Real Investments with Collar Options

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Abstract

A real collar option is devised for an active asset owner that guarantees a floor cash flow while imposing penalties on abnormally high cash flow levels. The embodied guarantee not only induces an earlier investment exercise compared with a without-collar arrangement but creates additional investment option value making the with-collar variant superior. The collar also incorporates a penalty. Although having no effect on the investment timing, the penalty reduces the option value with the result that costless collars can be engineered. Volatility increases make the collar less effective. In the event of the volatility increasing, the penalty has to be loosened to sustain with-collar superiority. The collar arrangement is compared with the reflecting boundary representation.

1 Introduction

Subsidy support policies may be offered by governments for promoting early investment in infrastructure and other public projects. Particularly PPP (Public Private Partnership) projects are introduced as part of an economic and political reform movement facilitating a deeper and wider private sector participation in public service provision because of their superiority in corporate management and private finance over existing public administration and bureaucracy, Hood (1991). Seen as growth enhancing, these policies are interpreted as a viable solution for a world following the 2007 global financial crash and subsequent EU sovereign debt crisis where unsustainably high public indebtedness prevails, de-leveraging is crucial and the economic outlook is very uncertain. They are designed to improve the attractiveness of projects by offering pricing arrangements that significantly reduce risk, reduce inputs below cost, or both, resulting in a significant decrease in the investment threshold and increase in the opportunity value. In this way, private capital is motivated not only to undertake these projects but to implement them early. However, these policies are alleged to distort the risk-return profile in favour of the private party and may be seen to be too generous. According to Shaoul et al. (2012), PPPs are expensive and have failed to deliver value for public money. Any analysis on the provision of subsidies and similar support programmes should consider not only the desirable outcome of inducing investment but also their potential to claw-back abnormally high returns.

We present the collar option as a suitable policy device for a government to induce investment by guaranteeing a floor in the face of adverse circumstances, and simultaneously capturing abnormally high returns when the circumstances are sufficiently favourable. Previous studies focus on offering an immediate investment subsidy to spur investor commitment, which is partly or fully reimbursed through the taxation of future profits. Implementing a collar similarly results in an earlier exercise due to the guarantee while its cost may be partially recouped by penalizing significantly high profits. The analysis of collars adopts a real option formulation because the implied guarantee and penalty are expressible as real options, the sunk cost is partly irretrievable, deferral flexibility is present, and uncertainty prevails. Using an American perpetuity model, we show that while the guarantee enhances the attractiveness of the with-collar compared to the without-collar opportunity and reduces its threshold resulting in an earlier exercise, the existence of the penalty is manifested only through a reduction in the real option value (ROV). This finding produces a straightforward method for engineering a collar because the guarantee level can first be ascertained from knowing the desired threshold prompting exercise, and the penalty level can then be determined from deriving the appropriate ROV (which may, or may not, be paid by the concession investor to the government). American perpetuity and European fixed maturity collars share the characteristic of involving the buying and selling of puts and calls, but the former is designed as an investment timing model.

There exist two relevant strands of studies on subsidies. The first strand considers the impacts of investment subsidy and profit taxation on investor behaviour. Its motivation is the determination of the optimal combination of subsidy and taxation under a variety of different contexts that result in inducing investment. The second strand examines the effects of particularly cash flow guarantees on the investment option value notably using a Monte-Carlo or similar simulation methods. The aim of this paper is to formulate and solve an analytical real option investment model representing both guarantees and penalties that yields an earlier timely investment, but where the value created is shared, not necessarily equally, between the investor and government.

By considering the combined effects of a subsidy and taxable future profits on the investment timing decision, Pennings (2000) shows using an analytical real option model that the investment threshold decreases with a subsidy even for a zero expected net revenue position. When extending the context to foreign direct investment, Pennings (2005) shows that investors are stimulated to make investments earlier in host countries where the governments offered subsidies that significantly offset the deterring effects of profit taxation and uncertainty, while Yu et al. (2007) establish that an entry cost subsidy is more effective than a tax rate reduction. These authors conclude that between the two policy alternatives for incentivizing investment, it is the subsidy rather than the tax cut that is dominant. The comparative merits of investment subsidy versus tax reduction are further examined by Danielova and Sarkar (2011) and Sarkar (2012). The former argues that in the presence of leverage, tax payments are influenced by borrowing levels, which in turn are influenced by the subsidy level. By enlarging the model to include leverage, the authors establish that an optimal policy for the government is to offer a combination of investment subsidy and tax reduction for inducing investors to commit earlier. Sarkar (2012) argues that differential borrowing rates are instrumental and shows that if a government uses a different discount rate from the market and has to borrow funds, then an optimal policy is to offer investment subsidy while charging a positive profit tax. Barbosa et al. (2016) re-interpret the role of government as an active player instead of a passive agent, who can undertake the investment but less efficiently and set differential taxes. In this extended model that also captures the multiplier effect of investing, the authors show that the subsidy acts more effectively than a tax reduction in inducing investment, but only up to some maximum level.

A second strand of investigation formulates a contractual relationship, usually underpinning a PPP deal, as a set of real options embedded in an active project. Most of these formulations adopt numerical techniques like Monte-Carlo simulation approach sometimes in conjunction with a binomial lattice for obtaining their findings, but some base their conclusions on an analytical real option framework. By evaluating numerically an actual toll road concession involving both a guarantee and penalty, Rose (1998) shows that the government guarantee contributes significant value to the project because returns are conserved at a minimum level. This is replicated using an alternative formulation by Alonso-Conde et al. (2007), who show that these guarantee not only act as incentives but also have the potential of generously transferring significant value to the investor. Cheah and Liu (2006) adopt a similar methodology to reach a similar finding in their investigation of a toll crossing concession. Garvin and Cheah (2004) discuss the advocacy of a real option formulation for capturing the value from deferment and guarantees embedded in PPP deals. The implied value of several interacting flexibilities for a rail concession are investigated by Bowe and Lee (2004), while Huang and Chou (2006) appraise minimum revenue guarantees and abandonment rights for a similar concession using a European-style framework. Blank et al. (2009) investigate the role of a graduated series of guarantees and penalties incurred when operating a toll road concession as a risk transfer device for avoiding bankruptcy that benefits both the investor and lender. Besides these numerical investigations, there are two key analytical studies. Takashima et al. (2010) design a PPP deal involving government debt participation that incorporates a floor on the future maximum loss level where the investor has the right to sell back the project whenever adverse conditions emerge. Using an analytical model, they show the effect of such deals on the investment timing decision. Also, Armada et al. (2012) make an analytical comparison of various subsidy policies and a demand guarantee scheme to reveal their differentiated qualities.

This paper is organized in the following way. The fundamental investment opportunity model (without a collar) is reproduced to act as a benchmark for comparing the qualities of the with-collar model. We then proceed to formulate the with-collar model analytically and examine its key properties. This requires developing the collar representation for an active project and incorporating its features within an investment opportunity model. The next section compares the collar formulation with a reflecting boundary model because they share a similar structure. Finally, further insights are gained from performing a numerical sensitivity analysis. The last section is a conclusion.

2 Fundamental Model

For a firm in a monopolistic situation confronting a single source of uncertainty due to output price variability, and ignoring operating costs and taxes, the opportunity to invest in an irretrievable project at cost K depends solely on the price evolution, which is specified by the geometric Brownian motion process:

$$\mathrm{d}P = \alpha P \mathrm{d}t + \sigma P \mathrm{d}W , \tag{1}$$

where α denotes the expected price risk-neutral drift, σ the price volatility, and $\mathrm{d}W$ an increment of the standard Wiener process. Using contingent claims analysis, the option to invest in the project $F(P)$ follows the risk-neutral valuation relationship:

$$\tfrac{1}{2}\sigma^2 P^2 \frac{\partial^2 F}{\partial P^2} + (r-\delta) P \frac{\partial F}{\partial P} - rF = 0 , \tag{2}$$

where $r > \alpha$ denotes the risk-free interest rate and $\delta = r - \alpha$ the rate of return shortfall. The generic solution to (2) is:

$$F(P) = A_1 P^{\beta_1} + A_2 P^{\beta_2} , \tag{3}$$

where A_1, A_2 are to be determined generic constants and β_1, β_2 are, respectively, the positive and negative roots of the fundamental equation, which are given by:

$$\beta_1, \beta_2 = \left(\tfrac{1}{2} - \frac{r-\delta}{\sigma^2} \right) \pm \sqrt{\left(\tfrac{1}{2} - \frac{r-\delta}{\sigma^2} \right)^2 + \frac{2r}{\sigma^2}} . \tag{4}$$

In (3), if $A_2 = 0$ then F, a continuously increasing function of P, represents an American perpetual call option, Samuelson (1965), while if $A_1 = 0$ then it is a decreasing function and represents a put option, Merton (1973), Merton (1990).

In the absence of other forms of optionality and a fixed output volume Y, a firm optimally invests when the value matching relationship linking the call option value and the net proceeds $PY/\delta - K$ is in balance:

$$A_0 P^{\beta_1} = PY/\delta - K. \tag{5}$$

Following standard methods, the without-collar optimal price threshold level triggering investment $\hat{P}_0$ is:

$$\hat{P}_0 = \frac{\beta_1}{\beta_1 - 1} \frac{\delta}{Y} K \tag{6}$$

and the value function is:

$$F_0(P) = \begin{cases} = \dfrac{K}{\beta_1 - 1}\left(\dfrac{P}{\hat{P}_0}\right)^{\beta_1} & \text{for } P < \hat{P}_0 \\ = \dfrac{PY}{\delta} - K & \text{for } P \geq \hat{P}_0, \end{cases} \tag{7}$$

with:

$$A_0 = \frac{\hat{P}_0^{1-\beta_1} Y}{\delta \beta_1} = \frac{K \hat{P}_0^{-\beta_1}}{\beta_1 - 1}. \tag{8}$$

3 Investment and Collar Option

3.1 Real Collar Option

A collar option is designed to confine the output price for an active project to a tailored range, by restricting its value to lie between a floor level P_L and a ceiling level P_H. Whenever the price trajectory falls below the floor, the received output price is assigned the value P_L, and whenever it exceeds the ceiling, it is assigned the value P_H. By restricting the price to this range, the firm is benefiting by receiving a price that never falls below P_L and is obtaining protection against adverse price movements, whilst at the same time, it is being forced never to receive a price exceeding P_H by sacrificing the upside potential. Protection against downside losses are mitigated in part by sacrificing upside gains. If as part of its subsidy policy, a government offers a firm a price collar in its provision of some output Y, the government compensates the firm by a positive amount equalling $(P_L - P)Y$ whenever

$P < P_L$, but if the ceiling is breached and $P > P_H$, then the firm reimburses the government by the positive amount $\left(P - P_H\right)Y$. It follows that for an active project, the revenue accruing to the firm is given by $\pi_C\left(P\right) = \min\left\{\max\left\{P_L, P\right\} P_H\right\} \times Y$ and its value V_C is described by the risk-neutral valuation relationship:

$$\tfrac{1}{2}\sigma^2 P^2 \frac{\partial^2 V_C}{\partial P^2} + \left(r - \delta\right) P \frac{\partial V_C}{\partial P} - rV_C + \pi_C\left(P\right) = 0. \tag{9}$$

The relationship (2) and (9) are identical in form except for the revenue function.

The valuation of an active project with a collar is conceived over three mutually exclusive exhaustive regimes, I, II and III, specified on the P line, each with its own distinct valuation function. Regimes I, II and III are defined by $P \le P_L$, $P_L < P \le P_H$ and $P_H \le P$, respectively. Over Regime I, the firm is granted a more attractive fixed price P_L compared with the variable price P, but also possesses a call-style option to switch to the more favourable Regime II as soon as P exceeds P_L. This switch option increases in value with P and has the generic form AP^{β_1}, where A denotes a to be determined generic coefficient. Over Regime III, the firm is not only obliged to accept the less attractive fixed price P_H instead of P but also has to sell a put-style option to switch to the less favourable Regime II as soon as P falls below P_H. This switch option decreases in value with P and has the generic form AP^{β_2}. Over Regime II, the firm receives the variable price P, possesses a put-style option to switch to the more favourable Regime I as soon as P falls to P_L, but sells a call-style option to switch to the less favourable Regime III as soon as P attains P_H. The various switch options are displayed in Table 1.

Table 1: Various Switch Options

From - To	**Option Type**	**Value**	**Sign of A**
I - II	Call	AP^{β_1}	+
II - I	Put	AP^{β_2}	+
II - III	Call	AP^{β_1}	-
III - II	Put	AP^{β_2}	-

If the subscript C denotes the collar arrangement, then after paying the investment cost, the valuation function for the firm managing the active project is formulated as:

$$V_C\left(P\right)=\begin{cases}\dfrac{P_L Y}{r}+A_{C11}P^{\beta_1} & \text{for } P<P_L \\ \dfrac{PY}{\delta}+A_{C21}P^{\beta_1}+A_{C22}P^{\beta_2} & \text{for } P_L\leq P<P_H \\ \dfrac{P_H Y}{r}+A_{C32}P^{\beta_2} & \text{for } P_H\leq P.\end{cases} \tag{10}$$

In (10), the first numerical subscript for the coefficients denotes the regime $\{1,2,3\}$, while the second denotes a call if 2 or a put if 1. The coefficients A_{C11}, A_{C22} are expected to be positive because the firm owns the options and a switch is beneficial. In contrast, the A_{C21}, A_{C32} are expected to be negative because the firm is selling the options and is being penalized by the switch. The real collar is composed of a pair of both call and put options. The first pair facilitates switching back and forth between Regimes I and II, which results in the firm being advantaged, while the second pair facilitates switching back and forth between Regimes II and III, which results in the firm being disadvantaged. The real collar design differs from the typical European collar that only involves buying and selling two distinct options.

A switch between Regimes I and II occurs when $P=P_L$. It is optimal provided the value-matching relationship:

$$\frac{P_L Y}{r}+A_{C12}P^{\beta_2}=\frac{PY}{\delta}+A_{C21}P^{\beta_1}+A_{C22}P^{\beta_2}\,, \tag{11}$$

and its smooth-pasting condition expressed as:

$$\beta_2 A_{C12}P^{\beta_2}=\frac{PY}{\delta}+\beta_1 A_{C21}P^{\beta_1}+\beta_2 A_{C22}P^{\beta_2} \tag{12}$$

both hold when evaluated at $P=P_L$. Similarly, a switch between Regimes II and III occurs when $P=P_H$. It is optimal provided the value-matching relationship:

$$\frac{PY}{\delta}+A_{C21}P^{\beta_1}+A_{C22}P^{\beta_2}=\frac{P_H Y}{r}+A_{C31}P^{\beta_1} \tag{13}$$

and its smooth-pasting condition expressed as:

$$\frac{PY}{\delta}+\beta_1 A_{C21}P^{\beta_1}+\beta_2 A_{C22}P^{\beta_2}=\beta_1 A_{C31}P^{\beta_1} \tag{14}$$

both hold when evaluated at $P = P_H$. This reveals that:

$$
\begin{aligned}
&A_{C11} = \left[\frac{P_H Y}{P_H^{\beta_1}} - \frac{P_L Y}{P_L^{\beta_1}}\right] \times \frac{(r\beta_2 - r - \delta\beta_2)}{(\beta_1 - \beta_2) r\delta} > 0,\; A_{C21} = \frac{P_H Y (r\beta_2 - r - \delta\beta_2)}{P_H^{\beta_1} (\beta_1 - \beta_2) r\delta} < 0,\\
&A_{C22} = \frac{-P_L Y (r\beta_1 - r - \delta\beta_1)}{P_L^{\beta_2} (\beta_1 - \beta_2) r\delta} > 0,\; A_{C32} = \left[\frac{P_H Y}{P_H^{\beta_2}} - \frac{P_L Y}{P_L^{\beta_2}}\right] \times \frac{(r\beta_1 - r - \delta\beta_1)}{(\beta_1 - \beta_2) r\delta} < 0.
\end{aligned}
\tag{15}
$$

The signs of the four option coefficients comply with expectations. Other findings can also be derived. The coefficient A_{C22} for the option to switch from Regimes II to I, which depends on only P_L and not on P_H, increases in size with P_L. This switch option becomes more valuable to the firm as the floor level increases. Similarly, the coefficient A_{C21} for the option to switch from Regimes II to III, which depends on only P_H and not on P_L, decreases in magnitude with P_H. This switch option becomes less valuable to the government as the ceiling level increases. The coefficients A_{C11} and A_{C32} for the switch option from Regimes I to II and from Regimes III to II, respectively, depend on both P_L and P_H.

3.2 Investment Option

We conjecture that the with-collar optimal price threshold $\hat{P}_C$ triggering an investment lies between the floor and ceiling limits, $P_L \le \hat{P}_C \le P_H$. The floor limit holds because no optimal solution exists in its absence, that is for $\hat{P}_C < P_L$. We subsequently demonstrate that $\hat{P}_C$ attains a minimum of $P_L = rK/Y$ and a maximum of $\hat{P}_0$ for $P_L = 0$, so the introduction of a price floor always produces at least an hastening of the investment exercise and never its postponement. The ceiling limit holds because of the absence of any effective economic benefit from exercising at a price exceeding the ceiling. Initially the price can be presumed to be near zero and the investment option treated as out-of-the-money. With the passage of time, the price trajectory can be expected to reach the ceiling P_H before reaching some level exceeding P_H, and since the value outcome $P_H Y/r$ is the same for both $P = P_H$ and $P > P_H$, there is no gain in waiting. The following analysis treats the threshold $\hat{P}_C$ as lying between the lower and upper limits.

When $P_L \le \hat{P}_C \le P_H$, the optimal solution is obtained from equating the investment option value with the active project net value at the threshold $P = \hat{P}_C$. The optimal solution is determined from both the value-matching relationship:

$$A_{C0} P^{\beta_1} = \frac{PY}{\delta} + A_{C21} P^{\beta_1} + A_{C22} P^{\beta_2} - K \tag{16}$$

and its smooth-pasting condition expressed as:

$$\beta_1 A_{C0} P^{\beta_1} = \frac{PY}{\delta} + \beta_1 A_{C21} P^{\beta_1} + \beta_2 A_{C22} P^{\beta_2} \tag{17}$$

when evaluated for $P = \hat{P}_C$. This reveals that:

$$\frac{\hat{P}_C Y}{\delta} = \frac{\beta_1}{\beta_1 - 1} K - \frac{\beta_1 - \beta_2}{\beta_1 - 1} A_{C22} \hat{P}_C^{\beta_2}, \tag{18}$$

$$\begin{aligned} A_{C0} &= \frac{K \hat{P}_C^{-\beta_1}}{\beta_1 - 1} - \left(\frac{1 - \beta_2}{\beta_1 - 1} \right) A_{C22} \hat{P}_C^{\beta_2 - \beta_1} + A_{C21} \\ &= \frac{1}{\beta_1 - \beta_2} \left[\left(1 - \beta_2\right) \frac{\hat{P}_C Y}{\delta} + \beta_2 K \right] \hat{P}_C^{-\beta_1} + A_{C21}. \end{aligned} \tag{19}$$

The absence of a closed-form solution requires $\hat{P}_C$ to be solved numerically from (18), and A_{C0} from (19). The investment option value $F_{C0}\left(P\right)$ for the project is:

$$F_{C0}\left(P\right) = \begin{cases} A_{C0} P^{\beta_1} & \text{for } P < \hat{P}_C \\ \dfrac{PY}{\delta} - K + A_{C21} P^{\beta_1} + A_{C22} P^{\beta_2} & \text{for } \hat{P}_C \le P < P_H, \end{cases} \tag{20}$$

where $P_L \le \hat{P}_C \le P_H$.

From (18), the threshold $\hat{P}_C$ depends only on the floor P_L through A_{C22}, but not on the ceiling P_H. Adjusting the ceiling of the collar has no material impact on the threshold, so the timing decision is affected by the losses foregone by having a floor but not by the gains sacrificed by having a ceiling. Since A_{C22} is non-negative, the with-collar threshold $\hat{P}_C$ is always no greater than the without-collar threshold $\hat{P}_0$, and an increase in the floor produces an earlier exercise due to the reduced threshold level. However, the floor cannot increase without bound and consequently the with-collar threshold has a lower limit. In (18), if $P_L = 0$, then $A_{C22} = 0$ and $\hat{P}_C = \hat{P}_0$, the optimal investment threshold without a floor. Further, if $\hat{P}_C = P_L$, then $P_L Y = rK$ and consequently the investment threshold equals the zero NPV (Net Present Value) solution, since the project remains being viable whatever price trajectory emerges subsequent to exercise due to the presence of the floor level. It follows that the corresponding bounds for the optimal investment trigger level $\hat{P}_C$ and the price floor level P_L are $\left(P_L, \hat{P}_0\right)$ and $\left(0, rK/Y\right)$, respectively, and that $\hat{P}_C$ is a decreasing function of P_L.

An investment opportunity with a collar having only a floor is always more valuable than one without, and this value increases as the floor becomes increasingly more generous. We show this by establishing that the investment option coefficient A_{C0} with $A_{C21} = 0$, (19), is always at least greater then A_0, (8), and that A_{C0} is an increasing function of $\hat{P}_C$. Since $\hat{P}_0 \geq \hat{P}_C$ then from (19):

$$A_{C0} \geq \frac{1}{\beta_1 - \beta_2}\left[\left(1-\beta_2\right)\frac{\hat{P}_0 Y}{\delta} + \beta_2 K\right]\hat{P}_0^{-\beta_1} = \frac{K\hat{P}_0^{-\beta_1}}{\beta_1 - 1}.$$

In the absence of a ceiling, having a floor is always at least as valuable as not having a floor. Further, by differentiating (19) with respect to $\hat{P}_C$, A_{C0} is an increasing function. However, if a collar contains both a floor and a ceiling, then the sign and magnitude of the switch option coefficient A_{C21} have to be taken into account. This coefficient is negative and its magnitude decreases towards zero as P_H becomes increasingly large, so the negative effect of a ceiling on A_{C0} is strongest and most significant for relatively low P_H levels. This means that for sufficiently low P_H levels, $A_{C0} < A_0$ implying that an investment opportunity without a collar is more valuable than one with a collar despite the latter having a lower investment threshold and an earlier exercise time.

3.3 Price Floor Model

We use the additional subscript f to indicate a model with only a floor. From (10) the active project valuation function becomes:

$$V_{Cf}\left(P\right) = \begin{cases} \dfrac{P_L Y}{r} + A_{Cf11} P^{\beta_1} & \text{for } P \leq P_L \\ \dfrac{PY}{\delta} + A_{Cf22} P^{\beta_2} & \text{for } P_L \leq P, \end{cases} \tag{21}$$

with:

$$A_{Cf11}=\frac{-P_L Y\left(r\beta_2-r-\delta\beta_2\right)}{P_L^{\beta_1}\left(\beta_1-\beta_2\right)r\delta}\geq 0,\ A_{Cf22}=\frac{-P_L Y\left(r\beta_1-r-\delta\beta_1\right)}{P_L^{\beta_2}\left(\beta_1-\beta_2\right)r\delta}\geq 0. \tag{22}$$

The investment option value is specified by:

$$F_{C0f}\left(P\right)=\begin{cases}A_{Cf0}P^{\beta_1} & \text{for } P\leq\hat{P}_{Cf}\\ \dfrac{PY}{\delta}+A_{Cf22}P^{\beta_2}\text{ - }K & \text{for } \hat{P}_{Cf}\leq P,\end{cases} \tag{23}$$

with $\hat{P}_{Cf}=\hat{P}_C$ determined from (18) with A_{C22} replaced by A_{Cf22}, and the investment option coefficient by:

$$A_{Cf0}=\frac{1}{\beta_1-\beta_2}\left[\left(1-\beta_2\right)\frac{\hat{P}_C Y}{\delta}+\beta_2 K\right]\hat{P}_C^{-\beta_1}\geq A_0 \tag{24}$$

A feasible floor on the revenue for an active project induces a more valuable investment opportunity that is exercisable at an earlier time. Consequently, a floor represents a government granted subsidy, Armada et al. (2012).

3.4 Price Ceiling Model

We use the additional subscript c to indicate a model with only a ceiling. From (10) the active project valuation function becomes:

$$V_{Cc}\left(P\right)=\begin{cases}\dfrac{PY}{\delta}+A_{Cc21}P^{\beta_1} & \text{for } P<P_H\\ \dfrac{P_H Y}{r}+A_{C32}P^{\beta_2} & \text{for } P_H\leq P,\end{cases} \tag{25}$$

with:

$$A_{Cc21}=\frac{P_H Y}{P_H^{\beta_1}}\frac{r\beta_2-r-\delta\beta_2}{r\left(\beta_1-\beta_2\right)\delta}\leq 0,\quad A_{Cc32}=\frac{P_H Y}{P_H^{\beta_1}}\frac{r\beta_1-r-\delta\beta_1}{r\left(\beta_1-\beta_2\right)\delta}\leq 0\,. \tag{26}$$

The investment option value is specified by:

$$F_{C0c}\left(P\right)=\begin{cases}A_{Cc0}P^{\beta_1} & \text{for } P\leq\hat{P}_{Cc}\\ \dfrac{PY}{\delta}+A_{Cc21}P^{\beta_1}\text{ - }K & \text{for } \hat{P}_{Cc}\leq P\leq P_H,\end{cases} \tag{27}$$

with $\hat{P}_{Cc}=\hat{P}_0$ determined from (6), and investment option coefficient:

$$A_{Cc0} = \frac{K\hat{P}_{Cc}^{1-\beta_1}}{\beta_1 - 1} + A_{Cc21} \leq A_0 . \tag{28}$$

The imposition of a ceiling has no effect on the investment threshold and the timing, but it does produce a less valuable investment option. It is significantly less desirable than an opportunity without a ceiling, and consequently it is imposed by, for example, a government intent on offering a subsidy while reducing its cost, or by limits to growth due to firm or market characteristics.

4 Investment and Reflecting Boundary Model

4.1 Reflecting Boundary

In a way, two reflecting boundaries resemble a perpetual collar. A costless reflecting boundary is characterized by a lower or upper limit, which when hit downwards from above or upwards from below, respectively, reflects the price trajectory without cost. When the lower and upper boundaries are defined by $P = P_L$ and $P = P_H$, respectively, the two reflecting boundaries confine the price trajectory to only Regime II making Regimes I and III inaccessible. If the price hits the lower boundary P_L, it is reflected up with a price outcome of at least P_L, while if it hits the upper boundary P_H, its reflection results in a price of no more than P_H. This affects the value that can be generated by the active project, and consequently the investment threshold and the ROV. The real option analysis involving a reflecting boundary is examined by Dumas (1991). The analytical solution requires identifying the active project value for Regime II from (10) and determining the values of the relevant unknown coefficients by setting at the particular reflection boundary the first derivative of the project value equal to zero because the reflection is treated as costless:

$$\left.\frac{\partial V_C\left(P \middle| P_L \leq P \leq P_H\right)}{\partial P}\right|_{P=P_L} = \left.\frac{\partial V_C\left(P \middle| P_L \leq P \leq P_H\right)}{\partial P}\right|_{P=P_H} = 0. \tag{29}$$

Using the subscript R to denote the reflecting boundary model, (29) yields two equations that can be expressed as:

$$\frac{P_L Y}{\delta} + \beta_1 A_{R21} P_L^{\beta_1} + \beta_2 A_{R22} P_L^{\beta_2} = \frac{P_H Y}{\delta} + \beta_1 A_{R21} P_H^{\beta_1} + \beta_2 A_{R22} P_H^{\beta_2} = 0 , \tag{30}$$

which produces the following solutions for A_{R21} and A_{R22}:

$$A_{R21} = \left(\frac{P_L^{\beta_2} P_H - P_L P_H^{\beta_2}}{P_L^{\beta_1} P_H^{\beta_2} - P_L^{\beta_2} P_H^{\beta_1}}\right)\frac{Y}{\beta_1 \delta} \leq 0;\, A_{R22} = \left(\frac{P_L P_H^{\beta_1} - P_L^{\beta_1} P_H}{P_L^{\beta_1} P_H^{\beta_2} - P_L^{\beta_2} P_H^{\beta_1}}\right)\frac{Y}{\beta_2 \delta} \geq 0\,. \tag{31}$$

In (31), the signs of the reflecting boundary coefficient are as expected. In contrast to the collar model solution, (15), both A_{R21} and A_{R22} depend on both P_L and P_H. However, if $P_H = 0$, then $A_{R21} = 0$ and A_{R22} depends on only P_L while if $P_L = 0$, then $A_{R22} = 0$ and A_{R21} depends on only P_H.

4.2 Investment Option

If the price threshold signalling an optimal investment exercise is denoted by $\hat{P}_R$, from (16), the value matching relationship is specified by:

$$A_{R0}\hat{P}_R^{\beta_2} = \frac{\hat{P}_R Y}{\delta} + A_{R21}\hat{P}_R^{\beta_1} + A_{R22}\hat{P}_R^{\beta_2} - K\,. \tag{32}$$

Reproducing equivalent equations as (18) - (19) for the investment threshold and option coefficient for the reflecting boundary model yields, respectively:

$$\frac{\hat{P}_R Y}{\delta} = \frac{\beta_1}{\beta_1 - 1}K - \frac{\beta_1 - \beta_2}{\beta_1 - 1}A_{R22}\hat{P}_R^{\beta_2}\,, \tag{33}$$

$$A_{R0} = \frac{1}{\beta_1 - \beta_2}\left[(1-\beta_2)\frac{\hat{P}_R Y}{\delta} + \beta_2 K\right]\hat{P}_R^{-\beta_1} + A_{R21}\,. \tag{34}$$

There are certain similarities between the solutions for the collar and reflecting boundary models. In the absence of an upper reflecting boundary, $A_{R21} = 0$. Then, due to the similarity between (34) with (19), the investment option coefficient A_{R0} is at least as large as that for the fundamental model without a reflecting boundary, A_0, which means that between the two opportunities, the one having a lower reflecting boundary is the more valuable. However, when the upper reflecting boundary is also present, the option coefficient A_{R0} is effectively reduced by the amount A_{R21} and it is impossible to discern which of the two coefficients, A_0 and A_{R0} is the greater. The investment threshold solution (33) is similar to (18). Provided $A_{R22} > 0$ due to the presence of a lower reflecting boundary, the resulting threshold is less than $\hat{P}_0$, the threshold for the fundamental model, otherwise they are identical. Further, if

$P_L > 0$, then the threshold $\hat{P}_R$ is determined by both the P_L and P_H levels, but is always less than $\hat{P}_0$.

The investment option value for the project is:

$$F_{R0}(P) = \begin{cases} A_{R0}P^{\beta_1} & \text{for } P < \hat{P}_R \\ \dfrac{PY}{\delta} - K + A_{R21}P^{\beta_1} + A_{R22}P^{\beta_2} & \text{for } \hat{P}_R \le P, \end{cases} \tag{35}$$

where $P_L \le \hat{P}_R \le P_H$.

4.3 Lower Reflecting Boundary

The absence of an upper reflecting boundary implies that $A_{Rf\,21} = 0$ and $A_{Rf\,22} = -YP_L^{1-\beta_2} / \beta_2\delta > 0$, so:

$$\frac{\hat{P}_{Rf}Y}{\delta} = \frac{\beta_1}{\beta_1 - 1}K - \frac{\beta_1 - \beta_2}{\beta_1 - 1}A_{Rf\,22}\hat{P}_{Rf}^{\beta_2}, \tag{36}$$

$$A_{Rf\,0} = \frac{1}{\beta_1 - \beta_2}\left[(1-\beta_2)\frac{\hat{P}_{Rf}Y}{\delta} + \beta_2 K\right]\hat{P}_{Rf}^{-\beta_1}, \tag{37}$$

$$F_{Rf\,0}(P) = \begin{cases} A_{Rf\,0}P^{\beta_1} & \text{for } P < \hat{P}_{Rf} \\ \dfrac{PY}{\delta} - K + A_{Rf\,22}P^{\beta_2} & \text{for } \hat{P}_{Rf} \le P. \end{cases} \tag{38}$$

From (36), the investment threshold has to always exceed the lower boundary P_L to ensure that PY/δ is at least equal the investment cost and the project yields a non-negative NPV, a finding that contrasts with that for the collar option. However, both these two representations produce an investment threshold that is less than $\hat{P}_0$, the fundamental model level (6).

4.4 Upper Reflecting Boundary

Without a lower reflecting boundary, the coefficients for the upper reflecting boundary model are $A_{Rc21} = -YP_H^{1-\beta_1} / \beta_1\delta < 0$, $A_{Rc22} = 0$, so:

$$\frac{\hat{P}_{Rc}Y}{\delta} = \frac{\beta_1}{\beta_1 - 1}K, \tag{39}$$

$$A_{Rc0} = \frac{K\hat{P}_{Rc}^{-\beta_1}}{\beta_1 - 1} + A_{Rc21}, \tag{40}$$

$$F_{Rc0}(P) = \begin{cases} A_{Rc0}P^{\beta_1} & \text{for } P < \hat{P}_{Rc} \\ \dfrac{PY}{\delta} - K + A_{Rc21}P^{\beta_1} & \text{for } \hat{P}_{Rc} \le P. \end{cases} \tag{41}$$

The investment threshold, (39) is identical to that for the fundamental model, (6) and the option coefficient, (40) is always less than that for the fundamental model, (8), making any opportunity having an upper reflecting boundary less valuable.

5 Numerical Illustrations

Although the analytical results reveal some interesting properties, further insights into model behaviour is only obtainable from numerical evaluations. These are developed using the base case parameter values presented in Table 2. The evaluated power parameters for these values are $\beta_1 = -0.7369$ and $\beta_2 = 1.7369$ from (4), with $\hat{P}_0 = 9.4279$ and $A_{20} = 2.7547$, from (6) and (8), respectively. In this section, we consider first the behaviour of the switch options for the collar model before proceeding to the properties of the investment threshold and option value of the investment opportunity for the collar model, and ending with a comparative investigation of the reflecting boundary model.

Table 2: Base Case Parameter Values

K	100
Y	1
σ	0.25
δ	0.04
r	0.04

5.1 Collar Switch Option

Based on the Table 2 values, we illustrate in Table 3 the evaluated switch option coefficients, $A_{C11}, A_{C21}, A_{C22}, A_{C32}$ in Panels A-D, respectively, for various floor and ceiling levels. The floor levels are chosen to vary between a minimum $P_L = 0$ and a maximum $P_L = rK/Y = 4$, and the ceiling levels between a minimum $P_H = 10$, slightly in excess of $\hat{P}_0$, and a maximum

$P_H = \infty$. As expected, each of the 4 coefficients adopts the correct sign, A_{C21} is independent of P_L and A_{C22} of P_H, while A_{C11}, A_{C32} depend on both. Further, A_{C11}, the coefficient for the option to switch from Regime I to II, decreases with P_L but increases with P_H, since for any feasible Regime I price level, the switch option is more valuable for lower P_L levels because of the time value of money and that the price level is closer to P_L, and for higher P_H levels because less is being sacrificed. Similarly, A_{C32}, the coefficient for the option to switch from Regime III to II increases in magnitude with P_H because of the time value of money and decreases with P_L because less is being sacrificed. Finally, A_{C21}, the coefficient for the option to switch from Regime II to III decreases in magnitude with P_H because less is being sacrificed at higher P_H levels, while A_{C22}, the coefficient for the option to switch from Regime II to I increases with P_L because more is being gained for higher P_L levels. Note that the coefficients for the price floor are also available from Table 3 in the rows where $P_H = \infty$, while those for the price ceiling model are available from the columns where $P_L = 0$.

****Table 3 about here****

The switch option value $F_C(P)$ is derived from the active asset value $V_C(P)$, (10):

$$F_C(P) = \begin{cases} A_{C11} P^{\beta_1} & \text{for } P < P_L \\ A_{C21} P^{\beta_1} + A_{C22} P^{\beta_2} & \text{for } P_L \le P < P_H \\ A_{C32} P^{\beta_2} & \text{for } P_H \le P. \end{cases} \tag{42}$$

The difference between V_C and F_C is the long run value in the absence of any optionality. Since $V_C - F_C$ differs for each of the 3 regimes, F_C would normally experience a discontinuity jump at $P = P_L$ and $P = P_H$. However, in our case, since r and δ are selected to be equal, the discontinuity jumps are absent. Figures 1a and 1b illustrate the effect of P_L and P_H variations on $F_C(P)$ for constant P_H and P_L, respectively. These profiles tend to follow a similar pattern, being positive for P values around P_L where the owned option to switch between Regimes I and II dominates, and negative around P_H where the sold option to switch between Regimes II and III dominates. In Figure 1a where P_H is held constant, a P_L increase shifts the profile upwards for $P > P_L$ that reflects the enhanced switch option value due to the gain in downside protection. In Figure 1b where P_L is held constant, the switch option is more valuable around $P = P_L$ for higher P_H because of the lower probability of switching between Regimes II and III, but the switch option is less valuable around $P = P_H$ for higher P_H because of the higher probability of switching between Regimes III and II.

**** Figures 1a and 1b about here****

5.2 Investment Option

Based on Table 2 values, the solutions for variations in P_L and P_H, where P_L varies between 0 and rK/Y and P_H between 10 and infinity are illustrated in Table 4, where Panel A exhibits the threshold $\hat{P}_C$, (18), and Panel B the option coefficient A_{C0}, (19). As expected, the threshold declines as P_L increases within its allowable range, showing that an earlier exercise is achievable only for improvements in the floor. The locus relating the threshold $\hat{P}_C$ with the floor P_L defined by (18) is illustrated in Figure 2, which reveals not only the feasible limits of $\hat{P}_C$ and P_L, but also their negative relationship. In contrast, the choice of ceiling P_H has no effect on the threshold and the timing decision. In Panel B of Table 4, the option coefficient is observed to move in line with positive changes in P_L or P_H. A P_L increase raises the extent of the downside protection thereby making the investment option more attractive, while a P_H increase reduces the extent of the upside sacrifice thereby making it more valuable. In Table 4, the results for the floor model are obtainable from the row where $P_H = \infty$, and for the ceiling model from the column where $P_L = 0$.

****Table 4 and Figure 2 about here****

The relationship between the before and after exercise investment value, with and without a collar, and price is illustrated in Figure 3a and b. In Figure 3a, we select the collar levels as $P_L = 4.0$ and $P_H = 20.0$, which yield a threshold of $\hat{P}_C = 4.000$ and option coefficient $A_{C0} = 2.5270$. Despite having a higher threshold level, which suggests an earlier exercise for the collar variant if exercised, the collarless variant is always preferable by having a greater option coefficient. In contrast, in Figure 3b we select $P_L = 4.0$ and $P_H = 50.0$, which yields

$\hat{P}_C = 4.000$, the identical threshold as before, but a higher option coefficient $A_{C0} = 3.0726$. During the before exercise period when $P \leq \hat{P}_C$, the with-collar variant is a better choice than the without-collar variant, by having a greater option coefficient and therefore a greater investment option value. But, for price increases beyond $\hat{P}_C = 4.000$, the investment value for the with-collar variant begins to fade as the floor and ceiling options are activated, while that for the without-collar variant continues to increase and at a greater rate, with the consequence that the latter value begins to outstrip the former. Despite this, the with-collar variant remains a better choice because it has a greater investment option value and a lower price threshold. Governments can offer investors a collar as a form of subsidy, since if appropriately designed, it can advance the investment timing decision and create value for the investor. However, the collar does incur the government a cost equalling $\left(A_0 - A_{C0}\right)\hat{P}_C^{\beta_1}$. This cost is in fact a liability, and although its payable and receivable elements only materialize whenever $P < P_L$ or $P > P_H$, respectively, as such it may not appear as a sovereign debt at the time of its announcement and subsequent investment.

****Figures 3a and b about here****

The cost of the subsidy can be neutralized and the collar made "costless" by suitably engineering its floor and ceiling levels. It can be designed in the following way: (i) the without-collar option coefficient A_0 is evaluated from (8), (ii) for some pre-specified value of the collar threshold $\hat{P}_C$, perhaps equalling the prevailing price, the implied floor P_L can be determined from (18) because of its invariance with P_H, and finally (iii), by setting $A_{C0} = A_0$ the implied ceiling is determined from (19). Some illustrative "costless" P_L and P_H pairs are presented in Table 5. The pairs are inversely related, as expected, since for the collar to remain "costless", any increase in the floor and reduction in downside risk has to be compensated by an additional sacrifice in upside potential.

Table 5

Illustrative Pairs of P_L and P_H for a "Costless" Collar

P_L	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
P_H	9430.5	1780.0	649.94	307.02	164.71	93.963	54.011	27.302
$\hat{P}_C$	9.350	9.163	8.879	8.493	7.986	7.318	6.375	4.000
A_{C0}	2.755	2.755	2.755	2.755	2.755	2.755	2.755	2.755
A_{C21}	-0.012	-0.041	-0.085	-0.148	-0.235	-0.355	-0.534	-0.883
A_{C22}	3.032	10.106	20.437	33.685	49.632	68.123	89.038	112.280

We can make the following conclusions. In the presence of a stochastic output price, a collar option can be designed that protects the investor from downside risk by limiting adverse prices to a floor while simultaneously compelling the investor to forego favourable prices above a ceiling. This trade-off between upside potential and downside risk can be engineered to make the collar-variant to be more valuable as well as supporting an earlier exercise. The floor and ceiling affect the solution in distinct ways. Variations in P_H have no effect at all on the investment threshold, but the sacrifice of additional upside potential is reflected in decreases in the investment option coefficient. In contrast, an improvement in P_L and reduction in downside risk produces both a fall in investment threshold prompting an earlier exercise and a rise in the investment option coefficient making it more valuable. When designing a collar, initial attention focuses on the floor in determining the threshold for ensuring the investment has a timely exercise, and then on the ceiling in assessing the extent of the value created by the floor is to be sacrificed. While a viable floor increase for a collar motivates early exercise as well as enhancing its attractiveness, a ceiling decrease incurs a sacrifice leading to a reduction in its attractiveness.

5.2.1 Changes in Volatility

In the absence of a collar, a volatility increase is known to accompany a rise in both the investment threshold and investment option value, Dixit and Pindyck (1994). By using Table 2 values except that the volatility σ varies incrementally to 50%, we compare the impact of volatility changes on the with- and without-collar solutions for $P_L = 3$ and $P_H = 500$. Figures 4a and 4b illustrate the volatility effect on the threshold and option coefficient, respectively. The threshold for the without-collar variant is shown in Figure 4a to increase at a faster rate as expected because $P_L > 0$ and $\hat{P}_C < \hat{P}_0$, so the with-collar variant possesses a lower threshold and an earlier timing for all positive σ. In Figure 4b, although the option coefficient for the without-collar variant increases with volatility as expected, that for the with-collar variant tends to follow a S-shape. This means that the two profiles intersect. For $\sigma < 34\%$, the with-collar variant possesses the larger option coefficient and since $\hat{P}_C < \hat{P}_0$ it is exercised at an earlier time in preference to the without-collar variant. However, if $\sigma \geq 34\%$ then the without-collar variant has the larger option coefficient and is exercised despite the lower threshold of the with-collar variant.

The comparative timing decisions for the with- and without-collar variants remain essentially unaltered in the presence of a volatility change, because if $P_L > 0$ then $\hat{P}_C < \hat{P}_0$ while if $P_L = 0$ then $\hat{P}_C = \hat{P}_0$ for all positive σ. However, a volatility increase can produce a distinctive change in the with-collar option value, which can result in a change of the more preferred variant. If for low σ, the chance of a price trajectory penetrating the ceiling is insignificant, then the magnitude of the switch option coefficient A_{C21} is similarly insignificant and consequently the option coefficient is virtually unaffected. However, as σ increases, the chance of penetrating the ceiling becomes increasingly significant and likewise the coefficient A_{C21}, with the consequence that increases in the with-collar option coefficient begin to retard and falter enabling the without-collar option coefficient to assume dominance. In the design of a collar, if a government perceives a likely volatility increase to be imminent, then the ceiling has to be adjusted upwards to ensure its acceptance by the investor community.

**** Figures 4a and 4b about here ****

5.3 Reflecting Boundary Model

The reflecting boundary coefficients, A_{R21} and A_{R22}, are evaluated for the Table 2 values and illustrated in Table 6, Panel C and D, respectively. The feasible range of lower P_L and upper P_H boundaries are determined by generating the numerical relationship between the threshold $\hat{P}_R$ with P_L and P_H from (33). In contrast to the with-collar model, the threshold $\hat{P}_R$ depends on both P_L and P_H. An illustration of their feasible ranges are exhibited in Figure 5 for variations in $\hat{P}_R$ and P_L with the upper boundary fixed for $P_H = 10$ and $P_H = 50$. Similar to $\hat{P}_C$, the threshold $\hat{P}_R$ is confined to a minimum of $rK/Y = 4.0$ and a maximum of

$\hat{P}_0$, and the relationship between $\hat{P}_R$ and P_L is negative. Although the effect of variations in P_H on $\hat{P}_R$ and P_L is significant, the effect is modest, with the greatest effect occurring for $\hat{P}_R$ at the bottom of its range. Maintaining P_L constant, a P_H increase produces a $\hat{P}_R$ decrease because less is being sacrificed. In Table 6, the two coefficients adopt the correct sign, and A_{R21} decreases in magnitude with both P_L and P_H, while A_{R22} increases in magnitude with both P_L and P_H, as expected. Table 6 Panel A and B illustrate the investment threshold $\hat{P}_R$ and investment option coefficient A_{R0}, respectively. As expected, the threshold declines while the option coefficient rises for increases in both P_L and P_H, since any enhancement in P_L and relaxation in P_H lead to an earlier exercise and more valuable option.

**** Table 6 and Figure 5 about here ****

A contrast between the reflecting boundary and collar models can be constructed by comparing their results for the "costless" solution, which occurs when their option values equal that for the fundamental model while keeping their investment thresholds equal. The "costless" solution for the collar model is presented in Table 5. The "costless" solution for the reflecting boundary model is found by observing the similarity between (18) and (33), and between (19) and (34) so for $A_{R0} = A_{C0}$ with $\hat{P}_{R0} = \hat{P}_{C0}$ then $A_{R21} = A_{C21}$ and $A_{R22} = A_{C22}$. The implied floor and ceiling levels for the reflecting boundary model are evaluated from (31) based on the relevant Table 5 values and are presented in Table 7. The "costless" solutions for the two models share the common feature of the pairs of implied floor and ceiling levels being inversely related, since any improvement in the floor and reduction in downside risk has to be compensated by foregoing some of the upside potential. But, the floor and ceiling envelope is greater and less restricting for the reflecting boundary than for the collar model. If the price trajectory for the collar model penetrates the floor or ceiling limit, then the received price is maintained at that limit until the trajectory again penetrates the limit but from the opposite direction. In contrast, the price for the reflecting boundary model is reflected whenever it hits the floor or ceiling limit. Then because the solution is "costless", the limits for the collar model are more restrictive than those for the reflecting boundary model.

Table 7

Illustrative Pairs of P_L and P_H for a "Costless" Reflecting Boundary

P_L	0.249	0.498	0.749	1.002	1.259	1.524	1.803	2.126
P_H	15239.1	2876.3	1050.3	496.12	266.14	151.77	87.146	43.839
$\hat{P}_R$	9.350	9.163	8.879	8.493	7.986	7.318	6.375	4.000
A_{R0}	2.755	2.755	2.755	2.755	2.755	2.755	2.755	2.755
A_{R21}	-0.012	-0.041	-0.085	-0.148	-0.235	-0.355	-0.534	-0.883
A_{R22}	3.032	10.106	20.437	33.685	49.632	68.123	89.038	112.280

6 Conclusion

A collar is constructed using a switch option framework for an active project to establish that alternative policy devices exist capable of inducing investment and creating additional opportunity value for the investor as a viable alternative to the traditional combination of subsidy and profit taxation. By engineering the collar, its floor and ceiling properties can be ascertained from knowing the desired timing or investment threshold and the acceptable amount of additional opportunity value the government is willing to transfer to the investor. Collars can be designed to be "costless" by incurring no additional opportunity value to the government, but at the same time, the government benefits from investors committing at an earlier exercise time. One merit of the collar is that governments do not have to make upfront payments to investors to induce their commitment, unlike the subsidy-taxation policy, since all transfers between the investor and government are liabilities contingent on future cash flows of the active project.

Appendix: A Single Switch Collar Option

The real option collar presented above is designed in a way that allows multiple and continual switching between Regimes II and I and between Regimes II and III, since every switch entails creating an option to switch back. Alternatively, the collar design may impose a single switch constraint that forbids any recourse to the original regime. In this design, a sufficiently large negative or positive cumulative price movement causes an irreversible switch from Regime II to I or III, respectively. At exercise, the threshold price is assumed to lie between the floor and ceiling price. For limited price movements, the prevailing price is confined to Regime II, but if the price change is sufficiently great to cause a regime shift to I or III, then that particular regime rules irrespective of the future price trajectory. The absence of an opportunity to switch back and its potential opportunity loss suggest that the switch decision should be accompanied with a degree of caution and the price switch thresholds not be selected to coincide with the floor and ceiling levels. We define the price thresholds signalling a switch to Regime I and III by $\hat{P}_L$ and $\hat{P}_H$, respectively. The coefficients for the single switch model are designated by the subscript S.

From (11), the optimal switch between Regime II to I is governed by the value matching relationship:

$$\frac{P_L Y}{r} = \frac{\hat{P}_L Y}{\delta} + A_{S21}\hat{P}_L^{\beta_1} + A_{S22}\hat{P}_L^{\beta_2}, \tag{43}$$

with associated smooth pasting condition expressed as:

$$0 = \frac{\hat{P}_L Y}{\delta} + \beta_1 A_{S21}\hat{P}_L^{\beta_1} + \beta_2 A_{S22}\hat{P}_L^{\beta_2}. \tag{44}$$

From (13), the optimal switch between Regime II and III is governed by the value matching relationship:

$$\frac{\hat{P}_H Y}{\delta} + A_{S21}\hat{P}_H^{\beta_1} + A_{S22}\hat{P}_H^{\beta_2} = \frac{P_H Y}{r}, \tag{45}$$

with associated smooth pasting condition expressed as:

$$\frac{\hat{P}_H Y}{\delta} + \beta_1 A_{S21}\hat{P}_H^{\beta_1} + \beta_2 A_{S22}\hat{P}_H^{\beta_2} = 0. \tag{46}$$

Combining (43) - (46) yields:

$$A_{S21} = \frac{-\frac{\hat{P}_L Y}{\delta}\hat{P}_H^{\beta_2} + \frac{\hat{P}_H Y}{\delta}\hat{P}_L^{\beta_2}}{D\beta_1} = \frac{\left(\frac{P_L Y}{r} - \frac{\hat{P}_L Y}{\delta}\right)\hat{P}_H^{\beta_2} - \left(\frac{P_H Y}{r} - \frac{\hat{P}_H Y}{\delta}\right)\hat{P}_L^{\beta_2}}{D},$$

$$A_{S22} = \frac{-\frac{\hat{P}_L Y}{\delta}\hat{P}_H^{\beta_1} + \frac{\hat{P}_H Y}{\delta}\hat{P}_L^{\beta_1}}{D\beta_2} = \frac{\left(\frac{P_H Y}{r} - \frac{\hat{P}_H Y}{\delta}\right)\hat{P}_L^{\beta_1} - \left(\frac{P_L Y}{r} - \frac{\hat{P}_L Y}{\delta}\right)\hat{P}_H^{\beta_1}}{D},$$

where $D = \hat{P}_L^{\beta_1}\hat{P}_H^{\beta_2} - \hat{P}_L^{\beta_2}\hat{P}_H^{\beta_1} > 0$. Then:

$$\frac{\hat{P}_L Y}{\delta}\hat{P}_L^{-\beta_2} - \frac{\hat{P}_H Y}{\delta}\hat{P}_H^{-\beta_2} = \frac{\beta_1}{\beta_1 - 1}\left(\frac{P_L Y}{r}\hat{P}_L^{-\beta_2} - \frac{P_H Y}{r}\hat{P}_H^{-\beta_2}\right), \tag{47}$$

$$\frac{\hat{P}_H Y}{\delta}\hat{P}_H^{-\beta_1} - \frac{\hat{P}_L Y}{\delta}\hat{P}_L^{-\beta_1} = \frac{\beta_2}{\beta_2 - 1}\left(\frac{P_H Y}{r}\hat{P}_H^{-\beta_1} - \frac{P_L Y}{r}\hat{P}_L^{-\beta_1}\right). \tag{48}$$

Table 3

Switch Option Coefficients for the With-Collar Model for Variations in Floor and Ceiling Levels

Panel A: A_{C11}

P_H	$P_L = 0$	$P_L = 1$	$P_L = 2$	$P_L = 3$	$P_L = 4$
10	0.0000	8.2537	4.2116	2.6454	1.7862
20	0.0000	8.9944	4.9523	3.3862	2.5270
50	0.0000	9.5400	5.4979	3.9317	3.0726
100	0.0000	9.7663	5.7241	4.1580	3.2988
200	0.0000	9.9020	5.8599	4.2937	3.4346
500	0.0000	10.0020	5.9599	4.3937	3.5345
1000	0.0000	10.0435	6.0013	4.4352	3.5760
Infinity	0.0000	10.1057	6.0635	4.4974	3.6382

Panel B: A_{C21}

P_H	$P_L = 0$	$P_L = 1$	$P_L = 2$	$P_L = 3$	$P_L = 4$
10	-1.8520	-1.8520	-1.8520	-1.8520	-1.8520
20	-1.1112	-1.1112	-1.1112	-1.1112	-1.1112
50	-0.5656	-0.5656	-0.5656	-0.5656	-0.5656
100	-0.3394	-0.3394	-0.3394	-0.3394	-0.3394
200	-0.2036	-0.2036	-0.2036	-0.2036	-0.2036
500	-0.1037	-0.1037	-0.1037	-0.1037	-0.1037
1000	-0.0622	-0.0622	-0.0622	-0.0622	-0.0622
Infinity	0.0000	0.0000	0.0000	0.0000	0.0000

Panel C: A_{C22}

P_H	$P_L = 0$	$P_L = 1$	$P_L = 2$	$P_L = 3$	$P_L = 4$
10	0.000	10.106	33.685	68.123	112.280
20	0.000	10.106	33.685	68.123	112.280
50	0.000	10.106	33.685	68.123	112.280
100	0.000	10.106	33.685	68.123	112.280
200	0.000	10.106	33.685	68.123	112.280
500	0.000	10.106	33.685	68.123	112.280
1000	0.000	10.106	33.685	68.123	112.280
Infinity	0.000	10.106	33.685	68.123	112.280

Panel D: A_{C32}

P_H	$P_L = 0$	$P_L = 1$	$P_L = 2$	$P_L = 3$	$P_L = 4$
10	-551	-541	-518	-483	-439
20	-1838	-1828	-1804	-1770	-1726
50	-9027	-9017	-8994	-8959	-8915
100	-30090	-30080	-30057	-30022	-29978
200	-100299	-100289	-100265	-100231	-100187
500	-492596	-492586	-492562	-492528	-492484
1000	-1641948	-1641938	-1641914	-1641880	-1641836
Infinity	-Infinity	-Infinity	-Infinity	-Infinity	-Infinity

Table 4

Option Threshold and Coefficient Values for the With-Collar Model for Variations in Floor and Ceiling Levels

Panel A: $\hat{P}_C$

P_H	$P_L = 0$	$P_L = 1$	$P_L = 2$	$P_L = 3$	$P_L = 4$
10	9.4279	9.1627	8.4930	7.3178	4.0000
20	9.4279	9.1627	8.4930	7.3178	4.0000
50	9.4279	9.1627	8.4930	7.3178	4.0000
100	9.4279	9.1627	8.4930	7.3178	4.0000
200	9.4279	9.1627	8.4930	7.3178	4.0000
500	9.4279	9.1627	8.4930	7.3178	4.0000
1000	9.4279	9.1627	8.4930	7.3178	4.0000
Infinity	9.4279	9.1627	8.4930	7.3178	4.0000

Panel B: A_{C0}

P_H	$P_L = 0$	$P_L = 1$	$P_L = 2$	$P_L = 3$	$P_L = 4$
10	0.9028	0.9434	1.0513	1.2581	1.7862
20	1.6435	1.6842	1.7920	1.9989	2.5270
50	2.1891	2.2298	2.3376	2.5444	3.0726
100	2.4153	2.4560	2.5638	2.7707	3.2988
200	2.5511	2.5918	2.6996	2.9064	3.4346
500	2.6511	2.6917	2.7996	3.0064	3.5345
1000	2.6925	2.7332	2.8410	3.0479	3.5760
Infinity	2.7547	2.7954	2.9032	3.1101	3.6382

Table 6
Solution to the Reflecting Boundary Model for Varying Floor and Ceiling Levels

Panel A: $\hat{P}_R$

P_H	$P_L = 0.0$	$P_L = 0.5$	$P_L = 1.0$	$P_L = 1.5$	$P_L = 2.0$
10	9.428	9.191	8.668	7.902	6.815
20	9.428	9.179	8.597	7.678	6.175
50	9.428	9.170	8.543	7.498	5.514
100	9.428	9.166	8.520	7.419	5.134
200	9.428	9.164	8.506	7.372	4.831
500	9.428	9.162	8.496	7.336	4.508
1000	9.428	9.162	8.492	7.321	4.285
Infinity	9.428	9.161	8.486	7.299	4.016

Panel B: A_{R0}

P_H	$P_L = 0.0$	$P_L = 0.5$	$P_L = 1.0$	$P_L = 1.5$	$P_L = 2.0$
10	0.117	0.166	0.276	0.441	0.681
20	1.172	1.213	1.311	1.477	1.769
50	1.949	1.989	2.090	2.273	2.650
100	2.271	2.312	2.415	2.608	3.037
200	2.465	2.505	2.611	2.810	3.277
500	2.607	2.648	2.755	2.959	3.460
1000	2.666	2.707	2.815	3.021	3.539
Infinity	2.755	2.796	2.904	3.114	3.627

Panel C: A_{R21}

P_H	$P_L = 0.0$	$P_L = 0.5$	$P_L = 1.0$	$P_L = 1.5$	$P_L = 2.0$
10	-2.638	-2.625	-2.598	-2.563	-2.524
20	-1.583	-1.580	-1.575	-1.568	-1.559
50	-0.806	-0.805	-0.805	-0.804	-0.803
100	-0.483	-0.483	-0.483	-0.483	-0.483
200	-0.290	-0.290	-0.290	-0.290	-0.290
500	-0.148	-0.148	-0.148	-0.148	-0.148
1000	-0.089	-0.089	-0.089	-0.089	-0.089
Infinity	-0.000	-0.000	-0.000	-0.000	-0.000

Panel D: A_{R22}

P_H	$P_L = 0.0$	$P_L = 0.5$	$P_L = 1.0$	$P_L = 1.5$	$P_L = 2.0$
10	0.000	9.064	27.801	52.133	80.035
20	0.000	9.507	30.212	58.533	92.667
50	0.000	9.836	32.028	63.441	102.566
100	0.000	9.973	32.785	65.503	106.756
200	0.000	10.055	33.241	66.744	109.282
500	0.000	10.115	33.576	67.659	111.146
1000	0.000	10.140	33.716	68.038	111.919
Infinity	0.000	10.178	33.924	68.607	113.079

Figure 1a
Effect of Price on Switch Option Value
for Two Different Floor Levels

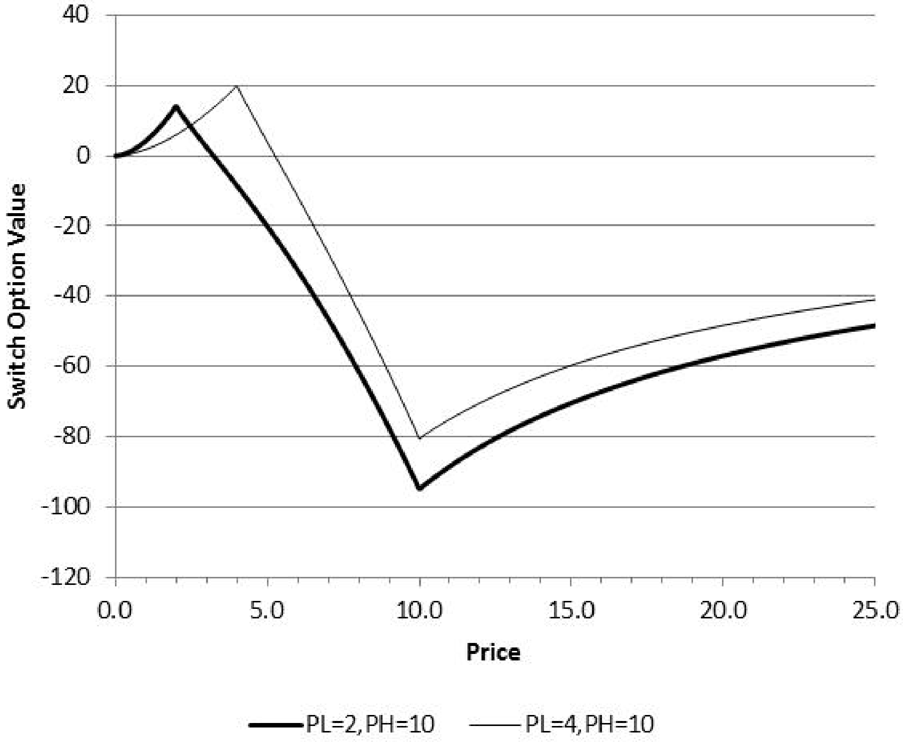


Figure 1b
Effect of price on Switch Option Value
for Two Different Ceiling Levels

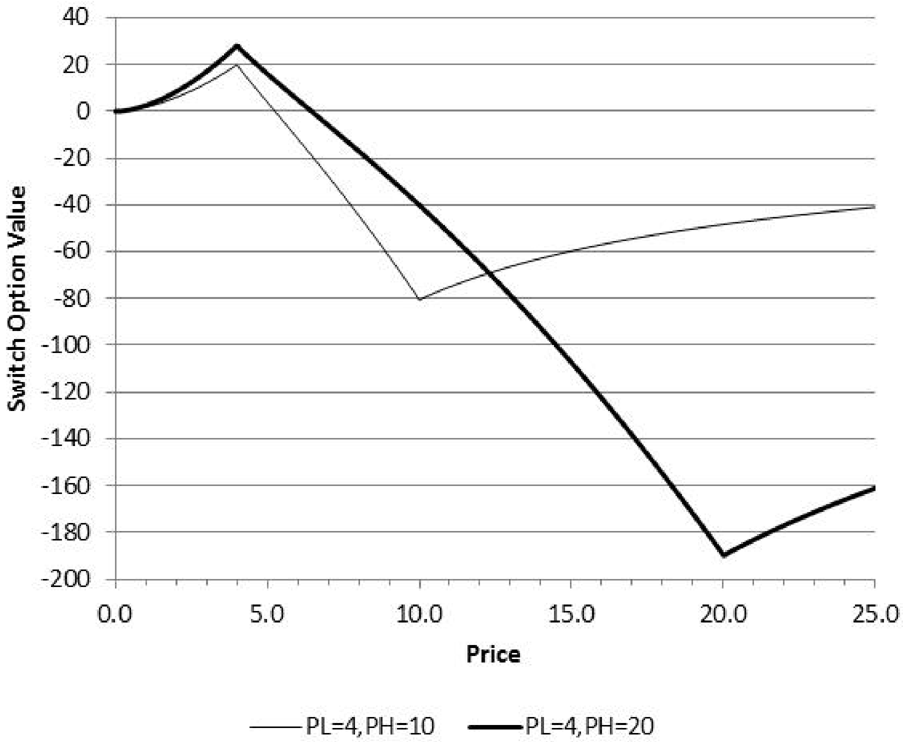


Table 2

Relationship between Floor and Threshold for the Collar Model

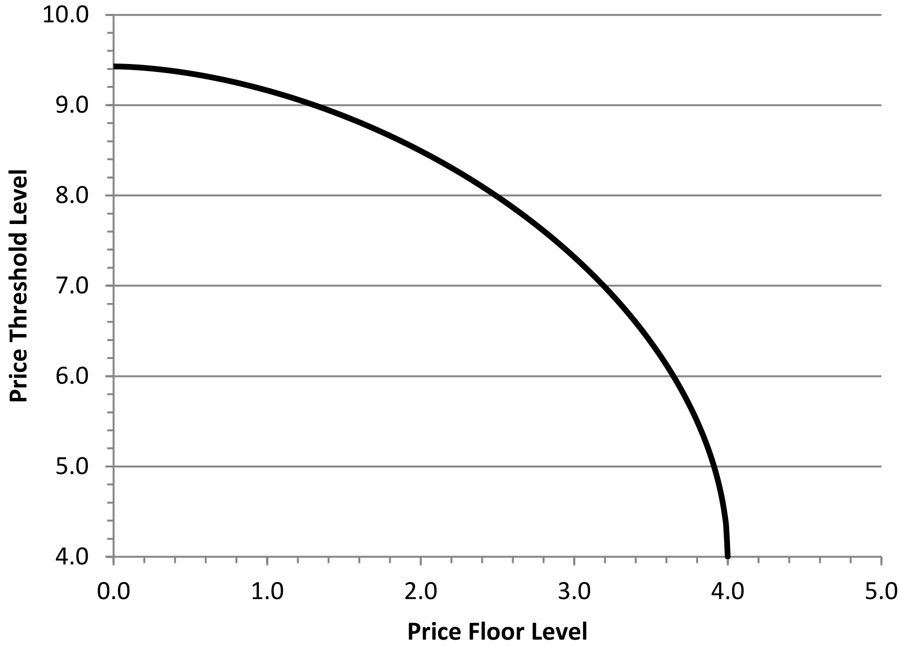
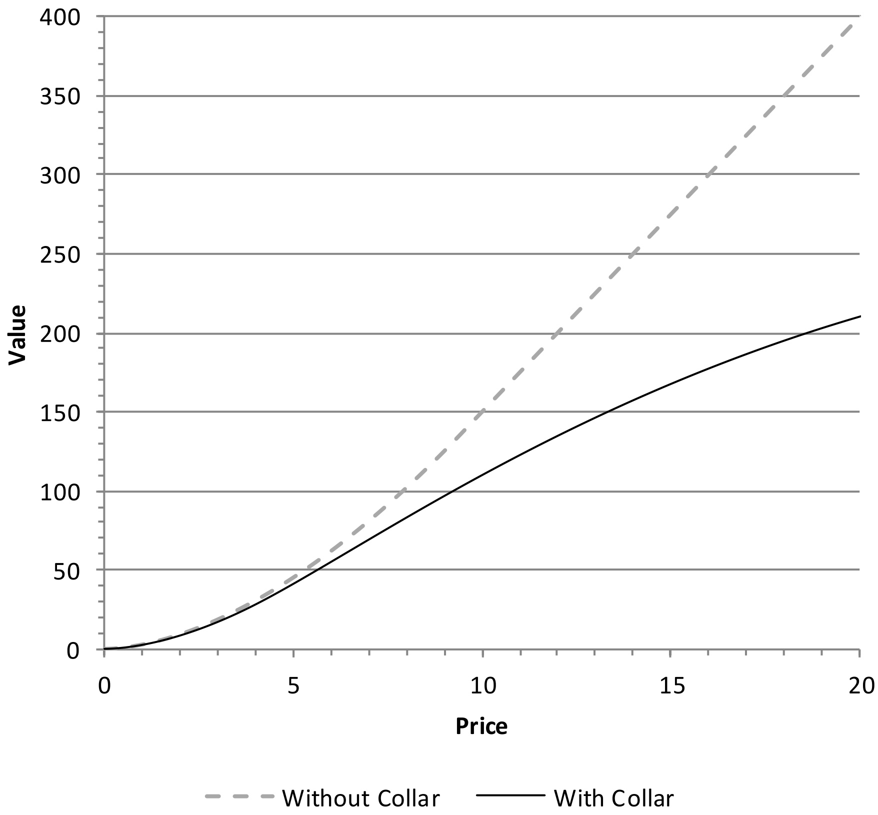
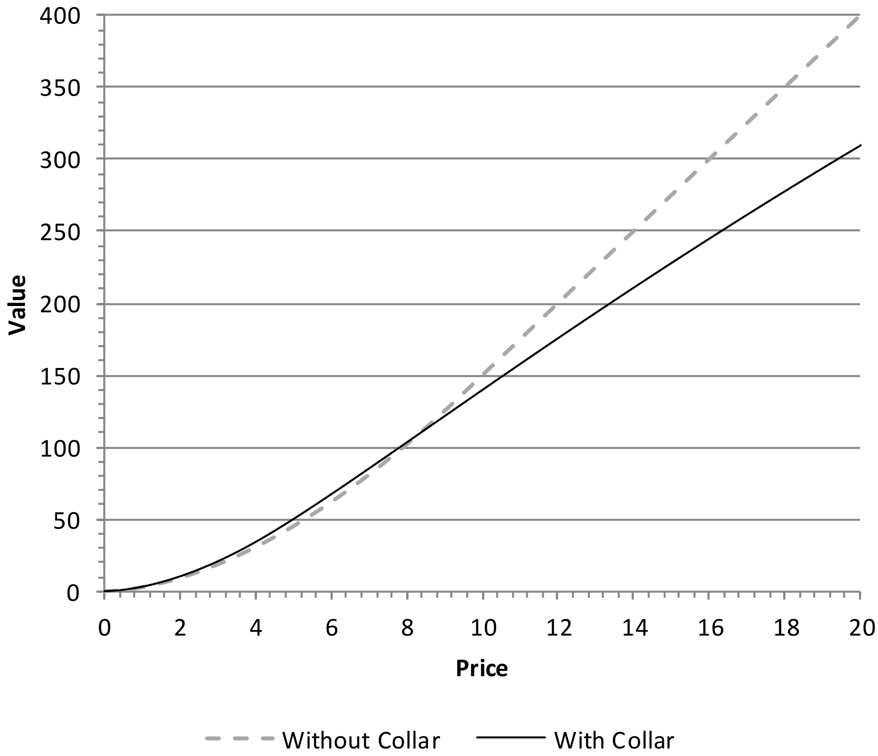


Figure 3a
The Effect of Price on the Investment Value
for the With- and Without-Collar Variants



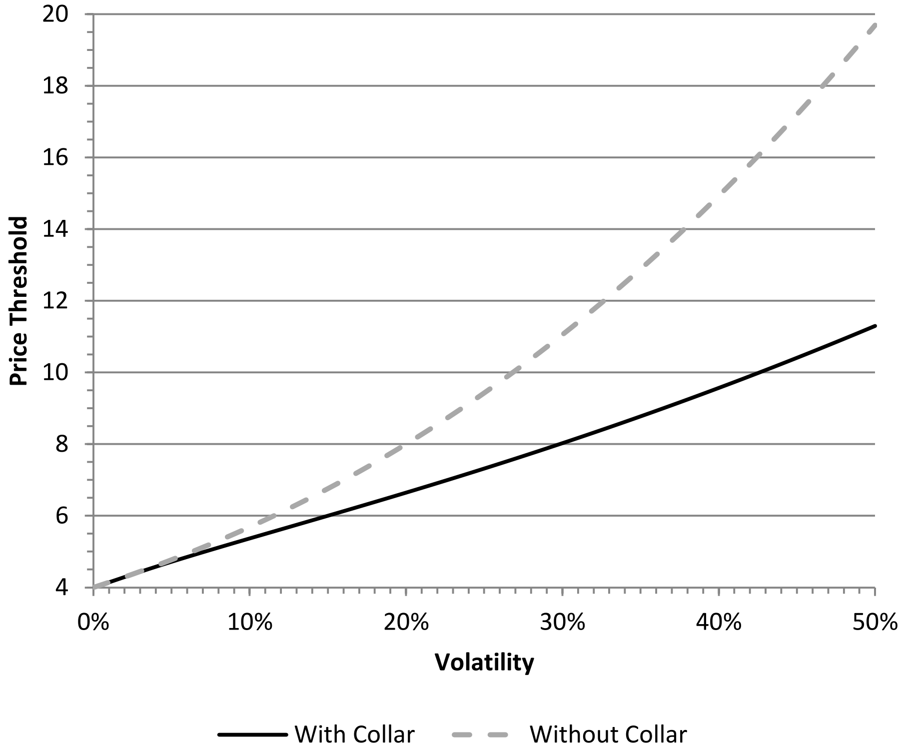
The floor and ceiling prices for the collar variant are $P_l = 4.0$ and $P_H = 50.0$, respectively. The evaluations for the two variants are based on Table 1 values. The solution values for the collarless variant are $A_0 = 2.7547$ and $\hat{P}_0 = 9.4273$, while those for the collar variant are drawn from Table 3 and 4.

Figure 3b
The Effect of Price on the Investment Value
for the With- and Without-Collar Variants



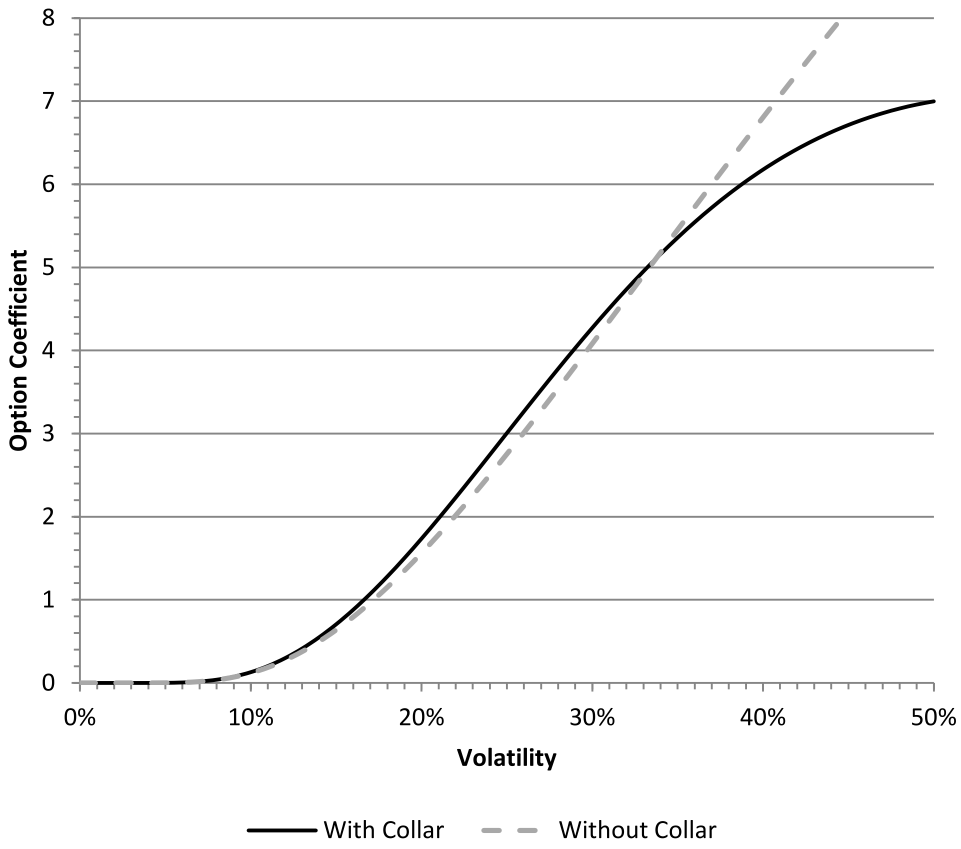
The evaluations for the two variants are based on Table 1 values, and the floor and ceiling prices for the collar variant are $P_l = 4.0$ and $P_H = 50.0$, respectively. The solution values for the collarless variant are $A_0 = 2.7547$ and $\hat{P}_0 = 9.4273$, while those for the collar variant are drawn from Table 3 and 4.

Figure 4a
Effect of Volatility Variations on the Price Thresholds for the
With- and Without-Collar Variants



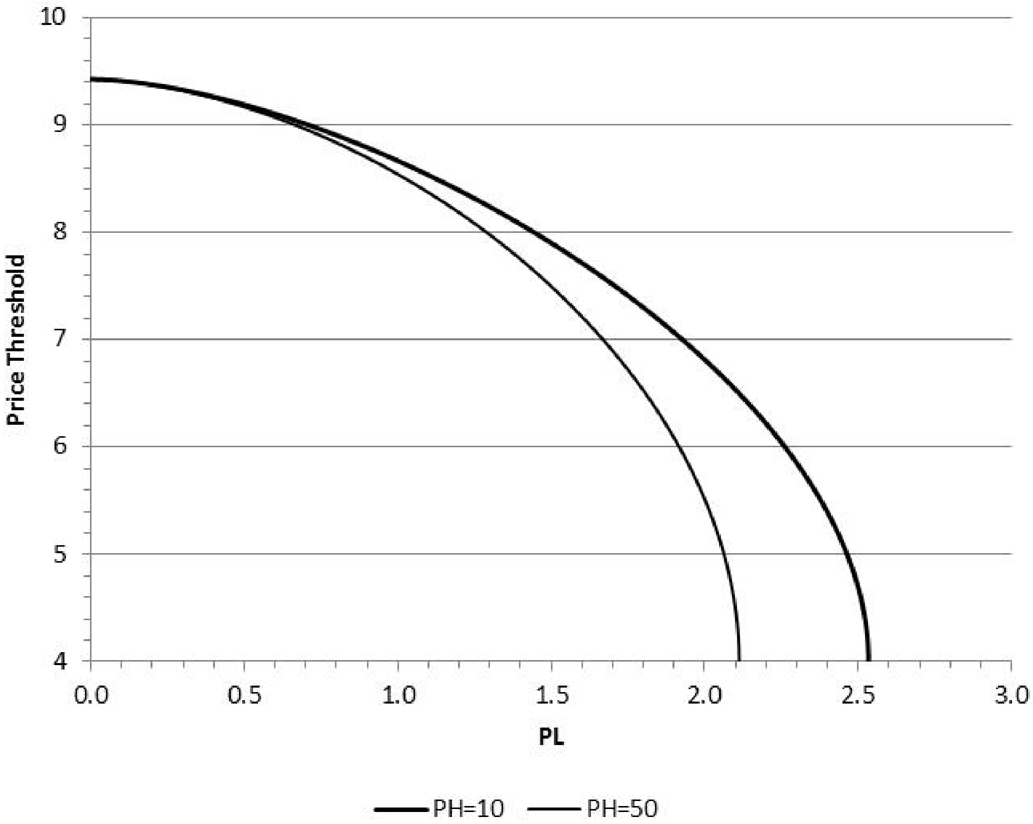
The evaluations for the two variants are based on Table 1 values, and the floor and ceiling prices for the collar variant are $P_l = 3.0$ and $P_H = 500.0$, respectively.

Figure 4b
Effect of Volatility Variations on the Option Coefficient for the
With- and Without-Collar Variants



The evaluations for the two variants are based on Table 1 values, and the floor and ceiling prices for the collar variant are $P_l = 3.0$ and $P_H = 500.0$, respectively.

Figure 5
Feasible Combinations of the investment Threshold, Floor and Ceiling
For the Reflecting Boundary Model



References

Alonso-Conde, A. B., C. Brown, and J. Rojo-Suarez. "Public private partnerships: Incentives, risk transfer and real options." *Review of Financial Economics* 16 (2007), 335-349.
Armada, M. R. J., P. J. Pereira, and A. Rodrigues. "Optimal subsidies and guarantees in public–private partnerships." *The European Journal of Finance* 18 (2012), 469-495.
Barbosa, D., V. M. Carvalho, and P. J. Pereira. "Public stimulus for private investment: An extended real options model." *Economic Modelling* 52, Part B (2016), 742-748.
Blank, F. F., T. K. N. Baidya, and M. A. G. Dias. "Real options in public private partnerships: A case of a toll road concession." In *Real Options Conference*. Braga, Portugal (2009).
Bowe, M., and D. L. Lee. "Project evaluation in the presence of multiple embedded real options: evidence from the Taiwan High-Speed Rail Project." *Journal of Asian Economics* 15 (2004), 71-98.
Cheah, C. Y. J., and J. Liu. "Valuing governmental support in infrastructure projects as real options using Monte Carlo simulation." *Construction Management and Economics* 24 (2006), 545-554.
Danielova, A., and S. Sarkar. "The effect of leverage on the tax-cut versus investment-subsidy argument." *Review of Financial Economics* 20 (2011), 123-129.
Dixit, A. K., and R. S. Pindyck. *Investment under Uncertainty*. Princeton, NJ: Princeton University Press (1994).
Dumas, B. "Super contact and related optimality conditions." *Journal of Economic Dynamics and Control* 15 (1991), 675-685.
Garvin, M. J., and C. Y. J. Cheah. "Valuation techniques for infrastructure investment decisions." *Construction Management and Economics* 22 (2004), 373-383.
Hood, C. "A public management for all seasons?" *Public Administration* 69 (1991), 3-19.
Huang, Y. L., and S. P. Chou. "Valuation of the minimum revenue guarantee and the option to abandon in BOT infrastructure projects." *Construction Management and Economics* 24 (2006), 379-389.
Merton, R. C. "Theory of rational option pricing." *Bell Journal of Economics and Management Science* 4 (1973), 141-183.
—. *Continuous-Time Finance*. Oxford: Blackwell Publishing (1990).
Pennings, E. "Taxes and stimuli of investment under uncertainty." *European Economic Review* 44 (2000), 383-391.
—. "How to maximize domestic benefits from foreign investments: The effect of irreversibility and uncertainty." *Journal of Economic Dynamics and Control* 29 (2005), 873-889.
Rose, S. "Valuation of interacting real options in a tollroad infrastructure project." *The Quarterly Review of Economics and Finance* 38 (1998), 711-723.
Samuelson, P. A. "Rational theory of warrant pricing." *Industrial Management Review* 6 (1965), 13-32.
Sarkar, S. "Attracting private investment: Tax reduction, investment subsidy, or both?" *Economic Modelling* 29 (2012), 1780-1785.
Shaoul, J., A. Stafford, and P. Stapleton. "The fantasy world of private finance for transport via public private partnerships." OECD Publishing, Discussion Paper No. 2012-6 (2012).
Takashima, R., K. Yagi, and H. Takamori. "Government guarantees and risk sharing in public–private partnerships." *Review of Financial Economics* 19 (2010), 78-83.
Yu, C.-F., T.-C. Chang, and C.-P. Fan. "FDI timing: Entry cost subsidy versus tax rate reduction." *Economic Modelling* 24 (2007), 262-271.