The Impact of Uncertain Revenues and Costs on Time-to-Build Projects: a Real Options Approach

Lauri Kauppinen

Department of Mathematics and Systems Analysis, Aalto University

Ahti Salo

Department of Mathematics and Systems Analysis, Aalto University

Afzal S. Siddiqui*

Department of Statistical Science, University College London and Department of Computer and Systems Sciences, Stockholm University

Abstract

Lagging public-sector investment in infrastructure and the deregulation of most industries mean that decisions will have to be made increasingly by the private sector under multiple sources of uncertainty. We enhance the traditional real options approach to analysing investment under uncertainty by accounting for both multiple sources of uncertainty and the time-to-build aspect. The latter feature arises in the energy and transportation sectors because investors can decide the rate at which the project is completed. Furthermore, two explicit sources of uncertainty represent the discounted cash inflows and outflows of the completed project. We use a finite-difference scheme to solve numerically for both the option value and the free boundary that characterises the optimal investment strategy. Somewhat counterintuitively, we find that with a relatively long time to build, a reduction in the growth rate of the operating cost may actually lower the investment threshold. This is contrary to the outcome when the time-to-build aspect is ignored in a model with uncertain price and cost. Hence, research and development efforts to enhance emerging technologies may be more relevant in infrastruc-

*Corresponding author
Email address: `afzal.siddiqui@ucl.ac.uk` (Afzal S. Siddiqui)
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ture projects with long lead times.

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1. Introduction

Public investment in infrastructure, such as power grids, telecommunications, and transport, in OECD countries has languished since the 1990s (OECD, 2011), dropping from a mean of over 4% of GDP in 1990 to 3% in 2007. In conjunction with a transition in many OECD countries towards service-based economies, an “infrastructure gap” has developed that could have serious consequences for the competitiveness of many OECD countries. Indeed, newly industrialising countries such as Brazil, China, and India have the comparative luxury of developing their infrastructure now with relatively generous public funding. By contrast, infrastructure in OECD countries is decades old in many sectors and faces a lack of public funding. In effect, the trend towards deregulation has in past thirty years has put more emphasis on private provision of infrastructure investment, which is confounded by the exposure to uncertain revenues and costs in decision making.

Due to the private sector’s greater role in handling infrastructure investments, concerns about managing uncertainty in the context of maximising profit have become more important. For example, John Laing PLC, a British private equity firm that develops and operates public infrastructure, has recently announced its intention to raise capital through a flotation on the London Stock Exchange to finance a fund for environmental infrastructure and is aiming to provide annual returns of 8% (FT, 2014). Thus, the introduction of private incentives into the public sphere necessitates the development and application of appropriate methods for decision making, viz., those that consider uncertainty in cash flows, managerial flexibility, and salient features of infrastructure projects. Such analysis may also provide insights to regulators in designing mechanisms that elicit desired outcomes, e.g., environmental or social, from a private sector that has profit maximisation as its principal motivation.

Using the real options approach (Dixit and Pindyck, 1994), we take a stylised view of an infrastructure project that is to be carried out by a private firm. Similar to Majd and Pindyck (1987), we assume that the firm has the discretion not only to decide when to launch the project but also to determine the rate at which its construction proceeds. Indeed, large infrastructure projects can take years or even decades to build, and once construction is initiated, the cash flows may fluctuate to the point where it is optimal for the firm to suspend progress. Thus, an optimal decision rule is characterised by a free boundary that indicates the minimum revenues from the completed project for every possible realisation of operating costs and remaining investment. If the current revenue level is above this threshold, then the next tranche of investment is undertaken; otherwise, it is optimal to suspend investment.

Motivated by the fact that next-generation infrastructure projects, e.g., for smart grids or electric vehicles, may have both uncertain and non-cointegrated revenues and operating costs, we extend Majd and Pindyck (1987) to the case of two sources of uncertainty. We find that this consideration has a non-monotonic effect on the optimal threshold boundary for investment when the operating cost is high. Intuitively, with almost no time to build, a reduction in the growth rate of the operating cost monotonically increases the investment threshold as it is optimal to wait for lower cash outflows before completing the project. However, with a relatively long time to build, a reduction in the growth rate of the operating cost may actually lower the investment threshold as the expected absolute decrease in the operating cost upon project completion is high. This effect is especially pronounced when the operating cost is high.

The rest of this paper is structured as follows:

- Section 2 provides a review of the relevant literature.
- Section 3 formulates the problem and provides a quasi-analytical solu-

tion to it.

- Section 4 consists of numerical examples that convey our main insights.
- Section 5 summarises the work, discusses its limitations, and points out directions for future research.

2. Literature Review

In contrast to the now-or-never net present value (NPV) approach, the real options framework reflects the value of managerial flexibility in response to unfolding uncertainties. For example, McDonald and Siegel (1986) examine the value of the deferral option in which a firm waits for the optimal time to invest when both revenues and investment costs are uncertain. Embedded options, such as the discretion to suspend and resume operations (McDonald and Siegel, 1985), expand or modify the project after initial investment (Pindyck, 1988), and determine the capacity of the project (Dixit, 1993; Dangl, 1999; Décamps et al., 2006), may also be handled. Such flexibility is often present in real projects and can affect the initial investment decision. Unlike the now-or-never NPV approach, the effects of these features on the value of the investment opportunity and optimal adoption thresholds may be assessed (Dixit and Pindyck, 1994).

A simplifying assumption in most of the literature is that the project is constructed immediately after the investment decision has been taken. In other words, the rate of investment is infinite, which is defensible only if lead times are low relative to the lifetime of the project. However, this assumption does not hold in most infrastructure projects, e.g., transmission lines for electricity may take several years to construct with several stages encompassing the initial planning permission to assembling the towers to restoring the land (Hydro-Québec, 2014). Relaxing the assumption of an infinite investment rate, Majd and Pindyck (1987) tackle the problem of a firm facing uncertain revenues with discretion over not only initiation of the investment cycle but also suspension and resumption of the investment process as each stage is completed. Thus, their decision rule encompasses an optimal revenue trigger for each stage that depends on the remaining investment until project completion. In effect, they have embedded options to manage the time to build and show that this additional flexibility reinforces the standard real options result that higher volatility and the effective growth rate of the revenues delay action. By contrast, Bar-Ilan and Strange (1996) have a model with investment lags and an embedded option to abandon the project costlessly. Because the marginal costs of waiting, i.e., the foregone revenues from not investing, are higher as a result of the lead time to receipt of cash flows and an abandonment option that puts a lower bound on the value of those cash flows, the standard real options result is weakened or even reversed: higher uncertainty may reduce the investment trigger. Aguerrevere (2003) extends the issue of investment lags to include competition.

Recent real options work develops quasi-analytical solutions to problems when both revenues and operating costs are uncertain (Adkins and Paxson, 2011). In contrast to McDonald and Siegel (1986), they relax the assumption that the project's payoff is homogenous in the revenues and costs because it may be that it is operating costs rather than the investment cost that is prone to uncertainty. Consequently, the dimension-reducing step of turning the partial differential equation (PDE) into an ordinary differential equation (ODE) no longer holds, and the optimal investment trigger is not a linear relationship between revenues and costs. Adkins and Paxson (2011) motivate their work in the context of renewal assets, while Dockendorf and Paxson (2013) apply a similar model to the case of commodity switching in a production plant. An important consideration of this strand of the literature is that often revenues and operating costs cannot be modelled together, i.e., as a single stochastic process describing the profit flow, because the two processes are not cointegrated. Indeed, for infrastructure projects concerning new technologies, e.g., smart grids or electric vehicles, there is not even a time series of the relevant revenues and operating costs from which to detect the presence of cointegration. Taking this point of view, our work also makes a methodological enhancement to real options by considering the time-to-build attribute together with multiple uncertain factors.

3. Analytical Model

3.1. Assumptions

We extend Majd and Pindyck (1987) by considering two stochastic variables that determine the value of the project. Likewise, our work could also be thought of as adding investment lags to the two-factor model of Adkins and Paxson (2011). Specifically, our aim is to examine how introducing a stochastic variable that represents the costs incurred by the finished project affects the investor's optimal investment policy. We denote the cash inflows of the finished project with V_t and the cash outflows with C_t, where $t \geq 0$ indicates time. Thus, the payoff of the finished project at time t is $\max(V_t - C_t, 0)$. Note that we implicitly assume that the finished project can be scrapped without cost as the payoff cannot yield negative values even if $V_t - C_t < 0$ when the project is finished. However, this assumption is not restrictive because a rational investor will never complete the investment program if the payoff is negative.

We assume that both the option to invest in the project and the project itself are perpetual.[2] This is not true in reality, but because the cash flows are discounted, the effect of distant cash flows is negligible. The benefit of assuming a perpetual option is that it relieves us from making the option value an explicit function of time. We also assume that V_t and C_t follow the following geometric Brownian motions (GBMs)

$$dV_t = \alpha_V V_t dt + \sigma_V V_t dz_t \tag{1a}$$

[2] The assumption that the project is perpetual could be easily relaxed by modifying the payoff of the investment option. However, it does not significantly alter our results.

$$dC_t = \alpha_C C_t dt + \sigma_C C_t dw_t \tag{1b}$$

where α_V and α_C are drift rates, $\sigma_V \geq 0$ and $\sigma_C \geq 0$ are volatilities, and dz_t and dw_t are increments of uncorrelated Wiener processes. We take the increments of the GBMs to be uncorrelated as they are assumed to arise from non-cointegrated time series. However, instantaneous correlation between the two increments would be straightforward to implement.

The assumption of constant drift and volatility parameters of GBMs in (1a) and (1b) implies that the investor cannot affect the evolution of V_t and C_t. In the case of V_t, this is essentially a perfect market assumption, i.e., the investor takes the market value of the output of the project as given. In the case of C_t, the interpretation depends on the situation. If $\alpha_C = 0$, then the interpretation is simply that the evolution of the cost variable is stochastic yet without a trend. When $\alpha_C > 0$, the interpretation is that the costs are expected to increase in the long run. For example, if the main cost determinant of the finished project is a diminishing natural resource, then the interpretation might be that because the price of this resource will increase in the future due to decreasing reserves; consequently, the costs of production will rise. By contrast, if $\alpha_C < 0$, then we can form an interesting interpretation. Consider the case of new technology adoption, e.g., electric vehicles (EVs) and a charging infrastructure. If the adoption of EVs is in line with the goals of policymakers, then they might support the R&D required to initiate private-sector investment and further accelerate the adoption process. In this case, it is also feasible for policymakers to make their information and progress available to the public so that the private sector can capitalise on the evolving technology, thereby fulfilling the goals behind the public investments.[3] This implies that the private investor in our case experiences

[3] According to the Joint Research Centre of the European Commission (European Commission, 2013), about 65% of the outstanding total European EV RD&D budget of €1.9 billion is from public funding. The report also finds that an increased exchange of information between the projects would result in a better societal return for the investments

an exogenous learning curve effect that decreases the costs of the finished project over time. Hence, by considering the case $\alpha_C < 0$, we can examine how an exogenous learning curve effect described above affects the actions of rational investors.

We model the investment process *à la* Majd and Pindyck (1987) and denote the capital investment left at time t with K_t, the investment rate with I, and the maximum investment rate with $k \geq 0$. Thus, the dynamics of K_t are as follows:

$$\frac{dK}{dt} = -I, \ I \leq k \tag{2}$$

We assume that the investor can continuously adapt the rate at which she invests as new information about the expected profitability of the finished project arrives. This implies that our framework is most relevant in modelling situations in which the investment is made in multiple stages and the investor can halt the investment between the stages. If the investor has an opportunity to halt the process during the stages, then our model is even more relevant. In fact, the more irreversible the investment process becomes, the less appropriate our model is in describing the optimal investment behaviour. With such irreversibility, a model such as the one in Bar-Ilan and Strange (1996) should be employed, which will lead to different results.

Since we will use the dynamic programming approach to value the investment option, we denote the firm's required rate of return with $\rho \geq 0$. As is typical with dynamic programming, ρ is interpreted as an exogenous parameter that represents the cost of maintaining the investment possibility. We assume $\rho > \alpha_V$ in order to rule out the case that it would be never optimal to exercise the option to invest.

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due to the exogenous learning effects described above.

3.2. Problem Formulation

Given initial values $V \equiv V_0$, $C \equiv C_0$, and $K \equiv K_0$, we denote the value of the option to invest as $F(V, C, K)$. The option value in $(V, C, K) \in \mathcal{X} \equiv (0, \infty) \times (0, \infty) \times (0, \infty)$ given the investment policy $I^*(V, C, K) \equiv I$ can be obtained from the following Bellman equation:

$$\rho F = \max_{I \in [0,k]} \left(\frac{\mathbb{E}[dF]}{dt} - I \right) \tag{3}$$

Note that dF is a function of I, and dt in the denominator means that the expression in the nominator is divided by the increment of time and not differentiated with respect to time. Intuitively, Eq. (3) states that the instantaneous return on the investment opportunity is equal to its net appreciation if it were managed optimally. By expanding dF using Itô's lemma and taking the expected value, we obtain:

$$\rho F = \max_{I \in [0,k]} \left(\frac{1}{2}\sigma_V^2 V^2 F_{VV} + \frac{1}{2}\sigma_C^2 C^2 F_{CC} + \alpha_V V F_V + \alpha_C C F_C - I F_K - I \right) \tag{4}$$

By noting that the expression to be maximised with respect to I is linear in I, we conclude that if it is optimal to invest at all, then it is also optimal to invest at the maximum rate k. Therefore, the optimal investment policy is "bang-bang" control as in Majd and Pindyck (1987).

Following Adkins and Paxson (2011), we use backward induction to obtain first the value of the option to invest when it is optimal to continue the investment program. This is separated from the option value in the waiting region by a unique continuous surface $V^*(C, K)$ in $\mathcal{X}$ so that it is optimal to invest if $V \geq V^*(C, K)$ and to wait otherwise. This assumption is based on the intuition that the option value is increasing in V. Thus, we denote the option value in the investment region $\mathcal{R} \equiv \mathcal{X} \cap \{V \geq V^*(C, K)\}$ with F and in the waiting region $\mathcal{W} \equiv \mathcal{X} \setminus \mathcal{R}$ with f. Under this assumption, the option value functions in the two regions are given by PDEs obtained by re-arranging Eq. (4):

$$\frac{1}{2}\sigma_V^2 V^2 F_{VV} + \frac{1}{2}\sigma_C^2 C^2 F_{CC} + \alpha_V V F_V + \alpha_C C F_C - k F_K - \rho F - k = 0 \text{ in } \mathcal{R} \tag{5a}$$

$$\frac{1}{2}\sigma_V^2 V^2 f_{VV} + \frac{1}{2}\sigma_C^2 C^2 f_{CC} + \alpha_V V f_V + \alpha_C C f_C - \rho f = 0 \text{ in } \mathcal{W} \tag{5b}$$

Note that only Eq. (5a) contains partial derivatives with respect to K as no investment occurs in $\mathcal{W}$.

The appropriate boundary conditions to the problem are:

$$F(V, C, 0) = \max(V - C, 0) \tag{6a}$$

$$\lim_{V \to 0} f(V, C, K) = 0 \tag{6b}$$

$$\lim_{C \to \infty} f(V, C, K) = 0 \tag{6c}$$

$$F(V^*(C, K), C, K) = f(V^*(C, K), C, K) \tag{6d}$$

$$F_V(V^*(C, K), C, K) = f_V(V^*(C, K), C, K) \tag{6e}$$

$$F_C(V^*(C, K), C, K) = f_C(V^*(C, K), C, K) \tag{6f}$$

Eq. (6a) is simply the payoff of the option when there is no investment requirement remaining, whereas Eq. (6b) states that when V reaches zero, the option becomes worthless. This is because zero is an absorbing barrier to the GBM given by Eq. (1a). Eq. (6c) means that the option value converges to zero as the operating costs of the finished project grow arbitrarily large. Eq. (6d) is the value-matching condition stitching together the two option values along the free boundary, $V^*(C, K)$. Eqs. (6e) and (6f) are the smooth-pasting conditions, which are first-order conditions for making optimal transitions across the free boundary. Note that now there are two smooth-pasting conditions as there are two stochastic variables.

3.3. Quasi-Analytical Solution

A general solution to Eq. (5b) is of the form

$$f(V, C, K) = A(K)V^{\beta(K)}C^{\eta(K)} \tag{7}$$

where coefficients $\beta(K)$ and $\eta(K)$ must satisfy the condition

$$\frac{1}{2}\sigma_V^2\beta(\beta-1) + \frac{1}{2}\sigma_C^2\eta(\eta-1) + \alpha_V\beta + \alpha_C\eta - \rho = 0 \tag{8}$$

for each value of K. We use short-hand notation for $\beta(K)$ and $\eta(K)$ here. By "general solution," we mean that any linear combination of functions of the form given by Eq. (7) satisfies the PDE given by Eq. (5b). Eq. (8) has solutions in all four quadrants of the (β, η)-plane (Adkins and Paxson, 2011). However, we can rule out three of the four quadrants by using the boundary conditions given by Eqs. (6b) and (6c). Doing so, we obtain that $\beta(K) > 0$ and $\eta(K) < 0$, i.e., the option value increases (decreases) with revenues (operating costs) in line with economic intuition. From now on, we will assume that the solution to PDE (5b) is $f(V, C, K) = A(K)V^{\beta(K)}C^{\eta(K)}$, where $(\beta(K), \eta(K)) \in (0, \infty) \times (-\infty, 0)\ \forall\ K \in (0, \infty)$ so that Eq. (8) holds. $A(K)$ must be solved for by using the other boundary conditions and the option value in $\mathcal{R}$.

Since the PDE in the investment region has no analytical solutions, we use a numerical approach based on an explicit finite-difference method to solve the rest of the investor's problem. However, now that we know the form of the analytical solution in the waiting region, we can write boundary conditions (6d)-(6f) in a more convenient form. By inserting the quasi-analytical solution given by Eq. (7) into the conditions mentioned above, we obtain that the following conditions must be met at the free boundary:

$$\frac{F(V^*(C,K), C, K)}{F_V(V^*(C,K), C, K)} = \frac{V^*(C,K)}{\beta(K)} \tag{9a}$$

$$\frac{F(V^*(C,K),C,K)}{F_C(V^*(C,K),C,K)} = \frac{C}{\eta(K)} \tag{9b}$$

where $\beta(K)$ and $\eta(K)$ satisfy Eq. (8). We will utilise conditions (9a) and (9b) to determine the free boundary numerically. Once the free boundary is obtained, we can solve for the values of $A(K)$, $\beta(K)$, and $\eta(K)$ for each discrete value of K. The numerical solution method is discussed in further detail in Appendix A.

4. Numerical Examples

We present the results of the model in two parts. First, we consider a base case and provide a discussion of the results in general. Next, we present the most interesting results by performing comparative statics to isolate the effects of individual parameters on the investor's optimal investment policy.

4.1. Base Case

For the base case, we assume that the total investment required to finish the investment program is $K = 6$ (M€) and the maximum investment rate is $k = 1$ (M€/year). This implies that the minimum time to complete the investment program is six years and that the unit of time is years. We set $\alpha_V = 0.04$ and $\sigma_V = 0.14$ in this section and consider at first a case in which the drift and volatility of C are the same as those of V ($\alpha_C = 0.04$ and $\sigma_C = 0.14$).[4] Finally, we assume that the discount rate is $\rho = 0.08$.

Figure 1 shows the level sets of the option value and the free boundary in the base case when $K = 6$. The option value is increasing in V and decreasing in C as intuition suggests, which is the case for other values of K as well.

[4] If we were considering an all-equity firm that consisted only of the investment opportunity studied here, then the base case values would imply that the volatility of the firm's stock is approximately $\sqrt{0.14^2 + 0.14^2} = 19.8\%$. Considering that the implied volatility of the S&P 500 index options sold on the Chicago Board Options Exchange is usually around 20%, the assumptions made on the volatilities of the processes are fairly realistic.

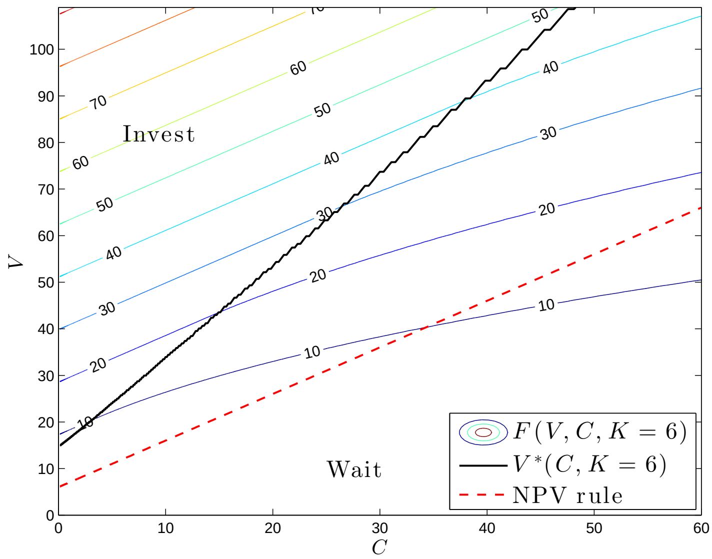

Figure 1: Option value, free boundary, and NPV threshold when $K = 6$

The black line indicates the position of the free boundary, $V^*(C, K = 6)$.[5] As expected, the investment threshold increases in C. Note also that the free boundary is not a level set of the option value. Therefore, we cannot, in general, draw a straight connection between the option value and the location of the investment threshold.

The red dashed line in Figure 1 shows the NPV investment threshold assuming that the entire investment is finished at the full rate if it is optimal to invest.[6] We can see that the NPV rule is to invest in cases when it is optimal to wait according to the real options rule. The NPV rule, by definition, is obtained by calculating the expected cash flows of the project

[5]The free boundary does not appear smooth because of the numerical finite-difference method used to solve the problem.

[6]In this case the NPV rule is to invest only if $e^{-\rho\frac{K}{k}}\left(Ve^{\alpha_V\frac{K}{k}} - Ce^{\alpha_C\frac{K}{k}}\right) - \frac{k}{\rho}\left(1 - e^{-\rho\frac{K}{k}}\right) \geq 0$.

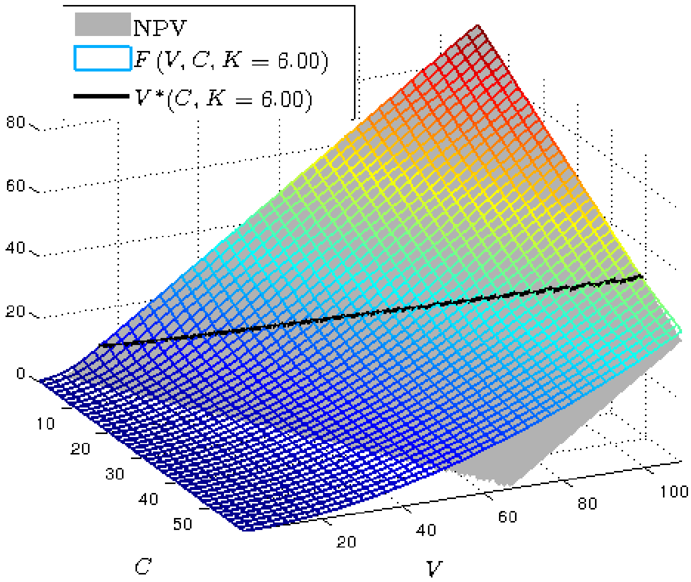

Figure 2: Option value surface, now-or-never NPV, and free boundary when $K = 6$

net of the initial investment costs. Therefore, there must be other reasons than the initial investment cost for the free boundary, $V^*(C, K = 6)$, to be above the NPV threshold. The reason is twofold. First, since both V_t and C_t evolve stochastically in time, there is a chance that the investment opportunity might increase in value over time. This implies that there are benefits to waiting that are not present in the NPV analysis. Second, as there is uncertainty in the value of the finished project due to the time-to-build aspect, it is optimal to wait longer than the NPV rule suggests in order to cover this uncertainty by waiting for the expected value of the finished project to rise well above the NPV rule.

Figure 2 shows the option value surface, the now-or-never NPV, and the projection of the free boundary onto the option value surface when $K = 6$. The value-matching and smooth-pasting conditions are satisfied by the numerical solution as the option values in the investing and waiting regions meet smoothly at the free boundary. Also, the option value is non-negative

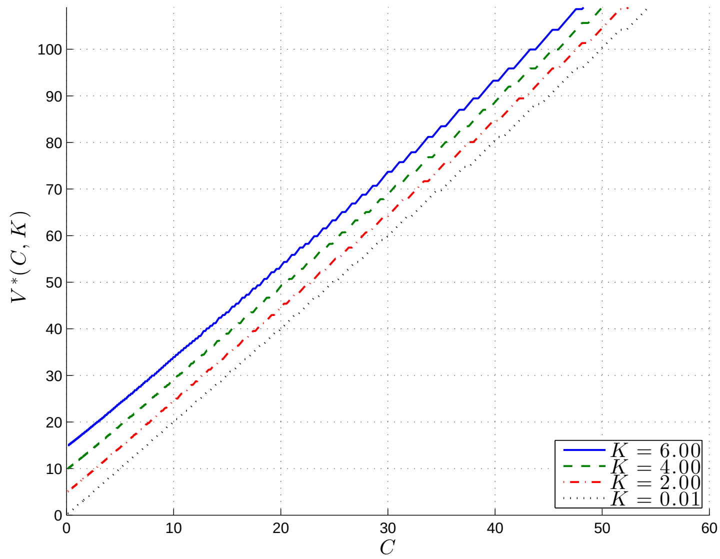

Figure 3: Free boundaries in the base case for different values of K

for all values of (V, C). By comparing the option value and the NPV, we observe that the option value is greater than the NPV for all values of (V, C), thereby reflecting the fact that unlike the NPV analysis, real options analysis considers also the value of waiting and the possibility to vary the investment rate. We also notice that the difference between the option value and the NPV converges to zero as V increases and C decreases. This happens because then the investment program will be completed almost certainly at full pace yielding, on average, a total payoff that equals the NPV.

Figure 3 shows the investment thresholds for various values of K in the base case. The threshold curves are increasing in C for each value of K as they should by the argument that the value of the finished project is decreasing in C. Also, in the base case, the investment thresholds increase in K. This is due to two reasons. First, the remaining initial investment cost increases in K. Second, the uncertainty over the value of the payoff when the investment program is completed is increasing in K since a large value of K indicates that the minimum time-to-build is large as well. Note that Figure 3 can be used as a decision rule: since we have implicitly assumed that the investor can observe V, C, and K at each point in time, she may use the investment thresholds at different values of K as a guide on how to proceed optimally with the investment program.

4.2. Comparative Statics

4.2.1. Sensitivity with Respect to α_C and α_V

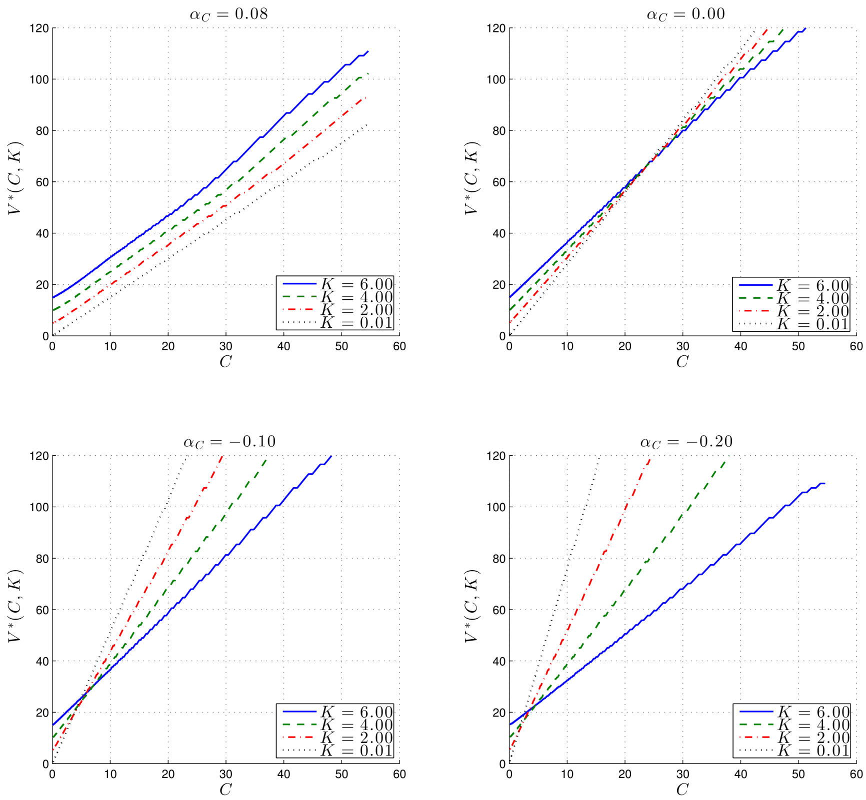

Figure 4: Sensitivity of the free boundary with respect to α_C

As the main motivation for this study is to gain insight into how the inclusion of C affects the investor's choices, we first discuss the mechanics behind the effects of α_C on the investor's optimal behaviour in detail. Figure 4 illustrates the investment thresholds at different values of K for various values of α_C while holding the other parameters the same as in the base case. For the smaller values of K, i.e., when the remaining time to build is negligible, the effect of α_C on the results is monotonic: a decrease in α_C shifts the investment threshold up and, thus, increases the incentive to wait. Intuitively, lowering α_C increases the value of the option to wait as the expected operating cost upon completion of the project will be lower. However, for $K = 6$ the effect is more subtle: when α_C decreases from 0.08 to -0.10, the investment threshold increases, but as α_C decreases further, $V^*(C, K = 6)$ actually reduces.

The effect of α_C on the results can be understood by considering the expected evolution of $V - C$. By using Eqs. (1a) and (1b), we obtain:

$$\mathbb{E}[(V - C)_{t+s}|\mathcal{F}_t] = V_t e^{\alpha_V s} - C_t e^{\alpha_C s}, \tag{10}$$

where $\mathcal{F}_t$ is a set containing all information on the evolution of V and C up to time t. Now, we observe explicitly that the expected evolution of the payoff depends on the value of α_C, i.e., specifically, it is decreasing in C and α_C.

We first assess how α_C affects the investment boundary when $K = 0.01$, i.e., the capital investment is nearly finished and the payoff can be obtained almost instantaneously. This will help us to understand the effect on the investment thresholds for larger values of K. The plots in Figure 4 show that the investment threshold $V^*(C, K = 0.01)$ increases monotonically as α_C decreases. In other words, when α_C decreases, the future probability distribution of the payoff shifts to a desirable direction, thereby creating incentives for waiting since the gap between the capital appreciation of $V - C$ and the required rate of return ρ narrows. This is also shown by McDonald and Siegel (1986) in the case $K \to 0$.[7]

Next, we consider the effect of α_C on the investment thresholds when $K > 0.01$ and the payoff cannot be obtained instantaneously on demand. Recall that the underlying idea behind the dynamic programming approach used to solve the investor's problem is that at each state (V, C, K), the optimal decision is derived assuming that the subsequent decisions are optimal as well. In our case, this means that the investor holding the option to invest with K amount of initial investment remaining knows the optimal investment rule for smaller values of K as well. Due to the fact that the cash flows are discounted, this implies that for larger values of K it is optimal to wait for V and C to reach such values that the remaining initial investment will be done with minimal pauses on average assuming that the investor follows the optimal investment rule.[8] In this way, the initial investment costs will be paid as late as it is reasonable while still allowing the investor to obtain the payoff as soon as it is optimal to do so in most cases. The drivers behind this logic are that, first, the discount factor implies that cash outflows paid in the future are less valuable than if they were paid now, and second, because of the discount rate, it is better to obtain the payoff now than in the future assuming that it would actually be optimal to obtain the payoff now. Therefore, the placement of the investment thresholds at larger values of K depends on both the placement of the investment threshold when $K \to 0$ and the stochastic evolution of the payoff.

To understand the logic above fully, we shall first consider the situation in the upper left plot of Figure 4, where $\alpha_V < \alpha_C$. By Eq. (10), this implies that the payoff is expected to increase in the future less than in the base case displayed by Figure 3. Therefore, as α_C increases from the base case value,

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[7]It can be shown numerically that $V^*(C, K)$ converges to the analytical results of McDonald and Siegel (1986) when $K \to 0$.

[8]It can be shown numerically that if $V - C$ evolves according to Eq. (10), then the investor will receive the payoff after the minimum time-to-build once it is optimal to invest at $K = 6$ by investing at full rate up to the completion of the investment program.

$\alpha_C = 0.04$, the investment threshold for $K = 0.01$ decreases for each value of C since the incentive to wait diminishes. Following the threshold $V^*(C, K = 0.01)$, the thresholds for larger values of K shift down as well since otherwise the investor would wait for too long to begin investing and the expected discounted payoff at the end of the investment program would decrease. Note, however, that the spread between the thresholds for different values of K increases in C. This is explained by Eq. (10): since the expected increase of C is linear in C, the investor will wait for V to increase further for larger values of C to offset the larger expected increase of C. This argument applies generally as the effects of α_C on the investment thresholds are amplified at large values of C.

Let us next consider what happens when α_C decreases from 0.08 to 0.00 by examining the upper plots in Figure 4. We notice that $V^*(C, K = 0.01)$ shifts upwards for each value of C in comparison to the same threshold curve for $\alpha_C = 0.08$ as α_C decreases. This reflects the increased incentive to wait since for $\alpha_C = 0.00$, C is not expected to increase at all, whereas the stochastic process of V remains the same as before.

However, we notice that the investment thresholds for large values of K shift up less than the thresholds for smaller values of K for each value of C as α_C decreases. The explanation for this is that the decrease of α_C from 0.08 to 0.00 increases the growth rate of the payoff. Therefore, since after the decrease of α_C the payoff is expected to increase more during the minimum time-to-build than in the case of $\alpha_C = 0.08$, it is optimal to start investing at lower values of V with respect to the investment threshold at $K \to 0$ given a value of C than in the case $\alpha_C = 0.08$. As explained above, this enables the investor to obtain the payoff as soon as it is optimal to do so on average.

Moreover, we observe that for small (large) values of C, the investment threshold is increasing (decreasing) in K. This is explained by the effect of the initial investment cost on the threshold. Since the investor needs to pay K amount of capital to obtain the payoff, it is not optimal to start investing if the expected payoff of the investment program exercised by the optimal policy does not at least exceed the discounted initial investment left. This is depicted by the fact that the intercept of the investment threshold and the vertical axis is positive in all cases where $K > 0$. Also, as K decreases, this intercept converges to zero as the initial investment left decreases and its effect on the investment threshold vanishes. Recall that the initial investment outflows are completely irreversible. Therefore, the initial investment outflows that are already paid are not taken into consideration in the subsequent investment decisions. This explains why for small values of C, the investor will ultimately settle for a payoff that is smaller than it is on the investment thresholds for larger values of K, as can be seen in Figure 4.

This threshold-increasing effect of K is present for all values of C, and its magnitude does not depend on the value of C. For small values of C, first, the expected payoff of the optimally completed investment program is not large in comparison to K for values of V that are near the investment threshold, and second, the absolute change in the value of the payoff during the time-to-build is on average small according to Eq. (10). Therefore, for small values of C, the investment threshold is increasing in K since the need to wait for V to reach such values that the expected payoff overcomes the initial investment dominates the investment-hastening effect of the expected growth of the payoff during the investment period.

By contrast, for larger values of C, the expected payoff of an optimally executed investment program is significantly larger than the initial investment and the expected absolute increase of the payoff during the investment program is substantial. Therefore, the effect of the expected growth of the payoff during the investment process dominates the effect of the initial investment, and, thus, the investment threshold is actually decreasing in K for large values of C.

Finally, we will consider the cases where α_C decreases below zero, i.e., C is expected to decrease in the future. We can see from Figure 4 that

$V^*(C, K = 0.01)$ will shift further up as α_C decreases. The effect on the thresholds at higher values of K is not as dramatic, however. We notice that, for example, $V^*(C, K = 6)$ stays the same as α_C changes from 0.00 to -0.10 and actually shifts down when α_C decreases further to -0.20. This happens as now C is expected to decrease in the future, whereas V is expected to increase as before. Therefore, when $K = 6$, it is optimal to start investing even if C is substantially larger than what it should be in order to exercise the option as $K \to 0$.

Our interpretation of the results above is that by starting to invest at larger values of K, the investor buys the right to be able to receive the payoff just as V and C reach such values that it is optimal to do so, rather than having to wait for the minimum time-to-build to receive the payoff once this happens. This interpretation justifies the observation that the investment threshold may be decreasing in K for large values of C since it is optimal to start investing even when the current value of the payoff is suboptimal in comparison to the threshold at $K \to 0$ if the payoff is expected to increase fast enough after the investment process begins. Also, the observation that $V^*(C, K = 6)$ shifts downwards as α_C decreases from -0.10 to -0.20 is then well explained by the fact that since C is expected to decrease at a higher rate when $\alpha_C = -0.20$, it is optimal to start investing at higher values of C given a value of V because the expected decrease of C is greater.

The effect of α_V on the results is similar to that of α_C. An increase in α_V increases the benefits of waiting and shifts the investment threshold $V^*(C, K = 0.01)$ upwards. Again, the investment thresholds at larger values of K are located in a way that once the first initial investment is made, the investor will, on average, be able to invest continuously at the maximum rate up to the end of the investment program. We should also note that the investment thresholds grow without boundaries as $\alpha_V \to \rho$ (assuming that $\alpha_V > \alpha_C$) since then the long-term capital rate of return of the payoff converges to ρ and the cost of waiting diminishes. Finally, as ρ represents the cost of waiting in our model, the effect of an increase in ρ is to shift the investment threshold down for all values of K and, thus, hasten investment.

4.2.2. Sensitivity with Respect to k

As our explanation for the results above relies on the logic that the investor holding the option considers both the expected evolution of $V - C$ during the investment period and the optimal investment policy at smaller values of K when making decisions on whether to invest or wait, we would assume that the results of the comparative statics above would be amplified for smaller values of k since this would imply a longer investment period. Consider, for example, the case where $\alpha_C < 0$ and C is expected to decrease while V is expected to increase. Now, if we decrease the maximum investment rate, then we expect that it is optimal to start investing at even higher values of C given a value of V since the minimum time-to-build is longer, thereby implying that the expected decrease of C during the investment period is larger as well. By generalising the logic above, we would assume that a decrease in k would amplify the results of the comparative statics above. Motivated by this, we will next analyse the results of the same comparative statics as above using a smaller maximum investment rate, $k = 0.5$. This doubles the minimum time-to-build for every value of K in comparison to the value $k = 1$ used above.

Figure 5 shows the results of the comparative statics with respect to α_C when $k = 0.5$ and the other parameters are the same as in the base case. We note that our intuition is correct as the smaller value of k amplifies the effects of α_C on the investment thresholds. Note that the investment thresholds are not affected by the change in the value of k when $K = 0.01$ since then the payoff can be received almost instantly. The reason why the other thresholds react more dramatically to changes in α_C than in the case above is that now the investor needs to look further ahead in time when making decisions for larger values of K as the minimum time-to-build is longer.

An interesting result occurs in the lower right case of Figure 5 where

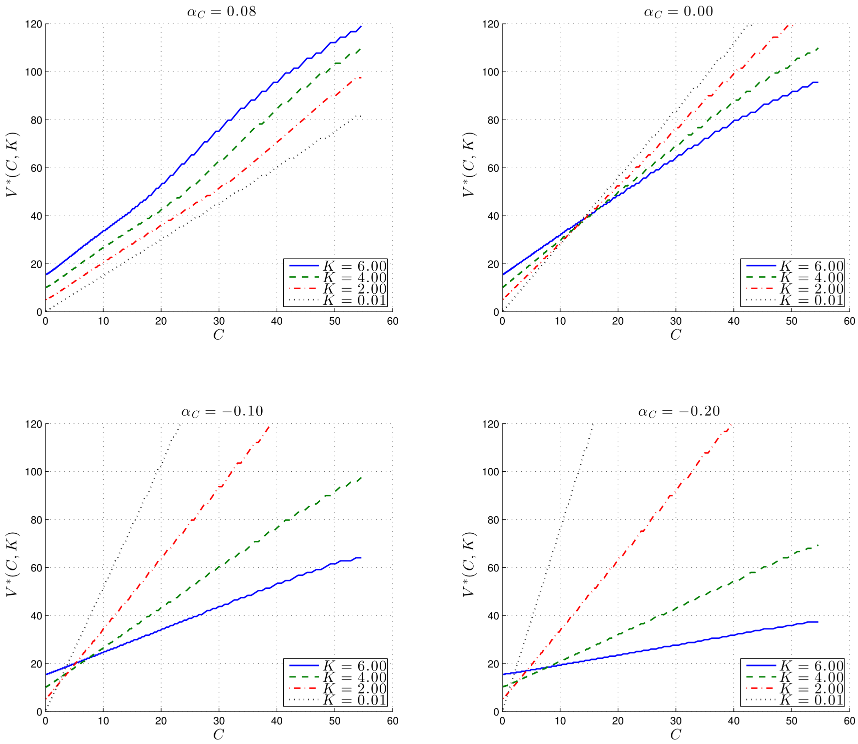

Figure 5: Sensitivity of the investment threshold with respect to α_C when $k = 0.5$

$\alpha_C = -0.20$. For large values of C and K, it is optimal to invest even if $V - C < 0$. However, this is well explained by the expected increase of $V - C$ during the investment program. Also, for each value of C, the NPV rule in this extreme situation is to invest at a smaller value of V than the real options rule suggests. The observation applies generally: the real options investment threshold is always larger than the now-or-never NPV threshold. This strengthens our explanation for why it might be optimal to invest even if the current value of the payoff is negative since the fact that the real options threshold is larger than the NPV threshold in all situations ensures that the average value of the investment program executed by the real options rule is positive in all cases.

Recall that Majd and Pindyck (1987) found the investment threshold to be increasing in K for all parameter values. On the contrary, in our two-factor model, the investment thresholds may be decreasing in K for certain values of C. What explains this difference? The answer is obvious: our model is built on different assumptions. Particularly, in our model there are two stochastic variables that determine the payoff whereas the model of Majd and Pindyck (1987) consists of only one. Therefore, the results are not completely comparable. This is also the explanation for why we find that in our model the investment threshold can be increasing in k, which might seem to contradict the results of Majd and Pindyck (1987). However, this effect is well explained by the evolution of the payoff as seen above.

4.2.3. Sensitivity with Respect to σ_V and σ_C

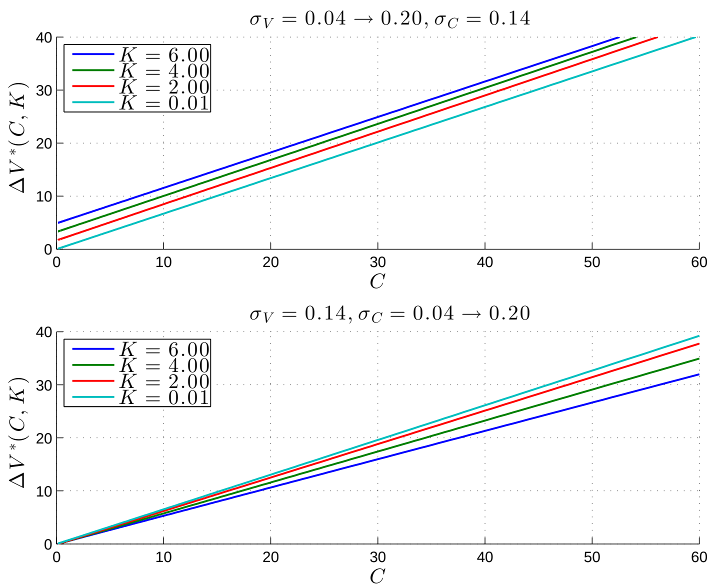

Figure 6: Sensitivity of the investment threshold with respect to σ_V and σ_C when $\alpha_V = \alpha_C = 0.04$, $k = 1.00$, and $\rho = 0.08$

Figure 6 indicates the sensitivity of the investment threshold with respect to the volatilities when the other parameters are as in the base case.[9] The upper graph shows how much the threshold changes given a value of (C, K) as σ_V changes from 0.04 to 0.20 and $\sigma_C = 0.14$, and the lower graph shows how much the threshold changes as σ_C changes from 0.04 to 0.20 and $\sigma_V = 0.14$. Both graphs reveal that the change in the threshold, i.e., $\Delta V^*(C, K)$, is positive for all values of (C, K). In fact, the observation holds for all tested parameter values: $V^*(C, K)$ is increasing in both σ_V and σ_C in all situations. This reflects the well-known property of options with convex payoffs: since the payoff $\max(V - C, 0)$ is bounded from below, an increase in the volatility of $V - C$ increases the benefits of waiting and, thus, increases the investment threshold (Majd and Pindyck, 1987; McDonald and Siegel, 1986).

A second observation common to both of the graphs in Figure 6 is that $\Delta V^*(C, K)$ is increasing in C for all values of K. Our interpretation is that this is due to the assumption that V and C follow GBMs. This assumption implies that the standard deviations of dV and dC are increasing in σ_V and σ_C by Eqs. (1a)-(1b). Therefore, the spread of the future values of the payoff $\max(V - C, 0)$ is more sensitive to the volatilities of V and C when the values of V and C are large. Thus, the sensitivity of the threshold with respect to the volatilities is increasing in C given a value of K since for a large value of C the value of V needs to be large as well in order to investment to occur as $V^*(C, K)$ is always increasing in C.

In the upper graph of Figure 6, $\Delta V^*(C, K)$ is increasing in K given a value of C, while in the lower graph, $\Delta V^*(C, K)$ is slightly decreasing with respect to K given any positive value of C. The evolution of $\Delta V^*(C, K)$ as a function of K given a value of C depends on other parameters than the volatilities as well. For example, Figure 7 displays $\Delta V^*(C, K)$ when $\alpha_V = 0.04$ and $\alpha_C = -0.20$. In this case, $\Delta V^*(C, K)$ is decreasing in K for all values of C when σ_C increases. Also, when σ_V increases, $\Delta V^*(C, K)$ is

[9] The lines in the graphs of this subsection are linearised to smooth out noise due to the numerical method used.

decreasing in K for large values of C.

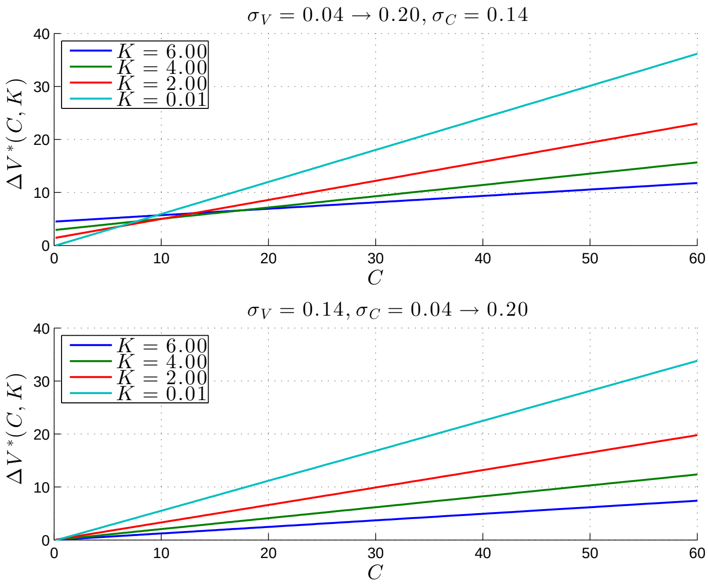

Figure 7: Sensitivity of the investment threshold with respect to σ_V and σ_C when $\alpha_V = 0.04$, $\alpha_C = -0.20$, $k = 1.00$, and $\rho = 0.08$

Next, we consider how the distribution of the total volatility of $V - C$ among the two variables affects the investment threshold. By total volatility, we mean the value of $\sigma_{total} = \sqrt{\sigma_V^2 + \sigma_C^2}$. Although σ_{total} is not an accurate measure of the standard deviation of $d(V - C) = dV - dC$ as this depends on the value of (V, C), we will use σ_{total} as an useful approximation of the volatility of $V - C$. Then, the question is that how does the investment threshold change as the value of (σ_V, σ_C) is varied so that σ_{total} remains constant. Figure 8 depicts such a sensitivity analysis when (σ_V, σ_C) evolves according to the chain $(0.20, 0.00) \to (0.14, 0.14) \to (0.00, 0.20)$, during which $\sigma_{total} = 0.20$, and $\alpha_V = \alpha_V = 0.04$, $k = 1.00$, and $\rho = 0.08$. The figure shows that for large values of K, $V^*(C, K)$ shifts down as σ_V decreases and σ_C increases. However, when $K = 0.01$, the investment threshold does not change visibly as (σ_V, σ_C) is varied so that σ_{total} remains constant, which is consistent with the result of McDonald and Siegel (1986). The behaviour of $V^*(C, K)$ with respect to the distribution of the volatilities described above is common to all tested parameter values. Hence, we conclude that when $K >> 0$, a situation in which most of the uncertainty is due to σ_V leads to higher investment thresholds than a situation in which most of the uncertainty stems from σ_C. Still, the threshold $V^*(C, K \to 0)$, which governs the completion of the investment program, depends only on σ_{total}.

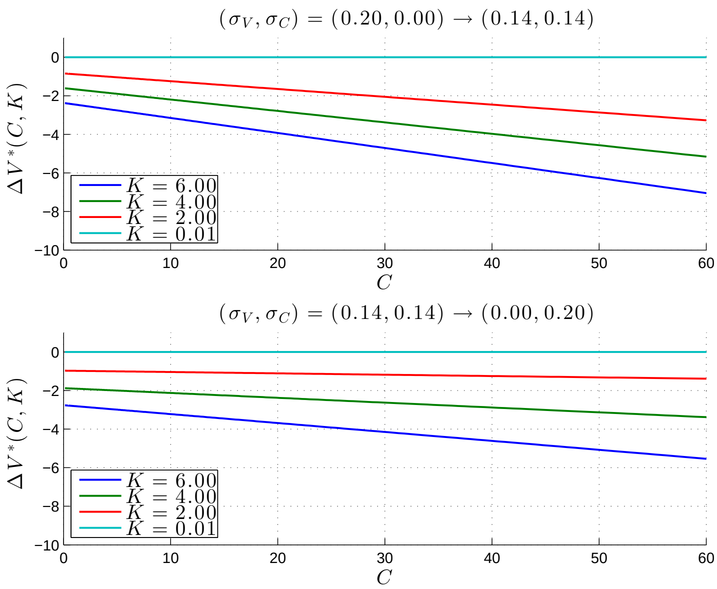

Figure 8: Sensitivity of the investment threshold with respect to the distribution of σ_{total} when $\alpha_V = 0.04$, $\alpha_C = 0.04$, $k = 1.00$, and $\rho = 0.08$

Finally, we consider what the effect of a non-zero correlation between dV and dC would be. By using Eqs. (1a) and (1b), we obtain that in the case of a non-zero correlation, the stochastic part of $d(V - C)$ has a variance of $(\sigma_V^2 V_t^2 - 2V_t C_t \sigma_V \sigma_C \mu + \sigma_C^2 C_t^2)\, dt$, where μ is the correlation between dz_t and dw_t. Therefore, a positive (negative) μ whould decrease (increase) the volatility of $V - C$ and, therefore, decrease (increase) the investment threshold in all situations since uncertainty is found to increase the threshold uniformly.

5. Conclusions

In this paper, we propose a method to compute the option value and the investment thresholds with an investor who sequentially invests in project opportunity, the payoff of which is a function of two stochastic variables. The sequential nature of the investment process is modelled by allowing the investor to choose the rate at which to invest continuously in time. As the investment rate is assumed to be bounded between zero and a positive constant, the investor cannot obtain the payoff instantly but has to wait for at least a minimum time-to-build.

The implications of the model are analysed via comparative statics. We find that when $K \to 0$, the investment threshold increases in α_V and decreases in α_C. For larger values of K, the placement of the investment thresholds depends on the expected stochastic evolution of the payoff $V - C$ alongside with the minimum time-to-build. As the value of k affects the minimum time-to-build, it affects the investment thresholds as well. We explain the results of the comparative statics by considering the investor's problem in the framework of dynamic programming. Uncertainty is found to postpone investment in all cases. Also, the real options investment threshold is found to be always larger than the now-or-never NPV threshold given a value of C.

Some of the outcomes that the model yields might seem to be in contradiction with the earlier results of Majd and Pindyck (1987). In particular, we find that if C is expected to decrease fast enough in the future compared to V, the minimum time-to-build is long, and the value of C is large compared to K, then it is optimal to start investing even if the current expected NPV of the payoff at the end of the investment program is negative. However, the difference is explained by the fact that in our model, the stochastic evolution of the payoff is different than in the model of Majd and Pindyck (1987) because of the inclusion of the second stochastic variable.

We choose V and C to represent the discounted cash in- and outflows of the completed project, respectively. We also assume that the value of the completed project is $\max(V - C, 0)$. However, the stochastic variables could have other interpretations depending on which particular investment situation is of interest. Also, the payoff could be generally any function of V and C in our framework.[10] In this sense, our model is general and can be used to analyse multiple investment situations that meet the assumptions made about the nature of the investment process and the stochastic variables.

One of the limiting assumptions of the model is that the investor can decide on whether to invest or wait continuously in time. As discussed above, this assumption might be an appropriate approximation in some situations. However, if the initial investment decision is completely irreversible, then our model does not apply. For example, the initial decision to build a coal power plant is practically completely irreversible once undertaken, and the construction time of the plant is substantial. Also, the major revenue and cost determinants of the power plant, i.e., the prices of electricity and coal, evolve stochastically in time. Therefore, the exercise of building and solving a two-factor model, in which the investment decision is modelled following the lead of Bar-Ilan and Strange (1996) could be interesting. In this case, the effect of the volatilities of the processes that V and C follow on the results could be in contrast to that in our model. In addition, it would be interesting to see if the effect of the drift rates would be similar to that in our model since even if the initial investment decision is made completely irreversible, then the investment lag implies that a rational investor considers how the payoff is expected to evolve during the lag when making investment decisions.

Acknowledgements

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[10]We do not take a stance on which conditions the payoff function should meet in order for the problem to have a solution.

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Appendix A. Numerical Solution Method

We first apply the transformation $F(V,C,K) = e^{-\rho\frac{K}{k}}G(X,Y,K)$, where $X = \ln V$ and $Y = \ln C$, to the PDE given by Eq. (5a) in order to modify the PDE to a simpler form and to ensure numerical stability. After the transformation, the PDE in $\mathcal{R}$ is:

$$\frac{1}{2}\sigma_V^2 G_{XX} + \frac{1}{2}\sigma_C^2 G_{YY} + \left(\alpha_V - \frac{1}{2}\sigma_V^2\right) G_X + \left(\alpha_C - \frac{1}{2}\sigma_C^2\right) G_Y - kG_K - ke^{\rho\frac{K}{k}} = 0 \tag{A.1}$$

Note that the coefficients of the PDE are now constant. After the transformation, the boundary conditions that solution for Eq. (A.1) must satisfy are:

$$G(X,Y,0) = e^{XY}, \tag{A.2a}$$

$$\frac{G(X^*(Y,K),Y,K)}{G_X(X^*(Y,K),Y,K)} = \frac{1}{\beta(K)} \tag{A.2b}$$

$$\frac{G(X^*(Y,K),Y,K)}{G_Y(X^*(Y,K),Y,K)} = \frac{1}{\eta(K)} \tag{A.2c}$$

where $\beta(K)$ and $\eta(K)$ solve Eq. (8) for each value of K.

Since we will solve the PDE numerically in a cubic grid, we need some additional boundary conditions that apply at the boundaries of the grid. For this purpose, we assume the following second-order boundary conditions:

$$\lim_{X\to\infty} G_{XX} = 0 \tag{A.3a}$$

$$\lim_{X\to-\infty} G_{XX} = 0 \tag{A.3b}$$

$$\lim_{Y\to\infty} G_{YY} = 0 \tag{A.3c}$$

$$\lim_{Y\to-\infty} G_{YY} = 0 \tag{A.3d}$$

These boundary conditions are chosen since they are known to work well with many financial options (Wilmott, 2007) as well as for our model. We will from now require that these conditions are approximately met at the boundaries of the lattice.

Let us denote $G(i\Delta X, j\Delta Y, \ell\Delta K) = G^{\ell}_{i,j}$, where $i \in \{i_{min}, i_{min}+1, ..., i_{max}\}$, $j \in \{j_{min}, j_{min}+1, ..., j_{max}\}$, and $\ell \in \{\ell_{min}, \ell_{min}+1, ..., \ell_{max}\}$. ΔX, ΔY, ΔK, and the minimum and maximum indices are predetermined constants that govern the dimensions of the lattice.[11] We use the following finite-difference approximations for the partial derivatives of G:

$$G_X(i\Delta X, j\Delta Y, \ell\Delta K) = \frac{G^{\ell}_{i+1,j} - G^{\ell}_{i-1,j}}{2\Delta X} \tag{A.4a}$$

$$G_Y(i\Delta X, j\Delta Y, \ell\Delta K) = \frac{G^{\ell}_{i,j+1} - G^{\ell}_{i,j-1}}{2\Delta Y} \tag{A.4b}$$

$$G_{XX}(i\Delta X, j\Delta Y, \ell\Delta K) = \frac{G^{\ell}_{i+1,j} - 2G^{\ell}_{i,j} + G^{\ell}_{i-1,j}}{(\Delta X)^2} \tag{A.4c}$$

$$G_{YY}(i\Delta X, j\Delta Y, \ell\Delta K) = \frac{G^{\ell}_{i,j+1} - 2G^{\ell}_{i,j} + G^{\ell}_{i,j-1}}{(\Delta Y)^2} \tag{A.4d}$$

$$G_K(i\Delta X, j\Delta Y, \ell\Delta K) = \frac{G^{\ell+1}_{i,j} - G^{\ell}_{i,j}}{\Delta K} \tag{A.4e}$$

By inserting the approximations above in the transformed PDE given by Eq. (A.1), we obtain the following difference equation:

$$G^{\ell+1}_{i,j} = a_+ G^{\ell}_{i+1,j} + a_- G^{\ell}_{i-1,j} + b_+ G^{\ell}_{i,j+1} + b_- G^{\ell}_{i,j-1} + cG^{\ell}_{i,j} - n_\ell \tag{A.5}$$

where

$$a_+ = \frac{\Delta K}{2k\Delta X}\left(\frac{\sigma_V^2}{\Delta X} + \alpha_V - \frac{\sigma_V^2}{2}\right) \tag{A.6a}$$

[11] i_{min} and j_{min} will be negative in order to obtain option values near the zero border in the (V, C)-world. The value of ℓ_{min} will be zero.

$$a_{-} = \frac{\Delta K}{2k\Delta X}\left(\frac{\sigma_V^2}{\Delta X} - \alpha_V + \frac{\sigma_V^2}{2}\right) \quad \text{(A.6b)}$$

$$b_{+} = \frac{\Delta K}{2k\Delta Y}\left(\frac{\sigma_C^2}{\Delta Y} + \alpha_C - \frac{\sigma_C^2}{2}\right) \quad \text{(A.6c)}$$

$$b_{-} = \frac{\Delta K}{2k\Delta Y}\left(\frac{\sigma_C^2}{\Delta Y} - \alpha_C + \frac{\sigma_C^2}{2}\right) \quad \text{(A.6d)}$$

$$c = 1 - \frac{\sigma_V^2 \Delta K}{k(\Delta X)^2} - \frac{\sigma_C^2 \Delta K}{k(\Delta Y)^2} \quad \text{(A.6e)}$$

$$n_\ell = \Delta K e^{\rho \frac{\ell \Delta K}{k}} \quad \text{(A.6f)}$$

If the lattice point considered is on the lattice boundary, then we discretise the boundary conditions given by Eqs. (A.4a)–(A.4e). Subsequently, the discretised boundary conditions can be inserted into Eq. (A.5) to compute the option value at the lattice point.

In terms of the computational method, first we calculate the values of G when $\ell = 0$ using Eq. (A.2a). Next, we calculate the values of option when $\ell = 1$ using Eq. (A.5) and the discretised versions of boundary conditions (A.4a)–(A.4e). Now that we know the preliminary option values at $\ell = 1$, the next task is to find the investment threshold. For this, we use boundary conditions (A.2b), (A.2c), and (8). By combining these conditions, the following equation must be met on the investment threshold:

$$\frac{1}{2}\sigma_V^2 \frac{G_X}{G}\left(\frac{G_X}{G} - 1\right) + \frac{1}{2}\sigma_C^2 \frac{G_Y}{G}\left(\frac{G_Y}{G} - 1\right) + \alpha_V \frac{G_X}{G} + \alpha_C \frac{G_Y}{G} - \rho = 0 \quad \text{(A.7)}$$

Our strategy is then to evaluate the left-hand side of this equation at every lattice point for $\ell = 1$ by using the finite-difference approximations in Eqs. (A.4a) and (A.4b).[12] The location of the investment threshold given a value of j is then the pair (i, j), for which the absolute value of the left-hand side

[12]The locations of the investment threshold at $\ell = \ell_{min}$ and $\ell = \ell_{max}$ are extrapolated.

of Eq. (A.7) is the smallest in $i \in \{i_{min}+1, i_{min}+1, ..., i_{max}-1\}$.[13] After we have numerically solved the free boundary for $\ell = 1$, we can solve the values of constants $A(\Delta K)$, $\beta(\Delta K)$, and $\eta(\Delta K)$ using Eqs. (A.2b) and (A.2c), the value-matching condition in Eq. (6d), the functional transformation, and the form of the analytical solution in the waiting region given by Eq. (7). We solve the values of these constants at each investment threshold for $\ell = 1$ and take the averages of these values to determine the final values.

After having calculated the initial option values, the placement of the investment threshold, and the constants of the analytical solution in the lower region, we should fill the waiting region for $\ell = 1$ with the values given by the analytical solution before repeating the procedure above for $\ell = 2$. However, as this proves to cause numerical instability, we update the option values after the initial option values and the investment thresholds have been determined for all values of ℓ. Once the iteration above has been completed for all values of ℓ and the option values in the waiting region are updated, the final solution for the investor's problem is obtained by using the functional and variable transformations in the opposite direction than what was initially done.

References

Adkins, R. and D. Paxson (2011), "Renewing Assets with Uncertain Revenues and Operating Costs," *Journal of Financial and Quantitative Analysis* 46(3):785–813.

Aguerrevere, F.L. (2003), "Equilibrium Investment Strategies and Output Price Behavior: A Real-Options Approach," *The Review of Financial Studies* 16(4):1239–1272.

[13]Here, we implicitly assume that the investment threshold is not at i_{min} or i_{max} for any value of Y or K. This assumption is met if the lattice dimensions are chosen properly.

Bar-Ilan, A. and W.C. Strange (1996), "Investment Lags," *The American Economic Review* 86(3):610–622.

Dangl, T. (1999), "Investment and Capacity Choice under Uncertain Demand," *European Journal of Operational Research* 117:1–14.

Décamps, J.-P., T. Mariotti, and S. Villeneuve (2006), "Irreversible Investment in Alternative Projects," *Economic Theory* 28:425–448.

Dixit, A.K. (1993), "Choosing Among Alternative Lumpy Investment Projects under Uncertainty," *Economic Letters* 43:281–285.

Dixit, A.K. and R.S. Pindyck (1994), *Investment under Uncertainty*, Princeton University Press, Princeton, NJ, USA.

Dockendorf, J. and D. Paxson (2013), "Continuous Rainbow Options on Commodity Outputs: What is the Real Value of Switching Facilities?," *European Journal of Finance* 19(7–8):645–673.

European Commission (2013), *Paving the Way to Electrified Road Transport: Publicly Funded Research, Development and Demonstration Projects on Electric and Plug-in Vehicles in Europe*, http://publications.jrc.ec.europa.eu/repository/bitstream/111111111/27868/1, Joint Research Centre, EC, Brussels, Belgium.

Financial Times (2014), "John Laing-Backed Infrastructure Fund to Float," http://www.ft.com/cms/s/0/9143ddbc-9223-11e3-9e43-00144feab7de.html, 10 February 2014.

Hydro-Québec (2014), "Power Transmission: Building a Line," http://www.hydroquebec.com/learning/transport/construction-ligne.html.

Majd, S. and R.S. Pindyck (1987), "Time to Build, Option Value, and Investment Decisions," *Journal of Financial Economics* 18(1):7–27.

McDonald, R. and D. Siegel (1985), “Investment and the Valuation of Firms When There is an Option to Shut Down,” *International Economic Review* 26:331–349.

McDonald, R. and D. Siegel (1986), “The Value of Waiting to Invest,” *The Quarterly Journal of Economics* 101(4):707–727.

OECD (2011), *Pension Funds Investment in Infrastructure: a Survey*, http://www.oecd.org/futures/infrastructureto2030/48634596.pdf, International Futures Programme, OECD, Paris, France.

Pindyck, R.S. (1988), “Irreversible Investment, Capacity Choice, and the Value of the Firm,” *The American Economic Review* 78(5):969–985.

Wilmott, P. (2007), *Paul Wilmott on Quantitative Finance*, John Wiley & Sons, Hoboken, NJ, USA.