Optimal Investment Strategy under Lévy ambiguity

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Abstract

This paper examines an optimal investment problem of Abel and Eberly (1997) and Imai and Tsujimura (2016) under higher degree of ambiguity. To that end we introduces an exponential Lévy process as the underlying risk process of the project. The ambiguity indicates a manager's disconfidence with respect to the underlying model. It can be formulated as allowing one to change the reference probability measure into a different equivalent probability measure. The difference between the reference measure and another equivalent measure indicates the manager's misspecification of the underlying model.

In the formulation, on the one hand, the firm's manager chooses the investment level to maximize the firm's value. On the other hand, Nature chooses the equivalent probability measure so that the firm's value is minimized. Consequently, the optimal investment problem developed in this paper can be formulated as a maxmin expected utility problem.

It is crucial to notice that an adoption of a Lévy process enables us express higher degree of ambiguity, that is, unlike the diffusion process, when the underlying asset follows an exponential Lévy process, a change of measure can affect not only its drift term of the diffusion but also its variance, skewness and kurtosis via changing the jump structure of the original Lévy process. Consequently, in this paper, we can express higher level of ambiguity, i.e, our model can describe potential misspecification with respect to its higher moments of the distribution.

Keywords: ambiguity, optimal investment, Lévy process,

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1. Introduction

Capital budgeting has been one of the central topics in corporate finance literature. A firm attempts to make the optimal decision to maximize shareholders' value of the firm. In particular, an investment decision under uncertainty has been thoroughly investigated. For representative examples, Hartman, Abel, and Eberly (Hartman, 1972; Abel, 1983; Abel and Eberly, 1994, 1997) investigated the impact of output price uncertainty on a firm's capital accumulation.

Most studies for investment decision-making under uncertainty so far have considered the optimal investment in the presence of risk, which implies that even though a firms manager recognizes unpredictable events in the future, he/she has perfect confidence about the distribution of the future events. Abel and Eberly (1997) investigated a capital investment problem and derived closed-form solutions under an output price risk. One of the difficulties in applying the theoretical analyses to an actual business is to estimate parameter values of the underlying process. In the case of financial options, it is relatively easier to statistically estimate the parameter values since in most cases the underlying assets are traded in liquid markets. On the other hand, in many cases of the real assets, the corresponding underlying assets are not traded in liquid market, sometimes they are not observable at all. These facts indicates that it is hard to obtain accurate estimation in many practical applications. Bloom et al. (2007) provided some empirical evidences, for instance. We refer the reader to Becker and Brownson (1964), Camerer and Weber (1992), Etner et al. (2012) and the references therein.

One way of resolving it is to admit the possibility of the misspecification, i.e., introducing the notion of ambiguity with respect to the underlying model. It enables one to formulate an optimization problem with allowing a possible misspecification whereas to derive the optimal strategy at the presence of the ambiguity. Imai and Tsujimura (2016) extended Abel and Eberly (1997) to a problem under ambiguity. They showed that the ambiguity produces a clear effect on the optimal decision. In particular, the existence of the ambiguity forces the firm's manager to have a pessimistic view with regard to the future output price, which could lead to decreasing the firm value. Furthermore, they showed that the trade-off between the acceptable degree of misspecification and the penalty of accepting misspecification determines the optimal level of distortion between the reference model and the approximation model.

It is critical to notice that all the existing models that take the ambiguity into account explicitly assume that the underlying risk follows a geometric Brownian motion. In this case, Girsanov theorem clearly indicates that any change of measure affects only the drift term of the process. In other words, under the assumption of the geometric Brownian motion, a manager's misspecification arises only on its expected rate of return, and never affects the volatility of the underlying risk. Note that this fact is also true for any diffusion process. It is against our intuition that a manager could misspecify not only the drift but also the volatility term of the underlying process. Consequently, in this sense, the standard assumption with respect to the underlying risk process used in many existing studies is not validated.

Motivated by these observations, in this paper, we examine the optimal investment problem of Abel and Eberly (1997) and Imai and Tsujimura (2016) under higher degree of ambiguity. To that end this paper introduces an exponential Lévy process as the underlying stochastic process of the project.

A Lévy process was first introduced in economics by Mandelbrot (1963), and since then it has long been used in the field of finance and actuarial science. It is known that any Lévy process can be decomposed into three parts, that is, a deterministic drift part, continuous diffusion (volatility) part, and a jump part. Hence, it can be considered as a natural extension of a Brownian motion and a compound Poisson process. Lévy processes have obtained a popularity in mathematical finance because it can provide a lot of flexibility for modeling an underlying risk.

The validity of a Lévy process in financial application is supported by many empirical tests. For example, Madan and Seneta (1990) reported that variance gamma Lévy process is better fit to Australian stock market data. Rydberg (1999) applied the Lévy process to U.S. stock prices. Prause (1999) compared the goodness of fit to German stock and US stock index market in terms of Kolmogorov distance and Anderson & Darling statistic, and concludes that the many infinite divisible distributions have better fit than normal distribution. A further details about Lévy process used in finance, we refer the reader to Raible (2000).

In this paper, the ambiguity indicates a manager's disconfidence with respect to the underlying risk process. It can be formulated as allowing one to change the reference probability measure into a different equivalent probability measure. The difference between the reference measure and another equivalent measure indicates the manager's misspecification of the underlying model. On the one hand, the firm's manager chooses the investment level to maximize the firm's value. On the other hand, the nature choose the equivalent probability measure so that the firm's value is minimized. Consequently, the optimal investment problem developed in this paper can be formulated as a minimax problem. This model is now called a maxmin (or multi-prior) expected utility (MEU) model. See Gilboa and Schmeidler (1989), for a rigorous introduction of the MEU model. The MEU model assumes that a decision-maker has a set of priors $\mathcal{P}$ that represents the fact that he/she cannot identify a particular set of parameter values for underlying stochastic process. A similar idea has been analyzed in the field of control theory. In order to keep control of physical systems under some ambiguous environment, they have developed a robust control theory. See a book by Hansen and Sargent (2008, Chapter 1) and reference therein for a review of robust control theory.

It is crucial to notice that an adoption of a Lévy process enables us express higher degree of ambiguity, that is, unlike the diffusion process, when the underlying asset follows an exponential Lévy process, a change of measure can affect not only its drift term of the diffusion but also its variance, skewness and kurtosis via changing the jump structure of the original Lévy process. Consequently, in this paper, we can express higher level of ambiguity, i.e, our model can describe potential misspecification with respect to its higher moments of the distribution. However, from a mathematical viewpoint, the set of equivalent measure for an exponential Lévy process is extremely large. First, there is no guarantee that the transformed process with a new equivalent probability measure belongs to the class of the exponential Lévy processes. Second, even though we restrict our attention to the set of equivalent probability measure that preserve the class of Lévy processes, it is still too large for us for discussing the ambiguity with respect to the model specification (see Cont and Tankov (2004b), for example).

For this reason we further restrict the set of equivalent probability measures on the following two aspects. First, we employ an Esscher transform for changing the equivalent probability measure. The Esscher transform has often been proposed and investigated in option pricing literature.

Gerber et al. (1994) proposed to use the Esscher transform for pricing options in incomplete markets. The Esscher transform can be also interpreted in terms of economic theory. It arises from a general equilibrium representative agent model with a utility in which a relative risk aversion is constant. See Keller (1997) for the detail of the discussion. Chan(1999) proved that the Esscher transform minimizes the relative entropy of the measure P and Q under all equivalent martingale transformation.

Second, we employ a generalized hyperbolic (GH) Lévy process as a representative of the Lévy process. The GH Lévy process, introduced by Barndorff-Nielsen (1977), is a special case of the Lévy process whose marginal distribution follows the GH distribution. It is thoroughly examined in Eberlein and Keller (1995), Prause (1999), Eberlein et al. (1998). Although a GH Lévy process is a special class of the Lévy processes, it has a rich structure that can offer a great variety of shapes. Many known infinitely divisible distributions can be considered as subclasses of the GH distribution, including normal, student-t, normal inverse, hyperbolic, variance gamma. In particular, we can examine the effect of additional degree of the ambiguity since the GH Lévy contains the geometric Brownian motion as a special case.

In this paper, we adopt a dynamic programming to formulate the problem. First, we introduce an iterative method for solving the maxmin problem numerically. Second, we use a value iteration algorithm for solving a fixed point to solve an infinite time horizon problem. Third, in order to implement the above algorithms, we utilize both a quasi-Monte Carlo method and a spline function approximation method to enhance numerical accuracy.

This paper is organized as follows. In Section 2, we describe the investment problem under ambiguity and formulate it as a maxmin problem. We also discuss an approximate dynamic programming for solving the optimization problem. In Section 3, we provide numerical results and discuss the effect of higher level of the ambiguity on the optimal decision. Finally, concluding remarks are given in Section 4.

2. Optimal investment problem

2.1. Modelling Investment Project

First, we take a quick review an infinite-time investment model under an output price uncertainty, that was originally developed in Abel and Eberly (1997) and extended in Imai and Tsujimura (2016). Let $F(L_t, K_t)$ denote a firm's production function that takes the Cobb–Douglas form:

$$F(L_t, K_t) = L_t^{\varpi} K_t^{1-\varpi}, \tag{1}$$

where K_t and L_t represent, respectively, the amount of capital stock and the labor at time t, and $\varpi \in (0, 1)$ represents the output elasticity of the labor. The amount of capital K_t is governed by

$$\mathrm{d}K_t = (I_t - \delta K_t)\mathrm{d}t, \qquad K_0 = k, \tag{2}$$

in which I_t is the amount of investment at time t and $\delta \in (0, 1)$ is the depreciation rate. We assume that the cost for obtaining additional amount of the capital, denoted by C, is a function of I_t, given by

$$C(I_t) = c_0 I_t + c_1 I_t^{\phi}, \tag{3}$$

where $c_0 > 0$ is the price of purchasing capital, $c_1 > 0$ is the conversion factor with $\phi > 1$ being the adjustment cost parameter. Equation (3) indicates that to obtain additional amount of the capital, the firm needs to pay the adjustment cost in addition to the price of the capital, and hence the total cost becomes convex with respect to the amount of the investment.

In the consideration of the labor cost, it can be shown that the net operating profit can be given by $\pi(K_t, P_t) - C(I_t)$ where $\pi(K_t, P_t)$ represents the operating profit given by

$$\pi(K_t, P_t) = \eta P_t^{\alpha} K_t,$$

with $\alpha = 1/(1-\varpi) > 1$ and $\eta = \alpha^{-\alpha}(\alpha-1)^{\alpha-1}w^{1-\alpha} > 0$. we assume that discount rate r is sufficiently large so that the accumulated profit will not explode.

Let us now introduce a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$. In this paper, we assume that the underlying output price process follows P_t an exponential Lévy process, i.e.,

$$P_t = p\mathrm{e}^{X_t}, \tag{4}$$

where X_t stands for a Lévy process with $X_0 = 0, a.s.$

A Lévy process refers to a cádlág stochastic process that has independent and stationary increments. The distribution of these increments are called infinitely divisible. Lévy processes are completely characterized by the Lévy triplet or the characteristic triplet, (γ, σ^2, ν) where γ represents the drift, σ is the volatility of the Brownian part, and ν is called the Lévy measure that governs the jump part of the process that satisfies $\nu(0) = 0$ and $\int \left(|x|^2 \wedge 1\right) d\nu(x) < \infty$. Lévy-Khinchin representation shows that there exists γ, σ, and ν such that the characteristic function χ_t of X_t has the representation

$$\chi_t(z) = \mathbb{E}\left[e^{izX_t}\right] = e^{t\psi(z)}$$

where

$$\psi(z) = i\gamma z - \frac{1}{2}\sigma^2 z^2 + \int_{\mathbb{R}} \left(e^{izx} - 1 - izh(x)\right)\nu(dx),$$

and $h(x) := 1_{[-1,1]}(x)$ is a truncation function. For a formal definition of a Lévy process, see Sato (1999), for example.

In the presence of the ambiguity, Equation (4) can be considered as a reference model, that is, it is a stochastic process based on the best possible estimate for the firm's manager. The firm's manager is, however, concerned about the robustness of his/her decisions to misspecification of the reference probability. In order to resolve the concern he/she prepares for a set of equivalent probability measures, $\mathcal{P}$, on $(\Omega, \mathcal{F})$ for expressing the possible misspecification. Then, the reference probability measure $\mathbb{P}$ could be replaced by another equivalent probability measure $\mathbb{Q} \in \mathcal{P}$. The penalty for a difference between the reference probability $\mathbb{P}$ and another equivalent probability $\mathbb{Q}$ is usually imposed to avoid choosing a too distant probability measure from the reference probability measure. We employ a relative entropy measure or KL divergence to measure the difference, denoted by $R_t(\mathbb{P}, \mathbb{Q})$, between the reference probability measure $\mathbb{P}$ and an alternative probability measure $\mathbb{Q}$ at time t. Let $Z_t = \frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t}$ represents the density process. The relative entropy is defined as

$$R_t(\mathbb{P}, \mathbb{Q}) = \mathbb{E}^{\mathbb{P}}\left[Z_t \log(Z_t)\right] = \mathbb{E}^{\mathbb{Q}}\left[\log(Z_t)\right]. \tag{5}$$

The firm attempts to choose the optimal investment rate at each time to maximize the expected firm's operating profit under output price ambiguity. To solve the problem, we adopt the robust control approach developed by Hansen and Sargent (2001), Hansen et al. (2002, 2006), which is based on maxmin expected utility (MEU) model of Gilboa and Schmeidler (1989). In the MEU model, it is proven that an ambiguity-averse decision-maker chooses a policy to maximize his/her utility in the possible worst case. Let $V(k,p)$ denote the value function with $P_0 = p$ and $K_0 = k$. It is formally formulated as follows:

$$\begin{aligned} V(k,p) &= \max_{I_t \in \mathcal{I}} \min_{\mathbb{Q} \in \mathcal{P}} \left\{ \mathbb{E}^{\mathbb{Q}} \left[\int_0^\infty e^{-rt} \left\{ \pi(K_t, P_t) - C(I_t) \right\} \mathrm{d}t \right] + \theta \int_0^\infty e^{-rt} R_t(\mathbb{P}, \mathbb{Q}) \, \mathrm{d}t \right\} \\ &= \max_{I_t \in \mathcal{I}} \min_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}^{\mathbb{Q}} \left[\int_0^\infty e^{-rt} Y_t \mathrm{d}t \right], \end{aligned} \tag{6}$$

where

$$Y_t := \pi(K_t, P_t) - C(I_t) + \theta \log Z_t, \tag{7}$$

and $\theta \geq 0$ is the multiplier on the relative entropy penalty. Note that Equation (6) is based on the new probability measure $\mathbb{Q}$. The formula can be written under the reference probability measure $\mathbb{P}$.

$$V(k,p) = \max_{I_t \in \mathcal{I}} \min_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} \left[\int_0^\infty e^{-rt} \hat{Y}_t \mathrm{d}t \right] \tag{8}$$

where

$$\hat{Y}_t := \{ \pi \left(K_t, \hat{P}_t \right) - C(I_t) \} Z_t + \theta Z_t \log Z_t. \tag{9}$$

Note that $\pi \left(K_t, \hat{P}_t \right) - C(I_t)$ is independent, and only Z_t depends on the choice of the new probability $\mathbb{Q}$.

2.2. A change of measure for a Lévy process

As mentioned in the introduction, the class of equivalent measure for an exponential Lévy process is extremely large and there is no guarantee that the transformed process with a new equivalent probability measure $\mathbb{Q}$ belongs to the class of the exponential Lévy processes. Jacod and Shiryaev (2013) and Raible (2000) proved the necessary and sufficient condition, respectively, for preserving the class of the exponential Lévy process by the change of measure, which is summarized as follows:

Proposition 2.1. *Let X be a Lévy process on $\mathbb{R}$ with Lévy triplet $(\gamma, \sigma^2, \nu)_{\mathbb{P}}$ under a measure $\mathbb{P}$. Then there exits an equivalent probability measure $\mathbb{Q}$ such that X is a Lévy process under $\mathbb{Q}$ with Lévy triplet $(\tilde{\gamma}, \tilde{\sigma}^2, \tilde{\nu})_{\mathbb{Q}}$, if and only if there exist β and a function y satisfying*

$$\int_{\mathbb{R}} |h(x)(1 - y(x))| \, \nu(dx) < \infty, \int_{\mathbb{R}} \left(1 - \sqrt{y(x)} \right)^2 \nu(dx) < \infty.$$

Furthermore, the corresponding Lévy triplet can be written by

$$\tilde{\gamma} = \gamma + \sigma^2 \beta + \int_R h(x)(1 - y(x)) \, \nu(dx), \tag{10}$$

$$\tilde{\sigma} = \sigma, \tag{11}$$

$$\frac{d\tilde{\nu}}{d\nu}(x) = y(x). \tag{12}$$

Proposition 2.1 indicates that a change of measure that preserves the Lévy process can be characterized by Girsanov quantities (β, y). Equation (10) indicates that the drift term of the diffusion part is shifted by a change of measure with the Girsanov quantities, whereas Equation (11) clearly shows that the volatility term of the diffusion part remains unchanged. On the other hand, Equation (12) indicates that the function y can change the jump structure under the $\mathbb{Q}$-measure. In other words, unlike the diffusion process, under the assumption that the underlying asset follows an exponential Lévy process, a change of measure can affect not only its drift term of the diffusion but also its variance, skewness and kurtosis of the distribution by changing the jump structure of the original Lévy process. Consequently, in this paper, we can express higher level of ambiguity, that is, our model can describe potential misspecification with respect to its higher moments of the distribution.

When an equivalent martingale measure $\mathbb{Q}$ is chosen, the relative entropy, denoted by $R_t(\mathbb{P}, \mathbb{Q})$, can be explicitly written by

$$R_t(\mathbb{P}, \mathbb{Q}) = \frac{t}{2\sigma^2}\left\{\tilde{\gamma} - \gamma - \int_{-1}^{1} x\,(\tilde{\nu} - \nu)\,dx\right\}^2 + t\int_{-\infty}^{\infty} (y\,(x)\ln y\,(x) - y\,(x) + 1)\nu\,(dx)\,. \tag{13}$$

For a derivation of Equation (13), see in Cont and Tankov (2004b)).

2.3. A generalized hyperbolic Lévy process

Even if we restrict our attention to the class of exponential Lévy processes, the set of equivalent probability measures is still too large to discuss the optimal investment under the ambiguity. In fact, in the book of Cont and Tankov (2004a), they state that in the presence of jumps even if we restrict our attention to structure preserving measures, the set of equivalent probability measures is surprisingly large. Hence, we consider a generalized hyperbolic (GH) Lévy process for analyzing the effect of the ambiguity on the optimal investment decision numerically.

Let us first review a basic facts about a generalized hyperbolic (GH) Lévy process. See Raible (2000) for a rigorous mathematical treatment and Predota (2005) for a comprehensive survey for financial applications. The probability density function (pdf) of the GH distribution has five parameters $(\lambda_g, \alpha_g, \beta_g, \mu_g, \delta_g)$ and is given by

$$f^{GH}(x; \lambda_g, \alpha_g, \beta_g, \mu_g, \delta_g) = \frac{\xi_g^{\lambda_g}}{\sqrt{2\pi}\alpha_g^{2\lambda_g - 1}\delta_g^{2\lambda_g} K_{\lambda_g}(\xi_g)} \eta_g(x)^{\lambda_g - \frac{1}{2}} K_{\lambda_g - \frac{1}{2}}\big(\eta_g(x)\big) e^{\beta_g(x - \mu_g)}, \tag{14}$$

where $\xi_g = \delta_g\sqrt{\alpha_g^2 - \beta_g^2}$, $\eta_g(x) = \alpha_g\sqrt{\delta_g^2 + (x - \mu_g)^2}$, and the parameters $\delta_g, \beta_g, \alpha_g$ and λ_g satisfy

$$\delta_g \quad \geq 0,\ \alpha_g > 0,\ |\beta_g| < \alpha_g, \quad \text{if } \lambda_g > 0, \tag{15}$$

$$\delta_g \quad > 0,\ \alpha_g > 0,\ |\beta_g| < \alpha_g, \quad \text{if } \lambda_g = 0, \tag{16}$$

$$\delta_g \quad > 0,\ \alpha_g \geq 0,\ |\beta_g| \leq \alpha_g, \quad \text{if } \lambda_g < 0. \tag{17}$$

Furthermore, K_{λ_g} is the modified Bessel function of the third kind with index λ_g and is given by[1]

$$K_{\lambda_g}(u) = \frac{1}{2}\int_0^\infty t^{\lambda_g - 1} \exp\left\{-\frac{1}{2}u\left(t + \frac{1}{t}\right)\right\} dt. \tag{18}$$

The characteristic function $\phi^{GH}(u)$and moment generating function $M^{GH}(u)$ of the GH distribution are given, respectively, by

$$\phi^{GH}(u) = e^{i\mu_g u}\left(\frac{\alpha_g^2 - \beta_g^2}{\alpha_g^2 - (\beta_g + iu)^2}\right)^{\frac{\lambda_g}{2}} \frac{K_{\lambda_g}\left(\delta_g\sqrt{\alpha_g^2 - (\beta_g + iu)^2}\right)}{K_{\lambda_g}\left(\delta_g\sqrt{\alpha_g^2 - \beta_g^2}\right)},$$

and

$$M^{GH}(u) = e^{\mu_g u}\left(\frac{\alpha_g^2 - \beta_g^2}{\alpha_g^2 - (\beta_g + u)^2}\right)^{\frac{\lambda_g}{2}} \frac{K_{\lambda_g}\left(\delta_g\sqrt{\alpha_g^2 - (\beta_g + u)^2}\right)}{K_{\lambda_g}\left(\delta_g\sqrt{\alpha_g^2 - \beta_g^2}\right)}, \qquad |\beta_g + u| < \alpha_g.$$

These results in turn allow us to derive moments of the GH distribution. The mean and variance of the distribution are given, respectively, by

$$E[X_1] = \mu_g + \frac{\beta_g\delta_g^2}{\zeta_g}\frac{K_{\lambda_g+1}(\zeta_g)}{K_{\lambda_g}(\zeta_g)}, \tag{19}$$

and

$$\mathrm{Var}[X_1] = \frac{\delta_g^2}{\zeta_g}\frac{K_{\lambda_g+1}(\zeta_g)}{K_{\lambda_g}(\zeta_g)} + \beta_g^2\frac{\delta_g^4}{\zeta_g^2}\left(\frac{K_{\lambda_g+2}(\zeta_g)}{K_{\lambda_g}(\zeta_g)} - \frac{K_{\lambda_g+1}^2(\zeta_g)}{K_{\lambda_g}^2(\zeta_g)}\right), \tag{20}$$

where $\zeta_g = \delta_g\sqrt{\alpha_g^2 - \beta_g^2}$. Furthermore, the skewness γ_1 and the kurtosis γ_2 can be derived as follows.

$$\begin{aligned} \gamma_1[X_1] &= Var[X_1]^{-2} \\ &\times \left[\frac{\beta_g^3\delta_g^6}{\zeta_g^3}\left(\frac{K_{\lambda_g+3}(\zeta_g)}{K_{\lambda_g}(\zeta_g)} - \frac{3K_{\lambda_g+2}(\zeta_g)K_{\lambda_g+1}(\zeta_g)}{K_{\lambda_g}^2(\zeta_g)} + \frac{2K_{\lambda_g+1}^2(\zeta_g)}{K_{\lambda_g}^3(\zeta_g)}\right)\right. \\ &+ \left.\frac{3\beta_g\delta_g^4}{\zeta_g^2}\left(\frac{K_{\lambda_g+2}(\zeta_g)}{K_{\lambda_g}(\zeta_g)} - \frac{K_{\lambda_g+1}^2(\zeta_g)}{K_{\lambda_g}^2(\zeta_g)}\right)\right], \end{aligned} \tag{21}$$

$$\begin{aligned} \gamma_2[X_1] &= -3 + Var[X_1]^{-2} \\ &\times \left[\frac{\beta_g^8\delta_g^4}{\zeta_g^4}\left(\frac{K_{\lambda_g+4}(\zeta_g)}{K_{\lambda_g}(\zeta_g)} - \frac{4K_{\lambda_g+3}(\zeta_g)K_{\lambda_g+1}(\zeta_g)}{K_{\lambda_g}^2(\zeta_g)} + \frac{6K_{\lambda_g+2}(\zeta_g)K_{\lambda_g+1}^2(\zeta_g)}{K_{\lambda_g}^3(\zeta_g)} - \frac{K_{\lambda_g+1}^4(\zeta_g)}{K_{\lambda_g}^4(\zeta_g)}\right)\right. \\ &+ \left.\frac{\beta_g^6\delta_g^2}{\zeta_g^3}\left(\frac{6K_{\lambda_g+3}(\zeta_g)}{K_{\lambda_g}(\zeta_g)} - \frac{12K_{\lambda_g+2}(\zeta_g)K_{\lambda_g+1}(\zeta_g)}{K_{\lambda_g}^2(\zeta_g)} + \frac{6K_{\lambda_g+1}^3(\zeta_g)}{K_{\lambda_g}^3(\zeta_g)}\right) + \frac{3\delta_g^4}{\zeta_g^2}\frac{K_{\lambda_g+2}(\zeta_g)}{K_{\lambda_g}(\zeta_g)}\right] \end{aligned} \tag{22}$$

[1]For the analytical details of the modified Bessel function, see Abramowitz and Stegun (1968).

Since GH distribution is infinitely divisible, the GH Lévy process denoted by $X^{GH}(t) = \left\{X^{GH}(t), t \geq 0\right\}$ can be defined by its characteristic function

$$\mathbb{E}\left[e^{iuX^{GH}(t)}\right] = \left[\phi^{GH}(u)\right]^t.$$

Each of the parameters of the GH distribution can be interpreted intuitively. The parameter λ_g determines the subclass of the GH distribution. α_g controls the steepness around the mode of the distribution and it also affects its tail behaviour. β_g indicates the ratio of asymmetry. μ_g indicates the location while δ_g indicates the scale. See Bibby and Sorensen (2003) for the shape of the GH distribution.

The Lévy measure of a GH distribution is explicitly given as follows.

$$\frac{d\nu(x)}{dx} = \begin{cases} \frac{e^{\beta_g x}}{|x|}\left(\int_0^\infty \frac{\exp\left\{-\sqrt{2y+\alpha_g^2}|x|\right\}}{\pi^2 y\left(J^2_{\lambda_g}\left(\delta_g\sqrt{2y}\right)+Y^2_{\lambda_g}\left(\delta_g\sqrt{2y}\right)\right)}dy + \lambda_g e^{-\alpha_g|x|}\right), & \lambda_g \geq 0, \\ \frac{e^{\beta_g x}}{|x|}\left(\int_0^\infty \frac{\exp\left\{-\sqrt{2y+\alpha_g^2}|x|\right\}}{\pi^2 y\left(J^2_{-\lambda_g}\left(\delta_g\sqrt{2y}\right)+Y^2_{-\lambda_g}\left(\delta_g\sqrt{2y}\right)\right)}dy + \lambda_g e^{-\alpha_g|x|}\right), & \lambda_g < 0. \end{cases}$$

where J_{λ_g} and Y_{λ_g} represent, respectively, Bessel functions. The GH distribution includes the normal distribution as a limiting case where $\beta_g = 0, \delta_g \to \infty$, $\alpha_g \to \infty$ with $\frac{\delta_g}{\alpha_g} \to const.$

In the case of the GH Lévy process, the class preserved change of measure can be simplified as follows.

Proposition 2.2. *Let X^{GH} be a GH Lévy process with parameters $(\lambda_g, \alpha_g, \beta_g, \delta_g, \mu_g)$ under the measure $\mathbb{P}$. Then, there is another locally equivalent martingale measure $\mathbb{Q}$ under which X^{GH} is again a GH Lévy process, with parameter $(\lambda'_g, \alpha'_g, \beta'_g, \delta'_g, \mu'_g)$ if and only if $\delta'_g = \delta_g$ and $\mu'_g = \mu_g$.*

This proposition means that if a GH Lévy process is preserved with some change of measure, the parameters of δ_g and μ_g are never changed.

2.4. An Esscher Transform

We introduce an Esscher transform in the following form. An Esscher transform with a parameter $\xi \in \mathbb{R}$ from the measure $\mathbb{P}$ to an equivalent measure $\mathbb{Q}$ with a density process $Z_t = \frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t}$ is defined as

$$Z_t = \frac{\exp(\xi X_t)}{M_X(\xi)^t}, \tag{23}$$

where M_X stands for a moment generating function of X_1.

It is shown that under the Esscher transform, for all $\xi \in \mathbb{R}$ such that $\mathbb{E}[\exp(\xi X_1)] < \infty$, X_t under the new measure $\mathbb{Q}$ becomes a Lévy process. The proof of this property is in Proposition 1.8 of Raible (2000). It is also known that the measure change function by the Esscher transform is given by the following form:

$$y(x) = e^{\xi x}. \tag{24}$$

It is known that the equivalent martingale measure that minimize the relative entropy can be attained via the Esscher transform. For technical details about the Esscher transform under an exponential Lévy process, see, for example, Fujiwara and Miyahara (2003).

Note that the relative entropy is a function of the Esscher parameter ξ. Notice that the relative entropy is convex functional of the measure $\mathbb{Q}$. See, Fujiwara and Miyahara (2003) and Cont and Tankov (2002) for the details. Raible (2000) proved that a GH distribution is closed under the Esscher transform with changing only one parameter values, which is summarized in the following proposition.

Proposition 2.3. *An Esscher transform of a GH distribution $GH(\lambda_g, \alpha_g, \beta_g, \delta_g, \mu_g)$ with Esscher parameter $\xi \in (-\alpha_g - \beta_g, \alpha_g - \beta_g)$ corresponds to a change of parameter from β_g to $\beta_g + \xi$.*

We refer the reader to Proposition 2.11 in Raible (2000) for the proof. Proposition 2.3 indicates that choosing an equivalent probability measure $\mathbb{Q}$ corresponds to choosing an Esscher parameter ξ if we focus our attention to the Esscher transform. Furthermore, Equations (19) to (22) indicate that a change of measure with the Esscher transformation changes not only its first moment but also higher moments of the GH distribution via β_g.

2.5. A Proposed Numerical Procedure

In this paper, we employ an approximate dynamic programming (ADP for abbreviation) for solving Equation (8) where the expectation is taken under the reference probability measure $\mathbb{P}$. Although a basic idea for the numerical procedure is almost the same as the one developed in Imai and Tsujimura (2016), we review it here for keeping this paper self-contained. The ADP is an extension of dynamic programming, which combines simulation and approximation in order to solve the optimization problem sufficiently accurately and efficiently. In the paper, we directly solve the maxmin problem by computing the expected value of the function V.

Let us assume that both an investment strategy I and an Esscher transformed strategy ξ are fixed. We define a gain function J for given I and ξ with the current output price p and the current capital k as

$$J_0\left(p, k, I, \xi\right) = \mathbb{E}^{\mathbb{P}}\left[\int_0^\infty \mathrm{e}^{-rt} \hat{Y}_t \mathrm{d}t\right]. \tag{25}$$

With the gain function, we can define the value function V_0, which is given by

$$V_0\left(p, k\right) = \max_{I \in \mathcal{I}} \min_{\xi \in \Xi} J_0\left(p, k, I, \xi\right). \tag{26}$$

Let I^* and ξ^* denote the optimal investment and the optimal Esscher parameter of Equation (26), respectively. Because the value function can be reached when we choose both the optimal investment strategy and the optimal distortion strategy, it can be written by

$$V_0\left(p, k\right) = J_0\left(p, k, I^*, \xi^*\right), \tag{27}$$

for any p and k.

We next apply the dynamic programming principle to the investment problem and derive the Bellman equation.

$$\begin{aligned} V_0(k,p) &= \max_{I\in\mathcal{I}}\min_{\xi\in\Xi}\mathbb{E}^{\mathbb{P}}\left[\int_0^{\Delta t}\mathrm{e}^{-rt}\hat{Y}_t\mathrm{d}t + \mathrm{e}^{-r\Delta t}V_{\Delta t}\left(\hat{Y}_{\Delta t}\right)\right] \\ &= \max_{I\in\mathcal{I}}\min_{\xi\in\Xi}\mathbb{E}^{\mathbb{P}}\left[\mathrm{e}^{-r\Delta t}\hat{Y}_{\Delta t}\Delta t + \mathrm{e}^{-r\Delta t}V_{\Delta t}\left(\hat{Y}_{\Delta t}\right)\right] \\ &= \max_{I\in\mathcal{I}}\min_{\xi\in\Xi}\left\{\mathrm{e}^{-r\Delta t}\mathbb{E}^{\mathbb{P}}\left[\hat{Y}_{\Delta t}\right]\Delta t + \mathrm{e}^{-r\Delta t}\mathbb{E}_{\mathbb{P}}\left[V_{\Delta t}\left(\hat{Y}_{\Delta t}\right)\right]\right\} \end{aligned} \tag{28}$$

Equation (28) is the Bellman equation we use in the numerical algorithm.

Let us now develop a numerical procedure utilized in this paper for deriving the value function defined in Equation (28). It mainly consists of two iterative methods. The first iterative method is called a value iteration in which the value function for p and k is obtained by solving a fixed point problem. For technical details about the convergence of the value iteration are given, for example, in Bertsekas (2012). The second method is for obtaining optimal solutions with respect to the investment I and the Esscher parameter ξ provided that p and k are given.

For solving the optimization problem numerically, we first discretize the state space (p,k) into equally spaced grid points. Let $[P_{\min}, P_{\max}]$ be a closed output price space, and N_p be the number of grid points. The grid points for the output price are given by $p_i = P_{\min} + i\Delta p, i = 0,\ldots,N_p$ with $\Delta p = \frac{P_{\max}-P_{\min}}{N_p}$. Similarly, let $[K_{\min}, K_{\max}]$ be a closed capital space, and N_k be the number of grid points. The grid points for the capital are given by $k_j = K_{\min} + j\Delta k, j = 0,\ldots,N_k$ with $\Delta k = \frac{K_{\max}-K_{\min}}{N_k}$.

Note that due to the constraint for the GH distribution in Equation (14), we assume that the set Ξ satisfies this condition. In addition we assume that the set of investment $\mathcal{I}$ is also bounded due to its practical reason. Accordingly, let $\mathcal{I} := [I_{\min}, I_{\max}]$ and $\Xi := [\xi_{\min}, \xi_{\max}]$. Note that from Proposition 2.3, $\xi_{\min} \geq -\alpha_g - \beta_g$, $\xi_{\max} \leq \alpha_g - \beta_g$ must be satisfied to reserve the class of the GH Lévy processes.

The first iterative procedure can be generated based on Equation (28). Let $v^m(k_j, p_i)$ denote the $m-$th iterative value function with the state vector of (k_j, p_i). Then, $m-$th iteration is given by

$$v^{m+1}(k_j,p_i) = \max_{I\in\mathcal{I}}\min_{\xi\in\Xi}\left\{\mathrm{e}^{-r\Delta t}\Delta t\mathbb{E}^{\mathbb{P}}\left[\hat{Y}_{\Delta t}\right] + \mathrm{e}^{-r\Delta t}\mathbb{E}^{\mathbb{P}}\left[v^m\left(p_{\Delta t}, k_{\Delta t}(I)\right)\right]\right\}, \tag{29}$$

for $i = 0,\ldots,N_p, j = 0,\ldots,N_k$, where

$$p_{\Delta t} = p_i \exp\left\{X_{\Delta t}^{GH}(\lambda,\alpha,\beta,\delta,\mu)\right\}, \tag{30}$$

under the reference probability measure $\mathbb{P}$ and

$$k_{\Delta t}(I) = (1-\delta)k_j + I. \tag{31}$$

To derive the value functions via the value iteration method, we need to set initial value function for any k and p. In the paper, we set

$$v^0(k,p) := 0,$$

for all $p \in [P_{\min}, P_{\max}]$ and any $k \in [K_{\min}, K_{\max}]$. In the $m-$th iteration to compute each $v^{m+1}(k_j, p_i)$, we need to solve the maxmin problem in Equation (29) for every i and j.

Let us begin with the value iteration when $m = 0$. Provided that the current output price p and the current capital k are given, the optimization problem turns out to become the following simple expression with ignoring constant terms.

$$\max_{I \in \mathcal{I}} \min_{\xi \in \Xi} \mathbb{E}^{\mathbb{P}} \left[\hat{Y}_{\Delta t} \right]. \tag{32}$$

Lemma **??** indicates that the maxmin problem becomes a convex-concave problem and the globally optimal value can be theoretically guaranteed.

It is important to notice that in order to calculate the expectation, approximation methods are required since we assume that both output price and the capital can take real values, while we discretize the state space to make the iteration feasible. To that end, this paper introduces two approximation methods, that is, a spline interpolation method and a quasi-Monte Carlo method.

We first propose an function approximation method via a tensor-product, two-dimensional spline method. It enables us to calculate value functions with off grid points when values on grid point (k_j, p_i) are provided. In the paper, we employ a method proposed in page 347 of De Boor (1978). Let $\tilde{v}^m(k,p)$ denote an approximation function of the $m-$th value function. In the proposed numerical algorithm, $v^m(k,p)$ in equation (29) is replaced by the approximated value function $\tilde{v}^m(k,p)$.

Second, we employ a quasi-Monte Carlo method to compute an expectation in an efficient manner. The quasi-Monte Carlo method is a well-known simulation method that is often replaced by a Monte Carlo method for enhancing numerical efficiency. Instead of using random sequence in the Monte Carlo method, the quasi-Monte Carlo method uses a low-discrepancy sequence for generating sample points. It is well-known that the quasi-Monte Carlo often substantially outperforms the Monte Carlo for a wide range of problems in economics, finance and actuarial science. See, for instance, Joy et al. (1996), for classical applications of the method.

Let $u_n \in (0,1)$ be the $n-$th realized point from a low-discrepancy sequence with $n = 1, \dots, N$, and let $x_n = F_{GH}^{-1}(u_n), n = 1, \dots, N$, where F_{GH} represents a cumulative distribution function of the GH distribution, and F_{GH}^{-1} is its inverse. By Equation (30), the $n-$th realized output price under $\mathbb{P}-$measure can be given by

$$p_{\Delta t,n} = p_i \exp\left\{ x_{\Delta t,n}^{GH}(\lambda, \alpha, \beta, \delta, \mu) \right\}, \tag{33}$$

where $x_{\Delta t,n}^{GH}(\lambda, \alpha, \beta, \delta, \mu)$ represents n-th sample from the GH distribution. In implementing the quasi-Monte Carlo method we employ a random number generator of Imai (2013) to efficiently obtain samples from the GH distribution.

Then, the expected value in Equation (29) can be approximated by an average of N function values, that is,

$$\mathbb{E}^{\mathbb{P}}\left[v^m(k_{\Delta t}, p_{\Delta t}) \right] \approx \frac{1}{N} \sum_{n=1}^{N} \tilde{v}^m(k_{\Delta t}, p_{\Delta t,n}). \tag{34}$$

and

$$\mathbb{E}^{\mathbb{P}}\left[\hat{Y}_{\Delta t}\right] \approx \frac{1}{N}\sum_{n=1}^{N} Y_{\Delta t}\left(k_{\Delta t}, p_{\Delta t}\right), \tag{35}$$

respectively. The boundary conditions for numerical procedure are given as follows.

$$v^{m}\left(k, 0\right) = 0,$$

for any k, and

$$v^{m+1}\left(k, p\right) = v^{m}\left(k, P_{\max}\right), \text{ if } p > P_{\max},$$
$$v^{m+1}\left(k, p\right) = v^{m}\left(K_{\max}, p\right), \text{ if } k > K_{\max}.$$

The second iteration method is used to obtain a numerical solution of the minimax problem in Equation (29).

It should be emphasized from a computational viewpoint that for both of the iteration procedures are equipped with convergence guarantee, namely, the true value can be obtained when the number of iterations tends to infinity. However, despite this theoretical fact, practical numerical efficiency is not always promised. In fact, our preliminary numerical experiment clearly indicated that the numerical procedure sometimes required unbearable computational time. In order to further enhance numerical efficiency of the method, we implement the following two methods.

The first one can be called an adaptive grid refinement method in which we begin with a relatively course grid, i.e., small N_k and N_p, and use a finer grid when the maximum error becomes relatively small. It enables us to reduce a computational time for early phase of the iteration. In addition, we carefully choose initial values for each iteration. The second one is with respect to the learning rates It has been recognized among researchers that the choice of the learning rate has a significant impact on the computational efficiency. In the paper, we employ an adaptive method called AdaGrad developed in Duchi et al. (2011) to improve numerical efficiency of the convergence. They depend not only on the number of iterations but also historical values of the learning rate.

3. A Numerical Example

This section provides numerical results under a GH Lévy ambiguity. We show the firm's value, the optimal investment I^* and the optimal Esscher parameter ξ^*. By comparing them with the results given in Imai and Tsujimura (2016) we can discuss the effect of the ambiguity level on the optimal investment decision. The numerical results will be presented in the conference.

4. Concluding remarks

In this paper, we examined an optimal investment problem in the presence of higher level of ambiguity. To describe the higher level of ambiguity we assume that the underlying risk process follows an exponential Lévy process. This assumption enables us to consider the manager's misspecification for not only its average but also its variance, skewness, and kurtosis. By applying the Esscher transform under an exponential generalize hyperbolic Lévy process, the problem becomes easier to deal with while it is still sufficiently general for our purpose. Numerical results revealed the impact of the high level of ambiguity on the optimal investment policy.

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