NORDIC NUGGET: MOSSIN MOTHBALLING MODEL

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Abstract

Forty-eight years ago Mossin published the first quantified real option scale model leading to other exit/entry models. Several aspects of this "Nordic nugget" are explored and expanded, showing the sensitivity of mothballing (temporary suspension) and reactivation thresholds and proxies for option value as a function of critical parameter values. A major contribution (and requirement) of Mossin's model is reflecting upper and lower barriers, which are missing from most current popular scale models. Twenty years ago Dixit recast the Mossin model in continuous time. The Dixit contributions and problems are analyzed and compared with the Mossin results. Both approaches present interesting challenges for future research.

JEL Classifications: B26, C41, G31, L91, R42
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1. INTRODUCTION

Jan Mossin published possibly the first real option scale model in 1968, with optimal thresholds for mothballing an active ship, and for reactivating a mothballed ship, assuming revenues follow a stationary random walk with reflecting barriers, and constant operating, maintenance, layup and reactivation costs. This characteristic stochastic process has not been commonly adopted in the real option literature, nor has Mossin been universally acknowledged as the Nordic godfather of real options. Indeed in the classic Nordic sponsored real options symposium of 1990, Mossin was not included in the author index or cited by any of the authors in Lund and Øksendal (1991)[1].

Perhaps Mossin conceived his mothballing and reactivation decisions as somewhat similar to transfers from securities to and from cash balances, given stochastic net cash flows. Miller and Orr (1966) (cited by Mossin) assume net cash flows for a firm follow a stationary random walk, characterized as a sequence of independent Bernoulli trials, with a probability p of an increase of "m" dollars, or a decrease with probability (1-p) over a fraction of a day (1/t). Securities earning an interest rate of "r" are sold and transferred instantaneously when the cash balances fall to a lower bound, and purchased when cash reaches an upper bound at a transfer cost "γ" in order to restore cash to the same average level. Similar discussions for random walks might have been available for Mossin in Cox and Miller (1965), along with transition matrixes and reflecting barriers. Mossin's critical solution for the decision thresholds is similar to Miller and Orr (1966) of multiples of $(\frac{3\gamma m^2 t}{4r})^{\frac{1}{3}}$ (see equation 8 below).

In any case, assuming geometric Brownian motion, the optimal thresholds as the simultaneous solution to a set of value matching and smooth pasting conditions for perpetual scale (entry/exit) real options were eventually developed by Tourinho in 1978, with instantaneous investments at a constant irrecoverable investment cost, or abandonment when prices reach a zero bound. Holding costs for the investment opportunity (perhaps similar to the mothballing maintenance cost in Mossin) were required

[1] However, Mossin (1968) is cited in Dixit and Pindyck (1994). But Wikipedia does not (yet) include the 1968 article in the partial bibliography of Jan Mossin. Google notes 4787 citations for the Mossin capital asset pricing article in Econometrica (1966) but only 46 citations for the lay-up decisions article, as of 13 Jan 2016.

for a solution, see Adkins and Paxson (2013) on Tourinho. Brennan and Schwartz (1985) extended this model further by considering initial idle states, then investment for an active state, which could be suspended, and then also reactivated. Tvedt (2000) provided analytical solutions to entry/exit problems in shipping assuming no switching costs, with implications for equilibrium freight rates. Paxson (2005) further extended these real scale options to a total of eight states, including expansion, which involves solving 16 equations simultaneously. Adkins and Paxson (2012) provided quasi-analytical solutions for start-up and shut-down switching options with stochastic inputs and outputs and constant switching costs.

While reflecting barriers might result in a type of mean reverting pattern, Bjerksund and Ekern (1995) suggested that freight rates follow an Ornstein-Uhlenbeck process, and provided analytical models for time charter contracts and European options on these contracts. Biekpe et al. (2003) demonstrated that power series expansions can be used for analytical solutions for optimal costly entry/exit thresholds based on mean reversion and other similar processes. Sødal et al. (2008) valued the flexibility to switch between dry and wet freight for combination carriers assuming a mean-reverting freight spread and a constant discount rate. Tsekrekos (2010) and several others have modeled entry/exit also assuming mean-reversion, but always at constant entry/exit costs. Adkins and Paxson (2016) consider stochastic abandonment costs, but in a geometric Brownian motion context.

The next section presents the Mossin model, along with spreadsheet solutions and sensitivity analysis. Section three reviews the contribution of Dixit (1988) with a less than satisfactory arithmetic Brownian motion approach, and also a standard geometric Brownian motion model (which is not easy to solve if the interest rate approaches zero, as in Mossin). The last section concludes.

2. THE MOSSIN MODEL

The advantage of Mossin's approach is its simplicity and completely analytical solution for the thresholds that justify laying-up of an operating ship, and reactivation from layup. The stochastic process for revenue (freight) is a stationary random walk (a Bernoulli process) with an equal probability of an increase or decrease for each unit of time (so there is no time varying drift), with a lower reflecting barrier "a" and an upper reflecting barrier "b". Although Mossin does not provide an economic reason for these reflecting barriers, it is reasonable that at some upper barrier laid-up ships will be reactivated and also new ships built, and at some lower barrier ships will be laid-up and older ships demolished for scrap. At the upper barrier increased supply of ships will tend to depress freight rates eventually, and at the lower barrier decreased supply will eventually result in increased freight rates. Mossin derives the upper and lower thresholds based on an average duration of lay-up and operating periods, which is combined with the average revenue during operation, and constant cost during lay-up, to derive the average profit "R" over any period.

$$R=(1-\alpha)(\xi-c_1)-c_2\alpha-c_3\mu \tag{1}$$

where α is the proportion of time laid up, ξ is the average revenue when operating, μ the average number of lay-ups per period, c_1 is the operating cost, c_2 the maintenance cost when laid up, and c_3 the combined mothballing and reactivation costs.

$$\alpha=\frac{\delta_1}{\delta_1+\delta_2} \tag{2}$$

$$\mu=\frac{1}{\delta_1+\delta_2} \tag{3}$$

where δ_1 is the average duration of lay-up periods, and δ_2 is the average duration of operating periods. Using a transition matrix for proceeding from "y" (the layup threshold) as a starting state with "z" as the reactivation threshold, Mossin shows that the sum of the mean number of times in each state is

$$\begin{aligned}\delta_1&=(z-y)(z+y-2a+1)\\ \delta_2&=(z-y)(2b-z-y+1)\end{aligned} \qquad (4)(5)$$

The undiscounted average revenue when operating is not exogenous but a function of the upper barrier and the thresholds:

$$\xi=\frac{b^2+b-\frac{1}{3}(z^2+y^2+zy)}{2b-z-y+1} \tag{6}$$

Substituting (4) and (5) into (2) and (3), and with (6) into (1) provides an alternative expression for R.

$$R=\frac{1}{2(b-a+1)}\left[b^2+b-(c_1-c_2)(2a-1)+\frac{1}{3}(z^2+y^2+zy)+(c_1-c_2)(z+y)-\frac{c_3}{z-y}\right]-c_1 \tag{7}$$

Taking the first derivative of (7) with respect to z and then y and setting each equal to zero, the optimal thresholds are

$$z = (c_1 - c_2) + (\frac{3}{4}c_3)^{1/3} \tag{8}$$

$$y = (c_1 - c_2) - (\frac{3}{4}c_3)^{1/3} \tag{9}$$

Finally with k=number of steps during a period (k=2 implies doubling of the variance in periodic changes)

$$z' = (c_1 - c_2) + k^{\frac{2}{3}}(\frac{3}{4}c_3)^{1/3} \tag{10}$$

$$y' = (c_1 - c_2) - k^{\frac{2}{3}}(\frac{3}{4}c_3)^{1/3} \tag{11}$$

These expressions are easily incorporated into a spreadsheet with simple analytical solutions, as shown in Figure 1, with k=1 or 2

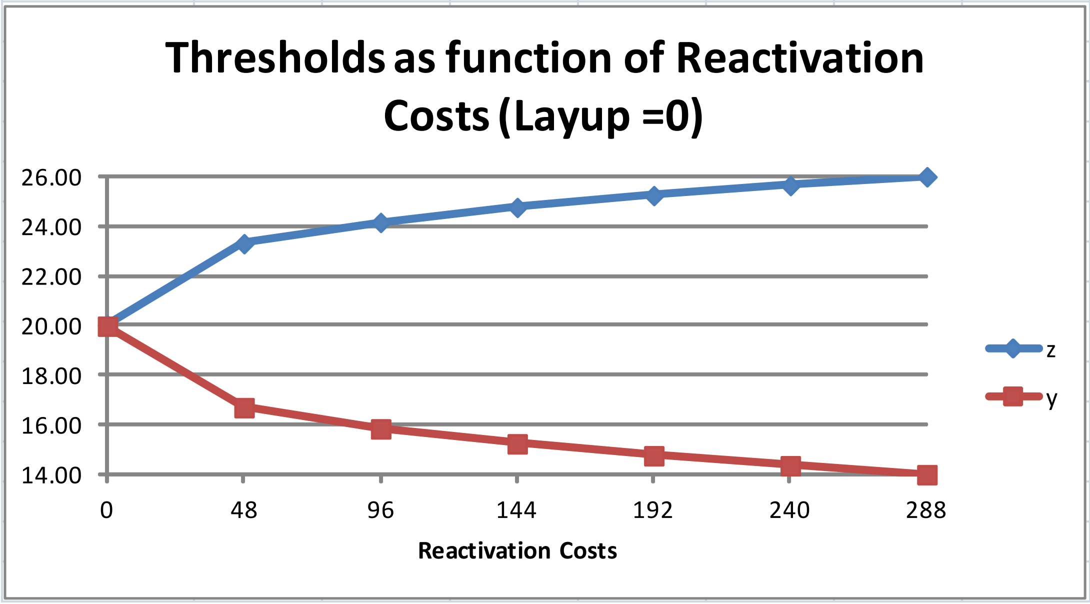
Figure 1

	A	B	C	D	E
1		**MOSSIN 1968**			
2	**INPUT**			EQ	
3					
4					
5	c_{3a}	Reactivation Cost	288.0000		
6	c_1	Operating Cost	25.0000		
7	c_{3b}	Mothballing Cost	0.0000		
8	r	Risk Free Rate			
9	δ	Asset Yield			
10	σ	Volatility			
11	c_2	Maintenance cost during lay-up	5.0000		
12					
13	a	Lower reflecting barrier	10.0000		
14	b	Upper reflecting barrier	50.0000		
15	k	Number of steps	2.0000		
16	**OUTPUT**				
17	α	Proportion time laid up	0.2561	2	C20/(C20+C21)
18	ξ	Average revenue when operating	35.0492	6	(C14^2+C14-(1/3)*(C25^2+C26^2+C25*C26))/(2*C14-C25-C26+1)
19	μ	Average number of lay ups	0.0010	3	1/(C20+C21)
20	$\delta 1$	Average duration of lay-ups	252.0000	4	(C25-C26)*(C25+C26-2*C13+1)
21	$\delta 2$	Average duration of operating	732.0000	5	(C25-C26)*(2*C14-C25-C26+1)
22	R	Average profit over any period	5.9024	1	(1-C17)*(C18-C6)-C11*C17-(C5+C7)*C19
23	R	Average profit over any period	5.9024	7	
24					
25	z	Threshold for reactivation	26.0000	8	MIN((C6-C11)+(0.75*(C5+C7))^(1/3),C14)
26	y	Threshold for lay-up	14.0000	9	MAX((C6-C11)-(0.75*(C5+C7))^(1/3),C13)
27	z k		29.5244	10	MIN((C6-C11)+(C15^(2/3))*(0.75*(C5+C7))^(1/3),C14)
28	y k		10.4756	11	MAX((C6-C11)-(C15^(2/3))*(0.75*(C5+C7))^(1/3),C13)
29	R	(1/(2*(C14-C13+1)))*(C14^2+C14-(C6-C11)*(2*C13-1)-(1/3)*(C25^2+C26^2+C25*C26)+(C6-C11)*(C25+C26)-(C5+C7)/(C25-C26))-C6			

These results are the same as in Mossin (page 176). So with a lower bound of 10, the threshold for lay-up is 14, and with an upper bound of 50, the threshold for reactivation is 26. Note that the upper and lower reflecting barriers do not enter into the threshold formulae (8) and (9), except that these barriers are also the upper and lower bounds on the thresholds.

As Mossin points out, the sensitivities of the thresholds to changes in the critical parameter values are intuitive. An increase in c_3 (combined lay-up and reactivation cost) increases the spread between z and y, as shown in Figure 2. Figure 2

c3a Reactivate	0	48	96	144	192	240	288
c3b Layup	0	0	0	0	0	0	0
z	20.02	23.30	24.16	24.76	25.24	25.65	26.00
y	19.98	16.70	15.84	15.24	14.76	14.35	14.00



Even though Mossin believes "the optimal values of z and y depend only upon the difference c_1-c_2", more precisely the difference between z and y depends only on $z - y = 2k^{\frac{2}{3}}(\frac{3}{4}c_3)^{1/3}$, as shown in Figure 3. Both z and y increase as the difference between c_1-c_2 increases, but z-y remains the same.

Figure 4 shows that the spread between thresholds, and also the level of the z threshold increases as the proxy for volatility increases (by a multiple of $k^{2/3}$). "Increased variability of revenues require revenue to fall to a lower level before lay-up is effected, and to rise higher before the ship is put back in operation" (page 177) a positive "vega" effect on thresholds (consistent with many other real option models).

Figure 3

c1-c2	20	22.5	25	27.5	30	32.5	35
z	26	28.5	31	33.5	36	38.5	41
y	14	16.5	19	21.5	24	26.5	29

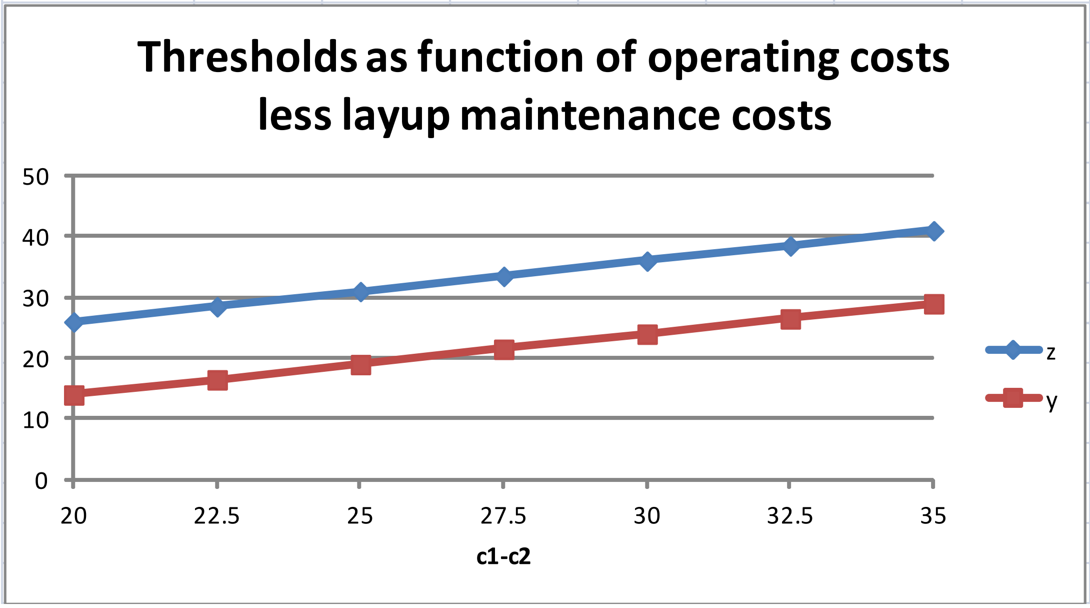
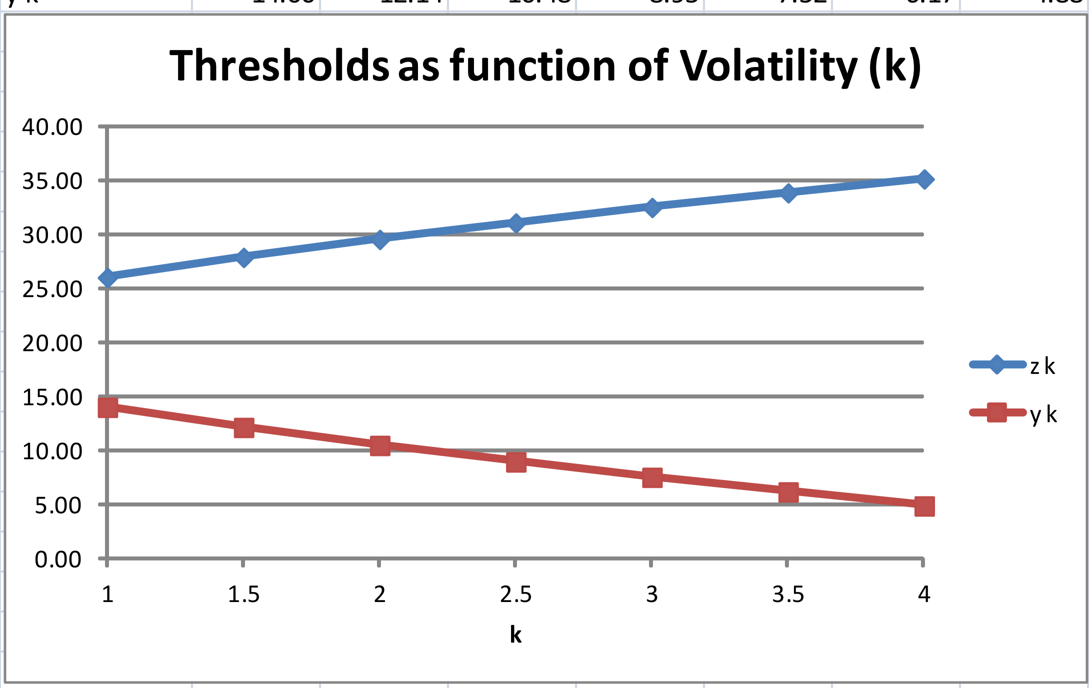


Figure 4

k	1	1.5	2	2.5	3	3.5	4
z k	26.00	27.86	29.52	31.05	32.48	33.83	35.12
y k	14.00	12.14	10.48	8.95	7.52	6.17	4.88

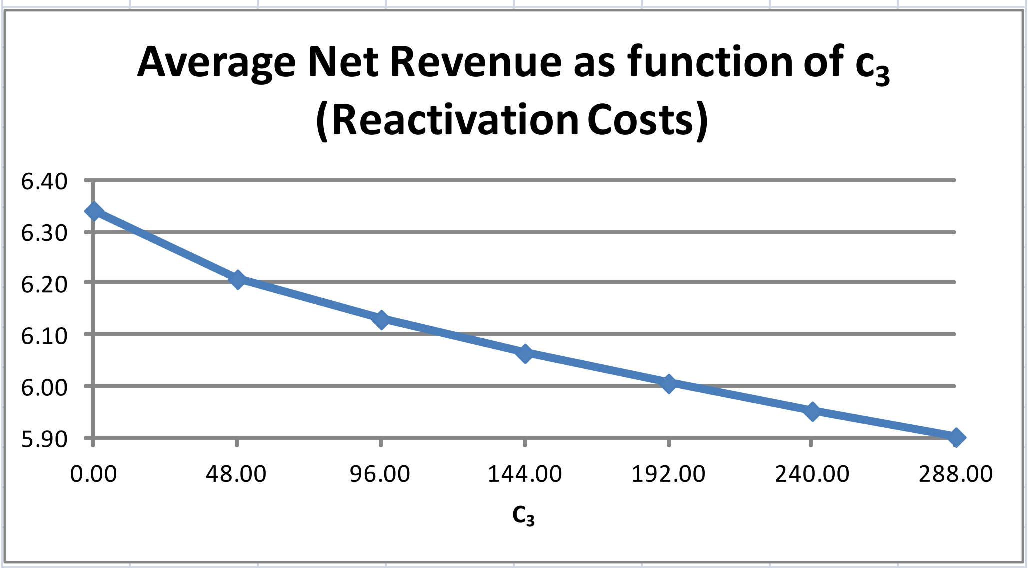


Other observations are that α is independent of c_3, and μ is independent of c_1-c_2, but entirely dependent on a and b, the lower and upper reflecting barriers.

Mossin's basic reflecting barriers model does not directly produce a real switching option value, but instead provides an "average profit per period" R. The "pure ROV" could be interpreted as the difference between the perpetual value of R when there is no switching cost (c_3=0) and an extremely high switching cost, that is a high level compared to operating costs and maintenance costs. But as Figure 5 shows R is not very sensitive to increases in c_3, here consisting solely of the reactivation costs with lay-up cost of zero, even though the thresholds change significantly. [Thus the additional value of a vessel that can mothballed at c_3=0 rather than c_3=288 is the present value of 6.34-5.90=.44 per period over the remaining lifetime of the vessel].

Figure 5

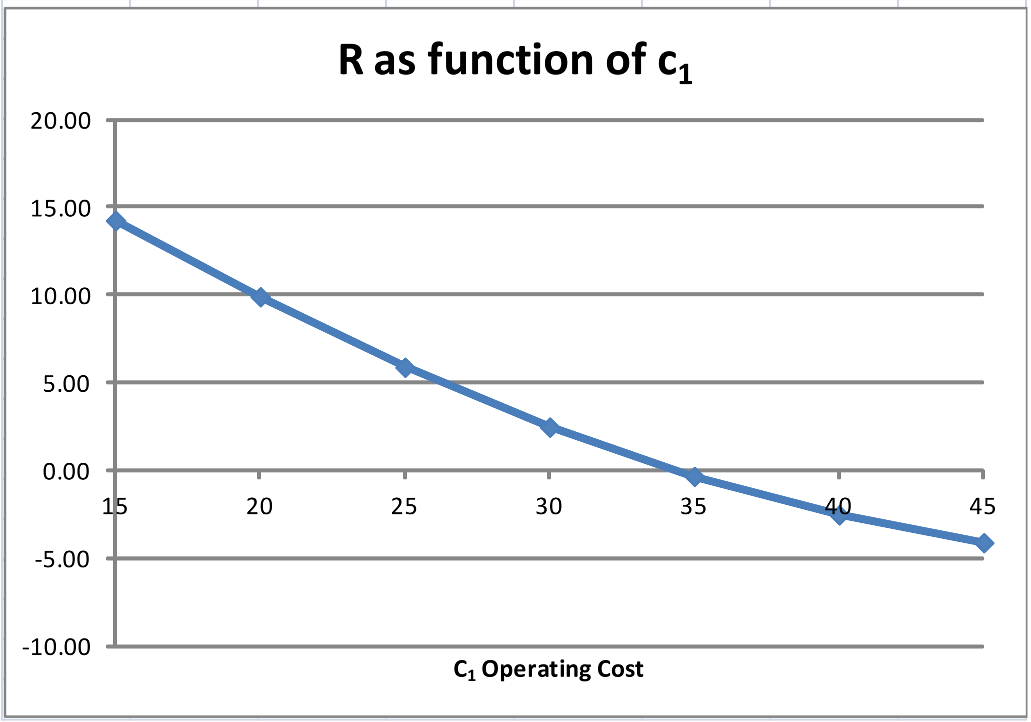
z	20.02	23.30	24.16	24.76	25.24	25.65	26.00
y	19.98	16.70	15.84	15.24	14.76	14.35	14.00
c_3	0.00	48.00	96.00	144.00	192.00	240.00	288.00
ξ	35.25	35.19	35.15	35.12	35.10	35.07	35.05
μ	0.312	0.002	0.001	0.001	0.001	0.001	0.001
δ_1	0.82	138.68	174.73	200.01	220.14	237.14	252.00
δ_2	2.39	402.84	507.54	580.99	639.46	688.84	732.00
R	6.34	6.21	6.13	6.06	6.01	5.95	5.90



As c_3 increases, the average profit when operating hardly changes, but the number of switches μ per period declines and the average duration of operating δ_2 increases substantially, and also the duration of lay-ups δ_1 increases. As expected R is highly sensitive to increases in operating costs c_1 as shown in Figure 6 as the proportion of lay-up time increases especially when operating costs are approaching or in excess of the average revenue when operating. ξ is a derived (endogenous) figure, rather than an exogenous freight rate. (Mossin does not consider that there is any effect of widespread industry mothballing on spot freight rates, a consideration also typically ignored in many other real option scale models).

Figure 6

c1	15	20	25	30	35	40	45
z	16	21	26	31	36	41	46
y	10	10	14	19	24	29	34
α	0.09	0.15	0.26	0.38	0.50	0.62	0.74
ξ	31.71	32.85	35.05	37.51	39.95	42.35	44.67
μ	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$\delta 1$	42.00	132.00	252.00	372.00	492.00	612.00	732.00
$\delta 2$	450.00	770.00	732.00	612.00	492.00	372.00	252.00
R	14.27	9.92	5.90	2.49	-0.32	-2.51	-4.10



R is a complex function of changes in c_2, maintenance costs during lay-up, since like c_1, these also affect most of the other parameter values, especially the thresholds and durations, as shown in Figure 7.

Figure 7

c_2	0	2.5	5	7.5	10	12.5	15
α	0.38	0.32	0.26	0.20	0.15	0.12	0.09
ξ	37.51	36.28	35.05	33.81	32.85	32.29	31.71
μ	0.00	0.00	0.00	0.00	0.00	0.00	0.00
δ_1	372.00	312.00	252.00	192.00	132.00	80.75	42.00
δ_2	612.00	672.00	732.00	792.00	770.00	616.25	450.00
R	7.49	6.62	5.90	5.34	4.92	4.58	4.27
z	31	28.5	26	23.5	21	18.5	16
y	19	16.5	14	11.5	10	10	10

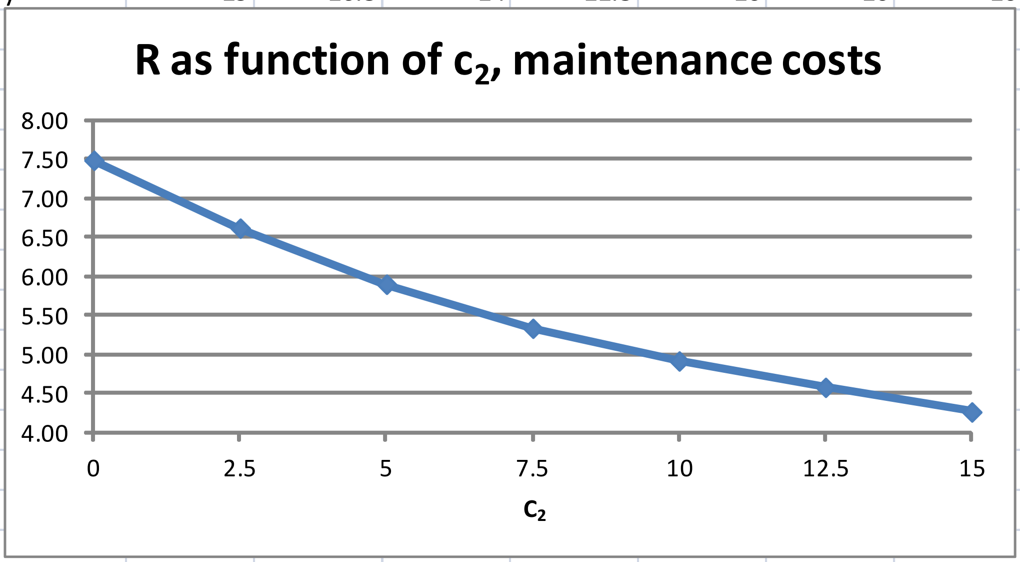
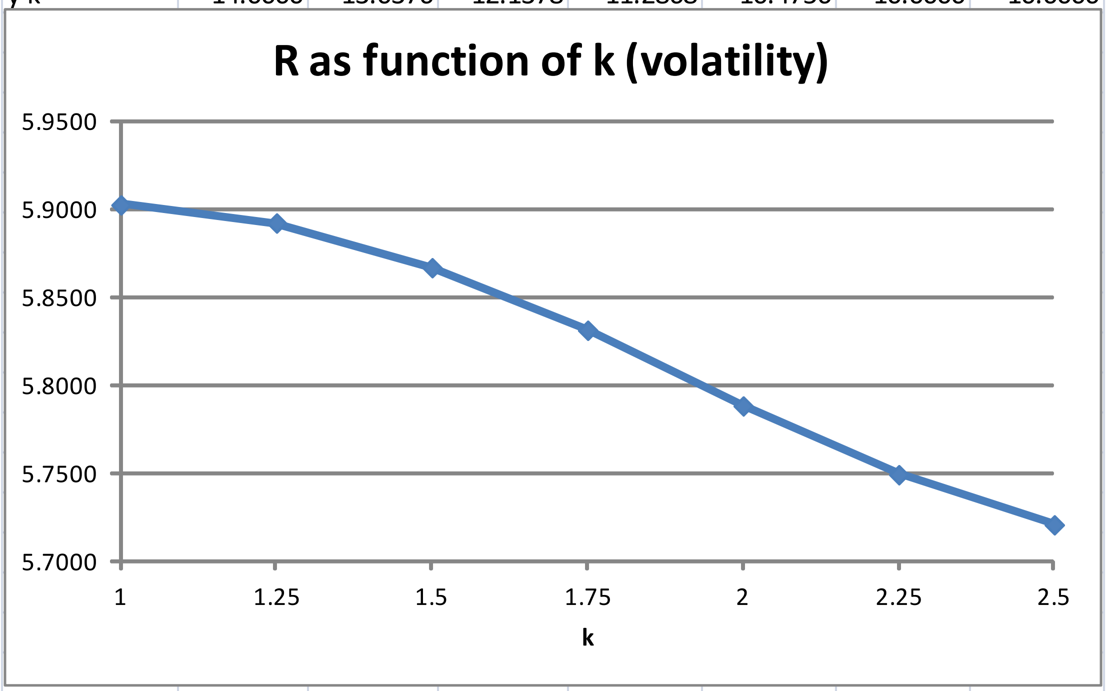


Figure 8 shows that R is a complex function of Mossin's proxy for volatility k, since changes in k affect the z and y thresholds, which in turn affect most of the other parameter values.

Figure 8

k	1	1.25	1.5	1.75	2	2.25	2.5
α	0.2561	0.2561	0.2561	0.2561	0.2561	0.2598	0.2689
ξ	35.0492	34.9810	34.9081	34.8310	34.7502	34.7556	34.8928
μ	0.0010	0.0009	0.0008	0.0007	0.0006	0.0006	0.0006
δ1	252.0000	292.4201	330.2134	365.9535	400.0251	432.4910	464.2428
δ2	732.0000	849.4108	959.1914	1063.0078	1161.9776	1232.3081	1262.0290
R	5.9024	5.8922	5.8668	5.8313	5.7883	5.7493	5.7209
z k	26.0000	26.9624	27.8622	28.7132	29.5244	30.3024	31.0521
y k	14.0000	13.0376	12.1378	11.2868	10.4756	10.0000	10.0000



R is more or less a linear function of both a and b, as shown in Figures 9 and 10, since the reflecting barriers do not affect the thresholds, as noted by Mossin. If these results represented a continuous time stochastic process, the “partial derivative” of R with respect to “a” would be:

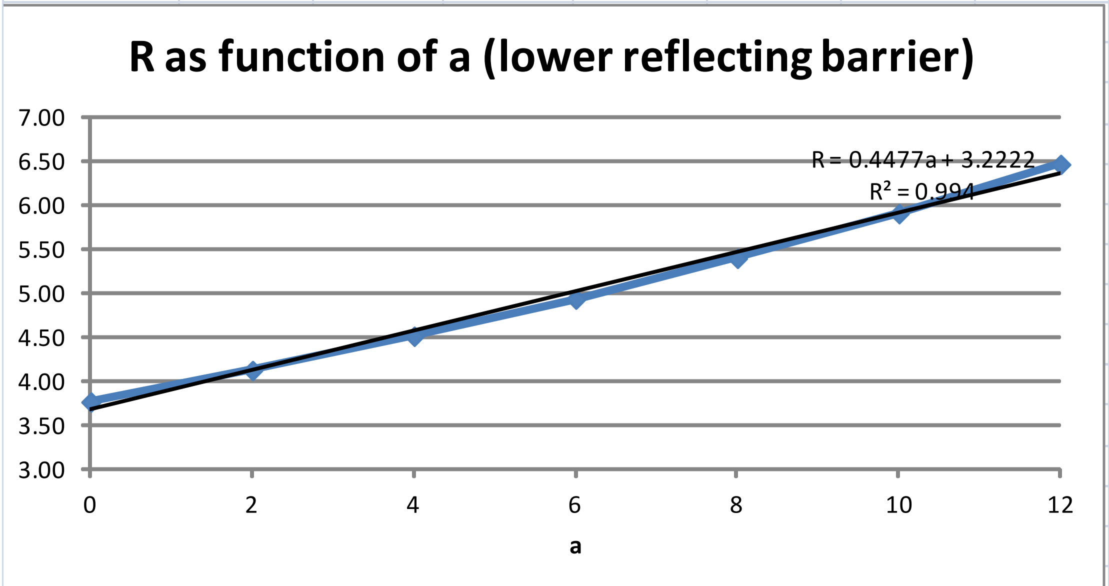
$$-\frac{c1-c2}{1-a+b}+\frac{b+b^{2}-(-1+2a)(c1-c2)-\frac{c3}{-y+z}+(c1-c2)(y+z)+\frac{1}{3}(y^{2}+z^{2}+zy)}{2(1-a+b)^{2}}$$

The “partial derivative” of R with respect to “b” would be:

$$\frac{1+2b}{2(1-a+b)}-\frac{b+b^{2}-(-1+2a)(c1-c2)-\frac{c3}{-y+z}+(c1-c2)(y+z)+\frac{1}{3}(y^{2}+z^{2}+zy)}{2(1-a+b)^{2}}$$

Figure 9

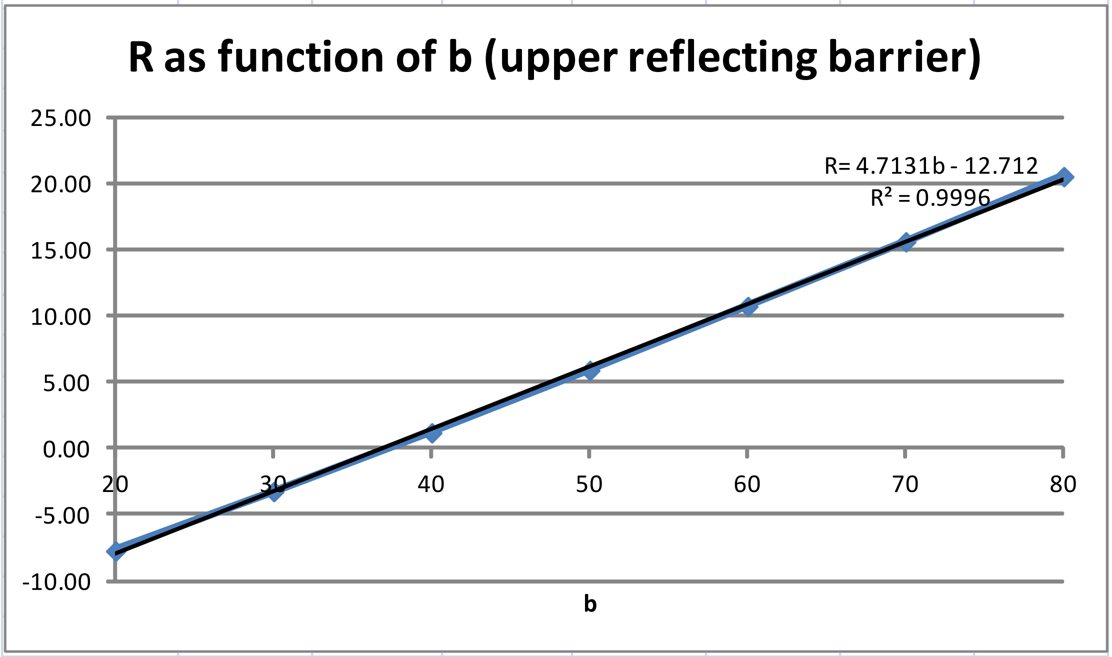
a	0	2	4	6	8	10	12
α	0.40	0.38	0.35	0.32	0.29	0.26	0.22
ξ	35.05	35.05	35.05	35.05	35.05	35.05	35.05
μ	0.00	0.00	0.00	0.00	0.00	0.00	0.00
δ1	492.00	444.00	396.00	348.00	300.00	252.00	204.00
δ2	732.00	732.00	732.00	732.00	732.00	732.00	732.00
R	3.7647	4.1224	4.5106	4.9333	5.3953	5.9024	6.4615



As shown on the Figures, R=3.22 +.45a and R=4.71b-12.71, with high R^2 in both cases. It is logical that a realistic lower bound is probably around the demolition price level, and an upper bound around the new building cost, but delays in new building for most types of vessels were surely the basis of second hand prices for five and even ten year tankers and dry bulk ships exceeding new build prices in 2007-2008.

Figure 10

b	20	30	40	50	60	70	80
α	0.68	0.50	0.34	0.26	0.21	0.17	0.15
ξ	18.29	24.67	29.95	35.05	40.10	45.13	50.15
μ	0.01	0.00	0.00	0.00	0.00	0.00	0.00
δ_1	90.00	252.00	252.00	252.00	252.00	252.00	252.00
δ_2	42.00	252.00	492.00	732.00	972.00	1212.00	1452.00
R	-7.73	-3.24	1.19	5.90	10.73	15.61	20.52



The advantages of the Mossin model are its simplicity and early vintage. Some disadvantages are: (i) that the real option value is not directly given, and so the extra value implicit in owning a vessel that can be mothballed compared to one that cannot (due to physical or contractual constraints) cannot be clearly quantified; (ii) the stochastic process is simple requiring a reversion to an unspecified level upon hitting the lower or upper barrier, with discrete steps; (iii) there is no guidance on how to calibrate these step sizes or frequency consistent with other common measures for volatility; (iv) no discounting is considered; (v) the stationary stochastic process is symmetric and with no drift bounds, although perhaps ignoring any bounds with the obvious supply and demand economics is also questionable; (vii) there are no further states, such as idle and scrapping, which are considered in Brennan and Schwartz (1985) and Dixit (1988); (viii) freight rates are discrete and given as more or less undiscounted endogenous averages; and (ix) perhaps other stochastic processes should be considered as in Dixit (1988). Further considerations in common with most other scale option models are the assumptions of constant operating, maintenance, mothballing and reactivation costs.

3. DIXIT ON MOSSIN

Twenty years after Mossin, Dixit (1988) recast this model in continuous time assuming both arithmetic (aBm) and geometric Brownian motion (gBm). Although Dixit first tried to approximate the Mossin results using gBm, his appendix "c" provided a continuous time model for assuming that revenue P is exogenous and follows a trendless aBm process.

$$dP = \theta dt + \sigma dz \quad (12)$$

where for Mossin $\theta = 0$. Following Shimko (1992), aBm is appropriate for net cash flows (Mossin's R) which may become negative but not for prices, or gross revenue. Following Dixit (1988) the differential equations for trendless aBm are:

$$\frac{1}{2}\sigma^2 \frac{\partial V_1^2}{\partial P^2} - rV_1 - c_1 = 0 \quad (13)$$

for a mothballed vessel with a reactivation option, and for a operating vessel with a mothballing opportunity

$$\frac{1}{2}\sigma^2 \frac{\partial V_2^2}{\partial P^2} - rV_2 + P - c_2 = 0 \quad (14)$$

The solutions are:

$$V_1 = H_1 e^{\gamma P} - \frac{c_1}{r} \quad (15)$$

$$V_2 = G_2 e^{-\gamma P} + \frac{P - c_2}{r} \quad (16)$$

$$\gamma = \sqrt{2r} / \sigma \quad (17)$$

(also see Dixit, 1993, page 41). The value matching equations are:

$$H_1 e^{\gamma R} - \frac{c_1}{r} - G_2 e^{-\gamma R} - \frac{R - c_2}{r} + c_{3a} = 0 \quad (18)$$

$$\gamma H_1 e^{\gamma R} + \gamma G_2 e^{-\gamma R} - \frac{1}{r} = 0 \quad (19)$$

$$-H_1 e^{\gamma L} + \frac{c_1}{r} + G_2 e^{-\gamma L} + \frac{L - c_2}{r} + c_{3b} = 0 \quad (20)$$

$$-\gamma H_1 e^{\gamma L} - \gamma G_2 e^{-\gamma L} + \frac{1}{r} = 0 \quad (21)$$

where R=z is the reactivation threshold and L=y is the mothballing threshold, c_{3a}=reactivation costs and c_{3b}=mothballing costs.

Figure 11

	A	B	C	D	E
1		**MOSSIN 1968**		EQ	Dixit aBm
2	V	Operating Value	-50000.0000		C3/C8-(C5/C7)
3	P	Revenue	20.0000		Revenue Per Ship
4	c_{3a}	Reactivation Cost	288.0000		Cost to reactivate from mothballed
5	c_2	Operating Cost	25.0000		Operating cost estimation
6	c_{3b}	Mothballing Cost	0.0000		C4-288
7	r	Risk Free Rate	0.000100		
8	δ	Asset Yield	0.000100		
9	σ	Volatility	0.0500		Freight rate spot market volatility
10	c_1	Maintenance cost during lay-up	5.0000		
11	γ1=		0.2828	17	(SQRT(2*C7))/C9
12	γ2=		-0.2828		-C11
13	NPV1		-50000.0000		-C10/C7
14	V1(P)	ROV1+NPV1	-32857.4618	15	C17*EXP(C3*C11)-C10/C7
15	V2(P)	ROV2+NPV2	-32717.2506	16	C18*EXP(C3*C12)+(C3-C5)/C7
16	NPV2		-50000.0000		(C3/C8)-(C5/C7)
17	H1		59.8873		
18	G2		4,947,131.0833		
19	R		**20.8345**		
20	L		**19.1943**		
21	**R-L**		1.6402		
22	Eq.M1		0.0000	18	C17*EXP(C19*C11)-C18*EXP(C19*C12)-C10/C7-(C19-C5)/C7+C4
23	Eq.M2		0.0000	19	C17*C11*EXP(C19*C11)-C18*C12*EXP(C19*C12)-1/C7
24	Eq.M3		0.0000	20	-C17*EXP(C20*C11)+C18*EXP(C20*C12)+(C20-C5)/C7+C10/C7+C6
25	Eq.M4		0.0000	21	-C17*C11*EXP(C20*C11)+C18*C12*EXP(C20*C12)+1/C7
26	SUM		0.0000		Set C26=0, Changing C17:C20

	A	B	C	D	E
34		IS $r<\sigma^2$?	yes		IF(C7<(C9^2),"yes","no")
35	MOSSIN	Y	20.0000		C5-C10
36		X	0.8143		(0.75*(C9^2)*(C4+C6))^(1/3)
37		R	20.8143		C35+C36
38		L	19.1857		C35-C36

Dixit notes that if $r<\sigma^2$, that is r is small, as r approaches 0 the approximate solution is the same as (8) and (9). Figure 11 shows the spreadsheet solution to the two value matching and two smooth pasting equations is consistent with the results for (8) and (9). Dixit assumes that Mossin's discrete random walk with steps of size 1 might be equivalent to a daily volatility of 5% (=1/20), (20 is the average of L=14 and R=26 base case in Figure 1). The apparent aBm solution is R=20.8 and L =19.2, which is consistent with Dixit's approximate solution of Mossin, shown in Cells C37:C38. Note the solution to these four nonlinear equations is very sensitive to very small changes in both r and σ, with an enormous G (mothballing option coefficient). It is not necessarily easy to obtain solutions using Excel Solver. Perhaps these calculations are problematical due to the interest rate assumption, realistic only in modern European times.

Dixit indicates greater success with replicating the Mossin results assuming gBm (he notes "Mossin's formulae are not rigorously valid for gBm and small r, in practice it seems to fit reasonably well", page 6).

It is assumed that there is a single factor P (revenue per capacity) which follows a geometric Brownian motion stochastic process:

$$\frac{dP}{P} = (\alpha_P - \delta_P)dt + \sigma_P dz \qquad (22)$$

where α_P is the drift rate over time, δ_P is the asset yield or convenience yield, σ_P is the instantaneous standard deviation of the P disturbance, and dz is the standardized Wiener process. Fixed cost of mothballing are c_{3b} and of reactivating c_{3a}. The constant operating cost is c_2, and maintenance cost c_1. The Brennan and Schwartz (1985) entry/exit models using stochastic dynamic programming solve two ordinary differential equations for the optimal entry/exit thresholds, which is very similar to the reactivation/mothballing problem.

All of the costs involved are known and constant, and the riskless rate of interest "r" is fixed, which for the Mossin case is assumed $r=\delta$, so $\alpha=0$, if $\alpha=r-\delta$. Moreover, the options to alter states are perceived to be perpetual, since the asset is assumed to last forever. Finally, the reactivation cost is considered irrecoverable, as are the one-off costs of mothballing. Output variables are $\beta_{1,2}$ = equations 27 and 28, R = P threshold that justifies immediate reactivation, and L = P threshold that justifies instantaneous mothballing.

In a stochastic model allowing for both reactivation and mothballing, there are two differential equations that the valuation functions must satisfy:

LAIDUP $$\tfrac{1}{2}\sigma^2 P^2 V_0''(P) + (r-\delta)PV_0'(P) - rV_0(P) - c_1 = 0 \qquad (23)$$

ACTIVE $$\tfrac{1}{2}\sigma^2 P^2 V_1''(P) + (r-\delta)PV_1'(P) - rV_1(P) + P - c_2 = 0 \qquad (24)$$

The solutions for each of these equations are:

$$V_0(P) = A_1 P^{\beta_1} - \frac{c_1}{r} \qquad (25)$$

$$V_1(P) = B_2 P^{\beta_2} + \frac{P}{\delta} - \frac{c_2}{r} \qquad (26)$$

The general solution for each state is of the form of some constant (to be determined) times P to the power of $\beta_{1,2}$ given by:

$$\beta_1 = \tfrac{1}{2} - (r-\delta)/\sigma^2 + \sqrt{[(r-\delta)/\sigma^2 - \tfrac{1}{2}]^2 + 2r/\sigma^2} \quad >1 \qquad (27)$$

$$\beta_2 = \tfrac{1}{2} - (r-\delta)/\sigma^2 - \sqrt{[(r-\delta)/\sigma^2 - \tfrac{1}{2}]^2 + 2r/\sigma^2} \quad <0 \qquad (28)$$

Each of the actions must meet value matching and smooth pasting conditions. The first term of equation (26) represents the value of the option to mothball, whereas the other two terms represent the perpetual value of operating the asset. Now, there are four unknowns that need to be determined, namely the two optimal thresholds R and L, and the two option value coefficients A_1 and B_2. At the optimal reactivation point R and at the optimal mothballing threshold L the value-matching and smooth pasting conditions need to be satisfied. For instance, L must satisfy:

$$V_1(L) = V_0(L) - c_{3b} \qquad V_1'(L) = V_0'(L) \qquad (29)$$

After substitutions and simplifications, there are four equations to be solved simultaneously.

$$A_1 R^{\beta_1} - B_2 R^{\beta_2} - \frac{R}{\delta} + \frac{c_2 - c_1}{r} + c_{3a} = 0 \qquad (30)$$

$$\beta_1 A_1 R^{\beta_1 - 1} - \beta_2 B_2 R^{\beta_2 - 1} - 1/\delta = 0 \qquad (31)$$

$$-A_1 L^{\beta_1} + B_2 L^{\beta_2} + \frac{L}{\delta} - \frac{c_2 - c_1}{r} + c_{3b} = 0 \tag{32}$$

$$-\beta_1 A_1 L^{\beta_1 - 1} + \beta_2 B_2 L^{\beta_2 - 1} + \frac{1}{\delta} = 0 \tag{33}$$

Using the Mossin inputs but assuming a very small r (perhaps appropriate only recently), the results are close to Mossin, as shown in Figure 12, in this case splitting the base case c_3 equally between reactivation and mothballing costs. These results are very sensitive to very small changes in r and σ.

`

Figure 12

	A	B	C	D	E
1		**DIXIT 1988**		EQ	
2	V	Operating Value	-5494.5055		C3/C8-(C5/C7)
3	P	Revenue	20.0000		Revenue Per Ship
4	c_{3a}	Reactivation Cost	144.0000		Cost to reactivate from mothballed
5	c_2	Operating Cost	25.0000		Operating cost estimation
6	c_{3b}	Mothballing Cost	144.0000		
7	r	Risk Free Rate	**0.000910**		
8	δ	Asset Yield	0.000910		
9	σ	Volatility	**0.0500**		Freight rate spot market volatility
10	c_1	Maintenance cost during lay-up	5.0000		
11	β_1		1.4889	27	
12	β_2		-0.4889	28	
13	NPV0		-5494.5055		-C10/C7
14	$V_0(P)$	ROV0+NPV0	5266.5435	25	C17*(C3^C11)-C10/C7
15	$V_1(P)$	ROV1+NPV1	5245.6726	26	C18*(C3^C12)+(C3/C8)-(C5/C7)
16	NPV1		-5494.5055		(C3/C8)-(C5/C7)
17	A1		124.3657		
18	B2		46,466.03		
19	**R**		**27.4199**		
20	**L**		**15.0039**		
21	**R-L**		12.4160		
22	VM1		0.0000	30	(-C18*(C19^C12))+(C17*(C19^C11))-(C19/C8)+((C5-C10)/C7)+C4
23	SP1		0.0000	31	(-C12*C18*(C19^(C12-1)))+(C11*C17*(C19^(C11-1)))-(1/C8)
24	VM2		0.0000	32	(-C17*(C20^C11))+(C18*(C20^C12))+(C20/C8)-((C5-C10)/C7)+C6
25	SP2		0.0000	33	(-C17*C11*(C20^(C11-1)))+(C18*C12*(C20^(C12-1)))+1/C8
26	SUM		0.0000		Set C26=0, Changing C17:C20
27					
28	ROV0		10761.0490		C17*(C3^C11)
29	ROV1		10740.1781		C18*(C3^C12)
30	β_1	0.5-((C7-C8)/(C9^2))+SQRT(((((C7-C8)/(C9^2))-0.5)^2)+(2*(C7/(C9^2))))			
31	β_2	0.5-((C7-C8)/(C9^2))-SQRT(((((C7-C8)/(C9^2))-0.5)^2)+(2*(C7/(C9^2))))			
32					
33	Idle	DE Laid-up Ship	0.0000	23	0.5*(C9^2)*(C3^2)*C36+(C7-C8)*C3*C35-C7*C14-C10
34	Active	DE Active Ship	0.0000	24	0.5*(C9^2)*(C3^2)*C38+(C7-C8)*C3*C37-C7*C15-C5+C3
35		Δ V0 P	801.1272		C11*C17*(C3^(C11-1))
36		Γ V0 P	19.5851		C11*(C11-1)*C17*(C3^(C11-2))
37		Δ V1 P	836.3366		C12*C18*(C3^(C12-1))+(1/C8)
38		Γ V1 P	19.5471		C12*(C12-1)*C18*(C3^(C12-2))

A more conventional approach is using interest rates not approaching zero. Dixit uses r=5%, where the thresholds are much lower for high mothballing costs, and higher with high reactivation costs, as in Figures 13 and 14. "The split of the fixed costs between lay-up and re-start makes a big difference...intuitive, since with a higher interest rate the costs immediately incurred matter more" (page 7). With only mothballing costs, both thresholds are low, but especially the mothballing threshold (high abandonment costs are an incentive to avoid abandonment), but with only reactivation costs, both thresholds are higher, but especially the reactivation threshold (high investment costs deter immediate investment).

Figure 13 Only Mothballing Costs

	A	B	C	D
1		**DIXIT 1988**		gBm
2	V	Operating Value	-100.0000	C3/C8-(C5/C7)
3	P	Revenue	20.0000	Revenue Per Ship
4	c_{3a}	Reactivation Cost	0.0000	Cost to reactivate from mothballed
5	c_2	Operating Cost	25.0000	Operating cost estimation
6	c_{3b}	Mothballing Cost	288.0000	
7	r	Risk Free Rate	0.0500	
8	δ	Asset Yield	0.0500	
9	σ	Volatility	0.0500	Freight rate spot market volatility
10	c_1	Maintenance cost during lay-up	5.0000	
11	β_1		6.8443	
12	β_2		-5.8443	
13	NPV0		-100.0000	
14	$V_0(P)$	ROV0+NPV0	-76.7810	C17*(C3^C11)-C10/C7
15	$V_1(P)$	ROV1+NPV1	-99.9962	C18*(C3^C12)+(C3/C8)-(C5/C7)
16	NPV1		-100.0000	(C3/C8)-(C5/C7)
17	A1		0.000000029	
18	B2		153,339	
19	**R**		**23.4220**	
20	**L**		**4.7819**	
21	**R-L**		18.6401	
22	Eq.30		0.0000	(-C18*(C19^C12))+(C17*(C19^C11))-(C19/C8)+((C5-C10)/C7)+C4
23	Eq.31		0.0000	(-C12*C18*(C19^(C12-1)))+(C11*C17*(C19^(C11-1)))-(1/C8)
24	Eq.32		0.0000	(-C17*(C20^C11))+(C18*(C20^C12))+(C20/C8)-((C5-C10)/C7)+C6
25	Eq.33		0.0000	(-C11*C17*(C20^(C11-1)))+(C12*C18*(C20^(C12-1)))+(1/C8)
26	SUM		0.0000	Set C26=0, Changing C17:C20
27				
28	ROV0		23.2190	
29	ROV1		0.0038	
30	β_1	0.5-((C7-C8)/(C9^2))+SQRT(((((C7-C8)/(C9^2))-0.5)^2)+(2*(C7/(C9^2))))		
31	β_2	0.5-((C7-C8)/(C9^2))-SQRT(((((C7-C8)/(C9^2))-0.5)^2)+(2*(C7/(C9^2))))		
32				
33	Eq. 23	DE Laid-up Ship	0.0000	0.5*(C9^2)*(C3^2)*C36+(C7-C8)*C3*C35-C7*C14-C10
34	Eq. 24	DE Active Ship	0.0000	0.5*(C9^2)*(C3^2)*C38+(C7-C8)*C3*C37-C7*C15-C5+C3
35		Δ V0 P	7.9459	C11*C17*(C3^(C11-1))
36		Γ V0 P	2.3219	C11*(C11-1)*C17*(C3^(C11-2))
37		Δ V1 P	19.9989	C12*C18*(C3^(C12-1))+(1/C8)
38		Γ V1 P	0.0004	C12*(C12-1)*C18*(C3^(C12-2))

Figure 14 Only Reactivation Costs

	A	B	C	D
1		**DIXIT 1988**		gBm
2	V	Operating Value	-100.0000	C3/C8-(C5/C7)
3	P	Revenue	20.0000	Revenue Per Ship
4	c_{3a}	Reactivation Cost	288.0000	Cost to reactivate from mothballed
5	c_2	Operating Cost	25.0000	Operating cost estimation
6	c_{3b}	Mothballing Cost	0.0000	
7	r	Risk Free Rate	0.0500	
8	δ	Asset Yield	0.0500	
9	σ	Volatility	0.0500	Freight rate spot market volatility
10	c_1	Maintenance cost during lay-up	5.0000	
11	β_1		6.8443	
12	β_2		-5.8443	
13	NPV0		-100.0000	
14	$V_0(P)$	ROV0+NPV0	-99.0210	C17*(C3^C11)-C10/C7
15	$V_1(P)$	ROV1+NPV1	-76.6481	C18*(C3^C12)+(C3/C8)-(C5/C7)
16	NPV1		-100.0000	(C3/C8)-(C5/C7)
17	A1		0.000000001	
18	B2		937,383,845	
19	**R**		**40.2435**	
20	**L**		**17.1090**	
21	**R-L**		23.1345	
22	Eq.30		0.0000	(-C18*(C19^C12))+(C17*(C19^C11))-(C19/C8)+((C5-C10)/C7)+C4
23	Eq.31		0.0000	(-C12*C18*(C19^(C12-1)))+(C11*C17*(C19^(C11-1)))-(1/C8)
24	Eq.32		0.0000	(-C17*(C20^C11))+(C18*(C20^C12))+(C20/C8)-((C5-C10)/C7)+C6
25	Eq.33		0.0000	(-C11*C17*(C20^(C11-1)))+(C12*C18*(C20^(C12-1)))+(1/C8)
26	SUM		0.0000	Set C26=0, Changing C17:C20
27				
28	ROV0		0.9790	
29	ROV1		23.3519	
30	β_1	0.5-((C7-C8)/(C9^2))+SQRT(((((C7-C8)/(C9^2))-0.5)^2)+(2*(C7/(C9^2))))		
31	β_2	0.5-((C7-C8)/(C9^2))-SQRT(((((C7-C8)/(C9^2))-0.5)^2)+(2*(C7/(C9^2))))		
32				
33	Eq. 23	DE Laid-up Ship	0.0000	0.5*(C9^2)*(C3^2)*C36+(C7-C8)*C3*C35-C7*C14-C10
34	Eq. 24	DE Active Ship	0.0000	0.5*(C9^2)*(C3^2)*C38+(C7-C8)*C3*C37-C7*C15-C5+C3
35		Δ V0 P	0.3350	C11*C17*(C3^(C11-1))
36		Γ V0 P	0.0979	C11*(C11-1)*C17*(C3^(C11-2))
37		Δ V1 P	13.1762	C12*C18*(C3^(C12-1))+(1/C8)
38		Γ V1 P	2.3352	C12*(C12-1)*C18*(C3^(C12-2))

These Figures replicate the Dixit (1988) results, but show in addition that (23) and (24) are solved, using the apparent first and second derivatives of the solutions (25) and (26), and also show the real option value at each stage. Note with high mothballing costs, the real option to mothball ROV_1 is small, while with no reactivation costs in Figure 13, the ROV_0 is large. Note with zero mothballing costs, the real option to mothball ROV_1 is large, while with high reactivation costs in Figure 14, the ROV_0 is small. As noted by Dixit, this illustrates a disadvantage of the Mossin model, where R and L are insensitive to the distribution between these investment type once-off irrecoverable charges. Additional comparisons are in the Appendix.

4. CONCLUSION

Forty-eight years ago Mossin published the first quantified real option model leading to exit/entry and other scale models. The primary contributions of this inaugural "Nordic nugget" are formulating the real scale option problem and providing a simple analytical solution for the optimal thresholds. Extension of this model show that the spread between mothballing (temporary suspension) and reactivation thresholds is highly sensitive to the combined reactivation and mothballing irrecoverable once-off costs and to the proxy for revenue volatility, but invariant to the differences between the operating costs and the maintenance costs during mothballing. Another extension of this model is that the average net revenue is not very sensitive to the combined reactivation and mothballing irrecoverable once-off costs or to the proxy for revenue volatility (in contrast to more conventional models), but very sensitive to changes in both operating costs and the maintenance costs during mothballing, and especially to changes in the reflecting barrier levels. A major contribution (and requirement) of Mossin's model is considering reflecting upper and lower barriers, which are missing from most current popular scale models.

Twenty-eight years ago Dixit recast the Mossin model in continuous time. Approximating Mossin with an arithmetic Brownian motion with zero drift and an interest rate approaching zero, does not apparently result in the Mossin thresholds using the same parameter values However, assuming geometric Brownian motion with zero drift and an interest rate approaching zero does produce results close to the Mossin thresholds, even though the very small discounting rate is problematical in calculations. Using normal interest rate levels results in quite different thresholds and real option values, which are sensitive to the distribution of reactivation and mothballing costs.

What are some lessons for future research? One consideration is the attractiveness of simple real option models, even based on somewhat tenuous assumptions. Surely these models can be adapted to emphasize some of the advantages of analytical solutions, while minimizing some of the disadvantages especially the distribution of irrecoverable investment type costs. Economically based reflecting upper and lower barriers for revenues could be an important Mossin contribution, complimenting the many other scale models (especially mean reverting). Finally stochastic operating, mothballing and reactivation costs should not be too difficult to incorporate in some scale models.

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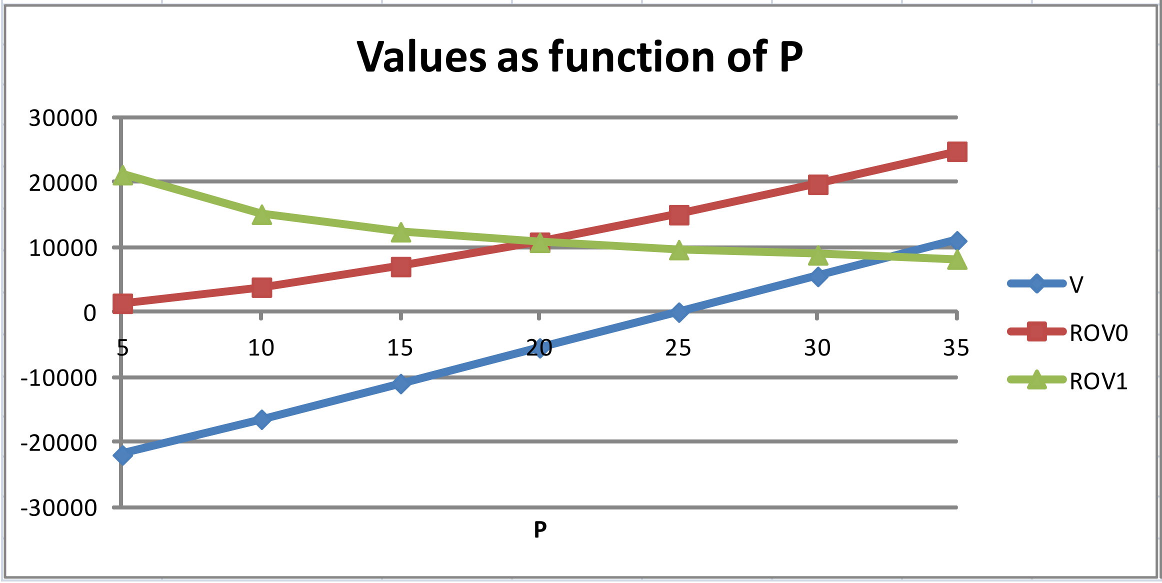
APPENDIX

Figure 15

Dixit Mothballing and Reactivation gBm Model							
P	5	10	15	20	25	30	35
V	-21978.02	-16483.52	-10989.01	-5494.51	0.00	5494.51	10989.01
ROV_0	1365.92	3833.89	7011.78	10761.05	15001.95	19680.84	24758.45
ROV_1	21153.49	15072.90	12362.29	10740.18	9630.05	8808.74	8169.22
Δ V0 P	406.75	570.84	696.01	801.13	893.48	976.79	1053.25
Γ V0 P	39.78	27.91	22.69	19.59	17.47	15.92	14.71
Δ V1 P	-969.65	361.93	695.94	836.34	910.56	955.34	984.78
Γ V1 P	615.99	109.73	40.00	19.55	11.22	7.13	4.85

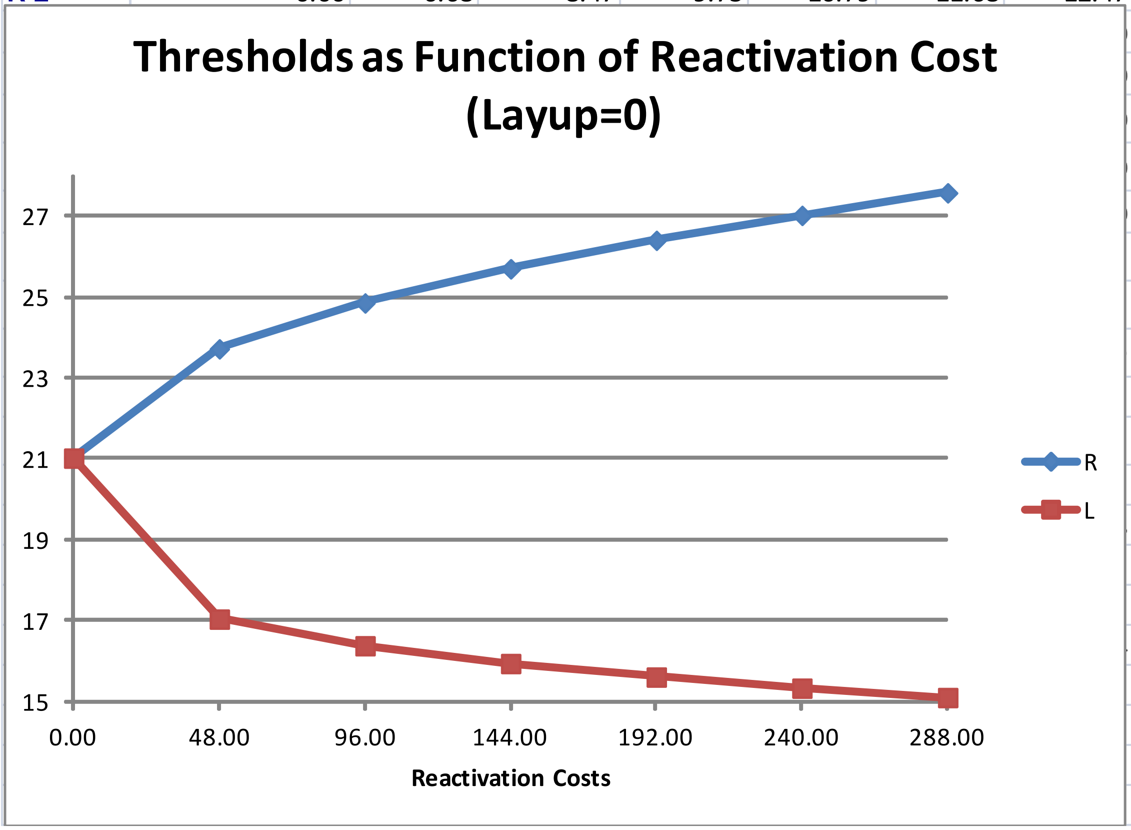


ROV_0 and ROV_1 are the LHS of (25) and (26), V the last two terms of (26) as P increases from 5 to 35, having derived the real option coefficients A_1 and B_2 and thresholds R and L as the solution for (30)-(33). Δ and Γ are the first and second derivatives of (25) and (26) with respect to P used in confirming that (23) and (24) are solved. Inputs are from Figure 12, except for P.

Logically, the mothballing option value declines and the reactivation option value increases as the freight rate P increases.

Figure 16

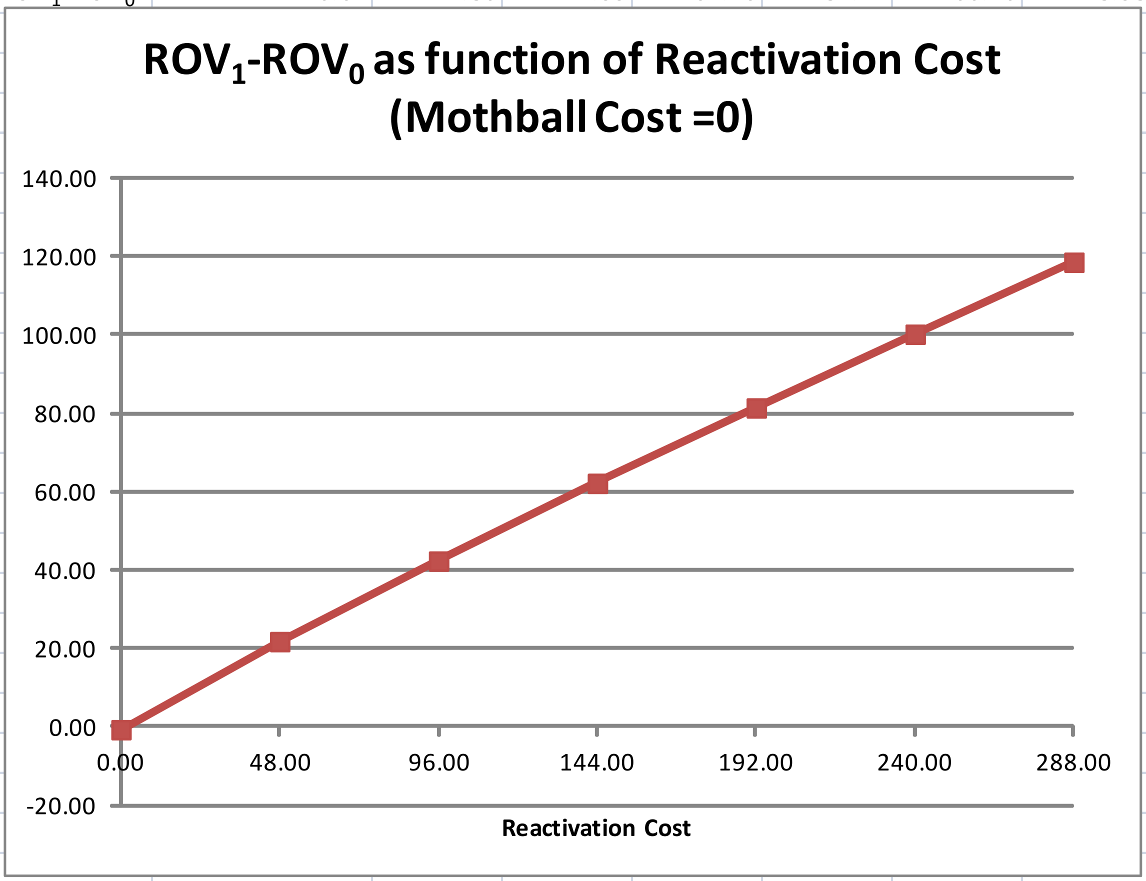
R	21.025	23.726	24.859	25.704	26.407	27.023	27.580
L	21.025	17.047	16.388	15.950	15.616	15.343	15.111
R-L	0.00	6.68	8.47	9.75	10.79	11.68	12.47



Having derived the real option coefficients A_1 and B_2, the thresholds R and L are the solution for (30)-(33). Inputs are from Figure 12, except for c_3.

Figure 17

ROV_0	11102.14	10997.12	10928.60	10870.97	10819.53	10772.29	10728.21
ROV_1	11101.48	11018.93	10970.94	10933.13	10900.94	10872.49	10846.79
R-L	0.00	6.68	8.47	9.75	10.79	11.68	12.47
ROV_1-ROV_0	-0.67	21.80	42.35	62.16	81.41	100.20	118.58

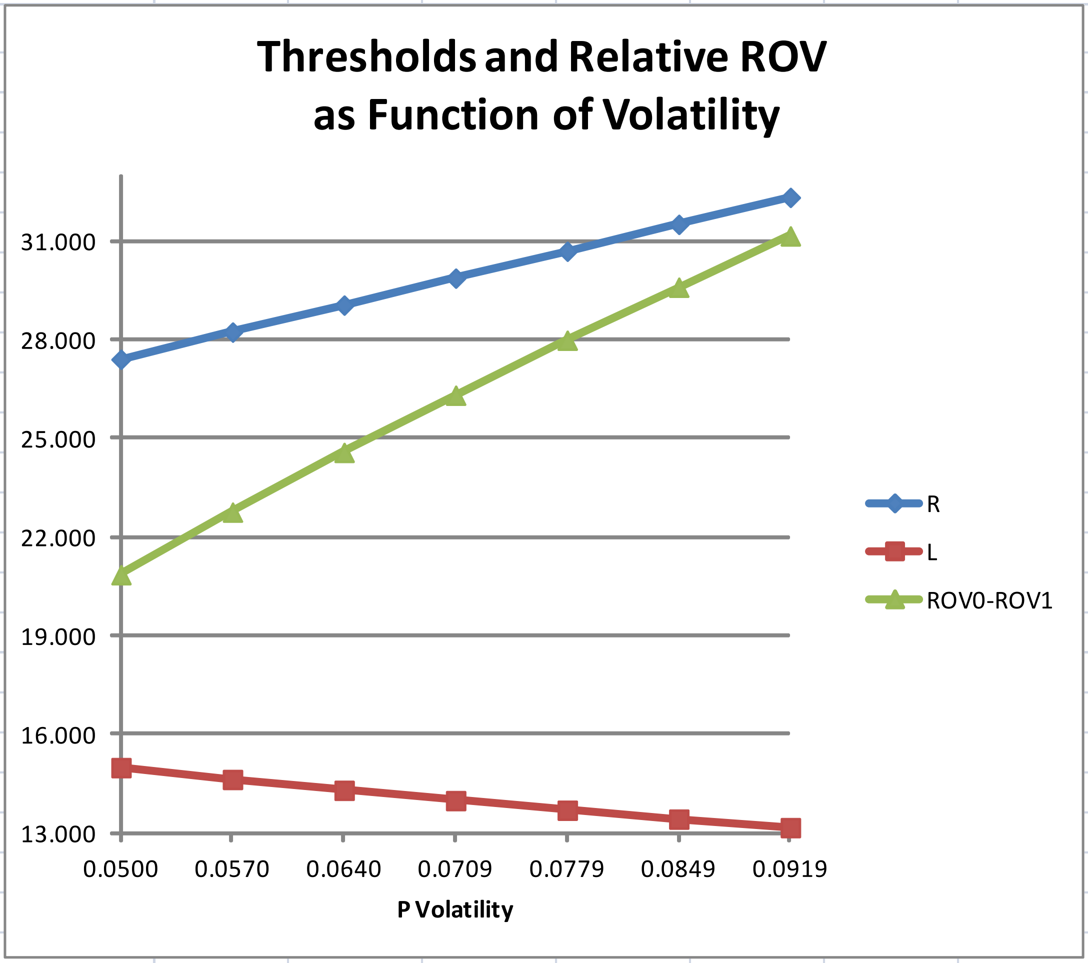


ROV_0 and ROV_1 are the LHS of (25) and (26) as the reactivation cost c_{3a} increases from 0 to 288 having derived the real option coefficients A_1 and B_2 and thresholds R and L as the solution for (30)-(33). Inputs are from Figure 12, except for c_3.

The reactivation option value ROV_0 decreases more than the mothballing option value decreases as the reactivation cost increases. Not shown is that the ROV_0 decreases and the mothballing option value increases as the reactivation cost increases (if the mothballing cost decreases so the combined cost remains constant).

Figure 18

σ_P	**0.0500**	**0.0570**	**0.0640**	**0.0709**	**0.0779**	**0.0849**	**0.0919**
R	27.420	28.246	29.067	29.885	30.702	31.521	32.341
L	15.004	14.645	14.312	14.002	13.711	13.438	13.180
ROV_0-ROV_1	20.871	22.773	24.587	26.326	27.998	29.610	31.168
ROV_0	10761.049	11852.916	12830.249	13701.079	14474.728	15160.928	15769.233
ROV_1	10740.178	11830.144	12805.661	13674.753	14446.730	15131.318	15738.065
R-L	12.42	13.60	14.75	15.88	16.99	18.08	19.16



ROV_0 and ROV_1 are the LHS of (25) and (26) as the volatility of the revenue increases from 5% to 9.2% having derived the real option coefficients A_1 and B_2 and thresholds R and L as the solution for (30)-(33). Inputs are from Figure 12, except for σ_P.

The reactivation option value ROV_0 increases slight more than the mothballing option value increases as volatility increases. The spread between R and L increases as volatility increases, consistent with the Mossin basic model, Figure 4.