DANILO NOGUEIRA DE PAULA

CARLOS BASTIAN-PINTO

# Convergence Study of the LSM Algorithm applied to Real Options Valuation

Yvonne: Where were you last night?
Rick: That's so long ago, I don't remember.
Yvonne: Will I see you tonight?
Rick: I never make plans that far ahead.

Captain Renault: What in heaven's name brought you to Casablanca?
Rick: My health. I came to Casablanca for the waters.
Captain Renault: The waters? What waters? We're in the desert.
Rick: I was misinformed.

Casablanca

## Abstract

LSM (Least Squares Monte Carlo) is an algorithm proposed by Lonstaff & Schartz (2001) for pricing options that, as the name implies, uses the Monte Carlo method, and values the early exercice of america type options. The procedure is simple: after simulating possible paths of movement of the underlying asset, the cashflow of the last period is calculate. With this cashflow information from the last period, it is possible to estimate the expected cash flow in the previous period given the state of the asset in the subsequent period. Making use of this estimative, we can determine when the exercise the option is interesting in the mentioned period and thus are able to determine the cashflow of the penultimate period. This information is used to calculate a new regression, this time estimating the expected cash flow in the penultimate sentence given the state of the asset in the period before. Finally, this process is repeated until the desired period is reached.

The object of this paper is to study the LSM for applications, targeting in particular the pricing of real options. For this, relevant characteristics of the algorithm - such as convergence, optimal exercise boundary and applicability to real options - will be assessed. Concomitantly, the results of the algorithm will be compared to results obtained by other models, such as the Bjerksund-Stensland model and the binomial model of Cox, Ross and

Rubinstein, in the case where the uncertainty is modeled with a Geometric Browning Motion. One of the significant advantages of the LSM algorithm is the possibility of use of virtually any stochastic model for the underlying uncertainty. Therefore the article will also access the convergence of the LSM method or algorithm for stochastic models such as mean reversion, two factor processes such as Schwartz & Smith (2000) and also processes combined with Poison jumps.

# 1 Introduction

Although options are a rather elusive financial instrument for most people, they have existed for centuries. Options are mentioned in writings as antique as Aristoteles although not with this nomenclature. They were present in the Dutch Tulip fever. In fact whenever there is the right, but not the obligation to do something, real options are generally present. Although such instruments might have such a simple definition, determining its value is no simple task and has been the object of studies for the last decades. Complexity in pricing of options, and therefore of real options is so great, when compared to other financial assets, that most of these do not have a close form solutions such as the Black & Sholes (1973) formula. Even for the ones, simple enough so the can be priced with these approaches, the differential stochastic calculus, known as Itô´s Calculus, is fundamental. In such context, alternative approaches have been developed for the pricing of options, typically using numerical approaches and simulations. Of such we can mention the lattice methods and Monte Carlo simulation.

Through these it is possible to price a wide range of complex different options. Another advantage of numerical approaches being that they do not fully require knowledge of Stochastic Calculus making these techniques more propagated through practitioners. And since they involve a simpler mathematical knowledge, several concepts relative to options can be approached in a more intuitive form, enlarging the range of option applications, specially of real options, when the optimal exercise rule might be as important as the value of such an option.

This paper aims at studying the option pricing algorithm known as Least-Squares Monte Carlo (LSM), compare it to possible alternatives, specially the binomial lattice approach of Cox, Ross e Rubinstein (1979) (CRR), not only in option value but in exercise boundary, as well as with other stochastic processes beyond the Geometric Brownian Motion which the CCR approach models, as well as the Black & Sholes formula. Also will be studied the characteristics generally present in real options contrary to financial options, such as: longer time span or maturity, variable strike values, etc. and how to adapt the LSM methodology to these, since it is originally designed to price financial type options with fixed strike values.

# 2 Theoretical Background

Among the varying techniques for option pricing, three large groups are clearly distinguishable in which these fit themselves: analytical techniques approached using decision trees and lattices, and simulation approaches.

## 2.1 Analytical methods

In this group the Black & Sholes (1973) formulation allows to find the exact value of European (with exercise only at maturity) call and put options on no dividends paying assets. Merton (1973) adapts their formula for dividends.

$$C = SN(d_1) - Xe^{-rT}N(d_2)$$

With

$$d_1 = \frac{\ln(S/X) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \quad and \quad d_2 = d_1 - \sigma\sqrt{T}$$

Where C is the price of the options, S the underlying asset, T the time to maturity, r the risk free rate, and σ is the volatility or the standard deviation of the log-returns of the underlying asset. The expression N() refers to the cumulative normal distribution of the argument in parenthesis..

Although widely used, its limitation to European type options narrows the model applicability in real world applications.

An analytical approach that minimizes this problem is the Bjerksund-Stensland (1993) model - BJS. These authors use a simplified form for the exercise boundary curve, or “trigger curve”. Specifically they divide the curve in two periods and use one exercise rule for each on of these periods. From this supposition they are able to formulate a model which prices approximately American put and call options with dividends.

## 2.2 Decision threes and Lattices

These approaches do not require complex mathematical tools. They suppose that the underlying asset can be determined by a weighted average of two (or more) possible states of the future, discounted properly. From this idea these models transform a continuous of possible states in a limited number of representative states capable of emulating the real cash flows when correctly weighted. It is essential to distinguish between weighting metrics based in real probabilities, which must be discounted the a risk adjusted rate, and weighting metrics which adjust the cash flows to an “equivalent certainty” and whish must be discounted at the risk free rate. Typically these numerical approaches using decision these or lattices use the risk neutral approach. The process starts by definition of the multiplicative or additive factors that determine the future up and down scenarios. Also is estimated the adjusted probabilities of occurrence of each scenario. Frequently only one up and one down scenarios are estimated so from one specific state only two other are possible, in a model defined as binomial. Repeating this procedure for each new future state a structure of states is built which is frequently called three, or lattice if the structure has a recombining characteristic. In order to value an option on an asset described by such a three, starting from the last group of states, the option is exercised at each state where applicable, and moving backwards from the last states in the three, the process is repeated, but considering also the discounted values of the future states weighed by the risk neutral probabilities.

One of the more frequently used and referenced binomial models for options and real options is the Cox, Ross & Rubinstein (1979) – CRR, model. In it the multiplying factors of up (u), down (d) and up risk neutral probability are expressed by these formulas.

$$u = e^{\sigma\sqrt{\Delta t}}$$

$$d = {}^{1}\!/_{u}$$

$$p = \frac{e^{r*\Delta t} - d}{u - d}$$

Where σ is the volatility of the returns of the underlying asset, Δt is the time increment between two state moments in the three, and r is the risk free rate. When Δt converges to 0, the final distribution of this model approaches those of the Black & Sholes model.

## 2.3 Simulation Approach Methodology

For options with a higher lever of complexity frequently analytical and binomial approaches are not applicable. In such cases computational simulations can be utilized to estimate the option value. The most popular way of doing this is through Monte Carlo simulation or Monte Carlo methodology. These approaches simply repeat thousands of times individual simulations of the phenomenon of interest changing randomly one variable or parameter at each period and or  simulation. The great flexibility of this approach allows its use in a wide range of knowledge areas, not limited to finance.

For option pricing this technique is primarily used to generate examples of possible paths for the underlying asset as well as for the pertinent state of the option. It is important to notice that this technique divide the pricing algorithm in two different parts: the first is the simulation per see, which depends of the asset to be modeled and the probability distribution attributed to it. The second part is the pricing technique itself. One of the Monte Carlo option pricing models is the LSM proposed by Longstaff & Schwartz (2001). This model utilizes simple minimum least square regressions to estimate the conditional expexctations of cash flows at each period and will be explained further on.

A forma mais comum de se modelar a distribuição, ou o “movimento”, do ativo é supor que ele segue o Movimento Geométrico Browniano (MGB). Essa suposição embasa os modelos de Black-Scholes, Bjerksund-Stensland e Cox-Ross-Rubinstein. Dizer que um ativo segue o MGB significa dizer que ele segue a seguinte equação:

The most comon form of distribution modeling of the underlying asset assumes that it follows a Geometric Brownian Motion (GBM). This is the underlying assumption with the Black & Scholes, Bjerksund & Stensland and Cox, Ross & Rubinstein models. It is described by the following equation:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

Where St is the asset price at time t, μ is the drift or growth rate of the process, σ the process volatility and Wt s the known Wiener process,. Therefore the asset price value at a given time t is:

$$S_t = S_{t-1} * e^{[\left(r - \sigma^2/_2\right) * \Delta t + \sigma * \sqrt{\Delta t} * N \sim (0,1)]}$$

**3.** Analysis of the LSM Algorithm

The LSM algorithm as developed by Longstaff & Schwartz (2001) is generic as it is not intended for pricing a a specific type of option and no code is given for its implementation, only a description of how the methodology works. Therefore the LSM method can be implemented in different ways and with any computing language allowing the necessary simulations and the calculus of recursive regressions which are the models differential,

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## 3.1 Description of the LSM Algorithm

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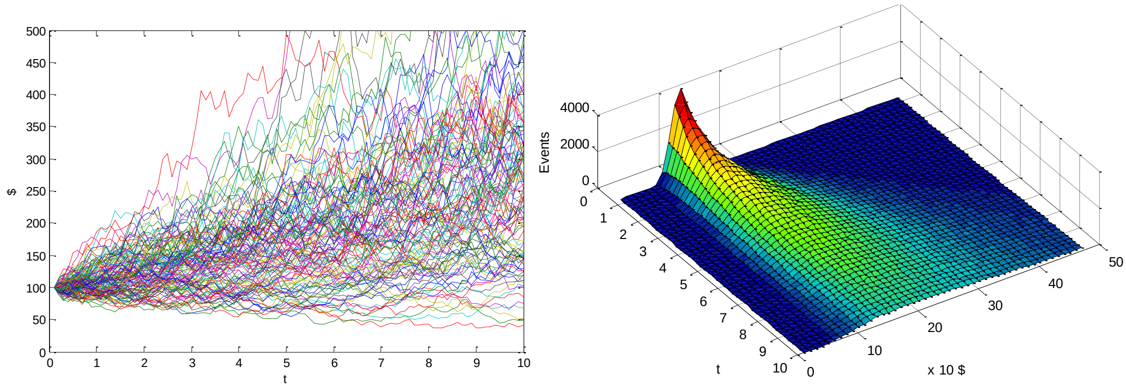


**Figure 1 – GBM Price simulation 120 trajectories (left), with: Δ$t$ = 0.1; α = 0.1 and σ = 0.2; and corresponding density distributions for 20,000 simulations (right).**

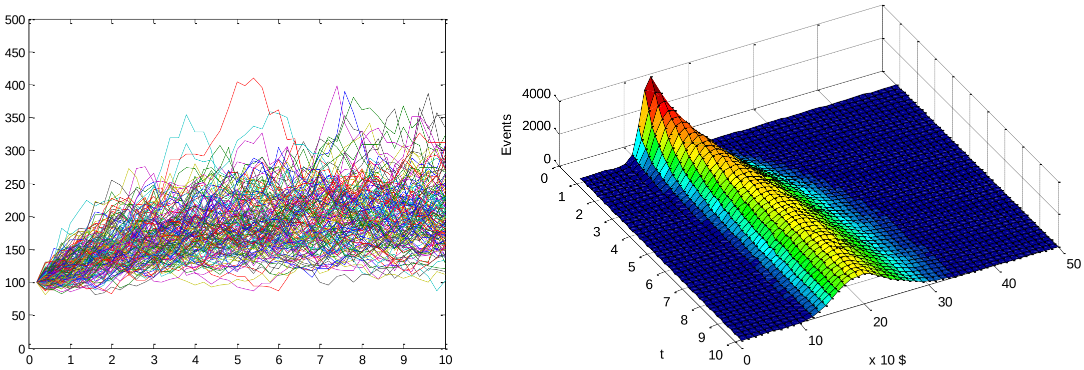


**Figure 2 –MRM Price simulation 120 trajectories (left), with: Δ$t$ = 0.1; $\eta$ = 0.5;σ = 0.2; $x_0$ =100; $\overline{x}$= 200; and corresponding density distributions for 20,000 simulations (right).**

## 3.2 Convergence tests for the LSM algorithm

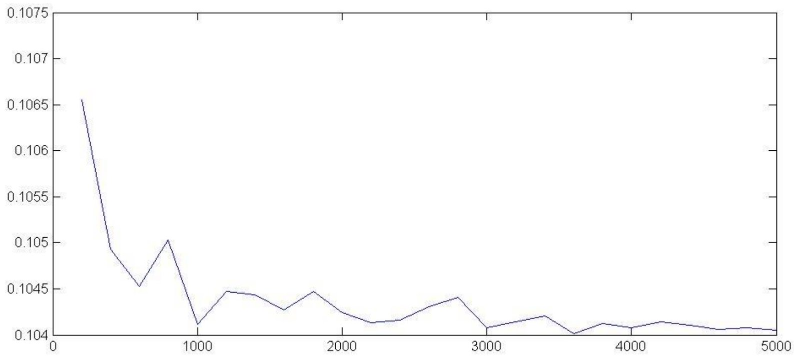


**Figura 3 – Convergence of the average of results –limited test**

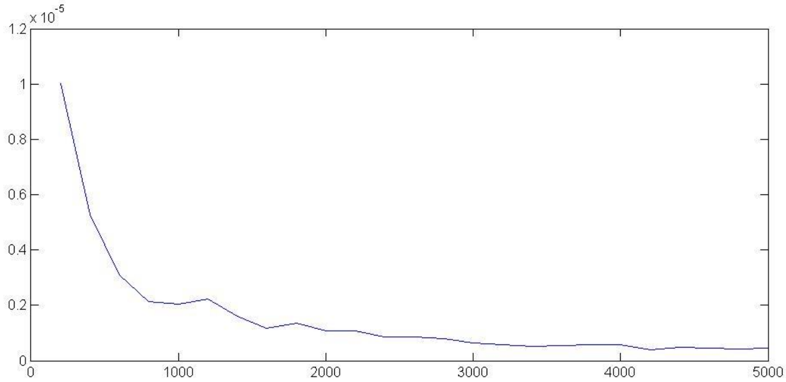


**Figura 4 – Variance – limited test**

## 3.3 Comparing results

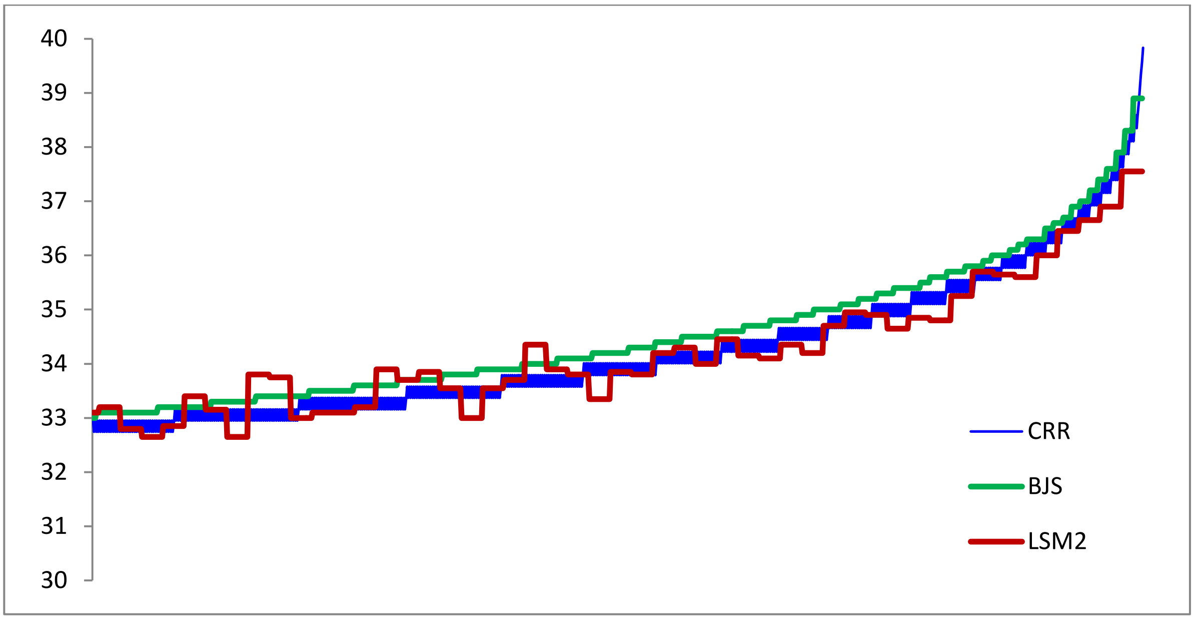


**Figura 1– Comparing trigger curves for CRR, LSM and BJS for Put Option**

# 4 Application of LSM to a Call Option with variable exercise value

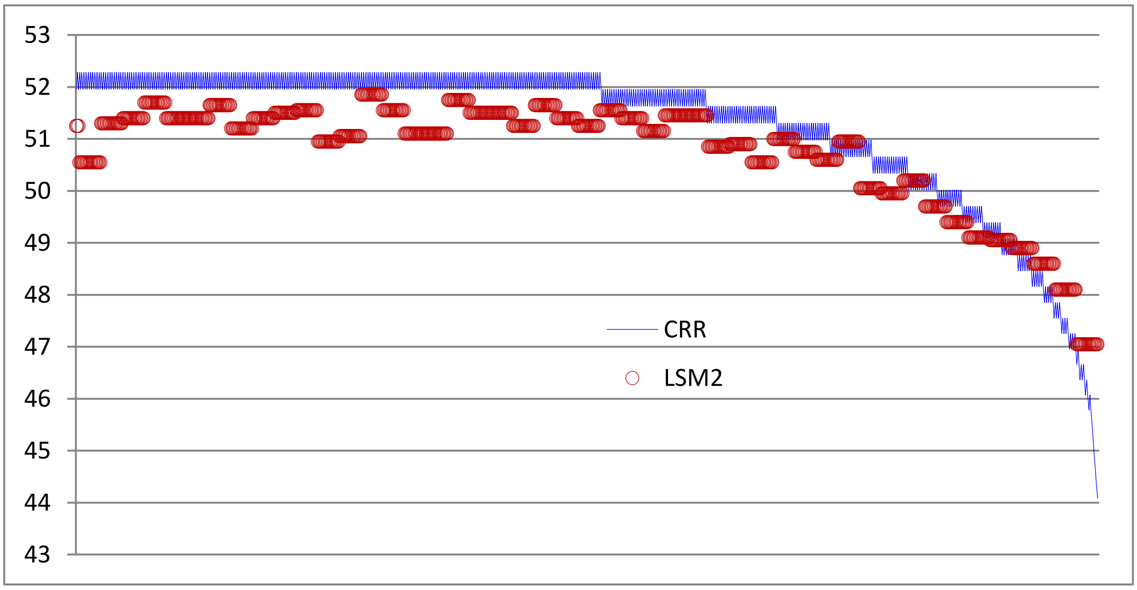


**Figura 6– Trigger Curves CCR and LSM for Call Option with growing strike**

# 5 Other Stochastic Processes

# 6 Conclusion

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