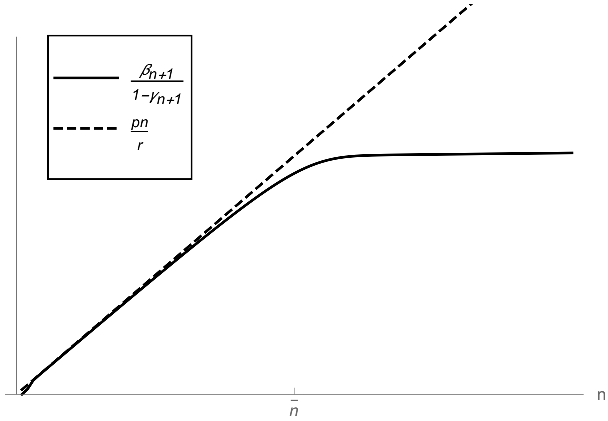


Figure 5: General shape of the fraction $\frac{\beta_{n+1}}{1-\gamma_{n+1}}$, for different values of $n = 1, \ldots, N$.

additional consumer only reduces the expected time until investment slightly. This brings up the question whether this is a general result and whether this also holds for $n^* > \bar{n}$. As we will show, it is not. Intuitively, for $n^* > \bar{n}$, getting closer to n^* for values of $n > \bar{n}$ yields a lot of value, since it was expected that the process goes down and this increase makes investment more probable. Generally speaking, one expects the function F to be concave for $n \leq \bar{n}$ and to be convex for $\bar{n} < n < n^*$. Let $G : \mathbb{N} \to \mathbb{R}$ be an increasing function. Then G is called *convex at* n if

$$G(n+1) - G(n) > G(n) - G(n-1).$$

Proposition 8 *Let $n \in \{2, 3, \ldots, n^* - 1\}$. Then the value function F is convex at n if and only if*

$$\frac{r + \lambda n(1-\gamma_n)}{r + \lambda(n-1)(1-\gamma_{n-1})} > \frac{\eta n(N-n)}{r + \eta(n-1)(N-n+1) + \lambda(n-1)(1-\gamma_{n-1})}.$$

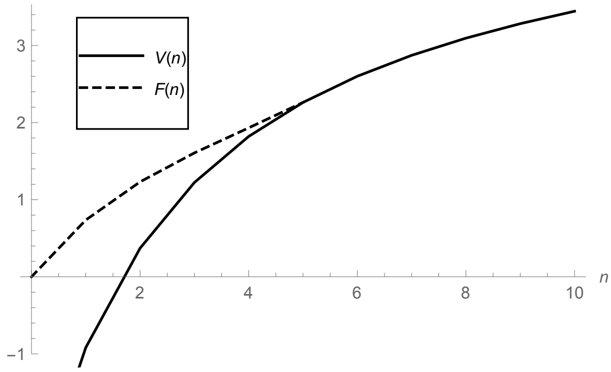


Figure 6: Value function $V(n)$ and value of waiting $F(n)$ for different values of n, with $N = 10$, $\lambda = 6$, $\eta = 1$, $n^* = 5$, and $I = 3$.

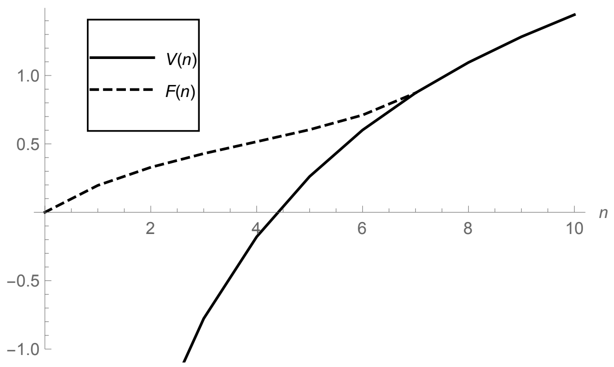


Figure 7: Value function $V(n)$ and value of waiting $F(n)$ for different values of n, with $N = 10$, $\lambda = 6$, $\eta = 1$, $n^* = 7$, and $I = 5$.

One can show that this condition is equivalent to

$$\gamma_{n+1} < \frac{1 - \gamma_{n+1}}{1 - \gamma_n}.$$

Since $\gamma_{n+1} \in [0, 1)$ one can easily see that this equation always holds if $\gamma_{n+1} \leq \gamma_n$, i.e. for when n is close to N. This is for cases where the investment threshold is large. The equation in the proposition indicates that for relatively large n this holds in general. The left hand-side is close to one, which makes that if the right hand-side is smaller than one, the equation is satisfied. Since the last term in the denominator is close to zero (for n not too close to N), we have that the denominator is definitely larger that the numerator if $n - 1 \geq \frac{1}{2}N$, for in that case $n(N - n)$ is decreasing in n.

Numerically one can check that this tipping point lies around $\bar{n}$ for $r = 0$. For $r > 0$ this point decreases so that for all $\bar{n} < n < n^*$ definitely the value of waiting behaves convexly.

5.2.1 Net present value

The following lemma relates the investment rule to the net present value.

Lemma 2 *Let β_n, γ_n and $V(n)$ be defined as in Proposition 3, then,*

$$\frac{\beta_{n+1}}{1 - \gamma_{n+1}} < V(n + 1)$$

for all n.

Investment is undertaken at $n = n^*$ for the smallest value of n such that

$$\frac{\beta_{n^*}}{1 - \gamma_{n^*+1}} \leq I.$$

However, according to Lemma 2, at $n = n^*$ it holds that $V(n^*+1) > I$ so that under the NPV rule investment would have been undertaken for some $n \leq n^*$. This shows that also under this paper's formulation investment of the investment problem, investment is undertaken for a larger trigger, compared to the NPV analysis.

5.2.2 Additional analyses

The following corollary shows us that the investment threshold converges to a constant when the number of consumers increases.

Proposition 9 *If $N \to \infty$, then*

$$n^* \to \frac{r}{p} I.$$

Larger investment cost, and similarly lower prices, increases the threshold since it requires more expected profits to cover the cost. When discounting is done under a higher rate expected profits are reduces. As a result one requires a higher number of active consumers to cover this. Notice that both λ and η play no role. Recall that the rate at which the process goes up equals $\eta n(N-n)$ and the rate at which the process goes down equals λn. This means that if N becomes very large, the rate the process goes up also becomes very large. As a result, λ is always dominated by N. Moreover, if N goes to infinity, then also $\bar{n}$ goes to infinity. Hence, when $N \to \infty$ one does not care about the probability that the process dies out anymore. Nevertheless, one does care about the relative costs.

Proposition 10 *Let T_{n_t} be the expected time until the process dies, given that there are currently n_t active consumers. Define*

$$\gamma_{n+1}^T = \frac{\eta n(N-n)}{\lambda n(1-\gamma_n^T) + \eta n(N-n)}$$
$$\beta_{n+1}^T = \frac{1 + \lambda n \beta_n^T}{\lambda n(1-\gamma_n^T) + \eta n(N-n)}$$

with $\gamma_1^T = \beta_1^T = 0$. Then the expected time until death equals

$$T_n = \frac{\eta n(N-n)T_{n+1} + 1 + \lambda n \beta_n^T}{\eta n(N-n) + \lambda n(1-\gamma_n^T)}$$

with

$$T_N = \frac{1 + \lambda N \beta_N^T}{\lambda N(1-\gamma_N^T)}.$$

5.2.3 Effect of parameters

Generally we see that firms delay their investment for larger values of λ, r and I, but accelerate for larger values of η, p and N. This implies that the threshold is increasing with θ. If the mean of the process n_t goes up, i.e. for larger N and η and smaller λ, the process is expected to stay longer around the mean before the process dies out. This means that, once a certain level is reached, investment is relatively safer for the same N when the mean increases. Therefore, the expected payoff increases and one can accelerate investment. Moreover, a larger η means that the process more quickly goes up, which makes investment more attrictive, decreasing the threshold. The opposite argument applies to λ. For larger r, one discounts more heavily so that future profits need to increase in order to make up for the investment cost which happens when investment is delayed. For a higher unit price p the same reasoning holds: it increases the profitability so that one invests in an earlier stage of the process. For I one obviously finds a positive relation with the investment threshold.

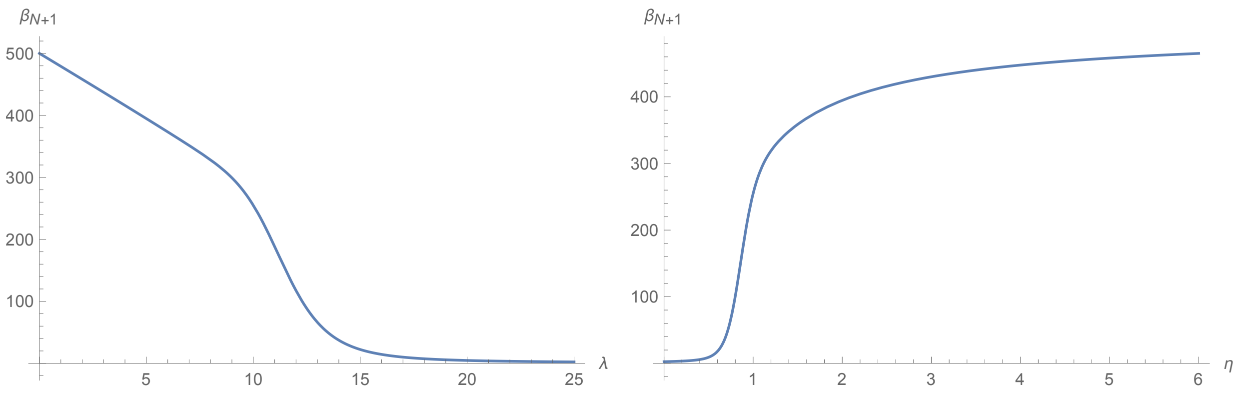
5.3 Feasibility of investment

As shown before, the investment trigger increases with I. That means that a larger value of I induces that investment is undertaken for a larger value of n. But since n is limited, $n \leq N$, this could mean that for sufficiently large values of I, as follows directly from equation (6), the investment cost is too large so that investment shall never be undertaken, this is, the investment in infeasible. After all, the fraction $\frac{\beta_{n+1}}{1-\gamma_{n+1}}$ does not depend on I. In this section we study this notion and, for a fixed level of the investment cost, how this is affected by the other parameter values.

It follows that investment is never undertaken if

$$I > V(N) = \beta_{N+1} = \frac{pN + \lambda N \beta_N}{r + \lambda N(1 - \gamma_N)}.$$

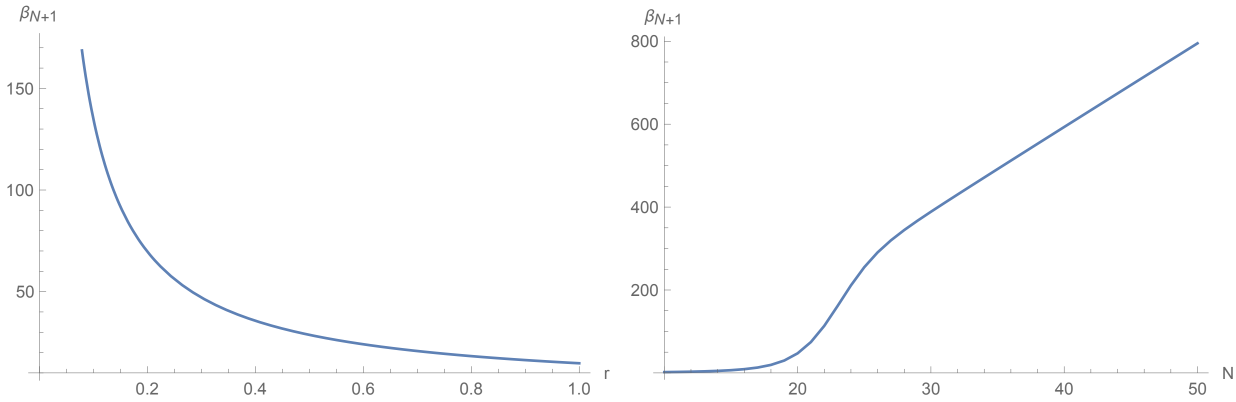
This means that for the existence of the investment trigger one must study the behavior of β_{N+1}. Assume $\eta = 1,\ \lambda = 10,\ N = 25,\ r = 0.05$ and $p = 1$. If λ increases the rate at which consumers turn from active to inactive increases so that not only the mean $\bar{n}$ decreases but also the rate at which the process goes down increases. Overall, profitability declines so that β_{N+1} also decreases, as illustrated by Figure 8a. The opposite is true for η: an increase boosts the rate at which consumer starts purchasing and it increases the mean $\bar{n}$. For extremely low values of η, the fraction $\theta = \frac{\lambda}{\eta}$ is blown up and the mean $\bar{n} = N - \theta$ becomes very small or even negative: profits are decimated. Contrarily, for large values of η the fraction θ goes to zero so that for large values of η an increase has little effect, for the mean had already converged to N. Discounting reduces the firm's profits, which has a clear effect on β_{N+1} as visualized in Figure 9a. Finally, an increase in N first makes the value of β_{N+1} increase concavely, but eventually a linear effect is observed. For small



(a) The function $V(N) = \beta_{N+1}$ for different values of λ.

(b) The function $V(N) = \beta_{N+1}$ for different values of η.

Figure 8: Effect on feasibility of the investment option under changes of λ and η.



(a) The function $V(N) = \beta_{N+1}$ for different values of r. (b) The function $V(N) = \beta_{N+1}$ for different values of N.

Figure 9: Effect on feasibility of the investment option under changes of r and N.

values of N the mean plays an important role, so that an additional consumer on the market also increases the mean by 1 making investment more interesting. However, since θ stays constant the effect becomes smaller the more consumers are present. Since only one additional consumer is expected to purchase the product a linear effect is realized.

Corollary 1 *If* $N < \frac{r}{p}I$*, then investment is never feasible.*

We saw that the investment trigger is decreasing in N, which means that if the lower bound to the threshold equals $\frac{r}{p}I$, see Proposition 9, then investment is only feasible for situations where $N \geq \frac{r}{p}I$.
$>$ *immediate investment if* $N > \bar{N}$?

Concludingly, for the investment to be feasible for any value of I, one requires, either, N to be sufficiently large, or, r to be sufficiently small.

5.4 Linear investment cost

In the baseline model it is assumed that the investment cost is fixed. However, when studying the effect of N, this might not be so reasonable as one can expect the investment cost to be much larger for larger groups of consumers. Therefore, in this section, an alternative cost structure is considered to account for these effects. In a set-up with N potential consumers, assume that the investment cost are linear in N, i.e.

$$I = cN,$$

for some $c \geq 0$. Then, we have the following result. For N large enough the threshold converges to a fraction of the number of consumers. More particularly,

$$\lim_{N\to\infty} \frac{n^*}{N} = \frac{cr}{p}.$$

In this case, thus, the investment threshold does not converges to a constant, but, contrarily, the percentage of active consumers at investment does. This means that for different pool sizes, the investment trigger is related to the relative group of potential consumers required to be active.

5.5 Absorbing state

In this final subsection we study the influence of the absorbing state $n = 0$ on the investment problem. In the baseline model, the upward rate is defined by

$$\eta(n) = \eta n(N - n),$$

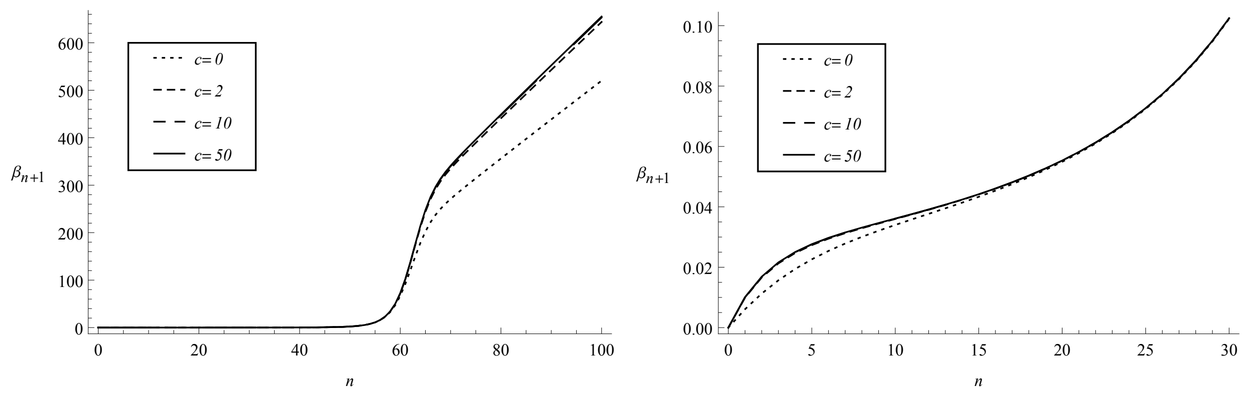
which implies that the process cannot go up in the states $n = 0$ and $n = N$. Since $\lambda(n) = \lambda n$, state $n = 0$ is an absorbing state. The question arises what happens if $\eta(n) > 0$ for $n = 0$, and how this changes the investment strategy of the firm. In this section we approach this by changing $\eta(n)$ in two different ways. First the situation is studied where, for some $c \geq 0$,

$$\eta(n) = \begin{cases} \eta n(N - n) & \text{if } n = 1, \dots, N, \\ c & \text{if } n = 0. \end{cases} \tag{7}$$

Then, a similar study is conducted for

$$\eta(n) = \eta(n + c)(N - n).$$

The functions γ_{n+1} and β_{n+1} directly influence the investment rule and the value function $V(n)$. Therefore, first these functions are studied. Intuitively, γ_{n+1} is the stochastic discount rate for the periode between n and $n + 1$. The difference between $V(n)$ and $\gamma_{n+1}V(n + 1)$ is then reflected by β_{n+1}, which can be interpreted for values of $n < \bar{n}$ as the intermediate profit between n and $n + 1$. As an example, we take $N = 100$,



(a) For all values of n. (b) For small values of n.

Figure 10: The function β_{n+1} for different values of $c \in \{0, 2, 10, 50\}$.

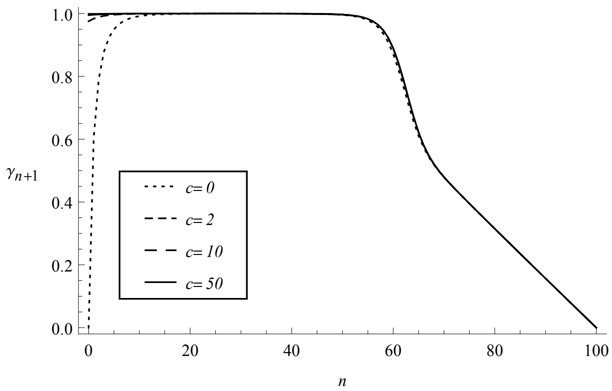
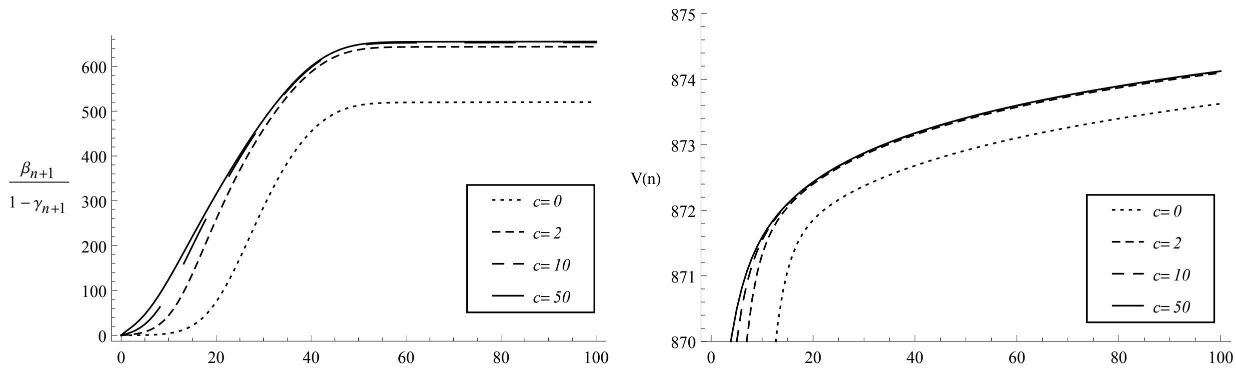


Figure 11: The function γ_{n+1} for different values of $c \in \{0, 2, 10, 50\}$.

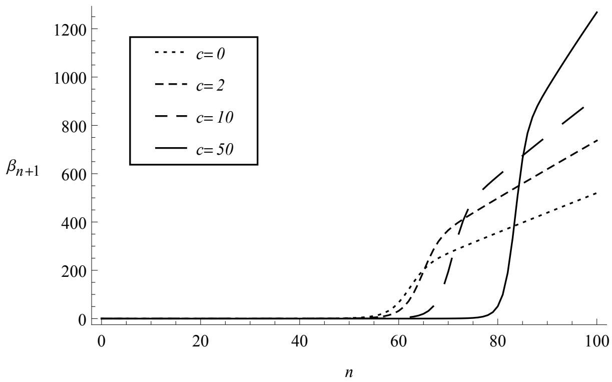
$\eta = 1$, $\lambda = 65$, $r = 0.05$ and $p = 1$. Figures 10 and 11 then display the functions β_{n+1} and γ_{n+1} respectively for different values of c. Most distinctively is the difference between $c = 0$ and $c = 2$. In the former case the state $n = 0$ is an absorbing state which means that once that state is reached, the process dies. However, for the latter case the process, although slowly, will eventually leave state $n = 0$. This creates value so that β_{n+1} goes up. Similarly, discounting is less strong, since the process is expected to leave $n = 0$ for positive values of c. It is clear from the figures that for the magnitude of c has little influence on the course of the functions, as long as its positive. For these examples the mean equals $\bar{n} = 35$. Around the mean, the functions are relatively flat so that for larger values of the mean the functions more or less are shifted to the right and the tails shrink. For γ_{n+1} this implies that the descend will take off for a larger value of n and for β_{n+1} this implies that the region with the large function values becomes smaller.



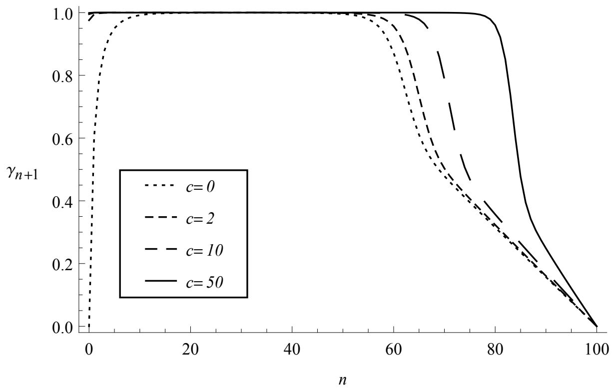
(a) The stopping and continuation region with $\lambda = 65$.

(b) The value function $V(n)$ with $\lambda = 55$.

Figure 12: The stopping/continuation region and value function $V(n)$ for $c \in \{0, 2, 10, 50\}$.



(a) β_{n+1} for all values of n.



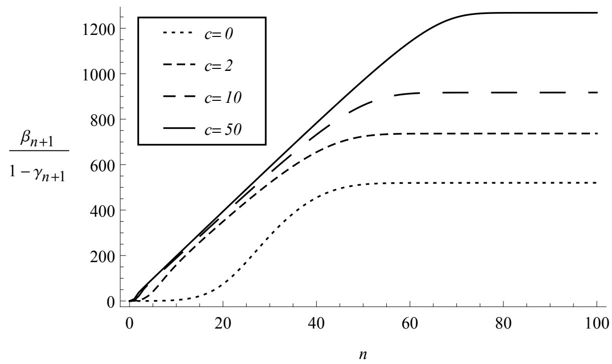
(b) γ_{n+1} for all values of n.

Figure 13: The functions β_{n+1} and γ_{n+1} for different values of $c \in \{0, 2, 10, 50\}$ with $\eta(n) = \eta(n+c)(N-n)$.

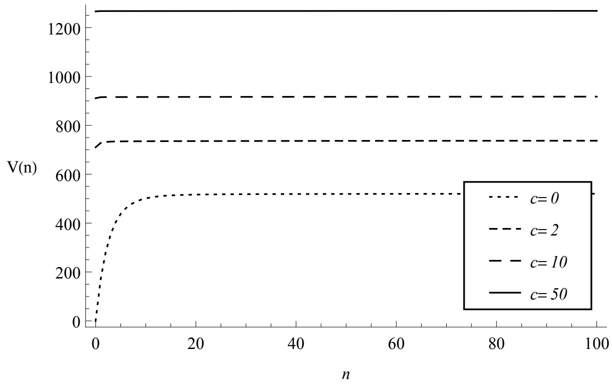
The stopping region was defined as all values of n such that

$$\frac{\beta_{n+1}}{1-\gamma_{n+1}} \geq I.$$

The fraction on the left-hand side is shown in Figure 12a. Figure 12b displays the value function $V(n)$. A similar pattern is observed here. A positive value of c shifts the functions upwards. For different positive values of c the functions are close. Again, this is for a relatively small mean. For larger means the difference between the functions become smaller, since the number of times the state $n = 0$ is reached is smaller so that the smaller profits obtained in that period are less prevalent. For a mean closer to zero, the difference between the functions becomes larger so that even between the functions associated with $c = 2$, $c = 10$ and $c = 50$ there is a relatively large difference.



(a) The stopping and continuation region with $\lambda = 65$.



(b) The value function $V(n)$ with $\lambda = 55$.

Figure 14: The stopping/continuation region and value function $V(n)$ with $\eta(n) = \eta(n+c)(N-n)$.

When considering

$$\eta(n) = \eta(n + c)(N - n),$$

one should notice that the greatest difference with (7) is that $\eta(n)$ changes for all n when changing c. This is then also reflected in the outcomes. Compared to (7), qualitatively all functions undergo the same transformation when increasing c. However, since $\eta(n)$ is affected for all n a second effect comes in, visible in Figures 13 and 14. Overall, one can conclude that the presence of an absorbing state does not change the outcomes qualitatively.

6 General form

In this section we analyze the general case where the rate at which the process, at state n, goes down equals $\lambda(n)$ and the rate the process goes up at state n equals $\eta(n)$. As an example, consider three forms of each rate. In the first case people's decisions to stop or start purchasing are not influenced by the decisions of other agents. In that case each of the $N - n$ inactive consumers decides to switch at rate λ leading to an overall rate of $\eta(n) = \eta(N - n)$ at which the process goes up. Similarly at rate $\lambda(n) = \lambda n$ the process goes down. Secondly, on the other hand, people's decisions could be based upon the current number of consuming agents, i.e. $\lambda(n) = \lambda n^2$ and $\eta(n) = \eta n(N - n)$. Alternatively, people are influenced by the number of people not purchasing the product which results in $\lambda(n) = \lambda n(N - n)$ and $\eta(n) = \eta(N - n)^2$. This leads to various combinations and, as a result, different types of processes. Table 2 shows for different combinations the trend of the resulting process n_t, i.e. $\frac{\mathbb{E}dn}{dt}$. This table also includes $\eta(n) = \eta n$ and $\lambda(n) = \lambda(N - n)$ to study, for instance, a birth-death process.

	λn	λn^2	$\lambda(N-n)$	$\lambda n(N-n)$
ηn	$n(\eta - \lambda)$	$\eta n(1 - \theta n)$	$(\lambda + \eta)(n - \frac{\lambda}{\lambda+\eta}N)$	$\lambda n(\frac{1}{\theta} - (N - n))$
$\eta(N-n)$	$\eta(N - n(1 + \theta))$	$\eta(N - n - \theta n^2)$	$(N - n)(\eta - \lambda)$	$\eta(N - n)(1 - \theta n)$
$\eta n(N-n)$	$\eta n(N - n - \theta)$	$\eta n(N - n(1 + \theta))$	$\eta(N - n)(n - \theta)$	$\eta n(N - n)(1 + \theta)$
$\eta(N-n)^2$	$\eta(N - n)^2 - \lambda n$	$\eta(N - n)^2 - \lambda n^2$	$\eta(N - n)(N - n - \theta)$	$\eta(N - n)(N - n(1 + \theta))$

Table 2: Trend of the process

In line with Proposition 1, define by $P_i(n)$ the probability of reaching state n when $n(0) = i$. Let $\theta(k) = \frac{\lambda(k)}{\eta(k)}$ with $\theta(0) = 1$, then for the general case A_i is defined as

$$A_i = \sum_{j=1}^{i} \prod_{k=0}^{j-1} \theta(k).$$

Example 3 *In the case of a birth-death process, i.e.* $\lambda(n) = \lambda n$ *and* $\eta(n) = \eta n$*, this probability is defined as* $P_i(n) = \frac{\theta^i - 1}{\theta^n - 1}$*. The same is obtained for any combination such that* $\theta(k) = \theta$*. For the combination*

$\bar{n}$	λn	λn^2	$\lambda(N-n)$	$\lambda n(N-n)$
ηn		$\frac{1}{\theta}$	$\frac{\lambda}{\lambda+\eta}N$ (MD)	$N - \frac{1}{\theta}$ (MD)
$\eta(N-n)$	$\frac{N}{1+\theta}$	$\frac{1}{2\theta}(\sqrt{1+4\theta N}-1)$		$\frac{1}{\theta}$
$\eta n(N-n)$	$N-\theta$	$\frac{N}{1+\theta}$	θ (MD)	
$\eta(N-n)^2$	$N+\frac{1}{2}\theta-\sqrt{\frac{1}{4}\theta^2+\theta N}$	$\bar{n} > N$	$N-\theta$	$\frac{N}{1+\theta}$

Table 3: Mean in case of a mean reverting process (MD in case of a mean diverting process).

of $\lambda(n) = \lambda(N-n)$ and $\eta(n) = \eta n(N-n)$ one can show that $P_i(n)$ is the cdf of a conditional Poisson distribution with expectation θ, i.e. $P_i(n) \sim F_{i\leq n}(i-1)$ where $F \sim POI(\theta)$, or

$$P_i(n) = F_{i\leq n}(i-1) = \frac{\sum_{k=0}^{i-1} \frac{\theta^k}{k!}e^{-\theta}}{\sum_{k=0}^{n-1} \frac{\theta^k}{k!}e^{-\theta}}.$$

Notice that for Proposition 1 it holds that $\theta(k) = \frac{\theta}{N-k}$.

Table 3 shows the resulting means $\bar{n}$ in case of a mean reverting process. MD stands for mean diversion. Here it is shown that if the incentives for both switching options for the consumers are affected by the same group of (in)active consumers, i.e. the combinations described above, the process is always mean reverting with mean $\bar{n} = \frac{N}{1+\theta} = \frac{\eta}{\eta+\lambda}N$. For asymmetric cases this is not the case. In, e.g., the basic model in this paper the process was mean reverting around $\bar{n} = N - \theta$. This distinction plays an important role when considering a large group of consumers. In the former, when mean reverting around $\frac{\eta}{\eta+\lambda}N$, the process converges in expectation to a fraction of the total amount of consumers in the market, whilst the in the latter case the fixed amount of consumers not purchasing the product is a fixed number. A third type of mean also assumes the mean to remain fixed for different values of N, but defines it as a fixed amount of active consumer as with, e.g., $\lambda(n) = \lambda n^2$ and $\eta(n) = \eta n$ with $\bar{n} = \theta^{-1}$.

A second way to distinguish the different cases is the following. In most of the combinations one can speak of a mean reverting process, this is, at every instance t the process is expected to move closer to the mean $\bar{n}$ in case of a jump in the interval $[0, \Delta t)$. In some cases the trend of the process is alway positive or negative, see, e.g., $\lambda(n) = \lambda n$ and $\eta(n) = \eta n$. Nevertheless, in 3 cases the process, in expectation, moves away from $\bar{n}$. We will call this *mean diversion*. This happens in, for example, the case where $\lambda(n) = \lambda(N-n)$ and $\eta(n) = \eta n$. Here the rate at which people switch from active to inactive solely depends on the size of the group of the currently inactive consumers and the opposite holds for $\eta(n)$. This implies that if one of the two respective groups is large it excellerates the speed at which the group grows, as opposed to the situation where the growth of the group only depends on the size of the other group imposing a mean revering effect. This *magnet effect* causes the process to either converge in expectation to $n = 0$ or $n = N$.

6.1 Investment problem

The value at investment is defined by the recurrence relation that, for the general case, be written as

$$V(n) = \frac{np}{\lambda(n)+\eta(n)+r} + \frac{\eta(n)}{\lambda(n)+\eta(n)+r}V(n+1) + \frac{\lambda(n)}{\lambda(n)+\eta(n)+r}V(n-1).$$

As in the base case this equation is solved by using the Tridiagonal Matrix Algorithm, which leads to

$$V(n) = \gamma_{n+1}V(n+1) + \beta_{n+1} = \frac{\eta(n)V(n+1) + pn + \lambda(n)\beta_n}{r+\eta(n)+\lambda(n)(1-\gamma_n)}$$

with

$$\begin{aligned}\gamma_{n+1} &= \frac{\eta(n)}{r+\eta(n)+\lambda(n)(1-\gamma_n)},\\ \beta_{n+1} &= \frac{pn+\lambda(n)\beta_n}{r+\eta(n)+\lambda(n)(1-\gamma_n)}.\end{aligned}$$

where $\gamma_1 = 0$, $V(N) = \beta_{N+1}$, and $\beta_1 = 0$. The investment rule remains the same, so that the continuation region consists of all points such that

$$I \leq \frac{pn+\lambda(n)\beta_n}{r+\lambda(n)(1-\gamma_n)} = \frac{\beta_{n+1}}{1-\gamma_{n+1}}.$$

Example 4 *Consider a pure birth process where $\lambda(n) = 0$ for all n. Then,*

$$V(n) = \frac{\eta(n)}{r+\eta(n)}V(n+1) + \frac{pn}{r+\eta(n)},$$

with $V(N) = \frac{p}{r}N$ and investment is undertaken as soon as $n+1 \geq \frac{r}{p}I$. Notice that the latter expression is the same as the limit in Proposition 9.

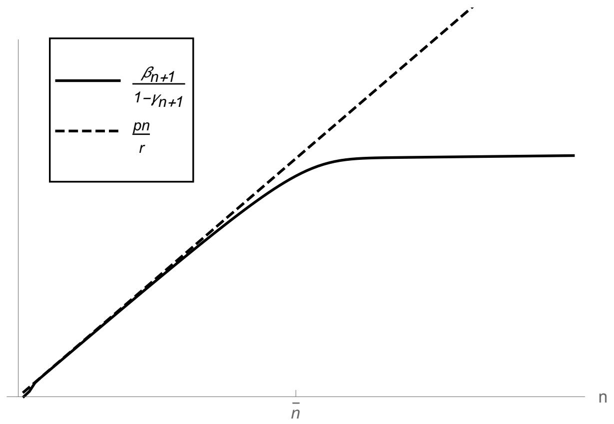
One can show that, for the case where $\eta(n) = \eta$ for all $0 < n < N$, one obtains,

$$V(n) = \begin{cases} \left(\frac{\eta}{r+\eta}\right)^{N-n} \frac{p}{r}N + p\sum_{k=1}^{N-n}\left(\frac{\eta}{r+\eta}\right)^{N-n-k}\frac{N-k}{r+\eta} & \text{if } 0 < n < N, \\ \frac{p}{r}N & \text{if } n = N, \\ 0 & \text{if } n = 0. \end{cases}$$

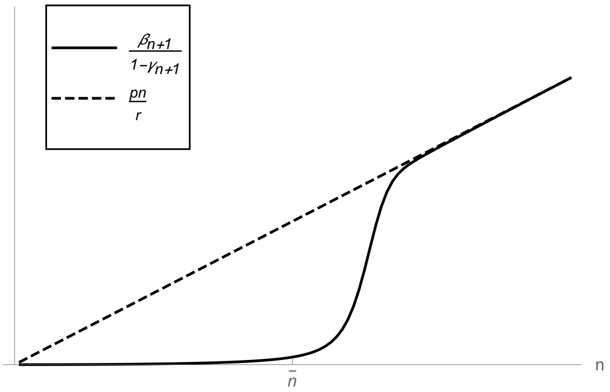
For the case of $\lambda(n) = \lambda(N-n)$ and $\eta(n) = \eta n(N-n)$ we see that for very large values of N the function γ_n converges to $\gamma_n = 1$ for sufficiently large n. The function β_n is non-monotonic, however, remains small relative to pn for large values of n so that $I \leq \frac{1}{r}(pn + \lambda(N-n)\beta_n) \to \frac{pn}{r}$. This implies that Proposition 9 still holds for sufficiently large I.

Although the different cases treated above imply different characteristics for each corresponding process, the investment problem is generally the same for all mean reverting processes and mean diverting processes. Figure 15a show how mean reverting processes generally can be translated into an investment rule. Here, the fraction

$$\frac{\beta_{n+1}}{1-\gamma_{n+1}}$$



(a) Mean reverting processes.



(b) Mean diverting processes.

Figure 15: General shape of the fraction $\frac{\beta_{n+1}}{1-\gamma_{n+1}}$, for different values of $n = 1, \ldots, N$.

is shown so that investment is undertaken when n is sufficiently large so that this fraction reaches the bound I for the first time. Here one can see that the function changes its shape for values of n larger than $\bar{n}$. Once the process has crossed the mean $\bar{n}$ it is expected to return to its mean so that one additional current consumer does not bring in much value for the project so that it would not cover much of extra expenses with respect to I. Differently, the value of waiting for an additional consumer is much less for $n > \bar{n}$ than for $n < \bar{n}$. Therefore the fraction does not increase as rapidly for large values of n as it does for values below $\bar{n}$. This explains the shape of the curve. For a mean diverting process this changes. Since the process is expected to converge to $n = 0$ for $n < \bar{n}$ the value of investment is relatively small and investment shall only be optimal for these values when I is very close to zero. However, for values above $\bar{n}$ the process is expected to increase at every instance. Since each jump is uncertain one still faces the risk of the process dying out when being close to the mean. Therefore, as Figure 15b illustrates, only for values of n sufficiently above $\bar{n}$ investment is considered.

Nevertheless, when considering large consumers groups, i.e. large values of N, the investment problem changes as a result of their mean. When considering a mean diverting process, investment is not undertaken for $n < \bar{n}$. It is then crucial to see how this mean changes when $N \to \infty$. In the case where $\lambda(n) = \lambda(N-n)$ and $\eta(n) = \eta n(N-n)$, the process finds its mean at θ so that for large value of N this mean remains constant. This implies that the investment decision is not altered. Notice that a downward jump happens with probability $\frac{\lambda}{\lambda+\eta n}$ so that only the speed of the process is affected but not the jump distribution. However, for the mean diverting processes with $\eta(n) = \eta n$ the mean converges with the size of N. This happens since the downward rate is dependent on N but the upward rate is not, i.e. leaving the group of active consumers is related to the size of the inactive consumer group, which becomes large, whilst entering as inactive is only dependent on the current size of the actives. Hence, for these cases investment becomes infeasible for large groups of consumers.

The remaining three cases where one cannot speak of a mean, have a monotone trend. When assuming that this trend is positive, the fraction either increases linearly or convexly. In this way, investment shall always be undertaken, since, as in the case of a geometric Brownian motion, for sufficiently large values of N, there is always a value with respect to the fraction larger than I.

Example 5 *Consider a birth-death process, i.e.* $\lambda(n) = \lambda n$ *and* $\eta(n) = \eta n$*. Then if* $r \to 0$*, then* $\gamma_n \to \frac{\eta}{\lambda}$ *and* $\beta_n \to \frac{p}{\lambda}(n-1)$*. As a result,* $n^* = \frac{\lambda - \eta}{p} I$*.*

6.1.1 Absorbing states

Suppose, in addition to our standard model, one assumes state $n = N$ to be absorbing as well, i.e. $\lambda(N) = \eta(N) = 0$. Then, both functions β_n and γ_n are not affected for $n = 1, \ldots, N$. Hence, the investment rule is not altered. The only change to the model is the function $V(N)$, through which all $V(n)$ are affected. This concludes that changing N into an absorbing state does not change the investment problem but only shifts the pay-offs. This means that one does not care about the upward potential as much as one cares about the probability of death.

6.2 Birth-death process

Assume that there are N consumers of which n are currently buying the product. Then assume that with rate ηn a new consumer starts buying the product and with rate λn one consumer drops out. Then, (2) can be rewritten as

$$V(n) = \frac{np}{n\lambda + n\eta + r} + \frac{n\eta}{n\lambda + n\eta + r} V(n+1) + \frac{n\lambda}{n\lambda + n\eta + r} V(n-1).$$

This equation is solved for

$$V(n) = \gamma_{n+1} V(n+1) + \beta_{n+1} = \frac{\eta n V(n+1) + pn + \lambda n \beta_n}{r + \eta n + \lambda n(1 - \gamma_n)}$$

with

$$\gamma_{n+1} = \frac{\eta n}{r + \eta n + \lambda n(1 - \gamma_n)},$$
$$\beta_{n+1} = \frac{pn + \lambda n \beta_n}{r + \eta n + \lambda n(1 - \gamma_n)}.$$

where $\gamma_1 = 0$, $V(N) = \beta_{N+1}$, and $\beta_1 = 0$. Since $1 - \varepsilon \approx \frac{1}{1+\varepsilon}$ for small ε it holds that

$$\frac{\lambda n}{\lambda n + r} \approx 1 - \frac{r}{\lambda n}.$$

Then one can check that, by approximation, it holds that

$$\gamma_n \approx \frac{\eta n}{r + \lambda n},$$
$$\beta_n \approx \frac{p(n-1)}{r + \lambda}.$$

This approximation leads to the following value function,

$$V(n-1) = \frac{\eta n}{r+\lambda n}V(n) + \frac{p(n-1)}{r+\lambda},$$

with $V(N) = \frac{pN}{r+\lambda}$ and $V(0) = 0$. Investment is undertaken when

$$I \leq \frac{pn}{r+(\lambda-\eta)(n+1)}\frac{r+\lambda(n+1)}{r+\lambda}.$$

7 Conclusions

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Appendix A Proofs

In the proof of Proposition 3 Itô's Lemma is applied. For processes with discrete jumps Itô's Lemma is defined as follows. Let

$$\mathrm{d}X_t = \mu_t \mathrm{d}t + \eta_t \mathrm{d}Y_t,$$

with $\mathrm{d}Y_t = Z_{n_t}\mathrm{d}n_t$. Define

$$f'(X)\eta \mathrm{d}Y_t \equiv [f(X_t) - f(X_{t^-})]\mathrm{d}n = [f(X_{t^-} + \eta_t Z_{n_t}) - f(X_{t^-})]\mathrm{d}n_t.$$

Then,

$$\begin{aligned}\mathrm{d}f(X_t) &= f'(X_t)\mathrm{d}X + \tfrac{1}{2}f''(X)\mathrm{d}\langle X, X\rangle,\\ &= \mu_t f'(X_t)\mathrm{d}t + [f(X_{t^-} + \eta_t Z_{n_t}) - f(X_{t^-})]\mathrm{d}n_t.\end{aligned}$$

Proof of Proposition 1:

First notice that

$$\frac{(N-i)i\eta}{(N-i)i\eta+\lambda i}P_i+\frac{\lambda i}{(N-i)i\eta+\lambda i}P_i=P_i=\frac{(N-i)i\eta}{(N-i)i\eta+\lambda i}P_{i+1}+\frac{\lambda i}{(N-i)i\eta+\lambda i}P_{i-1}.$$

So that

$$P_{i+1}-P_i=\frac{\lambda}{(N-i)\eta}(P_i-P_{i-1}).$$

Since it holds that $P_0=0$ we have

$$P_2-P_1=\frac{\lambda}{(N-1)\eta}P_1$$

which, recursively, becomes

$$P_{i+1}-P_i=\frac{\lambda^i}{\eta^i\frac{(N-1)!}{(N-i-1)!}}P_1.$$

Then,

$$P_{i+1}-P_1=\sum_{l=1}^{i}(P_{l+1}-P_l)=\sum_{l=1}^{i}\frac{\lambda^l}{\eta^l\frac{(N-1)!}{(N-l-1)!}}P_1,$$

which is the same as

$$P_{i+1}=P_1\left[1+\sum_{l=1}^{i}\left(\frac{\lambda}{\eta}\right)^l\frac{(N-l-1)!}{(N-1)!}\right]\equiv P_1A_{i+1}.$$

Notice that if we substitute $l=j-1$,

$$A_i=1+\sum_{l=1}^{i-1}\left(\frac{\lambda}{\eta}\right)^l\frac{(N-l-1)!}{(N-1)!}=1+\sum_{j=2}^{i}\left(\frac{\lambda}{\eta}\right)^{j-1}\frac{(N-j)!}{(N-1)!}=\sum_{j=1}^{i}\left(\frac{\lambda}{\eta}\right)^{j-1}\frac{(N-j)!}{(N-1)!}.$$

From here it follows that, since $P_n=1$ that $1=P_1A_n$ so that $P_{i+1}=A_n^{-1}A_{i+1}$. □

Proof of Proposition 2:

First notice that Q_0=1 by definition. Moreover,

$$Q_i=\frac{(N-i)i\eta}{(N-i)i\eta+\lambda i}Q_{i+1}+\frac{\lambda i}{(N-i)i\eta+\lambda i}Q_{i-1}.$$

For Q_N one obtains,

$$Q_N=\frac{0}{\lambda N}Q_{N+1}+\frac{\lambda N}{\lambda N}Q_{N-1},$$

so that $Q_N=Q_{N-1}$. Similarly one can show that $Q_{N-1}=Q_{N-2}$. By induction one finds $Q_{N-2}=\ldots=Q_1$. Finally,

$$Q_1=\frac{(N-1)\eta}{(N-1)\eta+\lambda}Q_2+\frac{\lambda}{(N-1)\eta+\lambda}Q_0.$$

Which has a unique solution, $Q_0=Q_1=Q_2=1$. Hence $Q_i=1$ for all i. □

Proof of Proposition 3:

Method 1 Notice that

$$V(n_t) = pn_t\mathrm{d}t + (1 - r\mathrm{d}t)\mathbb{E}[V(n_{t+\mathrm{d}t})|n_t] + \mathcal{O}(\mathrm{d}t).$$

This can be rewritten as

$$(r + \eta n_t(N - n_t) + \lambda n_t)V(n_t)\mathrm{d}t = pn_t\mathrm{d}t + \eta(n_t)V(n_t + 1)\mathrm{d}t + \lambda(n_t)V(n_t - 1)\mathrm{d}t + \mathcal{O}(\mathrm{d}t),$$

since

$$V(n_{t+\mathrm{d}t}) = \begin{cases} V(n_t + 1) & \text{with probability } \eta n_t(N - n_t)\mathrm{d}t \\ V(n_t - 1) & \text{with probability } \lambda n_t\mathrm{d}t \\ V(n_t) & \text{with probability } 1 - \eta n_t(N - n_t)\mathrm{d}t - \lambda n_t\mathrm{d}t. \end{cases}$$

Then, if $\mathrm{d}t \to 0$ one obtains (2).

Method 2: Bellman First notice, from Itô, that,

$$\mathrm{d}V(n_t) = V'(n_t)Y_t\mathrm{d}I_t = [V(n_{t^-} + Y(n_t)) - V(n_{t^-})]\mathrm{d}I_t.$$

It follows

$$\begin{aligned}\mathbb{E}\mathrm{d}V(n_t) &= (p_u + p_d)\mathrm{d}t\left[\frac{p_u}{p_u + p_d}V(n_{t^-} + 1) + \frac{p_d}{p_u + p_d}V(n_{t^-} - 1) - V(n_{t^-})\right] \\ &= \left[\eta n(N - n)V(n + 1) + \lambda nV(n - 1) - (\eta n(N - n) + \lambda n)V(n)\right]\mathrm{d}t\end{aligned}$$

Which concludes

$$\begin{aligned}rV(n) &= \pi(n) + \frac{1}{\mathrm{d}t}\mathbb{E}\mathrm{d}V \\ &= pn + \eta n(N - n)V(n + 1) + \lambda nV(n - 1) - (\eta n(N - n) + \lambda n)V(n).\end{aligned}$$

Rewriting leads to (2). □

Proof of Proposition 4:

Follows directly from the tridiagonal matrix algorithm with

$$\begin{aligned}a_n &= -\lambda n, \\ b_n &= r + \lambda n + \eta n(N - n), \\ c_n &= -\eta n(N - n).\end{aligned}$$

Since $r + \lambda n + \eta n(N - n) > \lambda n + \eta n(N - n)$ the condition is satisfied. □

Proof of Lemma 1:

From the definition it follows that $\gamma_{n+1} < \gamma_n$ if and only if

$$\frac{\eta n(N - n)}{r + \eta n(N - n) + \lambda n(1 - \gamma_n)} < \gamma_n.$$

Rewriting leads to

$$(\gamma_n)^2\lambda n - \gamma_n(r + \eta n(N-n) + \lambda n) + \eta n(N-n) < 0.$$

This parabola has two roots. It can easily be shown that the largest root is larger than 1 and one can check that the smallest root is in between 0 and 1, as is γ_n. □

Corollary 2 *Let N be the number of consumers on the market. Then $V(N) < V(N-1)$ if and only if*

$$pN(1-\gamma_N) < r\beta_N$$

As will be shown later, the fraction $\frac{\beta_{n+1}}{1-\gamma_{n+1}}$ is increasing, so that

$$\frac{p}{r}N > V(N) = \beta_{N+1} = \frac{\beta_{N+1}}{1-\gamma_{N+1}} > \frac{\beta_N}{1-\gamma_N}.$$

Hence, the inequality in Corollary 2 never holds, and therefore $V(N-1) < V(N)$.

Proof of Corollary 2:

From Proposition 4 is follows that $V(N-1) = \gamma_N V(N) + \beta_N$ so that $V(N-1) > V(N)$ if and only if

$$\frac{\beta_N}{1-\gamma_N} > V(N) = \frac{pN + \lambda N\beta_N}{r + \lambda N(1-\gamma_N)}.$$

Rewriting leads to the equation. □

Proof of Proposition 10:

The expected time until death can be written as,

$$\begin{aligned} T_{N_t} &= \mathrm{d}t + \mathbb{E}[T_{N_{t+\mathrm{d}t}}|N_t] \\ &= \mathrm{d}t + p_u\mathrm{d}tT_{N_t+1} + p_d\mathrm{d}tT_{N_t-1} + (1-(p_u+p_d)\mathrm{d}t)T_{N_t}, \end{aligned}$$

which is the same as

$$\eta N(K+\theta-N)T_N = 1 + \eta N(K-N)T_{N+1} + \lambda N T_{N-1}.$$

This can be solved using the tridiagional matrix algorithm, as before. □

Proof of Proposition 5:

As before, using the tridiagonal matrix algorithm, the solution is defined by

$$F(n) = \frac{\eta n(N-n)F(n+1) + \lambda n\beta_n^F}{r + \lambda n + \eta n(N-n) - \lambda n\gamma_n^F}$$

with $F(n^*) = V(n^*) - I$. The variables β^F and γ^F are defined as

$$\gamma_{n+1}^F = \frac{\eta n(N-n)}{r + \lambda n + \eta n(N-n) - \lambda n\gamma_n^F}$$

$$\beta_{n+1}^F = \frac{\lambda n\beta_n^F}{r + \lambda n + \eta n(N-n) - \lambda n\gamma_n^F}$$

where $\gamma_1^F = 0$ and $\beta_1^F = 0$. This implies that $\beta_n^F = 0$ for all n. Finally, one can check that $\gamma_n^F = \gamma_n \ \forall_n$. □

Lemma 3 *Let β_n and γ_n be defined as in , then,*

$$\frac{\beta_n}{1-\gamma_n} \leq \frac{pn}{r} \;\Rightarrow\; \frac{\beta_{n+1}}{1-\gamma_{n+1}} < \frac{p(n+1)}{r}$$

Proof of Lemma 3:

First,

$$\frac{\beta_{n+1}}{1-\gamma_{n+1}} < \frac{p(n+1)}{r} \;\Leftrightarrow\; \frac{pn+\lambda n\beta_n}{r+\lambda n(1-\gamma_n)} < \frac{p(n+1)}{r}$$

which is equivalent to

$$\lambda n(1-\gamma_n)\left[\frac{\beta_n}{1-\gamma_n} - \frac{pn}{r}\right] < p + \frac{p}{r}\lambda n(1-\gamma_n).$$

Since the right-hand side is always positive, it is sufficient to have that the left-hand side is nonpositive. □

Lemma 4 *Let β_n and γ_n be defined as in , then,*

$$\frac{\beta_n}{1-\gamma_n} < (\leq)\frac{\beta_{n+1}}{1-\gamma_{n+1}} \;\Leftrightarrow\; \frac{\beta_n}{1-\gamma_n} < (\leq)\frac{pn}{r}$$

Proof of Lemma 4:

Notice that

$$\frac{\beta_{n+1}}{1-\gamma_{n+1}} - \frac{\beta_n}{1-\gamma_n} = \frac{pn+\lambda n\beta_n}{r+\lambda n(1-\gamma_n)} - \frac{\beta_n}{1-\gamma_n} = \frac{(1-\gamma_n)pn - \beta_n r}{(1-\gamma_n)(r+\lambda n(1-\gamma_n))}$$

It follows that this fraction is positive if and only if $(1-\gamma_n)pn > \beta_n r$. □

Proof of Proposition 7:

First, since $\beta_1 = \gamma_1 = 0$ we have that

$$\frac{\beta_n}{1-\gamma_n} = \frac{p}{r+\lambda} < \frac{p}{r}$$

for $n = 2$ so that

$$\frac{\beta_n}{1-\gamma_n} \leq \frac{pn}{r}$$

for $n = 1$ and $n = 2$. Then, from Lemma 3, by induction, we know that this holds for all $n \geq 2$. Then, the rest of the proof follows from Lemma 4. □

Lemma 5 *Let β_n and γ_n be defined as in , then,*

$$\frac{\beta_{n+1}}{1-\gamma_{n+1}} < \frac{pn}{r}.$$

Proof of Lemma 5:

Since

$$\frac{\beta_{n+1}}{1-\gamma_{n+1}} - \frac{pn}{r} = \frac{\lambda n(1-\gamma_n)}{\lambda n(1-\gamma_n)+r}\left[\frac{\beta_n}{1-\gamma_n} - \frac{pn}{r}\right],$$

it follows that if $\frac{\beta_n}{1-\gamma_n} < \frac{pn}{r}$ then it also holds that $\frac{\beta_{n+1}}{1-\gamma_{n+1}} < \frac{pn}{r}$. Proposition 7 and Lemma 4 show that this condition holds. □

Proof of Proposition 8:

Let $n \in \{2, 3, \ldots, n^* - 2\}$. Notice that,

$$F(n+1) - F(n) = (V(n^*) - I) \prod_{k=n+1}^{n^*-1} \frac{\eta k(N-k)}{r + \eta k(N-k) + \lambda k(1-\gamma_k)} \left[1 - \frac{\eta n(N-n)}{r + \eta n(N-n) + \lambda n(1-\gamma_n)}\right],$$

$$F(n) - F(n-1) = (V(n^*) - I) \prod_{k=n}^{n^*-1} \frac{\eta k(N-k)}{r + \eta k(N-k) + \lambda k(1-\gamma_k)} \left[1 - \frac{\eta(n-1)(N-n+1)}{r + \eta(n-1)(N-n+1) + \lambda(n-1)(1-\gamma_{n-1})}\right].$$

Then $F(n+1) - F(n) > F(n) - F(n-1)$ can be rewritten as

$$\frac{r + \lambda n(1-\gamma_n)}{r + \eta n(N-n) + \lambda n(1-\gamma_n)} > \frac{\eta n(N-n)}{r + \eta n(N-n) + \lambda n(1-\gamma_n)} \frac{r + \lambda(n-1)(1-\gamma_{n-1})}{r + \eta(n-1)(N-n+1) + \lambda(n-1)(1-\gamma_{n-1})}.$$

This leads to the equation in the proposition. For $n = n^* - 1$ the same analysis holds, but then without the product in the middle, so that it leads to the same conclusion. □

Proof of Lemma 2:

Rewriting

$$V(n+1) > V(n) = \gamma_{n+1} V(n+1) + \beta_{n+1}$$

gives the corresponding equation. □

Proof of Proposition 6:

At $n = n^*$ it holds that $F(n) = V(n) - I$. However, for $n = n^* - 1$ we have

$$\begin{aligned} F(n) &= \frac{\eta n(N-n)}{r + \lambda n(1-\gamma_n) + \eta n(N-n)} (V(n^*) - I) \\ V(n) &= \frac{\eta n(N-n)}{r + \lambda n(1-\gamma_n) + \eta n(N-n)} V(n^*) + \frac{pn + \lambda n \beta_n}{r + \lambda n(1-\gamma_n) + \eta n(N-n)} - I. \end{aligned}$$

Firms wait as long as $F(n) > V(n)$, which implies

$$\frac{\eta n(N-n)}{r + \lambda n(1-\gamma_n) + \eta n(N-n)} (-I) > \frac{pn + \lambda n \beta_n}{r + \lambda n(1-\gamma_n) + \eta n(N-n)} - I.$$

Rewriting gives

$$\begin{aligned} I &> \frac{pn + \lambda n \beta_n}{r + \lambda n(1-\gamma_n)} \\ &= \frac{p(n^*-1) + \lambda(n^*-1)\beta_{n^*-1}}{r + \lambda(n^*-1)(1-\gamma_{n^*-1})} \\ &= \frac{\beta_{n^*}}{1-\gamma_{n^*}}. \end{aligned}$$

Differently, firms do not wait for $n+1$ when $F(n) \leq V(n)$, i.e.

$$I \leq \frac{pn + \lambda n \beta_n}{r + \lambda n(1-\gamma_n)} = \frac{\beta_{n+1}}{1-\gamma_{n+1}},$$

which concludes the proof. □

Proof of Corollary 1:

The investment trigger is defined as n^* such that

$$I(r + \lambda n^*(1-\gamma_{n^*})) = pn^* + \lambda n^* \beta_{n^*}.$$

One can check that $\gamma(n) \to 1$ and $\beta(n) \to 0$ for a fixed $n < \infty$ if $N \to \infty$. This means that the equation converges to $rI = pn^*$. □

Appendix B Thomas Algorithm

The recurrence relation in (2) can be solved using the tridiagonal matrix algorithm, or sometimes called the Thomas algorithm after Thomas (1949). This algorithm recursively solves the set of equations defined by

$$a_i x_{i-1} + b_i x_i + c_i x_{i+1} = y_i$$

for $i = 1, \ldots, n$ where $a_1 = c_n = 0$. This is equivalent to

$$Ax = \begin{bmatrix} b_1 & c_1 & 0 & \dots & 0 \\ a_2 & b_2 & \ddots & & \vdots \\ 0 & \ddots & \ddots & & 0 \\ \vdots & & & \ddots & c_{n-1} \\ 0 & \dots & 0 & a_n & b_n \end{bmatrix} \begin{bmatrix} x_1 \\ \\ \vdots \\ \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ \\ \vdots \\ \\ y_n \end{bmatrix} = y.$$

Matrices of the kind of A are called tridiagonal matrices. Define

$$\gamma_{i+1} = \frac{-c_i}{a_i \gamma_i + b_i}$$
$$\beta_{i+1} = \frac{y_i - a_i \beta_i}{b_i + a_i \gamma_i},$$
$$x_n = \beta_{n+1} = \frac{y_n - a_n \beta_n}{b_n + a_n \gamma_n},$$

with $\gamma_1 = \beta_1 = 0$. The echelon form of the matrix $[Ay]$ is row equivalent to

$$\begin{bmatrix} 1 & -\gamma_2 & 0 & 0 & \dots & 0 & \beta_2 \\ 0 & 1 & -\gamma_3 & \ddots & & \vdots & \beta_3 \\ 0 & 0 & \ddots & \ddots & & 0 & \vdots \\ \vdots & & \ddots & 1 & -\gamma_{n-1} & 0 & \vdots \\ \vdots & & \ddots & 0 & 1 & -\gamma_n & \beta_n \\ 0 & \dots & 0 & 0 & 0 & 1 & x_n \end{bmatrix}.$$

As a result, it follows that, if $|b_i| > |a_i| + |c_i|$ for all i, then

$$x_i = \gamma_{i+1} x_{i+1} + \beta_{i+1} = \frac{-c_i}{a_i \gamma_i + b_i} x_{i+1} + \frac{y_i - a_i \beta_i}{a_i \gamma_i + b_i}$$

is the solution of the set of equations with x_n as defined before.