Strategic Technology Adoption and Hedging under Incomplete Markets*

Markus Leippold[†] and Jacob Stromberg[‡]

December 6, 2014

Abstract

We investigate the implications of technological innovation and non-diversifiable risk on entrepreneurial entry and optimal portfolio choice. In a real options model where two risk-averse individuals strategically decide on technology adoption, we show that the impact of non-diversifiable risk on the option timing decision is ambiguous and depends on the frequency of technological change. Compared to the complete market case, non-diversifiable risk may accelerate or delay the optimal investment decision. Moreover, strategic considerations regarding technology adoption play a central role for the entrepreneur's optimal portfolio choice in the presence of non-diversifiable risk.

JEL Classification: G11; G31; E2;
Keywords: Real Options; Incomplete Markets; Technology Adoption; Optimal Portfolio Choice; Hedging.

*The authors thank Marc Chesney, Zhigang Feng, Jianjun Miao, Anca Pana, Luca Taschini, and Alexandre Ziegler for helpful discussions. Financial support from the Swiss Finance Institute, Bank Vontobel, and the National Centre of Competence in Research Financial Valuation and Risk Management is gratefully acknowledged.

[†]Department of Banking and Finance, University of Zurich, Switzerland and Swiss Finance Institute.

[‡]Department of Banking and Finance, University of Zurich, Switzerland and Swiss Finance Institute.

1 Introduction

Entrepreneurs are considered to be an engine of innovation and technological progress for the economy. Their behavior significantly influences aggregate economic fluctuations. They account for a substantial share of aggregate investment, production, and savings.[1] The decision to become an entrepreneur is driven by, among other things, strategic considerations regarding technology adoption, future innovations, and non-diversifiable risk in business projects. We study the implications of these factors on entrepreneurial entry and optimal portfolio choice. In a continuous-time model, we incorporate strategic interactions between two risk-averse agents within the real options paradigm.

For our model design, we focus on two important aspects: market incompleteness and technological change. Entrepreneurship is risky due to uncertain future income streams.[2] Moreover, entrepreneurship generates a non-diversifiable income risk, as it generally requires substantial ownership in the business.[3] Hence, the presence of non-diversifiable risk requires that we formulate our modeling approach for entreprenerial investment behavior under an incomplete market setup.

The second aspect is technological change. Because of private business owners' willingness to adopt and exploit new innovations, entrepreneurship is widely considered to be a driving factor for technological change and economic growth.[4]. Indeed, strategic aspects regarding technology adoption have important implications for both the valuation and timing of investment decisions.[5] Therefore, we include in our incomplete market model the evolution of technological change and we study its impact on entrepreneurial investment decisions.

In the standard real options model, the optimal time to invest is given by the moment at which productivity reaches a threshold such that the benefit of investment equals the direct cost plus the opportunity cost of investment. The general prediction in these models is that an increase in volatility

[1] See, e.g., Rampini (2004) and Cagetti and De Nardi (2006).

[2] Concerned about the low level of entrepreneurial activity in Europe, the European Commission published a survey in which the participants were asked a range of questions related to entrepreneurship, see European-Commission (2010). Asked about the greatest fears when starting up a business, the uncertainty of not having a regular income was mentioned by 40% of the Europeans (and by nearly 50% of the people in the US) as being the most important risk of becoming an entrepreneur.

[3] See, e.g., Hall and Woodward (2010), Gentry and Hubbard (2004) and Moskowitz and Vissing-Jorgensen (2002).

[4] See, e.g., Romer (1990) and Quadrini (2009). These findings support the general notion from Schumpeter (1934) that technological innovation is a central dimension of entrepreneurship.

[5] See Huisman and Kort (2004) for a discussion of this point in a complete market setting.

leads to a delay in investments. From Miao and Wang (2007) we know that this result does not hold for a lump-sum investment payoff under incomplete markets, but still holds for a flow-payoff investment. However, we show that the joint interaction of technological change and market incompleteness can cause an accelerated investment when volatility increases, even in a model with flow payoff. Hence, we complement the result of Miao and Wang (2007) in that idiosyncratic volatility not only generates a private equity premium, but that this premium has a crucial impact on the investment behavior also in a flow-payoff model, particularly so in innovative industries characterized by frequent technological changes.

It is useful to motivate our model setup in terms of a real world example. In the market for zero-emission vehicles, technological innovation plays a crucial role. While there is mounting political pressure to introduce zero-emission cars, it is not at all clear what these cars should look like:

> "The market outlook for electric vehicles seems bright [...] Yet the future of electric vehicles is far from assured. [...] Will other technologies—such as hybrid cars or vehicles powered by natural gas, ethanol, or hydrogen—emerge and win the competition against electric cars?" (Graham and Messer (2011))

Hence, the market is relatively uncertain regarding the technology that should power the next generation of cars. Hence, the large car companies are reluctant to invest on a large scale, leaving the market open to entrepreneurs. In 2007, the entrepreneur Shai Agassi founded the California-based company *Better Place*, an electric vehicles service provider with a vision of making zero-emission cars. In the *Harvard Business Review*, May 2009, Shai Agassi talks about technology adoption:

> "Every night I went to Wikipedia, picked a term like "ethanol" or "natural gas," and studied for hours. Eventually I wrote a white paper proposing a plan that relies on existing technology: cars that run on lithium-ion batteries recharged by renewable energy." (Akresh-Gonzales (2009))

Obviously, Shai Agassi decided to rely on an existing technology and not to wait for the arrival of a new technology. Upon the arrival of a more efficient technology, other entrepreneurs may invest and compete with the technology adopted by Shai Agassi. Since switching to another technology may involve substantial costs, we may view the above situation as a decision problem with a single investment opportunity either using the current available technology or waiting until a superior technology arrives. Furthermore, the decision to produce and develop zero-emission vehicles is exposed to risks unique to the business that cannot be completely hedged by trading in financial markets. For instance, it may include risks regarding potential suppliers' willingness to set up recharging stations or political initiatives that may foster investment and support for infrastructure in a particular technology. Hence, we believe that the interaction between strategic investments, the timing of technology adoption, and portfolio choice in an incomplete market setting is a highly relevant avenue of research.

In our paper, we first address the following question: What is the impact of non-diversifiable risk on the optimal investment timing decision compared to complete markets, in the presence of strategic considerations regarding technology adoption? To provide an answer, we extend the single-agent setting of Miao and Wang (2007) to a setting where two risk-averse entrepreneurs have access to an investment opportunity and technology may change. Each entrepreneur has to strategically decide when to invest and whether to adopt an existing technology for production or wait for a more efficient technology to become available for adoption.

In addition, we let our entrepreneurs hedge their investment risk in the financial market. Hence, they not only decide on the optimal time to exercise their real investment option, but they also have to make optimal intertemporal portfolio decisions as in Merton (1971). Hence, in our model, we also provide an answer to how the optimal portfolio choice is affected by strategic considerations regarding technology adoption. The joint presence of technological innovation and non-diversifiable risks has several important implications.

We obtain three main results. First, we show that the impact of non-diversifiable risk on the timing of the entrepreneurs' option is ambiguous, and depends on the frequency of technological change. Consequently, the presence of non-diversifiable risk may accelerate or delay the optimal investment timing compared to complete markets. This result complements the finding in Miao and Wang (2007). They show that the investment timing decision for a single agent should always be delayed in the presence of non-diversifiable risk compared to complete markets. Their finding has an intuitive explanation. Recalling the standard result from real options theory under complete markets that the option value of waiting is increasing in project volatility (e.g., Dixit and Pindyck (1994)), the presence of non-diversifiable risk should lead to delayed investment. However, when taking into account strategic aspects of future technological innovations, we show that this intuition may no longer be reliable.

Second, the model has empirical implications regarding optimal investment timing for individuals contemplating becoming entrepreneurs. We find that when it is optimal for one of the entrepreneurs to invest and become the leader, the predominant prediction is that technology adoption by an under-diversified risk-averse entrepreneur should occur sooner under incomplete markets compared to a well-diversified individual or company. This is consistent with other theoretical work, establishing that entrepreneurs tend to promote new innovations.[6] According to the model, a possible explanation for such behavior (at the micro-level) is driven by optimality concerns and risk-aversion: entrepreneurs may take strategic aspects of uncertain future technological innovations and their exposure to non-diversifiable income risk into account prior to investing.

Third, the model offers new insight into the determinants of optimal portfolio choice for both current and prospective entrepreneurs. The greater is the technological innovation and the higher the correlation between the operating net income and the risky asset, the more the prospective entrepreneur (follower) should reduce the portfolio allocation to the risky asset, In contrast, the current entrepreneur (leader) should increase the portfolio allocation to the risky asset, in anticipation that the follower optimally exercises their investment option, should the more efficient technology arrive: when the follower decides to exercise their investment option, the leader will experience a reduction in operating income from managing the business and also be less exposed to non-diversifiable income risk, which ceteris paribus induces a lower hedging demand. The precise effect depends on the relative profitability of operating in the market alone versus operating with an inferior technology. These findings have practical relevance for optimal portfolio choice for both current and prospective entrepreneurs in environments where technological innovation is important.

Two streams of the literature are related to our paper. The first is concerned with extending the real options paradigm to incomplete markets. Miao and Wang (2007) study the optimal consumption

[6]See, e.g., Romer (1990) or Schmitz (1989) by way of imitation by established companies.

and portfolio choice for a single entrepreneur who has a single investment opportunity. Henderson (2007) considers a single entrepreneur who has a single investment opportunity to receive a lump-sum payoff. Managerial investment behavior has been analyzed in Hugonnier and Morellec (2007). Evans, Henderson, and Hobson (2008) study the optimal time to sell an asset in the presence of wealth effects. Chen, Miao, and Wang (2010) combines a real options model under incomplete markets with Leland (1994)'s capital structure model. Bensoussan, Diltz, and Hoe (2010) study the mathematical properties of the differential equations defining the real options under incomplete markets for several decision makers. Wang, Wang, and Yang (2011) study the effects of non-diversifiable risk on the optimal investment and exit decisions of a single entrepreneur in the presence of financing and liquidity constraints.[7]

Except for Bensoussan, Diltz, and Hoe (2010), a common theme in the above papers is that they only consider the investment decision of a single entrepreneur. In reality, real investment opportunities can rarely be considered in isolation. The second stream of the literature to which our paper is related is concerned with strategic interactions in various forms. An early prominent contribution is Fudenberg and Tirole (1985), who, in a deterministic setting, present a theoretical formalization of games in continuous time. They study technology adoption for two identical firms and show that preemption should happen at the point where rent equalization occurs between the leader and the follower. Stenbacka and Tombak (1994) extend the model setup in Fudenberg and Tirole (1985) by introducing uncertainty into the length of time from the initial adoption of a technology until its successful implementation. Similarly, Hoppe (2000) extends the setting in Fudenberg and Tirole (1985) to consider uncertainty regarding the profitability of adopting a new technology. Recently, Thijssen, Huisman, and Kort (2002) have extended the Fudenberg and Tirole (1985) model to a stochastic setting and in a follow-up paper, Huisman and Kort (2004) extend the work by Thijssen, Huisman, and Kort (2002) to identical risk-neutral firms competing over technology adoption while taking into account future technological innovations. They show that the arrival of future technological innovations may introduce a second-mover advantage and turn the preemption game into a war of attrition (e.g., Hendricks, Weiss, and Wilson (1988)). However, a standard assumption in the

[7]Empirical papers concerned with entrepreneurship and non-diversifiable risk include Heaton and Lucas (2000), Moskowitz and Vissing-Jorgensen (2002) and Hall and Woodward (2010).

above mentioned papers is that the markets are complete. Hence, to our best knowledge, there is no research on optimal investment timing decisions under incomplete markets that takes technological innovation and strategic aspects of the technology adoption into account. This paper contributes to fill this gap in the literature.

We proceed as follows. Section 2 presents the economic setting. In Section 3, the model is solved for different scenarios. Section 4 presents a numerical analysis. Section 5 presents some conclusions. All proofs are relegated to the Appendix.

2 Notation and basic setup

We consider a frictionless financial market that consists of a risk-free bond P paying a constant interest rate $r > 0$ and a risky asset S. We can think of S as being a broad market index whose price process is exogenously determined. The dynamics of the risky asset is assumed to follow a geometric Brownian motion:

$$\begin{aligned} dS_t &= \mu_S S_t dt + \sigma_S S_t dB_t, \quad S_0 > 0, \\ &= (r + \eta\sigma_S)dt + \sigma_S S_t dB_t, \\ dP_t &= rP_t dt, \quad P_0 > 0, \end{aligned} \tag{1}$$

where μ_S and σ_S are constant parameters denoting the expected rate of growth, respectively, the volatility of the risky asset. We denote the market price of risk by

$$\eta = \frac{\mu_S - r}{\sigma_S},$$

Our economy is populated by two identical infinitely-lived risk-averse entrepreneurs, who can continuously invest in the financial market.[8] They also have to decide on an irreversible investment with which to enter a product market, which gives them an uncertain income stream. Not only do the entrepreneurs have to strategically decide on the time to invest, but also the production technology to adopt. They may either use the existing technology (technology 1) or they may wait until a

[8]Since our aim is to understand when investment should optimally take place, rather than to differentiate between which of the entrepreneurs will invest first, we assume for simplicity that each of the entrepreneurs will, with probability one-half, invest as the leader.

more efficient technology (technology 2) becomes available at some future point in time. We model the arrival of technology 2 as an exogenous Poisson process with constant parameter $\lambda > 0$. Thus, at the random time τ, which follows an exponential distribution with mean $1/\lambda$, the more efficient technology 2 will become available. We assume that each entrepreneur can only invest once, i.e., it is not possible for the entrepreneurs to start with technology 1 and then upgrade to technology 2.

We represent each of the entrepreneurs by the index $k \in \{i, j\}$. By $N_k \in \{0, 1, 2\}$, we indicate whether entrepreneur k has adopted technology 1 ($N_k = 1$), technology 2 ($N_k = 2$), or has not yet entered the market at all ($N_k = 0$). Irrespectively of the technology they use, the entrepreneurs can enter the market by paying a fixed investment cost $I > 0$, which the entrepreneur has to pay at the time of investment, τ. This cost is financed from the entrepreneurs' own wealth and if there is a shortage, they can borrow at the risk-free rate r. To analyze the impact of technology adoption, competition, and market incompleteness in the simplest possible setting, we set the investment cost equal for both entrepreneurs. Also, since our focus is on the interaction between market incompleteness and technological innovation, we do not consider any borrowing constraints from costly external financing.

After exercising the investment option at time τ, the entrepreneur receives operating net income or "flow payoff" from managing the business. The flow payoff, denoted by $D_{N_i N_j} Y_t$, is the product of two parts, an income factor and the operating profit. The income factor $D_{N_i N_j} > 0$ reflects the effectiveness of the technology adopted. We assume these factors to be constant. The process Y_t is the operating income following the dynamics

$$dY_t = \alpha_y dt + \sigma_y \rho dB_t + \sigma_y \sqrt{1-\rho^2} d\tilde{B}_t, \qquad Y_0 = y, \tag{2}$$

where $\{B_t\}_{t \geq 0}$ and $\{\tilde{B}_t\}_{t \geq 0}$ are independent standard Brownian motions. The parameter α_y denotes the expected drift and σ_y the volatility of the project value. Positive values of the process $\{Y_t\}_{t \geq 0}$ denote operating profits, negative values denote operating losses. The parameter ρ denotes the correlation between the project value and the risky asset in (1). We recall that whenever there is a non-perfect correlation ρ, the risks inherent in the investment project cannot be completely hedged by trading in the financial market.

For the income factors $D_{N_i N_j}$, we impose the same structure as in Huisman and Kort (2004).

A firm makes the highest profit for a given technology if the other company is not invested at all. Profits are lowest when the other company has invested in the superior technology. Finally, for a given technology of the competitor, the firm makes the highest profits when it invests in technology 2. The inequalities below summarize these assumptions.

$$\begin{array}{ccccc} D_{20} & > & D_{21} & > & D_{22} \\ \vee & & \vee & & \vee \\ D_{10} & > & D_{11} & > & D_{12}. \end{array}$$

The entrepreneurs receive zero operating income if they have not invested, i.e., $D_{0N_k} = 0$ for $N_k \in \{0, 1, 2\}$. The structure of the factors $D_{N_iN_j}$ implies that technology 2 is superior to technology 1. Furthermore, adopting either technology before the competitor yields a higher flow payoff than in a situation in which the competitor has adopted the same technology. The flow payoff accruing to each of the entrepreneurs depends on whether its competitor has already entered the market. The precise specification of the flow payoff for the follower-entrepreneur depends on what type of technology the other one has adopted and on the menu of technologies that are available at a particular time. Moreover, the future flow payoff for the leader will be reduced when the competitor decides to follow and enters the market as well.

Since we allow for a non-perfect correlation between the project value Y and the stock market S, our market model is incomplete. Hence, we cannot rely on the usual valuation framework from no-arbitrage option pricing theory. The standard replication argument no longer applies and no unique measure exists under which we can evaluate real asset investment opportunities. Therefore, we have to resort to other pricing methods suitable for incomplete markets.

To obtain a unique project price, we resort to utility indifference pricing and impose, via a utility function, explicit assumptions about the entrepreneurs' attitudes towards risk. Hence, the risk attitude will impact the option values characterizing the investment opportunities. We endow entrepreneurs with an exponential utility function over consumption c:

$$u(c) = -\frac{1}{\gamma}e^{-\gamma c}, \tag{3}$$

with $\gamma > 0$ denoting the coefficient of absolute risk-aversion. Both entrepreneurs have the same level of risk-aversion.[9] The objective for each of the entrepreneurs is to maximize the expected discounted utility from consumption over an infinite investment horizon,

$$\mathbb{E}\left[\int_0^{\infty} e^{-\beta s} u(c_s) ds\right], \tag{4}$$

where $\beta > 0$ denotes the entrepreneurs' subjective discount rate. The entrepreneurs optimize the objective function with respect to investment and consumption strategy, π and c, and strategically decide on the optimal time to undertake the investment project. To value these projects, we make use of dynamic programming techniques.[10] The entrepreneur's wealth dynamics is given by

$$dW_t = \pi_t \frac{dS_t}{S_t} + r(W_t - \pi_t)dt - c_t dt + \left(F(D_{N_i N_j}; Y_t)\mathbf{1}_{\{\tau \leq t\}} - I\delta(t-\tau)\right) dt, \tag{5}$$

where π denotes the fraction invested in the risky asset and W_t denotes the wealth of the entrepreneur at time t. The term $F(D_{N_i N_j}; Y_t)\mathbf{1}_{\{\tau \leq t\}}$ denotes a certain functional form of the flow payoff accruing to the entrepreneur from the time of investment τ. The term $\delta(\cdot)$ denotes the Dirac function.

In contrast to the classical portfolio optimization problem as in Merton (1969), the wealth dynamics in Equation (5) depend on additional circumstances. First, it depends on whether the competitor has already entered the new market. Second, in case the competitor has already invested, the type of flow payoff that the entrepreneur receives depends on what type of technology the competitor has adopted. Third, the dynamics also depends on what technologies are available at that particular time. Finally, if the entrepreneur becomes a first mover, the future flow payoff will be reduced when the competitor decides to enter the new market.

3 Derivation of the value functions

Our strategy to solve the model is to first derive the project value payoffs that the entrepreneurs obtain from exercising their investment options. Then, we derive the option value functions for the case when technology 2 is already available, and finally for the case when technology 2 is not yet

[9] Assuming different levels of risk-aversion for the entrepreneurs would imply that the one with the lowest risk-aversion would always invest before the other one. The same argument holds for differentiated investment costs, (e.g., Pawlina and Kort (2006). Without allowing for, e.g., incomplete information as in Lambrecht and Perraudin (2003), assuming different levels of risk-aversion and/or investment costs appears to be less interesting, since the order of investment is then already given a priori.

[10] See, e.g., Henderson (2004) or Birge and Linetsky (2007).

	T2 available		T2 not yet available
(S1):	Nobody invested in T1 Joint investement in T2 considered	**(S4):**	Leader invested in T1 Follower considers investing in T2
(S2):	Leader invested in T1 Follower considers investing in T2	**(S5):**	Leader invested in T1 Follower considers investing in T1
(S3):	Both invested in T1 Both consider investing in T2	**(S6):**	Both wait for investing in T2

Table 1: The different situations considered in the derivation of the value functions, depending on the availability of technology 2 (T2) and on prior investment in technology 1 (T1).

available. In total, we analyze six different scenarios, which we summarize in Table 1.

To derive the value functions in these different scenarios, we employ the following notation. We denote the value function by $G_{N_iN_j;TM}(y)$, where $G \in \{J, L, F\}$ indicates either joint investment (J), investment by the leader (L), or investment by the follower (F). The subscript N_iN_j refers to the income factor $D_{N_iN_j}$, the subscript $T \in \{A, B\}$ indicates whether the reference is to the situation before (B) or after (A) technology 2 has arrived. Finally, $M \in \{I, C\}$ indicates whether the market is incomplete (I) or complete (C).

3.1 Project value

In the presence of non-diversifiable risk and for a general income factor $D_{N_iN_j}$, we will now obtain the project value that the entrepreneurs obtain from investing.[11]

Proposition 1. *After investment, the project value associated with income factor $D_{N_iN_j}$ in the presence of non-diversifiable risk is given by*

$$f(y; D_{N_iN_j}) = \frac{D_{N_iN_j}}{r}y + \frac{(\alpha_y - \rho\sigma_y\eta)D_{N_iN_j}}{r^2} - \frac{\gamma\sigma_y^2(1-\rho^2)D_{N_iN_j}^2}{2r^2}. \tag{6}$$

The optimal portfolio and consumption policies of the entrepreneurs are given by

$$\pi^* = \frac{\eta}{\gamma r \sigma_S} - \frac{\rho\sigma_y}{r\sigma_S}D_{N_iN_j}, \tag{7}$$

$$c^* = r\left(w + f(y; D_{N_iN_j}) + \frac{\eta^2}{2\gamma r^2} + \frac{\beta - r}{\gamma r^2}\right). \tag{8}$$

To gain some intuition for the project value in Equation (6) of Proposition 1, we consider the

[11] We relegate the proof of this and all subsequent propositions to Appendix A.

situation in which the risk-aversion parameter tends to zero ($\gamma \to 0$). As in Miao and Wang (2007), we can assume that the agent can trade an additional risky asset to diversify the idiosyncratic risk stemming from the non-perfect correlation ρ. Under this specification, the operating net income evolves as

$$dY_t = (\alpha_y - \sigma_y \eta)\, dt + \sigma_y dB_t^{\mathbb{Q}}, \tag{9}$$

where $B^{\mathbb{Q}}$ is a Brownian motion under the risk-neutral probability measure $\mathbb{Q}$. Consequently, the project value can be computed as the expected present value of the flow payoff $D_{N_iN_j} Y_t$ under $\mathbb{Q}$ and discounted by the risk-free rate:

$$f(y; D_{N_iN_j}) \;=\; \mathbb{E}^{\mathbb{Q}}_{Y_0=y}\left(\int_0^\infty e^{-rt} D_{N_iN_j} Y_t dt\right) = \frac{D_{N_iN_j}}{r} y + \frac{(\alpha_y - \rho\sigma_y\eta)\, D_{N_iN_j}}{r^2}. \tag{10}$$

We find that the project value under complete markets in Equation (10) corresponds to the first two terms in the project value under incomplete markets in Equation (6). The third and last term in the project value under incomplete markets takes into account the risk attitude γ of the entrepreneurs and the non-diversifiable income risk $\sigma_y^2(1-\rho^2)$. The sign of this third term is always negative. Hence, for strictly positive risk aversion, the project value after investment is always smaller in an incomplete market setting than in a complete market setting. Increasing risk aversion leads to lower project value in an incomplete market setting, since the entrepreneurs face additional non-diversifiable income risk to which they are averse. An increase in the volatility of the flow-payoff also decreases the project value, irrespective of whether we work in a complete or incomplete market setting.

Concerning the portfolio and consumption policies in Equations (7) and (8) after investment, we find that the consumption policy is linear in y, while the portfolio policy is constant, since $f(y; D_{N_iN_j})$ is linear in y. If the correlation between the financial market and the operating profit is positive, the entrepreneur reduces investment in the stock market with increasing income factor $D_{N_iN_j}$. Once the entrepreneur has made an investment, changes in the operating profit have no influence on the portfolio allocation.

3.2 Technology 2 is available

We first consider the situation in which technology 2 is already available. There are three possible scenarios. In the first scenario **(S1)**, none of the entrepreneurs have invested before the technology's arrival time τ. In the second scenario **(S2)**, one entrepreneur has invested before time τ and has become the leader. In the third scenario **(S3)**, both entrepreneurs have invested before time τ. Below, we analyze each of these cases and derive the (option) value functions and investment thresholds.

(S1): Neither entrepreneur has invested before time τ. Both entrepreneurs will adopt the new technology 2, since by assumption it is superior to technology 1. Therefore, we can view this situation as one with a single entrepreneur, who has to decide at what time to invest in technology 2, given that this technology is available.

Proposition 2. *The value function at time $t \geq \tau$ for an investment in technology 2, when no entrepreneur has entered the market yet, is given by*

$$J_{22;AI}(y) = \begin{cases} g(y) & \text{if} \quad y \in (-\infty, \bar{y}_{22}) \\ f(y; D_{22}) - I & \text{if} \quad y \in (\bar{y}_{22}, +\infty) \end{cases},$$

where $\bar{y}_{22}$ denotes the optimal investment threshold. The function $g(y)$ satisfies the following non-linear ODE,[12]

$$rg(y) = \frac{\sigma_y^2}{2} g''(y) - \gamma r \frac{\sigma_y^2}{2}(1-\rho^2) g'(y)^2 + (\alpha_y - \rho\sigma_y\eta) g'(y) \tag{11}$$

for $y \in (-\infty, \bar{y}_{22})$ subject to $\lim_{y\to-\infty} g(y) = 0$ and the value matching and smooth pasting conditions,

$$\begin{aligned} g(\bar{y}_{22}) &= f(\bar{y}_{22}; D_{22}) - I, & (12) \\ g'(\bar{y}_{22}) &= f'(\bar{y}_{22}; D_{22}), & (13) \end{aligned}$$

where $f(\cdot)$ is given by Proposition 1. The optimal portfolio and consumption policies of the entrepreneurs before exercising the option are

$$\pi^* = \frac{\eta}{\gamma r \sigma_S} - \frac{\rho\sigma_y}{\sigma_S} g'(y), \quad c^* = r\left(w + g(y) + \frac{\eta^2}{2\gamma r^2} + \frac{\beta - r}{\gamma r^2}\right), \tag{14}$$

[12] We are not aware of a closed form expression for $g(y)$ satisfying the boundary conditions. Hence, we will have to resort to numerical methods. In particular, we use projection methods based on Chebyshev collocation to solve the differential equations numerically. This approach has proven to be far superior to, e.g., conventional finite difference methods when trying to numerically approximate the solution to non-linear differential equations subject to a free boundary (see, e.g., Judd (1992) and Dangl and Wirl (2004)).

and after exercising the option,

$$\pi^* = \frac{\eta}{\gamma r \sigma_S} - \frac{\rho \sigma_y}{r \sigma_S} D_{22}, \quad c^* = r\left(w + f(y; D_{22}) + \frac{\eta^2}{2\gamma r^2} + \frac{\beta - r}{\gamma r^2}\right), \tag{15}$$

respectively.

The function $g(y)$ represents the option value of joint investment using technology 2 in the continuation region. The term $f(y; D_{22}) - I$ denotes the intrinsic value of the option in the stopping region. When the operating net income approaches negative infinity, the option value becomes worthless. When the operating net income hits the endogenously determined threshold $\bar{y}_{22}$, both entrepreneurs exercise their option. They pay the investment costs I and receive the project value $f(y; D_{22})$.

The portfolio and consumption strategies before exercising the option have a similar form as after the exercise date. The only exception is that $f(y; D_{22})$ is replaced by $g(y)$. While from Proposition 1 $f(y; D_{22})$ is linear in y, the function $g(y)$ is no longer linear. Hence, the portfolio strategy before exercising the option depends on the operating profit y.[13]

We remark that the situation in Proposition 2 is equivalent, when we set $D_{22} = 1$, to the flow payoff model with hedging opportunities in Miao and Wang (2007) described in their proposition 4. Furthermore, if we set $\rho = 0$, we get their self-insurance flow payoff model analyzed in their proposition 3. Perturbing their model with self-insurance around $\sigma_y^2 = 0$, they find that the investment threshold increases in γ and delays the exercise of the option. When we do a perturbation in σ_y for our flow-payoff model, we get[14]

$$\bar{y}_{22} = \frac{Ir}{D_{22}} + \frac{\sigma_y^2}{2\alpha_y} + \frac{\eta \rho \sigma_y^3}{2\alpha_y^2} + \frac{\sigma_y^4\left(2\eta^2\rho^2 - r\right)}{4\alpha_y^3} + \frac{\sigma_y^4 D_{22}\gamma\left(1 - \rho^2\right)}{2\alpha_y^2} + O(\sigma_y^5). \tag{16}$$

Hence, under market incompleteness with hedging opportunities and after the arrival of the new technology, the critical threshold in Equation (16) is pushed upward compared to the complete market case, at least up to the fourth order term in σ_y, since $\frac{\sigma_y^4 D_{22}\gamma(1-\rho^2)}{2\alpha_y^2} > 0$. Furthermore, we can rewrite Equation (11):

$$rg(y) = \frac{\sigma_y^2}{2} g''(y) + (\alpha_y - \rho\sigma_y\eta - \zeta(y))g'(y), \tag{17}$$

[13]Given that the portfolio and consumption strategies differ only in terms of $f(y; D_{N_i N_j})$ and $g(y)$, we do not restate them in the following propositions. They can be found in Appendix A.

[14]For a derivation of this result, see Appendix B.

where the term

$$\zeta(y) = \gamma r \frac{\sigma_y^2}{2}(1-\rho^2)g'(y), \tag{18}$$

can be thought of as the private equity premium as in Miao and Wang (2007). The entrepreneur demands a higher risk premium when the idiosyncratic risk $\sigma_y(1-\rho^2)$ is large and when the entrepreneur is more prudent. Additionally, a higher risk premium is required when the option's delta is large, i.e., the option is sensitive to changes in the operating income y. Calculating the option's delta at $y = \bar{y}_{22}$ and plugging in the expression for the equity risk premium (18) into Equation (16), we get $\zeta(\bar{y}_{22}) = \zeta = \gamma D_{22}\frac{\sigma_y^2}{2}(1-\rho^2)$ and

$$\bar{y}_{22} = \frac{Ir}{D_{22}} + \frac{\sigma_y^2}{2\alpha_y} + \frac{\eta\rho\sigma_y^3}{2\alpha_y^2} + \frac{\sigma_y^4\left(2\eta^2\rho^2 - r\right)}{4\alpha_y^3} + \zeta\frac{\sigma_y^2}{\alpha_y^2} + O(\sigma_y^5). \tag{19}$$

Hence, an increase in the private equity premium leads to a delay in the optimal investment, which is in line with the classical result from the real options literature.

(S2): One entrepreneur has invested before time τ. When one entrepreneur has invested in technology 1 before time τ (the leader), the entrepreneur who has not yet invested (the follower) faces a situation of a single individual who has to decide when to invest in technology 2. We derive the follower's option value function and investment threshold in the next proposition.

Proposition 3. *The value function for the follower after time τ is*

$$F_{21;AI}(y) = \begin{cases} g(y) & \text{if} \quad y \in (-\infty, \bar{y}_{12}) \\ f(y; D_{21}) - I & \text{if} \quad y \in (\bar{y}_{12}, +\infty) \end{cases},$$

where $\bar{y}_{12}$ denotes the optimal investment threshold. The function $g(y)$ satisfies the non-linear ODE

$$rg(y) = \frac{\sigma_y^2}{2}g''(y) - \gamma r\frac{\sigma_y^2}{2}(1-\rho^2)g'(y)^2 + (\alpha_y - \rho\sigma_y\eta)g'(y), \tag{20}$$

for $y \in (-\infty, \bar{y}_{12})$ subject to $\lim_{y\to-\infty} g(y) = 0$ and the value matching and smooth pasting conditions

$$g(\bar{y}_{12}) = f(\bar{y}_{12}; D_{21}) - I, \tag{21}$$

$$g'(\bar{y}_{12}) = f'(\bar{y}_{12}; D_{21}), \tag{22}$$

where $f(\cdot)$ is given by Proposition 1.

The structure of the ODEs in Proposition 3 is similar to that in Proposition 2. Indeed, we get

$$\bar{y}_{12} = \frac{Ir}{D_{21}} + \frac{\sigma_y^2}{2\alpha_y} + \frac{\eta\rho\sigma_y^3}{2\alpha_y^2} + \frac{\sigma_y^4\left(2\alpha_y D_{21}\gamma\left(1-\rho^2\right)+2\eta^2\rho^2-r\right)}{4\alpha_y^3} + O(\sigma_y^5). \tag{23}$$

Again, market incompleteness (through the existence of a positive equity risk premium) leads to a delay in investment. The only difference between $\bar{y}_{12}$ and $\bar{y}_{22}$ is in the income factors. Since $D_{21} > D_{22}$, the follower invests earlier in technology 2 than would be the case under joint investments. This observation corresponds to our intuition, since the follower would be the first to profit from technology 2, while the leader has invested in the inferior technology 1.

Having determined the value function $F_{21;AI}(y)$ for the follower, we are in a position to determine the value function of the leader, i.e., for the entrepreneur who has invested in technology 1.

Proposition 4. *The value function for the leader after time τ is given by*

$$L_{12;AI}(y) = \begin{cases} g(y) & \textit{if} \quad y \in (-\infty, \bar{y}_{12}) \\ f(y; D_{12}) & \textit{if} \quad y \in (\bar{y}_{12}, +\infty), \end{cases}$$

where $\bar{y}_{12}$ denotes the follower's optimal investment threshold determined in Proposition 3. The function $g(y)$ satisfies the non-linear ODE

$$rg(y) = \frac{\sigma_y^2}{2}g''(y) - \gamma r\frac{\sigma_y^2}{2}(1-\rho^2)g'(y)^2 + (\alpha_y - \rho\sigma_y\eta)g'(y) + D_{10}y, \tag{24}$$

for $y \in (-\infty, \bar{y}_{12})$ subject to $\lim_{y\to-\infty} g(y) = f(y; D_{10})$ and the value matching condition

$$g(\bar{y}_{12}) = f(\bar{y}_{12}; D_{12}), \tag{25}$$

where $f(\cdot)$ is given by Proposition 1.

By inspection of the value matching condition in (25), we observe that the leader faces a reduction in the flow payoff from managing the business as soon as the follower invests in technology 2, since $D_{10} > D_{12}$.[15]

(S3): Both entrepreneurs have invested before time τ. When both entrepreneurs have invested in technology 1 before time τ, we get the following result.

[15]We remark that to determine the value function for the leader, we do not need to invoke any smooth pasting condition, because it is not a free boundary value problem. Moreover, since we assume the leader has already invested in technology 1, we do not subtract the investment costs in the value matching condition.

Proposition 5. *The value function for joint investment in technology 1 is given by* $J_{11;AI}(y) = f(y; D_{11})$*, where* $f(\cdot)$ *is given by Proposition 1.*

Hence, under the assumption that technology 2 is already available, we see that market incompleteness leads to a delay in exercising the investment option. This prediction is consistent with the model of Miao and Wang (2007). However, when technology 2 is not yet available, we will observe an explicit dependency in strategic interactions and technological innovation.

3.3 Technology 2 not yet available

Consider now the cases when technology 2 has not yet arrived. Since technology 2 is not yet available, the entrepeneurs' value function will, unlike in the previous section, become dependent on the parameter λ of the Poisson process of the arrival of the new technology. We first derive the follower's value function, then the leader's value function conditional on the follower's optimal investment behavior. This situation corresponds to scenarios **(S4)** and **(S5)** in Table 1. Finally, we derive the value functions when both entrepreneurs wait for an investment in technology 2 (scenario **(S6)**). We also derive the value functions if they jointly invest in the available technology 1.

(S4): The value function of the follower when waiting for technology 2. We identify two scenarios for which we can derive the option value functions for the follower. First, the follower waits for technology 2 before investing, given that the leader has invested in technology 1. Second, the follower considers investing in technology 1, given that the leader has invested in technology 1. The next proposition covers the first scenario.

Proposition 6. *The value function for the follower before time* τ *(given that the follower waits for technology 2) is given by* $F_{21;BI}(y) = G(y)$*, where* $G(y)$ *satisfies the non-linear ODE*

$$rG(y) = \frac{\sigma_y^2}{2}G''(y) - \gamma r\frac{\sigma_y^2}{2}(1-\rho^2)G'(y)^2 + (\alpha_y - \rho\sigma_y\eta)G'(y) + \lambda(F_{21;AI}(y) - G(y)), \tag{26}$$

for $y \in (-\infty, \infty)$ *subject to the boundary conditions*

$$\lim_{y\to-\infty} G(y) = 0 \tag{27}$$

$$\lim_{y\to+\infty} G(y) = Ay + B. \tag{28}$$

The expressions for A and B are given in Equations (A.29) *and* (A.30) *of Appendix A. The expression for $F_{21;AI}(\cdot)$ is given in Proposition 3. The optimal portfolio and consumption policy of the follower before investment are given by*

$$\pi^* = \frac{\eta}{\gamma r \sigma_S} - \frac{\rho\sigma_y}{\sigma_S} G'(y) \quad \textit{and} \quad c^* = r\left(w + G(y) + \frac{\eta^2}{2\gamma r^2} + \frac{\beta - r}{\gamma r^2}\right), \tag{29}$$

for $y \in (-\infty, \infty)$.

The important difference from the previous case, where technology 2 was already available, is that the ODE becomes dependent on the parameter λ, which is absent in the ODEs in Equations (11) and (20). Consequently, also the consumption and portfolio strategies become dependent on the technology's arrival rate.

The non-linear ODE for $G(y)$ has to be solved numerically.[16] The lower boundary condition (27) implies that the option to invest loses its value as the operating net income approaches negative infinity. The upper boundary condition (28) represents the expected net present value of the flow payoff $D_{21}Y_t$ accruing to the follower from investing at time τ and onwards using technology 2 adjusted for risk-aversion and incomplete hedging. We refer to the proof in Appendix A for additional details.

Next, we derive the value functions for the leader under the assumption of immediate investment in technology 1.[17]

Proposition 7. *The value function for the leader before time τ (given that the follower waits for technology 2) is $L_{12;BI}(y) = G(y) - I$, where $G(y)$ satisfies the non-linear ODE*

$$rG(y) = \frac{\sigma_y^2}{2} G''(y) - \gamma r \frac{\sigma_y^2}{2}(1-\rho^2) G'(y)^2 + (\alpha_y - \rho\sigma_y\eta) G'(y) + D_{10}y + \lambda\left(L_{12;AI}(y) - G(y)\right) \tag{30}$$

[16] We remark that to solve for $G(y)$ under complete markets, one would rely on the continuity and differentiability conditions around the threshold $\bar{y}_{12}$ (which is known under both complete and incomplete markets) to determine the relevant parameters. However, under incomplete markets, we do not know the value taken by $G(y)$ at this point and therefore we cannot use that information to determine $G(y)$ numerically since we have no closed form expression to guide us. Thus, we have to solve the differential equations subject to the boundary conditions given in the proposition.

[17] We simply subtract the investment costs at the end, since we assume immediate investment by the leader. They are not part of the problem, as was the case for the follower, since we are not explicitly concerned with the exact entry point by the leader at this stage. See also Pawlina and Kort (2006).

on $y \in (-\infty, \infty)$ subject to the boundary conditions

$$\lim_{y \to -\infty} G(y) = f(y; D_{10}) \tag{31}$$

$$\lim_{y \to \infty} G(y) = Ay + B. \tag{32}$$

The expressions for A and B are given in Equations (A.35) *and* (A.36) *of Appendix A. The function $L_{12;AI}(\cdot)$ is given by Proposition 4.*

The lower and upper boundary conditions have the same interpretation as in Proposition 6. The lower boundary condition implies that the leader is exposed to downside movements in the operating net income. As the operating net income approaches negative infinity, it is not optimal for the follower to invest regardless of the technology available and therefore the leader's value function converges to the project value $f(y; D_{10})$. The upper boundary condition captures the expected present value of receiving different flow payoffs during two different time periods. The flow payoff up to time τ is given by $D_{10}Y_t$ and arises from the immediate investment in technology 1. After time τ, the flow payoff is given by $D_{12}Y_t$, adjusted for risk-aversion and incomplete hedging.

(S5): The value function of the follower when considering investment in technology 1. Next, we consider the scenario where the follower considers investing in technology 1. Intuitively, such a strategy would only be attractive for the follower for sufficiently low values of λ.

Proposition 8. *The value function for the follower before time τ is*

$$F_{11;BI}(y) = \begin{cases} G(y) & \text{if} \quad y \in (-\infty, \bar{y}_{11}) \\ f(y; D_{11}) - I & \text{if} \quad y \in (\bar{y}_{11}, +\infty) \end{cases},$$

where $\bar{y}_{11}$ denotes the optimal investment threshold for the follower to invest in technology 1. The function $G(y)$ satisfies the non-linear ODE

$$rG(y) = \frac{\sigma_y^2}{2} G''(y) - \gamma r \frac{\sigma_y^2}{2}(1 - \rho^2) G'(y)^2 + (\alpha_y - \rho\sigma_y\eta) G'(y) + \lambda(F_{21;AI}(y) - G(y)), \tag{33}$$

for $y \in (-\infty, \bar{y}_{11})$ subject to $\lim_{y \to -\infty} G(y) = 0$ and the value matching and smooth pasting conditions

$$G(\bar{y}_{11}) = f(\bar{y}_{11}; D_{11}) - I, \tag{34}$$

$$G'(\bar{y}_{11}) = f'(\bar{y}_{11}; D_{11}). \tag{35}$$

The functions $f(\cdot)$ and $F_{21;AI}(\cdot)$ are given by Propositions 1 and 3, respectively.[18]

Since the value function can only be determined numerically, we perform again a perturbation around σ_y to analyze qualitatively the impact of market incompleteness on the optimal investment threshold. The details can be found in Appendix B. For the first-order approximation of $\bar{y}_{11}$, we get

$$\bar{y}_{11}^{(1)} = \frac{r}{D_{11}}I + \left(\frac{r}{D_{11}}I + \frac{\alpha_y}{r} - \frac{\rho\sigma_y\eta}{D_{11}r}\right)\sum_{i=1}^{\infty}\lambda^i\left(\frac{D_{21}-D_{11}}{D_{11}r}\right)^i. \tag{36}$$

Clearly, the presence of technological innovation ($\lambda \neq 0$) already has a first-order impact. The sign of the impact is affected by the correlation coefficient ρ. However, for reasonable parameter values, the impact of technological innovation on the optimal investment threshold is positive, i.e., it leads to an investment delay.[19]

To understand the combined effect of technological change and risk-aversion on the investment threshold, we need to solve for the second-order approximation, for which we obtain

$$\bar{y}_{11}^{(2)} = \bar{y}_{11}^{(1)} + \frac{\sigma_y^2}{2\alpha_y} - \gamma\sigma_y^2(1-\rho^2)\frac{D_{11}+D_{21}}{2r}\sum_{i=1}^{\infty}\lambda^i\left(\frac{D_{21}-D_{11}}{D_{11}r}\right)^i. \tag{37}$$

The sign of the last term is negative. Therefore, the presence of risk aversion decreases the investment threshold. If $\lambda \to 0$, the risk-aversion coefficient γ disappears from the Equation (37). Hence, in the presence of technological change, we get a remarkably different result for the second-order approximation of $\bar{y}_{11}$ in that market incompleteness leads to a decrease of the optimal investment threshold. Expressing the threshold in Equation (37) in terms of the private equity risk premium ζ, we get

$$\bar{y}_{11}^{(2)} = \bar{y}_{11}^{(1)} + \frac{\sigma_y^2}{2\alpha_y} - \zeta\sum_{i=1}^{\infty}\lambda^i\frac{(D_{11}+D_{21})(D_{21}-D_{11})^i}{(D_{11}r)^{i+1}}. \tag{38}$$

Hence, unlike the previous cases, the presence of a private equity risk premium leads to an acceleration of investment. For large values of λ, i.e., in innovative industries, the optimal threshold is further reduced. This result differs from the model in Miao and Wang (2007), in which investment is delayed in the model with flow payoff when idiosyncratic volatility increases. Here, because of the interaction between technological change and the private equity risk premium, we find their result reversed.

[18] We remark that we subtract the investment costs in the value matching condition. The operating net income process reaches (or is above) $\bar{y}_{11}$ and the follower invests in technology 1 and pays the investment costs.

[19] In particular, from an economic viewpoint, we would require that the risk premium η should provide a bound for the 'risk premium' α/σ_y of the flow payoff. Then, for $D_{11} \geq 1$, the impact of $\lambda > 0$ on $\bar{y}_{11}^{(1)}$ is always positive.

For the third-order expansion, we get an additional term, which does not depend on the risk aversion parameter γ:

$$\bar{y}_{11}^{(3)} = \bar{y}_{11}^{(2)} + \frac{\eta\rho}{2\alpha_y^2}\sigma_y^3. \tag{39}$$

The risk aversion parameter, however, does appear again in the fourth- and higher-order expansions. In particular, for the fourth-order expansion in σ_y, we get

$$\bar{y}_{11}^{(4)} = \bar{y}_{11}^{(3)} + \frac{2\eta^2\rho^2 - (r+\lambda)}{4\alpha^3}\sigma_y^4 + \frac{\gamma D_{11}\left(1-\rho^2\right)}{2\alpha^2}\sigma_y^4. \tag{40}$$

The last term in the optimal threshold $\bar{y}_{11}^{(4)}$ grows with γ. Hence, the fourth-order term increases the optimal threshold compared to the complete market case. Note that, in contrast to the second-order term in Equation (37), this effect holds irrespectively of the presence of technological change. Therefore, while for $\lambda = 0$ we have that market incompleteness leads to a delayed investment, increasing λ may eventually lead to an acceleration of investment under market incompleteness due to the second-order term.

In Appendix B, we further expand the optimal investment threshold up to order six. To get more insights into the combined effect of technological innovation and market incompleteness, we calculate the cross-derivative with respect to λ and γ:

$$\frac{\partial^2 \bar{y}_{11}^{(6)}}{\partial\gamma\partial\lambda} = -\frac{1}{4}\left(1-\rho^2\right)\sigma_y^2\left(\frac{\sigma_y^4(4D_{11}-D_{21})}{\alpha^4} + \frac{2D_{11}(D_{21}-D_{11})(D_{11}+D_{21})}{(D_{11}(\lambda+r)-D_{21}\lambda)^2}\right). \tag{41}$$

which clearly tends to be negative when λ is large.[20] Hence, when both the technological innovation parameter λ and the private equity risk premium are sufficiently large, the precautionary savings effect dominates the option effect and encourages the agent to exercise the option sooner, unlike the standard real options result and the results in Miao and Wang (2007). Furthermore, the impact not only depends on the level of risk-aversion, but also on the size (and sign) of the correlation and on the frequency of technological change λ. High frequencies will eventually lead to earlier investments. This effect emerges from the combined presence of market incompleteness and technological change.

Next, we state the result for the leader's value function for scenario **(S5)**, i.e., the follower considers

[20] To see this, recall that we require for the preemption equilibrium $D_{11}(\lambda+r) - D_{21}\lambda > 0$. Furthermore, $D_{21} > D_{11}$ by assumption. Hence, only the term $4D_{11} - D_{21}$ can turn negative. Together with the condition for the preemption equilibrium, we would need that $r > 3\lambda$. This condition requires unreasonably high interest rates when λ is large. Furthermore, even if we have $4D_{11} - D_{21}$, we still do not have a negative value for the cross-derivative in (41), since we also need to compensate for the second term, $\frac{2D_{11}(D_{21}-D_{11})(D_{11}+D_{21})}{(D_{11}(\lambda+r)-D_{21}\lambda)^2}$, which is strictly positive.

investing in technology 1.

Proposition 9. *The value function for the leader before time τ is*

$$L_{11;BI}(y) = \begin{cases} G(y) - I & \text{if} \quad y \in (-\infty, \bar{y}_{11}) \\ f(y; D_{11}) - I & \text{if} \quad y \in (\bar{y}_{11}, +\infty) \end{cases},$$

where $\bar{y}_{11}$ denotes the follower's optimal investment threshold for investment in technology 1 determined in Proposition 8. The function $G(y)$ satisfies the non-linear ODE

$$rG(y) = \frac{\sigma_y^2}{2}G''(y) - \gamma r\frac{\sigma_y^2}{2}(1-\rho^2)G'(y)^2 + (\alpha_y - \rho\sigma_y\eta)\, G'(y) + D_{10}y + \lambda\left(L_{12;AI}(y) - G(y)\right) \quad (42)$$

for $y \in (-\infty, \bar{y}_{11})$ subject to $\lim_{y\to-\infty} G(y) = f(y; D_{10})$ and the value matching condition $G(\bar{y}_{11}) = f(\bar{y}_{11}; D_{11})$. The functions $f(\cdot)$ and $L_{12;AI}(\cdot)$ are given by Proposition 1 and 4, respectively.

The upper boundary condition $G(\bar{y}_{11}) = f(\bar{y}_{11}; D_{11})$ captures the reduction in the flow payoff that the leader incurs at the moment when the follower invests in technology 1.

(S6): Joint waiting to invest until technology 2 arrives. Below, we derive the option value function when both entrepreneurs wait for technology 2.

Proposition 10. *The value function for joint waiting for the arrival of technology 2 is $J_{22;BI}(y) = G(y)$, where the function $G(y)$ satisfies the non-linear ODE*

$$rG(y) = \frac{\sigma_y^2}{2}G''(y) - \gamma r\frac{\sigma_y^2}{2}(1-\rho^2)G'(y)^2 + (\alpha_y - \rho\sigma_y\eta)G'(y) + \lambda(J_{22;AI}(y) - G(y)), \quad (43)$$

for $y \in (-\infty, \infty)$ subject to the boundary conditions $\lim_{y\to-\infty} G(y) = 0$ and $\lim_{y\to+\infty} G(y) = Ay + B$. The expressions for A and B are given in Appendix A. The function $J_{22;AI}(y)$ is given by Proposition 2.

The upper growth condition in Proposition 10 represents the expected net present value of the flow payoff $D_{22}Y_t$ accruing to both entrepreneurs from investing at time τ and onwards using technology 2 adjusted for risk-aversion and incomplete hedging. Finally, Proposition 11 states the value function for joint investment in technology 1.

Proposition 11. *The value function for joint investment in technology 1 before time τ is $J_{11;BI}(y) = f(y; D_{11}) - I$, where $f(\cdot)$ is given by Proposition 1.*

4 Discussion

We discuss the model's properties and its theoretical predictions based on a numerical analysis. We first introduce the complete market setting as the benchmark. We then analyze the strategic and optimal investment timing decisions under varying arrival intensities of technological innovation. We also discuss the implications for the optimal portfolio choice of the entrepreneurs.

4.1 Complete market benchmark

Since our incomplete market setting does not have closed form solutions, we have to resort to numerical methods to study how non-diversifiable risk and technological innovation affect the optimal strategic investment behavior of the risk-averse entrepreneurs. As a benchmark, we can use the complete market case, which we obtain either by restricting entrepreneurs to be risk-neutral or by setting the correlation ρ equal to ± 1.

Since the entrepreneurs' project value after investing is linear in the operating net income under both complete and incomplete markets, the upper growth conditions are also linear in the operating net income under the two market settings. These upper growth conditions determine to a large extent the critical values of λ that give rise to different types of equilibria. As in Huisman and Kort (2004), we find the following critical values:

$$\begin{aligned}
&\lambda \in [0, \tfrac{rD_{10}}{D_{21}-D_{12}}) && \rightarrow \text{ preemptive equilibrium,} \\
&\lambda \in [\tfrac{rD_{10}}{D_{21}-D_{12}}, \tfrac{rD_{10}}{D_{22}-D_{12}}) && \rightarrow \text{ attrition equilibrium,} \\
&\lambda \in [\tfrac{rD_{10}}{D_{22}-D_{12}}, \infty) && \rightarrow \text{ waiting equilibrium.}
\end{aligned} \tag{44}$$

For our numerical analysis, we set the correlation between innovations to the risky asset and the operating net income to be positive with $\rho = 0.15$. This value can be motivated by empirical evidence. For example, Heaton and Lucas (2000) find a positive correlation of 0.14 between the quarterly growth rate of real non-farm proprietary income for U.S. entrepreneurial businesses and the CRSP value-weighted stock returns.

4.2 Preemption equilibrium

We consider first the preemption equilibrium with $\lambda < \frac{rD_{10}}{D_{21}-D_{12}}$, which for our choice of parameter values is equivalent to $\lambda < 0.2667$. Panels A and C in Figure 1 show the (option) value functions for a low arrival intensity $\lambda = 0.02$ under complete and incomplete markets, respectively. Panels B and D show the cases when we increase the arrival intensity to a larger value by setting $\lambda = 0.15$.

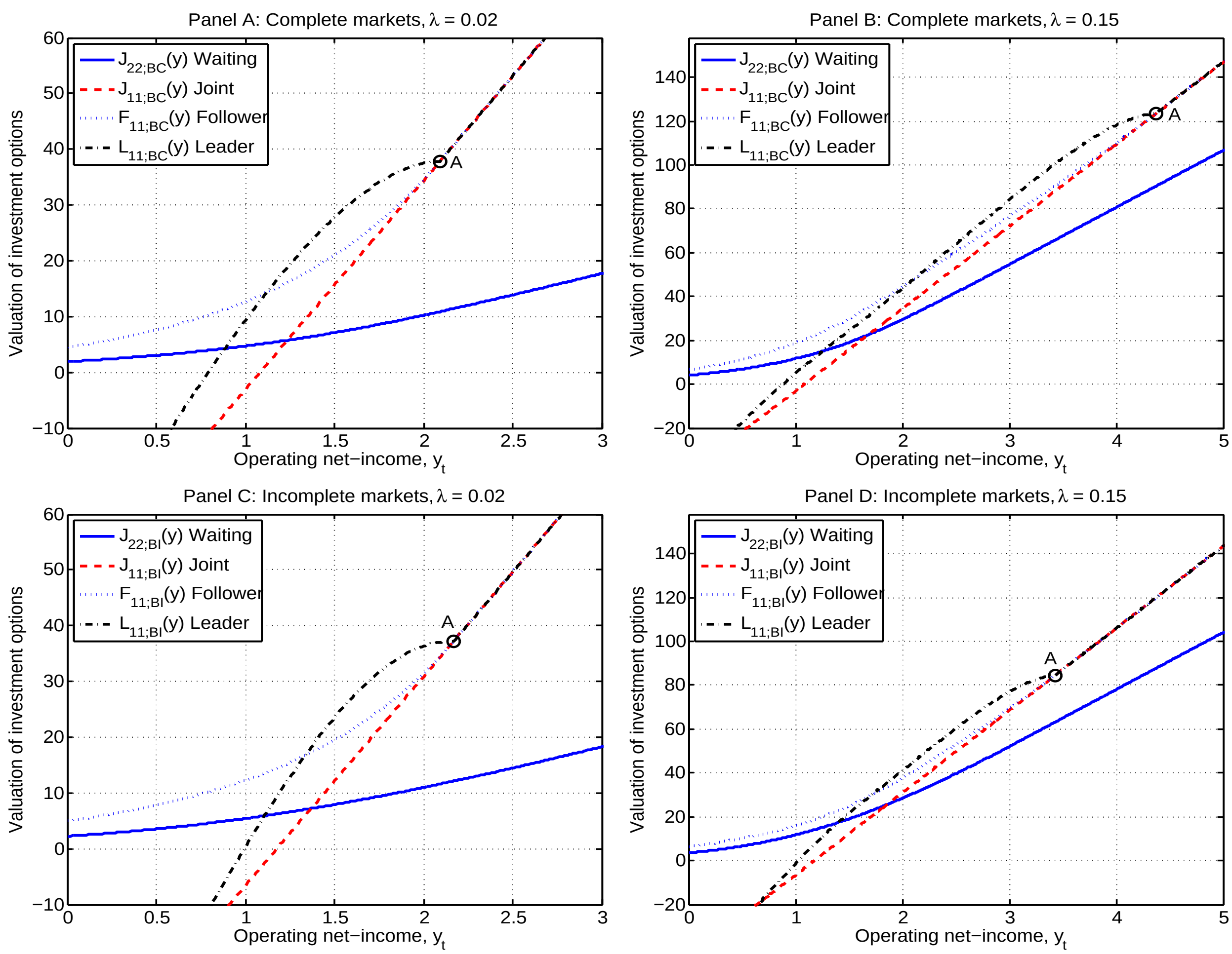

Figure 1: Value functions in the preemption equilibrium. Panels A and B show the value functions for complete markets for $\lambda = 0.02$ with $\lambda = 0.15$, Panels C and D for incomplete markets with $\lambda = 0.02$ and $\lambda = 0.15$. The dash-dotted line denotes the leader's value function, the dotted line the follower's value function, the solid line the value function for joint waiting for technology 2, and the dashed line denotes the value function for joint investment in technology 1. Parameters are set as follows: $\alpha_y = 0.05$, $\sigma_y = 30\%$, $r = 8\%$, $I = 50$, $\mu_s = 0.10$, $\sigma_s = 20\%$, $\gamma = 0.25$, and $\rho = 0.15$. The income factors are $D_{10} = 5$, $D_{12} = 2.5$, $D_{21} = 4$, $D_{11} = 3$ and $D_{22} = 3.2$.

4.2.1 The follower's investment threshold

The follower's optimal investment threshold $\bar{y}_{11}$ under incomplete markets is determined by trading off the relative changes of the project value $f(y; D_{11})$ against the option value $G(y)$. When the follower invests in technology 1, the value functions for the leader and follower coincide and equal the joint investment value $f(y; D_{11})$. This critical point, labelled as point A in Figure 1, lies at the intersection of the three curves: those of the leader, the follower, and the joint investment curve.

In Figure 1, Panels A and C, we plot the option value functions for a low arrival intensity of technological innovation ($\lambda = 0.02$). We find that the option's exercise should be delayed under incomplete markets relative to complete markets. This delay occurs because risk-aversion and incomplete hedging decrease the project value more than they decrease the option value. This result holds for small values of λ. Indeed, if we let the arrival intensity of technology 2 go to zero ($\lambda \to 0$), we essentially approach the model in Miao and Wang (2007). Hence, not surprisingly, our finding for small λ is consistent with theirs in that investing in a project with a flow payoff will always be delayed under incomplete markets compared to complete markets.

However, for sufficiently large values of λ, our model leads to different conclusions. Comparing Panels B and D, the investment in a project is not delayed, but accelerated under incomplete markets. Hence, the prediction of Miao and Wang (2007) is reversed for industries characterized by a high degree of technological innovation. This finding confirms our analysis of the approximative solutions in the previous section.

4.2.2 The leader's investment threshold

According to Fudenberg and Tirole (1985), preemption by the leader should occur when there is rent equalization between leader and follower (i.e., when the curves of the leader and the follower intersect each other). We therefore need to examine how the value functions change as we move from complete to incomplete markets.

When the operating net income decreases, the leader's value function is lowered more than the follower's value function as we move from complete to incomplete markets. The presence of risk-aversion and non-diversifiable risk reduces the value attained by the lower boundary condition in the leader's valuation problem, which results in a lower value function under incomplete markets. On the other hand, the follower has no direct exposure to movements in the operating net income, since the follower has not exercised the investment option.[21] As a result, the follower's value function converges to zero as the operating net income decreases regardless of the market setting.

From Panels A and C with $\lambda = 0.02$, we observe that preemption under complete markets occurs at that level of operating net income where the leader's value function under incomplete markets is lowered more than the follower's. Hence, investment under incomplete markets is delayed.

In contrast, when $\lambda = 0.15$, preemption should be accelerated under incomplete markets. In an incomplete market, the instantaneous expected drift decreases more in the follower's value function than it does in the leader's value function. As the operating net income increases, risk-aversion and non-diversifiable risk matter less for the leader, since the payoff flow from managing the business has an increasingly positive impact on the value function. Therefore, when the operating net income becomes sufficiently large, the follower's value function is lowered more than the leader's as we move from complete to incomplete markets.

In line with intuition, as technology 2 is more likely to arrive, preemption will occur at a higher level of operating net income. In Panels B and D, we observe that preemption under complete markets for $\lambda = 0.15$ occurs at a level of operating net income where it is more profitable to be the leader than to be the follower under incomplete markets. This leads to an accelerated investment by the leader under incomplete markets.[22] Hence, the implications of technological innovation on the optimal strategic investment timing behavior in the presence of non-diversifiable risk is ambiguous. Again, this observation differs from the predictions in Miao and Wang (2007) and stresses the importance of taking technological innovation into account in an incomplete market setting.

[21]However, because the follower is forward looking, the follower has an indirect exposure to movements in the operating net income through the option value function $G(y)$.

[22]The leader's valuation curve increases slightly as the arrival intensity of technology 2 increases. Since the leader is assumed to have invested in technology 1, an increased arrival intensity is mainly beneficial for the follower. As the arrival intensity of technology 2 increases, then so does $\bar{y}_{11}$, the follower's investment threshold for optimally investing in technology 1. Consequently, the leader receives the flow payoff $D_{10}Y_t$ for a longer period of time given that technology 2 does not arrive, moving the leader's valuation curve up.

4.3 Attrition equilibrium

When the arrival intensity of technology 2 ranges in the interval $\lambda \in [\frac{rD_{10}}{D_{21}-D_{12}}, \frac{rD_{10}}{D_{22}-D_{12}})$, the preemption game turns into a war of attrition. Panels A and C in Figure 2 show the value functions for $\lambda = 0.27$ under complete and incomplete markets, respectively. In Panels B and D we make the corresponding plots for $\lambda = 0.40$. For the given values of λ, the arrival likelihood of technology 2 is so large that the leader and the follower curves do not intersect. No preemption equilibrium will occur. For all values of the operating net income, it is optimal to invest as follower in technology 2 and it is suboptimal to invest as the first-mover in technology 1. Hence, there exists a second-mover advantage.

An attrition equilibrium arises when the leader and the waiting curves intersect each other. At point A of Figure 2, one of the entrepreneurs is better off investing in the existing technology (technology 1) and becoming the leader rather than engaging in a joint waiting for the arrival of technology 2. However, there is a second-mover advantage. Being a first-mover in technology 1 is suboptimal compared to being a second-mover in technology 2. An example of such a situation may well be the market for zero-emission vehicles. New innovations are constantly being developed. However, the time it takes for them to reach the market as operational technologies is highly uncertain. Eventually, one may be better off investing in the existing available technology.[23] As technology 2 is more likely to arrive over the next instant (Figure 2, Panels B and D), the waiting curve is pushed up relative to the leader curve. Hence, it becomes more valuable for the entrepreneurs to engage in joint waiting for technology 2. The attrition point occurs later under both complete and incomplete market settings.

We remark that the differential equation defining the value function for joint waiting in Proposition 10 is, apart from the income factor, identical to that defining the valuation of the follower's value function in Proposition 6. Therefore, the impact of risk-aversion and non-diversifiable risk on the value function for joint waiting is similar to that discussed in Section 4.2.1 for the follower's value function. Hence, as we move from complete to incomplete markets, the value function for joint waiting decreases more than the leader's value function for increasing values of the operating net income. For the follower's value function, the opposite situation occurs for decreasing values of the operating net

[23]When such a situation persists, market interventions may be necessary. This happened in California, where the California Air Resources Board enacted the CARB-ZEV program to promote the use of zero emission vehicles.

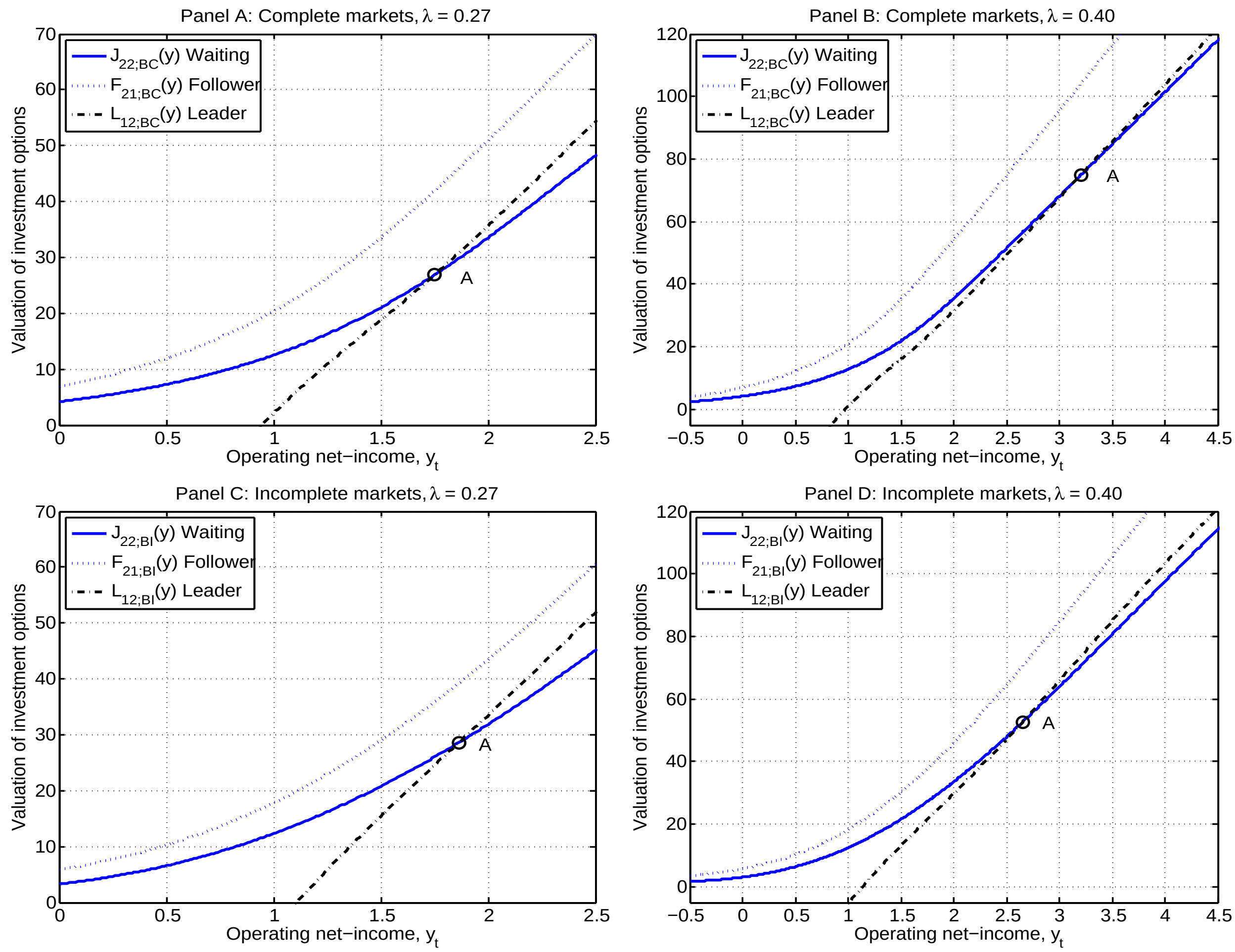

Figure 2: Value functions in the attrition equilibrium. Panels A and B show the value functions under complete markets for $\lambda = 0.27$ and $\lambda = 0.40$ and Panels C and D show the value functions under incomplete markets for $\lambda = 0.27$ and $\lambda = 0.40$, respectively. The dash-dotted line denotes the value function for the leader, the dotted line denotes the value function for the follower, the solid line denotes the value function for joint waiting for technology 2. Parameters are set as follows: $\alpha_y = 0.05$, $\sigma_y = 30\%$, $r = 8\%$, $I = 50$, $\mu_s = 0.10$, $\sigma_s = 20\%$, $\gamma = 0.25$, and $\rho = 0.15$. The income factors are $D_{10} = 5$, $D_{12} = 2.5$, $D_{21} = 4$, $D_{11} = 3$ and $D_{22} = 3.2$.

income.

For $\lambda = 0.27$, we obtain the attrition point at that level of operating net income where the leader's value function decreases more than the value function for joint waiting. As a result, investment is delayed under incomplete markets. In contrast, when $\lambda = 0.40$, the attrition point occurs at that level of operating net income at which the waiting curve suffers relatively more than the leader curve, thereby leading to accelerated investment under incomplete markets. Hence, we get again an ambiguous result on delaying investments when we move from complete to incomplete markets.

Whether or not investments are delayed depends crucially on the technology's arrival intensity, which may differ across different industries. In both the preemptive and the attrition equilibrium, market incompleteness leads to an acceleration of investments if the intensity of technological innovation is sufficiently large.

4.4 Waiting equilibrium

When the value of the arrival intensity is in the interval $\lambda \in [\frac{rD_{10}}{D_{22}-D_{12}}, \infty)$, we obtain a waiting equilibrium. Both entrepreneurs should optimally wait until technology 2 arrives, since the leader's value function is below the value function for joint waiting for all values of the operating net income. Panels A and B in Figure 3 illustrate the situation for $\lambda = 2.95$ under complete and incomplete markets, respectively.[24] The primary difference as we move from complete to incomplete markets is that the value functions decrease due to the presence of risk-aversion and non-diversifiable risk.

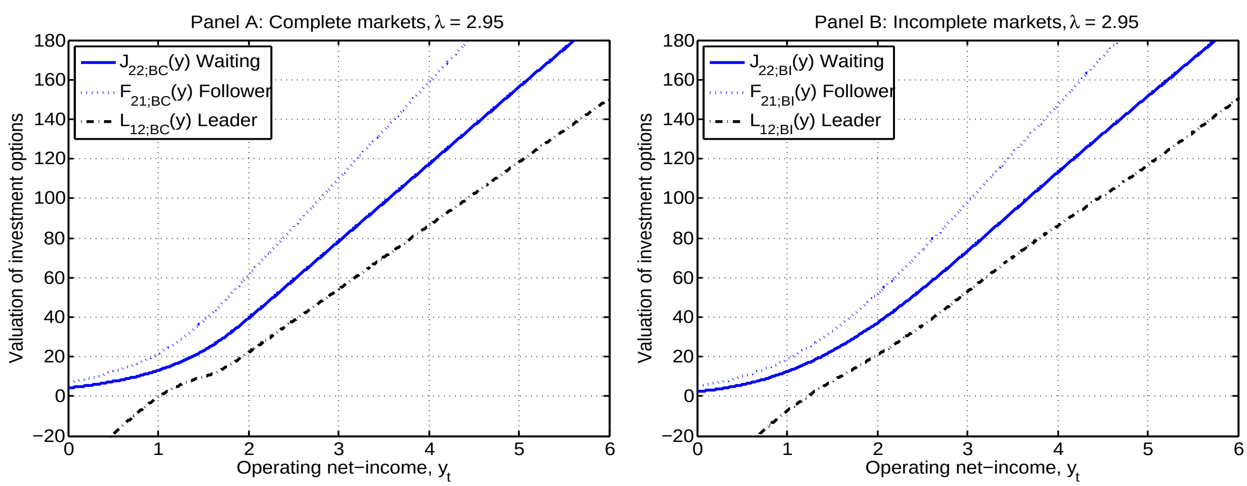

Figure 3: Value functions in the waiting equilibrium. Panels A and B show the option value functions under complete and incomplete markets. The dash-dotted line denotes the value function for the leader, the dotted line denotes the value function for the follower, the solid line denotes the value function for joint waiting for technology 2. Parameters are set as follows: $\alpha_y = 0.05$, $\sigma_y = 30\%$, $r = 8\%$, $I = 50$, $\mu_s = 0.10$, $\sigma_s = 20\%$, $\gamma = 0.25$, and $\rho = 0.15$. The income factors are $D_{10} = 5$, $D_{12} = 2.5$, $D_{21} = 4$, $D_{11} = 3$ and $D_{22} = 3.2$.

[24]The value of $\lambda = 2.95$ is chosen for illustrative purposes. For values close to the lower bound $\frac{rD_{10}}{D_{22}-D_{12}}$, the waiting curve and the leader curve will be very close, making it hard to visualize any differences in the figure.

4.5 Optimal portfolio policy

A number of studies in the literature have documented the importance of non-diversifiable risk for individuals' portfolio choices.[25] The common theme is that the presence of uninsurable risk forces individuals to adjust their risky asset holdings ("hedging demand") to partially hedge against unfavorable movements in this risk factor. We not only complement this literature, but we also shed new light on potential factors that might have an impact on the entrepreneur's optimal portfolio choice. Recalling the general form of the optimal portfolio strategy in our setup,

$$\pi^* = \frac{\eta}{\gamma r \sigma_S} - \frac{\rho \sigma_y}{\sigma_S} g'(y), \tag{45}$$

we see that in addition to the standard Merton mean-variance term, the strategy crucially depends on the value of the function $g'(y)$. This function depends on the business-specific characteristics of our economy. In particular, it depends on whether the competitor has already entered the market, on the specific technology adopted by the competitor, on the competitive market structure (through the income factors $D_{N_i N_j}$), and finally on the likelihood that a more efficient technology will arrive, represented by the arrival intensity λ.

In Figure 4, we illustrate the optimal portfolio choice as a function of the operating net income for the follower and the leader for the attrition equilibrium and compare it to the case when there is only one single entrepreneur and no technological change. This latter case corresponds to the setting of Miao and Wang (2007). From Panel A, we see that in anticipation of a more efficient technology, the the follower should reduce more their risky asset holdings as the operating net income increases. Specifically, as the option's sensitivity to changes in the operating net income increases, the follower optimally hedges the larger variations that can occur in the certainty-equivalent valuation of the non-traded investment opportunity by lowering the exposure to the risky asset even before exercising the investment option. Thus, the follower should reduce the amount allocated to the risky asset as the operating net income increases.

Similarly, the leader should also adjust the portfolio allocation in the risky asset. When the arrival intensity of technology 2 is large, the amount of hedging increases with increasing operating income

[25]See, e.g., Merton (1971), Bodie, Merton, and Samuelson (1992) and Viceira (2001).

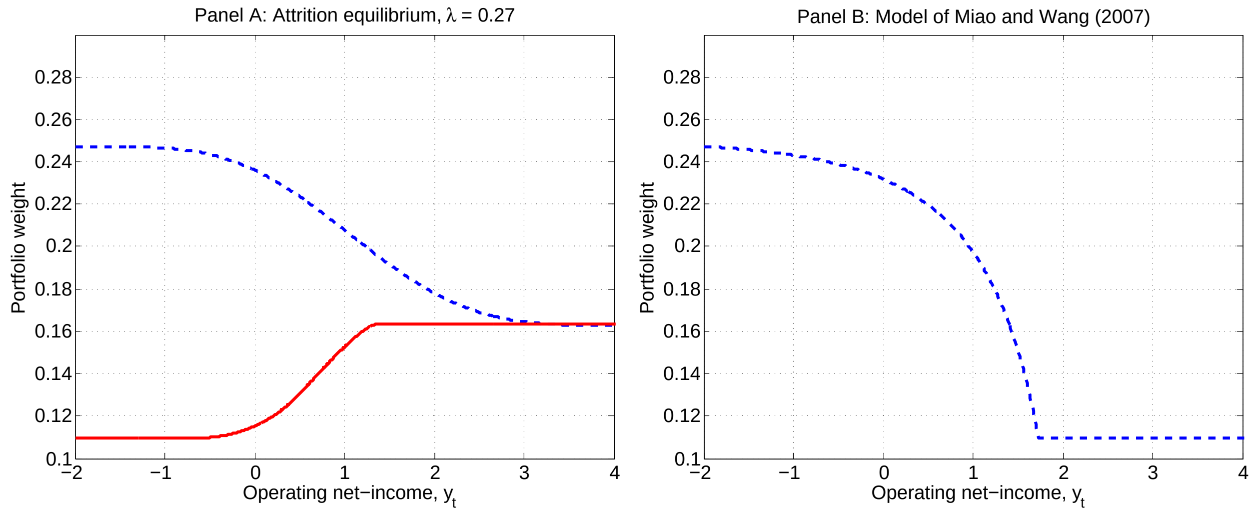

Figure 4: Optimal portfolios. Panel A plots the optimal portfolio in the attrition equilibrium for the follower and leader as a function of the operating net income respectively, for selected parameters. The solid line represents the optimal allocation in the stock for the leader and the dashed line represents the optimal allocation for the follower. In Panel B, we plot the optimal portfolio policy for the single-entrepreneur model of Miao and Wang (2007) when there is no technological change. For this case, we use the income factor $D_{10} = 5$. The parameters are set as follows: project value drift $\alpha_y = 0.05$, project value volatility $\sigma_y = 30\%$, risk-free interest rate $r = 8\%$, investment costs $I = 50$, drift risky asset $\mu_s = 0.10$, volatility risky asset $\sigma_s = 20\%$, risk-aversion $\gamma = 0.25$, initial wealth $W_0 = 100$. The income factors are $D_{10} = 5$, $D_{12} = 2.5$, $D_{21} = 4$, $D_{11} = 3$ and $D_{22} = 3.2$.

in anticipation of a sooner entry by the follower. Once the follower invests, the leader's operating net income from managing the business is adversely affected and the exposure to non-diversifiable risk is reduced. Therefore, the leader should increase the holdings in the risky asset even before investment by the follower, however still at levels below the myopic mean-variance portfolio. This observation is consistent with findings in Hongyan and Nofsinger (2009) and Heaton and Lucas (2000). They show that entrepreneurs exposed to uninsurable income risk hold less wealth in stocks than do similarly wealthy individuals or households.

In a setup with a single entrepreneur and no technological innovation, we do not observe these effects. For comparison, in Panel B of Figure 4, we plot the resulting portfolio for the entrepreneur in Miao and Wang (2007). Unlike our follower and leader, this entrepreneur adjusts the exposure to the risky asset as the entrepreneur knows at which level of operating income to invest. However, in our duopoly setting, the portfolio fraction for the risky asset becomes dependent on the action of the competitor in the product market. While it is still uncertain whether and when technology 2 will arrive, the follower will adjust the portfolio holding in anticipation of a potential investment.

At the same time, the leader adjusts the portfolio holdings as a strategic reaction to the potential investment of the follower.

4.6 Comparative statics analysis

To facilitate the discussion of the comparative statics, the set of parameters specified in Figures 1 to 4 are kept fixed, except for the parameter of interest discussed below.

First, if the entrepreneurs are more risk-averse ($\gamma = 0.45$), preemption will occur later relative to the setting in Figure 1 with $\lambda = 0.15$. But preemption will still occur sooner than with complete markets. This finding is intuitive. Higher risk-aversion implies that the operating net income has to be higher to make it profitable to invest as the leader. Furthermore, investment by the follower will occur sooner than in the situation with $\lambda = 0.15$. Higher risk-aversion implies that the certainty-equivalent gain under incomplete markets is reduced compared to the setting with $\gamma = 0.25$, i.e., the gain from waiting for arrival of technology 2 is further reduced, and that leads to accelerated investment compared to the setting with lower risk-aversion. For the situation where $\lambda = 0.02$, the predictions in Figure 1 continue to hold for complete markets. However, compared to the incomplete market setting, preemption when $\gamma = 0.45$ occurs later due to the higher risk-aversion, and the follower also invests later since the project value is reduced even further relative to the option value in this case.

A higher project volatility ($\sigma_y = 0.40$) implies that the entrepreneurs are also exposed to greater non-diversifiable risk, which has an overall negative impact on the value functions. Specifically, the leader's value function is lowered more for decreasing operating net income, resulting in later preemption than in the incomplete market setting with $\sigma_y = 0.30$ (for both cases $\lambda = 0.02$ and $\lambda = 0.15$). Similar to increasing risk-aversion, greater project volatility ($\sigma_y = 0.40$) reduces the certainty-equivalent gain from the arrival of technology 2 and this has a larger negative impact on the option value compared to the project value when $\lambda = 0.15$. This results in earlier investment by the follower in technology 1 relative to the incomplete market setting in Figure 1. In contrast, for $\lambda = 0.02$, the certainty-equivalent gain matters less for the option valuation and the project value is therefore reduced more than the option value, resulting in delayed investment.

Similar to the optimal investment timing, the optimal portfolio choice of the entrepreneurs is also sensitive to the profitability of adopting a given technology, reflected by the income factors D_{ij}s. A higher income factor implies a higher expected drift in the operating net income, but it also implies a higher exposure to non-diversifiable risk. For instance, from the follower's perspective, a higher income factor D_{21} would lead to a higher valuation of being the follower. It would also lead to a higher delta for the option, which would generate a larger hedging demand and thus a smaller amount of wealth invested in the risky asset.

Hence, when investing in the financial markets, it is important to take into account the strategic considerations regarding the technology adoption by other individuals and the arrival of future technological innovations. But the relative profitability of managing the business with a given technology compared to its competitors also matters profoundly when deciding how much wealth to allocate to the financial markets in order to hedge the non-diversifiable risk.

5 Conclusion

We study the implications of technological innovation and strategic considerations of technology adoption on the optimal timing of investment and on the valuation of the associated investment opportunities. According to the model, before individuals (contemplating becoming entrepreneurs) decide whether to invest in a business project using a current available technology or whether to wait until a more efficient technology may be available for adoption, they should take into account at least three aspects: the likelihood of such a better technology's arriving; the potential entrance by other individuals and its impact on profitability; and any non-diversifiable (income) risk surrounding the business project.

Failure to take into account these elements when deciding to invest may lead individuals to overestimate the value attached to the investment opportunity and subsequently lead to suboptimal investment behavior. An important result of our analysis is that by taking into account strategic aspects and the arrival of future technological innovations, the presence of non-diversifiable risk may accelerate investment compared to complete markets. This finding contrasts with the result in Miao and Wang (2007) for the setting of a single entrepreneur with a flow payoff and contrasts with the standard prediction from the traditional real options analysis.

Our model generates a rich set of empirical predictions. Significant differences can occur regarding the optimal strategic investment behavior between under-diversified individuals such as entrepreneurs who own a substantial share of their entrepreneurial business and well-diversified individuals or companies. The degree of technological innovation as well as risk attitudes in conjunction with non-diversifiable risk appear to play an important role in determining the optimal investment timing. This leads to empirical implications regarding optimal investment timing for individuals contemplating becoming entrepreneurs.

We have studied the interaction of technological change and market incompleteness in the simplest possible setting. Hence, our model has a number of limitations that offer the opportunity for some interesting extensions. First, we could consider more involved utility functions that are more realistic than the exponential utility function, so as to include, e.g., wealth effects and other important aspects. Furthermore, one could introduce financing decisions, e.g., as in Chen, Miao, and Wang (2010) and Wang, Wang, and Yang (2011). We leave these avenues of theoretical extensions and empirical investigation for future research.

References

Akresh-Gonzales, Josette, 2009, Energy CEO Shai Agassi on Recognizing a "Sliding Doors" Moment, *Harvard Business Review.*

Bensoussan, Alain, David J. Diltz, and SignRu Hoe, 2010, Real Options Games in Complete and Incomplete Markets with Several Decision Makers, *SIAM Journal on Financial Mathematics* 1, 666–728.

Birge, John, and Vadim Linetsky, 2007, *Handbooks in Operations Research and Management Science: Financial Engineering, Chapter 12, Incomplete Markets, Handbook of Financial Engineering* (Elsevier).

Bodie, Zvi, Robert M. Merton, and William F. Samuelson, 1992, Labor Supply Flexibilitz and Portfolio Choice in a Life Cycle Model, *Journal of Economic Dynamcis and Control* 16, 427–449.

Cagetti, Marco, and Mariacristina De Nardi, 2006, Entrepreneurship, frictions, and wealth, *Journal of Political Economy* 114, 835–870.

Chen, Hui, Jianjun Miao, and Neng Wang, 2010, Entrepreneurial Finance and Non-diversifiable Risk, *Review of Financial Studies* 23, 4348–4388.

Dangl, Thomas, and Franz Wirl, 2004, Investment under Uncertainty: Calculating the Value Function when the Bellman Equation cannot be Solved Analytically, *Journal of Economic Dynamics and Control* 28, 1437–1460.

Dixit, Avinash K., and Robert S. Pindyck, 1994, *Investment under Uncertainty* (Princeton University Press: Princeton) 1. edn.

European-Commission, 2010, Entrepreneurship in the EU and Beyond, *Flash EB Series* 283.

Evans, Jonathan, Vicky Henderson, and David Hobson, 2008, Optimal timing for an indivisible asset sale, *Mathematical Finance* 18, 545–567.

Fudenberg, Drew, and Jean Tirole, 1985, Preemption and Rent Equalization in the Adoption of New Technology, *Review of Economic Studies* 52, 383–401.

Gentry, William M., and Glenn R. Hubbard, 2004, Entrepreneurship and Household Saving, *Advances in Economic Analysis and Policy* 4.

Graham, John, D., and Natalie Messer, 2011, Can Electric Vehicles Take Off? A Roadmap to Find the Answer, *Yale Environment 360*.

Hall, Robert E., and Susan E. Woodward, 2010, The Burden of Non-diversifiable Risk, *American Economic Review* 100, 1163–1194.

Heaton, John C., and Deborah J. Lucas, 2000, Portfolio Choice and Asset Prices: The Importance of Entrepreneurial Risk, *The Journal of Finance* 55, 1163–1198.

Henderson, Vicky, 2004, *Utility Indifference Pricing - An Overview* (To appear in Volume of Indifference Pricing, Princeton University Press).

——— , 2007, Valuing the Option to Invest in an Incomplete Market, *Mathematics and Financial Economics* 1, No. 2, 103–128.

Hendricks, Ken, Andrew Weiss, and Charles A Wilson, 1988, The War of Attrition in Continuous Time with Complete Information, *International Economic Review* 29, 663–680.

Hongyan, Fang, and John R. Nofsinger, 2009, Risk Aversion, Entrepreneurial Risk, and Portfolio Selection, *The Journal of Entrepreneurial Finance* 13, 25–55.

Hoppe, Heidrun C., 2000, Second-mover Advantages in the Strategic Adoption of New Technology under Uncertainty, *International Journal of Industrial Organization* 18, 315–338.

Hugonnier, Julien, and Erwan Morellec, 2007, Corporate Control and Real Investment in Incomplete Markets, *Journal of Economic Dynamics Control* 31, 1781–1800.

Huisman, Kuno J.M., and Peter M. Kort, 2004, Strategic Technology Adoption taking into Account Future Technological Improvements: A Real Options Approach, *European Journal of Operational Research* 159, 705–728.

Judd, Kenneth L., 1992, Projection Methods for Solving Aggregate Growth Models, *Journal of Economic Theory* 58, 410–452.

Lambrecht, Bart, and William Perraudin, 2003, Real Options and Preemption under Incomplete Information, *Journal of Economic Dynamics and Control* 27, 619–643.

Leland, Hayne E., 1994, Corporate Debt Value, Bond Covenants, and Optimal Capital Structure, *Journal of Finance* 49, 1213–1252.

Merton, Robert C., 1969, Lifetime portfolio selection under uncertainty: The continuous-time case, *The Review of Economics and Statistics* 51, No. 3, 247–257.

———, 1971, Optimum Consumption and Portfolio Rules in a Continuous-time Model, *Journal of Economic Theory* 3, 373–413.

Miao, Jianjun, and Neng Wang, 2007, Investment, Consumption, and Hedging under Incomplete Markets, *Journal of Financial Economics* 86, 608–642.

Moskowitz, Tobias J., and Annette Vissing-Jorgensen, 2002, The Returns to Entrepreneurial Investment: A Private Equity Premium Puzzle, *American Economic Review* 92, 745–778.

Pawlina, Grzegorz, and Peter M. Kort, 2006, Real Options in an Asymmetric Duopoly: Who Benefits from your Competitive Disadvantage, *Journal of Economics and Management Strategy* 15 (1), 1–35.

Quadrini, Vincenzo, 2009, Entrepreneurship in Macroeconomics, *Annals of Finance* 5, 295–311.

Rampini, Adriano A, 2004, Entrepreneurial activity, risk, and the business cycle, *Journal of Monetary Economics* 51, 555–573.

Romer, Paul M., 1990, Endogenous Technological Change, *Journal of Political Economy* 98, 71–102.

Schmitz, James A., 1989, Imitation, Entrepreneurship, and Long-Run Growth, *Journal of Political Economy* 97, 721–739.

Schumpeter, Joseph A., 1934, *The Theory of Economic Development* (Cambridge, MA: Harvard University Press).

Stenbacka, Rune, and Mihkel M. Tombak, 1994, Strategic Timing of Adoption of New Technologies under Uncertainty, *International Journal of Industrial Organization* 12, 387–411.

Thijssen, Jacco J.J., Kuno J.M. Huisman, and Peter M. Kort, 2002, Symmetric Equilibrium Strategies in Game Theoretical Real Option Models, *CentER Discussion Paper, Tilburg University* 2002-81.

Viceira, Luis M., 2001, Optimal Portfolio Choice for Long-Horizon Investors with Nontradable Labor Income, *The Journal of Finance* 56, 433–470.

Wang, Chong, Neng Wang, and Jinqiang Yang, 2011, Dynamics of Entrepreneurship under Incomplete Markets, *Working Paper, Columbia Business School.*

A Proofs

Proof of Proposition 1

We begin by considering the situation after investment has taken place. Denote by $J(w, y)$ the value function of the entrepreneur after option exercise. After option exercise, the wealth dynamics w but also the project value y affect the value function, because the entrepreneurs are exposed to non-diversifiable risk from managing the business, which cannot be hedged by trading in the risky financial asset.[26] Then, we get the following Hamilton–Jacobi–Bellman (HJB) equation,

$$\begin{aligned} \beta J(w,y) &= \max_{\pi,C} U(c) + J_w(w,y)\left(rw + \pi(\mu_S - r) + D_{N_iN_j}y - c\right) + J_y(w,y)\alpha_y \\ &\quad + \frac{\pi^2\sigma_S^2}{2}J_{ww}(w,y) + \frac{\sigma_y^2}{2}J_{yy}(w,y) + \rho\sigma_y\pi\sigma_S J_{wy}(w,y), \end{aligned} \tag{A.1}$$

subject to the transversality condition $\lim_{\tau\to\infty} E[e^{-\beta\tau}J(W_\tau, Y_\tau)] = 0$. We omit the time index for simplicity. We conjecture that the value function takes the form

$$J(w,y) = -\frac{1}{\gamma r}\exp\left(-\gamma r\left(w + f(y; D_{N_iN_j}) + \frac{\eta^2}{2\gamma r^2} + \frac{\beta - r}{\gamma r^2}\right)\right), \tag{A.2}$$

for some function $f(y; D_{N_iN_j})$ and where $\eta = \frac{\mu_S - r}{\sigma_S}$ denotes the Sharpe ratio of the risky asset. Deriving the first-order conditions with respect to the portfolio and consumption gives

$$\begin{aligned} 0 &= J_w(w,y)(\mu_S - r) + \sigma_S^2\pi J_{ww}(w,y) + J_{wy}(w,y)\rho\sigma_y\sigma_S \\ \pi^* &= \frac{-J_w(w,y)(\mu_S - r)}{J_{ww}(w,y)\sigma_S^2} + \frac{-J_{wy}(w,y)\rho\sigma_y}{J_{ww}(w,y)\sigma_S} \end{aligned} \tag{A.3}$$

and

$$U'(c) = J_w(w,y) \Leftrightarrow c^* = -\frac{1}{\gamma}\log(J_w(w,y)). \tag{A.4}$$

[26] As shown in Miao and Wang (2007), this situation differs from the lump-sum case, in which the exercise of the investment option generates an exit from incomplete markets.

Given our conjecture, we can write the derivatives of the value function as

$$\begin{aligned}
J_w(w,y) &= -\gamma r J(w,y) \qquad J_{ww}(w,y) = (\gamma r)^2 J(w,y), \\
J_{wy}(w,y) &= (\gamma r)^2 f'(y; D_{N_iN_j}) J(w,y), \\
J_y(w,y) &= -\gamma r f'(y; D_{N_iN_j}) J(w,y), \\
J_{yy}(w,y) &= -\gamma r f''(y; D_{N_iN_j}) J(w,y) + (\gamma r)^2 f'(y; D_{N_iN_j})^2 J(w,y), \qquad \text{(A.5)}
\end{aligned}$$

and we obtain the optimal portfolio and consumption policy:

$$\pi^* = \frac{\eta}{\gamma r \sigma_S} - \frac{\rho \sigma_y}{\sigma_S} f'(y; D_{N_iN_j}) \quad \text{and} \quad c^* = r\left(w + f(y; D_{N_iN_j}) + \frac{\eta^2}{2\gamma r^2} + \frac{\beta - r}{\gamma r^2}\right). \qquad \text{(A.6)}$$

Plugging these expressions back into the HJB equation and simplifying, we obtain the following non-linear ordinary differential equation (ODE) for $f(y; D_{N_iN_j})$:

$$\begin{aligned}
&\frac{\sigma_y^2}{2} f''(y; D_{N_iN_j}) - \gamma r \frac{\sigma_y^2}{2}(1-\rho^2) f'(y; D_{N_iN_j})^2 + (\alpha_y - \rho\sigma_y\eta) f'(y; D_{N_iN_j}) \\
&- r f(y; D_{N_iN_j}) + D_{N_iN_j} y = 0, \qquad \text{(A.7)}
\end{aligned}$$

subject to the transversality condition stated above. Ruling out speculative bubbles in the project value (see, e.g., Dixit and Pindyck (1994)), we obtain the solution given in the proposition. □

Proof of Proposition 2

We obtain the situation after option exercise from Proposition 1 by setting $D_{N_iN_j} = D_{22}$. Thus, the project value is

$$f(y; D_{22}) = \frac{D_{22}}{r} y + \frac{(\alpha_y - \rho\sigma_y\eta) D_{22}}{r^2} - \frac{\gamma \sigma_y^2 (1-\rho^2) D_{22}^2}{2r^2}, \qquad \text{(A.8)}$$

and the optimal portfolio and consumption policy are

$$\pi^* = \frac{\eta}{\gamma r \sigma_S} - \frac{\rho \sigma_y}{\sigma_S} f'(y; D_{22}) \quad \text{and} \quad c^* = r\left(w + f(y; D_{22}) + \frac{\eta^2}{2\gamma r^2} + \frac{\beta - r}{\gamma r^2}\right). \qquad \text{(A.9)}$$

To analyze the situation before option exercise, we note that the wealth dynamics before investment is

$$dW_t = (rW_t + \pi(\mu_S - r) - c_t)\, dt + \pi \sigma_S dB_t. \qquad \text{(A.10)}$$

Before investing, neither of the entrepreneurs receives a flow payoff since $D_{0N_j} = 0$ by assumption. Similar to the derivations for the project value in Proposition 1, we conjecture that the value function takes the form

$$V(w,y) = -\frac{1}{\gamma r}\exp\left(-\gamma r\left(w + g(y) + \frac{\eta^2}{2\gamma r^2} + \frac{\beta - r}{\gamma r^2}\right)\right), \tag{A.11}$$

for some function $g(y)$ and where $\eta = \frac{\mu_s - r}{\sigma_S}$ again denotes the Sharpe ratio of the risky asset. The derivations are similar to those in the proof of Proposition 1. Hence, we obtain the optimal portfolio and consumption,

$$\pi^* = \frac{\eta}{\gamma r \sigma_S} - \frac{\rho\sigma_y}{\sigma_S} g'(y) \quad \text{and} \quad c^* = r\left(w + g(y) + \frac{\eta^2}{2\gamma r^2} + \frac{\beta - r}{\gamma r^2}\right). \tag{A.12}$$

The function $g(y)$ satisfies the non-linear ODE

$$\frac{\sigma_y^2}{2} g''(y) - \gamma r \frac{\sigma_y^2}{2}(1-\rho^2) g'(y)^2 + (\alpha_y - \rho\sigma_y\eta) g'(y) - r g(y) = 0. \tag{A.13}$$

The function $g(y)$ represents the option value function of the entrepreneurs. The function has to be studied numerically subject to the value matching and smooth pasting conditions

$$g(\bar{y}_{22}) = f(\bar{y}_{22}; D_{22}) - I, \tag{A.14}$$

$$g'(\bar{y}_{22}) = f'(\bar{y}_{22}; D_{22}) = \frac{D_{22}}{r}, \tag{A.15}$$

and the lower boundary condition $\lim_{y \to -\infty} g(y) = 0$. The optimal investment threshold $\bar{y}_{22}$ also has to be found numerically. The reason for the three boundary conditions is that we have to jointly determine the endogenous investment threshold $\bar{y}_{22}$ together with the option value function $g(y)$. The option value function can take the following form,[27]

$$J_{22;AI}(y) = \begin{cases} g(y) & \text{if } y \in (-\infty, \bar{y}_{22}) \\ f(y; D_{22}) - I & \text{if } y \in (\bar{y}_{22}, +\infty). \end{cases}.$$

The optimal portfolio and consumption policy after investment are

$$\pi^* = \frac{\eta}{\gamma r \sigma_S} - \frac{\rho\sigma_y}{\sigma_S} f'(y; D_{22}) \quad \text{and} \quad c^* = r\left(w + f(y; D_{22}) + \frac{\eta^2}{2\gamma r^2} + \frac{\beta - r}{\gamma r^2}\right). \tag{A.16}$$

□

[27] We denote the value function by $J_{22;AI}(y)$, where J refers to joint investment, the subscript "22" refers to the income factor D_{22} to the operating net income process the entrepreneurs receive, A indicates that technology 2 has already arrived, and I indicates incomplete markets.

Proof of Proposition 3

We obtain the situation after option exercise from Proposition 1 by setting $D_{N_iN_j} = D_{21}$. Thus the project value is

$$f(y;D_{21}) \quad = \quad \frac{D_{21}}{r}y + \frac{(\alpha_y - \rho\sigma_y\eta)D_{21}}{r^2} - \frac{\gamma\sigma_y^2(1-\rho^2)D_{21}^2}{2r^2}, \tag{A.17}$$

and the optimal portfolio and consumption policy after investment are

$$\pi^* = \frac{\eta}{\gamma r\sigma_S} - \frac{\rho\sigma_y}{\sigma_S}f'(y;D_{21}) \quad \text{and} \quad c^* = r\left(w + f(y;D_{21}) + \frac{\eta^2}{2\gamma r^2} + \frac{\beta - r}{\gamma r^2}\right). \tag{A.18}$$

Before investment, they take the form

$$\pi^* = \frac{\eta}{\gamma r\sigma_S} - \frac{\rho\sigma_y}{\sigma_S}g'(y) \quad \text{and} \quad c^* = r\left(w + g(y) + \frac{\eta^2}{2\gamma r^2} + \frac{\beta - r}{\gamma r^2}\right). \tag{A.19}$$

The derivations to obtain the follower's option value function when technology 2 is available follow the arguments in the proof of Proposition 2 on replacing the income factor D_{22} with D_{21} and where the optimal investment threshold for the follower in technology 2 is denoted by $\bar{y}_{12}$. The boundary conditions are given analogously. □

Proof of Proposition 4

After option exercise, the leader receives the flow payoff $D_{10}Y_t$ up to the moment when the follower invests in technology 2. From that point onwards, the leader's flow payoff is reduced to $D_{12}Y_t$. Before the follower optimally invests in technology 2 at the threshold $\bar{y}_{12}$, the leader has wealth dynamics given by

$$dW_t \quad = \quad (rW_t + \pi(\mu_S - r) + D_{10}Y_t - c_t)\,dt + \pi\sigma_S dB_t. \tag{A.20}$$

Deriving the HJB equation following similar steps as in the earlier proofs, the value function for the leader satisfies the non-linear ODE

$$\frac{\sigma_y^2}{2}g''(y) - \gamma r\frac{\sigma_y^2}{2}(1-\rho^2)g'(y)^2 + (\alpha_y - \rho\sigma_y\eta)g'(y) - rg(y) + D_{10}y = 0, \tag{A.21}$$

subject to the lower boundary condition

$$\lim_{y\to-\infty} g(y) = f(y;D_{10}), \tag{A.22}$$

and the value matching condition $g(\bar{y}_{12}) = f(\bar{y}_{12}; D_{12})$ where $f(\cdot)$ is given by Proposition 1. The reduction in the flow payoff to the leader when the follower invests in technology 2 is captured in the value matching condition. The optimal portfolio and consumption policy of the leader before the follower has invested are

$$\pi^* = \frac{\eta}{\gamma r \sigma_S} - \frac{\rho \sigma_y}{\sigma_S} g'(y) \quad \text{and} \quad c^* = r\left(w + g(y) + \frac{\eta^2}{2\gamma r^2} + \frac{\beta - r}{\gamma r^2}\right). \tag{A.23}$$

After the follower has invested, they become

$$\pi^* = \frac{\eta}{\gamma r \sigma_S} - \frac{\rho \sigma_y}{\sigma_S} f'(y; D_{12}) \quad \text{and} \quad c^* = r\left(w + f(y; D_{12}) + \frac{\eta^2}{2\gamma r^2} + \frac{\beta - r}{\gamma r^2}\right). \tag{A.24}$$

□

Proof of Proposition 5

The proof is similar to that for Proposition 1 on inserting the income factor D_{11}. The optimal portfolio policy of the entrepreneurs is similar to that in Proposition 2. □

Proof of Proposition 6

We denote the value function to the follower before option exercise by $V(w, y)$ and we write it as

$$V(w, y) = -\frac{1}{\gamma r} \exp\left(-\gamma r\left(w + G(y) + \frac{\eta^2}{2\gamma r^2} + \frac{\beta - r}{\gamma r^2}\right)\right), \tag{A.25}$$

where $G(y)$ represents the follower's option value before technology 2 has arrived. The wealth dynamics of the follower before investment is given by

$$dW_t = (rW_t + \pi(\mu_S - r) - c_t)\, dt + \pi \sigma_S dB_t + \lambda\left(F_{21;AI}(y) - G(y)\right) dt. \tag{A.26}$$

Note that in each interval dt, technology 2 may arrive with probability λdt, then making technology 1 inferior. Therefore, in each dt, the entrepreneur may experience an expected capital gain of $F_{21;AI}(y) - G(y)$ with probability λdt. This expected positive gain at some future time τ is reflected by the term $\lambda\left(F_{21;AI}(y) - G(y)\right) dt$.

Including the possibility of the arrival of technology 2 with intensity λ we obtain the HJB equation

$$\begin{aligned}\beta V(w,y) &= \max_{\pi,C} U(c) + V_w(w,y)\left(rw + \pi(\mu_S - r) + \lambda\left(F_{21;AI}(y) - G(y)\right) - c\right)\\ &\quad + V_y(w,y)\alpha_y + \frac{\pi^2\sigma_S^2}{2}V_{ww}(w,y) + \frac{\sigma_y^2}{2}V_{yy}(w,y) + \rho\sigma_y\pi\sigma_S V_{wy}(w,y), \qquad \text{(A.27)}\end{aligned}$$

subject to the transversality condition $\lim_{\tau\to\infty} E[e^{-\beta\tau}V(W_\tau, Y_\tau)] = 0$. Following the same steps as in the previous proofs, we get the non-linear ODE

$$\frac{\sigma_y^2}{2}G''(y) - \gamma r\frac{\sigma_y^2}{2}(1-\rho^2)G'(y)^2 + (\alpha_y - \rho\sigma_y\eta)G'(y) - (r+\lambda)G(y) + \lambda F_{21;AI}(y) = 0, \qquad \text{(A.28)}$$

where $F_{21;AI}(y)$ is derived in Proposition 3.

By inserting the expression for $F_{21;AI}(y)$, we can derive the boundary conditions. The lower boundary condition $\lim_{y\to-\infty} G(y) = 0$ implies that the follower's option value loses its value as the operating net income approaches negative infinity. The upper growth condition follows as the particular solution to the system of ODEs we obtain by inserting the expression for $F_{21;AI}(y)$. Therefore, the upper growth condition becomes $\lim_{y\to+\infty} G(y) = Ay + B$, with

$$\begin{aligned} A &= \frac{\lambda D_{21}}{r(r+\lambda)}, \qquad \text{(A.29)}\\ B &= -\frac{\gamma r\sigma_y^2(1-\rho^2)\lambda^2 D_{21}^2}{2r^2(r+\lambda)^3} + \frac{(\alpha_y - \rho\sigma_y\eta)\lambda D_{21}}{(r+\lambda)^2 r} + \frac{\lambda(\alpha_y - \rho\sigma_y\eta)D_{21}}{(\lambda+r)r^2}\\ &\quad -\frac{\lambda\gamma\sigma_y^2(1-\rho^2)D_{21}^2}{(r+\lambda)2r^2} - \frac{\lambda}{(r+\lambda)}I. \qquad \text{(A.30)}\end{aligned}$$

To get some insight into the upper boundary condition, it is useful to consider it under complete markets. Let $\lim_{y\to\infty} G^c(y)$ denote the expected value under complete markets of receiving the flow payoff $D_{21}Y_t$ in perpetuity, starting from time τ (when technology 2 arrives) and paying the investment costs I. It can be computed as follows:

$$\begin{aligned}\lim_{y\to\infty} G^c(y) &= E_{Y_0=y}\left[e^{-r\tau}\left(\frac{D_{21}}{r}Y_\tau + \frac{(\alpha_y - \sigma_y\eta)D_{21}}{r^2} - I\right)\right]\\ &= \frac{\lambda D_{21}}{r(r+\lambda)}y + \frac{(\alpha_y - \sigma_y\eta)\lambda D_{21}}{(r+\lambda)^2 r} + \frac{\lambda(\alpha_y - \sigma_y\eta)D_{21}}{(\lambda+r)r^2} - \frac{\lambda}{(r+\lambda)}I,\end{aligned}$$

since $E_{Y_0=y}(Y_t) = y + (\alpha_y - \sigma_y\eta)t$ under complete markets and because τ is exponentially distributed with intensity λ. The expression for $\lim_{y\to\infty} G^c(y)$ resembles the upper growth condition in Proposition 6, except for the two remaining terms that incorporate the notion of risk-aversion and incomplete hedging. Thus, the upper growth condition in Proposition 6 represents the expected net present-value of the flow payoff $D_{21}Y_t$ accruing to the follower from investing at time τ and onwards using technology 2 adjusted for risk-aversion and incomplete hedging. The optimal portfolio and consumption policy of the follower before investment are

$$\pi^* = \frac{\eta}{\gamma r \sigma_S} - \frac{\rho\sigma_y}{\sigma_S} G'(y) \quad \text{and} \quad c^* = r\left(w + G(y) + \frac{\eta^2}{2\gamma r^2} + \frac{\beta - r}{\gamma r^2}\right) \tag{A.31}$$

for $y \in (-\infty, \infty)$. □

Proof of Proposition 7

Denote the value function for the leader (before the follower optimally exercises the option to invest in technology 2 at the threshold $\bar{y}_{12}$) by $V(w, y)$, where

$$V(w,y) \quad = \quad -\frac{1}{\gamma r}\exp\left(-\gamma r\left(w + G(y) + \frac{\eta^2}{2\gamma r^2} + \frac{\beta - r}{\gamma r^2}\right)\right) \tag{A.32}$$

and where $G(y)$ represents the value function. The leader's wealth dynamics takes the form

$$dW_t = (rW_t + \pi(\mu_S - r) + D_{10}Y_t - c_t)\,dt + \pi\sigma_S dB_t + \lambda\left(L_{12;AI}(y) - G(y)\right)dt \tag{A.33}$$

Note, that we assume immediate investment in technology 1 by the leader which means that the leader receives a flow payoff equal to $D_{10}Y_t$ accruing to the wealth at any given time. Moreover, the leader may experience (at any given point in time with probability λdt) an expected shift in wealth corresponding to $L_{12;AI}(y) - G(y)$ similar to that of the follower which happens when technology 2 arrives. Deriving the usual HJB equation we obtain that the non-linear ODE defining the leader's value function is

$$\begin{aligned} rG(y) \quad &= \quad D_{10}y + \lambda\left(L_{12;AI}(y) - G(y)\right) + (\alpha_y - \rho\sigma_y\eta)G'(y) \\ &\quad -\gamma r\frac{\sigma_y^2}{2}(1-\rho^2)G'(y)^2 + \frac{\sigma_y^2}{2}G''(y) \end{aligned} \tag{A.34}$$

and the expressions for $L_{12;AI}(y)$ are determined in Proposition 4.

As the operating net income approaches negative infinity, it is not optimal for the follower to invest regardless of the technology available and therefore the value function for the leader converges to the project value $f(y; D_{10})$. By inserting the expressions for $L_{12;AI}(y)$ into the non-linear ODE above we can derive the upper growth condition as the particular solution. The upper growth condition equals $\lim_{y\to\infty} G(y) = Ay + B$ where

$$A = \frac{D_{10}}{(r+\lambda)} + \frac{\lambda D_{12}}{r(r+\lambda)} \tag{A.35}$$

$$\begin{aligned} B = & -\frac{\gamma r \sigma_y^2(1-\rho^2)\left(\frac{D_{10}}{(r+\lambda)} + \frac{\lambda D_{12}}{r(r+\lambda)}\right)^2}{2(r+\lambda)} + \frac{(\alpha_y - \rho\sigma_y\eta)\left(\frac{D_{10}}{(r+\lambda)} + \frac{\lambda D_{12}}{r(r+\lambda)}\right)}{(r+\lambda)} \\ & + \frac{\lambda(\alpha_y - \rho\sigma_y\eta)D_{12}}{(\lambda+r)r^2} - \frac{\lambda\gamma\sigma_y^2(1-\rho^2)D_{12}^2}{(r+\lambda)2r^2} \end{aligned} \tag{A.36}$$

To obtain an intuition about the upper boundary condition, it is again useful to consider it under complete markets. Under complete markets, the growth condition for the leader contains the expected present value of receiving the flow payoff $D_{12}Y_t$ in perpetuity from time τ when the follower invests in technology 2. The leader invest immediately and thus receives a flow payoff $D_{10}Y_t$ but only up to the moment when the follower invests in technology 2. Hence, the growth condition also contains the expected present value of receiving the flow payoff $D_{10}Y_t$ up to time τ. Denoting said quantity by $\lim_{y\to\infty} G^c_{D_{10}}(y)$ it follows that

$$\begin{aligned} \lim_{y\to\infty} G^c_{D_{10}}(y) &= E_{Y_0=y}\left[\int_0^\tau e^{-rt} D_{10} Y_t dt\right] = E\left(E_{Y_0=y}\left(\int_0^k e^{-rt} D_{10} Y_t dt \middle| \tau = k\right)\right) \\ &= \frac{D_{10}}{(r+\lambda)} y + \frac{(\alpha_y - \sigma_y\eta)D_{10}}{(r+\lambda)^2} \end{aligned}$$

since $E_{Y_0=y}(Y_t) = y + (\alpha_y - \sigma_y\eta)t$ under complete markets and because τ is exponentially distributed with intensity λ. To sum up, the upper growth condition in Proposition 8 captures the expected present value of receiving the flow payoff $D_{10}Y_t$ (from immediate investment in technology 1) up to time τ and from that point onwards the flow payoff $D_{12}Y_t$ in perpetuity adjusted for risk-aversion and incomplete hedging.

The optimal portfolio and consumption policy of the leader before any investment by the follower are given by

$$\pi^* = \frac{\eta}{\gamma r \sigma_S} - \frac{\rho\sigma_y}{\sigma_S} G'(y) \quad \text{and} \quad c^* = r\left(w + G(y) + \frac{\eta^2}{2\gamma r^2} + \frac{\beta - r}{\gamma r^2}\right) \tag{A.37}$$

for $y \in (-\infty, \infty)$.

Proof of Proposition 8

The problem is similar to that in Proposition 6 except that the upper boundary condition has changed. The follower invests at the threshold $\bar{y}_{11}$, which has to be determined as part of the problem. Therefore the value matching and smooth pasting conditions become $G(\bar{y}_{11}) = f(\bar{y}_{11}; D_{11}) - I$ and $G'(\bar{y}_{11}) = f'(\bar{y}_{11}; D_{11})$. The optimal portfolio and consumption policy of the follower before investment are

$$\pi^* = \frac{\eta}{\gamma r \sigma_S} - \frac{\rho \sigma_y}{\sigma_S} G'(y) \quad \text{and} \quad c^* = r\left(w + G(y) + \frac{\eta^2}{2\gamma r^2} + \frac{\beta - r}{\gamma r^2}\right) \tag{A.38}$$

for $y \in (-\infty, \bar{y}_{11})$ and after investment in technology 1 they are

$$\pi^* = \frac{\eta}{\gamma r \sigma_S} - \frac{\rho \sigma_y}{\sigma_S} f'(y; D_{11}) \quad \text{and} \quad c^* = r\left(w + f(y; D_{11}) + \frac{\eta^2}{2\gamma r^2} + \frac{\beta - r}{\gamma r^2}\right) \tag{A.39}$$

for $y \in (\bar{y}_{11}, \infty)$.

Proof of Proposition 9

The proof is similar to that in Proposition 7 except that the upper boundary condition changes. The follower invests at the threshold $\bar{y}_{11}$ and that results in the following value matching condition, $G(\bar{y}_{11}) = f(\bar{y}_{11}; D_{11})$. The optimal portfolio and consumption policy of the leader before the follower has invested in technology 1 are

$$\pi^* = \frac{\eta}{\gamma r \sigma_S} - \frac{\rho \sigma_y}{\sigma_S} G'(y) \quad \text{and} \quad c^* = r\left(w + G(y) + \frac{\eta^2}{2\gamma r^2} + \frac{\beta - r}{\gamma r^2}\right) \tag{A.40}$$

for $y \in (-\infty, \bar{y}_{11})$, and after the follower has invested in technology 1, they are

$$\pi^* = \frac{\eta}{\gamma r \sigma_S} - \frac{\rho \sigma_y}{\sigma_S} f'(y; D_{11}) \quad \text{and} \quad c^* = r\left(w + f(y; D_{11}) + \frac{\eta^2}{2\gamma r^2} + \frac{\beta - r}{\gamma r^2}\right) \tag{A.41}$$

for $y \in (\bar{y}_{11}, \infty)$.

Proof of Proposition 10

The proof is similar to that for Proposition 6 after replacing the income factor D_{21} with D_{22} and the value function when technology 2 is available (to be inserted in the non-linear ODE) by $J_{22,AI}(y)$

instead of $F_{21,AI}(y)$. The option value function $G(y)$ therefore satisfies the non-linear ODE

$$rG(y) = \frac{\sigma_y^2}{2}G''(y) - \gamma r\frac{\sigma_y^2}{2}(1-\rho^2)G'(y)^2 + (\alpha_y - \rho\sigma_y\eta)G'(y) + \lambda(J_{22;AI}(y) - G(y)), \qquad \text{(A.42)}$$

for $y \in (-\infty, \infty)$, where $J_{22;AI}(y)$ was determined in Proposition 2. By inserting the expression for $J_{22;AI}(y)$, we can derive the boundary conditions as the particular solutions to the system of non-linear ODEs. The lower boundary condition is similar to that in Proposition 6. The upper growth condition is $\lim_{y\to+\infty} G(y) = Ay + B$ where

$$\begin{aligned} A &= \frac{\lambda D_{22}}{r(r+\lambda)}, \\ B &= -\frac{\gamma r\sigma_y^2(1-\rho^2)\lambda^2 D_{22}^2}{2r^2(r+\lambda)^3} + \frac{(\alpha_y - \rho\sigma_y\eta)\lambda D_{22}}{(r+\lambda)^2 r} + \frac{\lambda(\alpha_y - \rho\sigma_y\eta)D_{22}}{(\lambda+r)r^2} \\ &\quad -\frac{\lambda\gamma\sigma_y^2(1-\rho^2)D_{22}^2}{(r+\lambda)2r^2} - \frac{\lambda}{(r+\lambda)}I. \end{aligned}$$

Similar to the motivation for the boundary condition in Proposition 6, the upper growth condition in Proposition 10 represents the expected net present value of the flow payoff $D_{22}Y_t$ accruing to both entrepreneurs from investing at time τ and onwards using technology 2 adjusted for risk-aversion and incomplete hedging. The optimal portfolio and consumption policy of the entrepreneurs before investment in technology 2 are

$$\pi^* = \frac{\eta}{\gamma r\sigma_S} - \frac{\rho\sigma_y}{\sigma_S}G'(y) \quad \text{and} \quad c^* = r\left(w + G(y) + \frac{\eta^2}{2\gamma r^2} + \frac{\beta - r}{\gamma r^2}\right), \qquad \text{(A.43)}$$

for $y \in (-\infty, +\infty)$.

Proof of Proposition 11

The proof is similar to that for Proposition 5.

B Approximate Solutions

To gain some intuition about the impact of technology adoption, we use an asymptotic approximation method to compute an approximate solution for the implied option values and the investment thresholds. We first obtain an expression for $F_{21;AI}(y)$ of Proposition 3. Then, we look at the situation before the arrival of technology 2 and consider the evaluation of $F_{11;BI}(y)$. Hence, we look at the scenario where the follower considers investing in technology 1 (Proposition 8).

Technology 2 is available

We consider a perturbation of $F_{21;AI}(y)$ in σ_y. In particular, we are interested in the value of the function for $y \in (-\infty, \bar{y}_{12})$. For this region, we have $F_{21;AI}(y) = g(y)$, where $g(y)$ satisfies the ODE in Proposition 3. For $g(y)$, we assume the following fourth-order approximation in σ:

$$\tilde{g}(y) = g^{(0)}(y) + \sigma_y g^{(1)}(y) + \frac{1}{2}\sigma_y^2 g^{(2)}(y) + \frac{1}{6}\sigma_y^3 g^{(3)}(y) + \frac{1}{24}\sigma_y^4 g^{(4)}(y). \tag{B.44}$$

The function $g^{(0)}(y)$ satisfies

$$rg^{(0)}(y) = \alpha_y g^{0\prime}(y).$$

Since $g^{(0)\prime}(\bar{y}_{12}^{(0)}) = D_{12}/r$, we get

$$g^{(0)}(y) = e^{\frac{r}{\alpha_y}(y-\bar{y}_{12}^{0})} D_{21}\frac{\alpha}{r^2}, \tag{B.45}$$

and from

$$g^{(0)}(\bar{y}_{12}^{(0)}) = \bar{y}_{12}^{(0)}\frac{D_{21}}{r} + \frac{\alpha_y D_{21}}{r^2},$$

we get

$$\bar{y}_{12}^{(0)} = \frac{r}{D_{21}} I. \tag{B.46}$$

Next, we consider the ODE for $g^{(1)}(y)$:

$$0 = -rg^{(0)}(y)\frac{\eta\rho}{\alpha} - rg^{(1)}(y) + \alpha_y g^{(1)\prime}(y). \tag{B.47}$$

Assuming that we can write

$$\bar{y}_{12}^{(1)} = \bar{y}_{12}^{(0)} + \delta\sigma_y,$$

we must have $g^{(1)\prime}(\bar{y}_{12}^{(1)}) = -D_{21}\delta/\alpha_y$, which gives

$$g^{(1)}(y) = e^{\frac{r}{\alpha_y}(y-\bar{y}_{12}^{(1)}} \frac{D_{21}}{r^2\alpha_y}\left(r\alpha_y\delta + ((y-\bar{y}_{12}^{(1)})r - \alpha_y)\eta\rho\right), \tag{B.48}$$

Furthermore, we require

$$g^{(0)}(y_{12}^{(1)}) + \sigma_y g^{1}(y_{12}^{(1)}) = f(y; D_{21}) - I, \tag{B.49}$$

where for $f(y; D_{21})$ we have to consider only terms up to first order in σ_y. This equation can only be fulfilled if we have $\delta = 0$. Hence,

$$y_{12}^{(0)} = y_{12}^{(1)}, \tag{B.50}$$

which means that up to first order in σ_y, market incompleteness does not alter the investment threshold. Furthermore, we have

$$g^{(1)}(y) = e^{\frac{r}{\alpha_y}(y-\bar{y}_{12}^{(1)})} \frac{D_{21}}{r^2 \alpha_y} \Big((y - \bar{y}_{12}^{(1)})r - \alpha)\eta\rho \Big). \tag{B.51}$$

We can now move on to the second-order approximation. Plugging in the solutions of $g^{(0)}(y)$ and $g^{(1)}(y)$ to the ODE in Proposition 3 for $\tilde{g}(y)$, we obtain an ODE for $g^{(2)}(y)$. Defining $\bar{y}_{12}^{(2)} = \bar{y}_{12}^{(0)} + \Delta\sigma^2$, one boundary condition to be fulfilled is $\tilde{g}'(\bar{y}_{12}^{(2)}) = 0$. In addition, we must have $\tilde{g}'(\bar{y}_{12}^{(2)}) = f(\tilde{g}'(\bar{y}_{12}^{(2)}); D_{21})$. Solving for Δ, we get $\Delta = 1/(2\alpha_y)$ and

$$\bar{y}_{12}^{(2)} = \bar{y}_{12}^{(0)} + \frac{\sigma_y^2}{2\alpha_y}. \tag{B.52}$$

Furthermore,

$$g^{(2)}(y) = e^{\frac{r}{\alpha_y}(y-\bar{y}_{12}^{(0)})} D_{21} \left[\frac{y - \bar{y}_{12}^{(0)}}{\alpha_y^3} \Big((y - \bar{y}_{12}^{(0)})\eta^2\rho^2 - \alpha_y \Big) + D_{21} \frac{\gamma(1-\rho^2)}{r^2} \Big(e^{\frac{r}{\alpha_y}(y-\bar{y}_{12}^{(0)})} - 2 \Big) \right] \tag{B.53}$$

Hence, when we move from complete to incomplete markets, the investment threshold will be increased. Therefore, the presence of undiversifiable risk will eventually lead to a delay of a project investment. This result is the same as in Miao and Wang (2007) for their self-insurance model with flow payoffs (their Model III). Therefore, in the first-order approximation, the investment threshold $\bar{y}_{12}^{(2)}$ increases in volatility σ_y. The agent's risk attitude does not affect the timing of the investment. This prediction is thus, to first order, qualitatively the same as in the standard real options models.

Repeating the same arguments for deriving the critical investment thresholds, we get a third-, fourth-, and fifth-order expansion for $\bar{y}_{12}$ as follows:

$$\begin{aligned}
\bar{y}_{12}^{(3)} &= \bar{y}_{12}^{(2)} + \frac{\eta\rho}{2\alpha_y^2}\sigma_y^3 && \text{(B.54)} \\
\bar{y}_{12}^{(4)} &= \bar{y}_{12}^{(3)} + \frac{2\eta^2\rho^2 - r + 2D_{11}\alpha_y\gamma(1-\rho^2)}{4\alpha_y^3}\sigma_y^4. && \text{(B.55)} \\
\bar{y}_{12}^{(5)} &= \bar{y}_{12}^{(4)} + \eta\rho\frac{2\eta^2\rho^2 - 3r + 4D_{11}\alpha_y\gamma(1-\rho^2)}{4\alpha_y^4}\sigma_y^5. && \text{(B.56)}
\end{aligned}$$

Technology 2 is not yet available

We start again by using a perturbation of the solution for the ODE in Equation (33) of Proposition 8:

$$\tilde{G}(y) = G^{(0)}(y) + \sigma_y G^{(1)}(y) + \frac{1}{2}\sigma_y^2 G^{(2)}(y) + \frac{1}{6}\sigma_y^3 G^{(3)}(y) + \frac{1}{24}\sigma_y^4 G^{(4)}(y). \tag{B.57}$$

As in Proposition 8, we need to find $\tilde{G}(y)$ that satisfies

$$r\tilde{G}(y) = \frac{\sigma_y^2}{2}\tilde{G}''(y) - \gamma r \frac{\sigma_y^2}{2}(1-\rho^2)\tilde{G}'(y)^2 + (\alpha_y - \rho\sigma_y\eta)\tilde{G}'(y) + \lambda(\tilde{F}_{21;AI}(y) - \tilde{G}(y)), \tag{B.58}$$

for $y \in (-\infty, \bar{y}_{11})$ subject to $\lim_{y\to-\infty} G(y) = 0$ and the value matching and smooth pasting conditions

$$\begin{aligned} \tilde{G}(\bar{y}_{11}^2) &= f(\bar{y}_{11}; D_{11}) - I, & \text{(B.59)} \\ \tilde{G}'(\bar{y}_{11}^2) &= f'(\bar{y}_{11}; D_{11}), & \text{(B.60)} \end{aligned}$$

where $\bar{y}_{11}^{(2)}$ is the critical threshold level approximated to second-order terms in σ^2. Furthermore, since we look at the situation $\bar{y}_{12}^{(2)} < \bar{y}_{11}^{(2)}$, the function $\tilde{F}_{21;AI}(y)$ is equal to $f(y, D_{21})$. We first solve the ODE in (B.58) for $g^{(0)}(y)$, which gives us

$$G^{(0)}(y) = \frac{\alpha_y r\left(D_{11}(\lambda+r) - D_{21}\lambda\right) e^{\frac{\left(y-\bar{y}_{11}^0\right)(\lambda+r)}{\alpha_y}} + \lambda\left(D_{21}\left(\alpha_y\lambda + r^2 y + 2\alpha_y r + \lambda r y\right) - I r^2(\lambda+r)\right)}{r^2(\lambda+r)^2}. \tag{B.61}$$

Compared to $g^{(0)}(y)$ when technology 2 is available, Equation (B.61) explicitly depends on the parameter λ and on the differences between D_{21} and D_{11}. Indeed, we can show that $\lim_{\lambda\to 0, D_{11}\to D_{21}} G^{(0)}(y) = g^{(0)}(y)$. For the critical value $\bar{y}_{11}^{(0)}$, we get

$$\bar{y}_{11}^{(0)} = \frac{(D_{21} - D_{11})\,\alpha_y\lambda + I r^3}{r\left(D_{11}(\lambda+r) - D_{21}\lambda\right)}. \tag{B.62}$$

Note that we have $D_{11}(\lambda+r) - D_{21}\lambda > 0$. To clarify the effect of λ on $\bar{y}_{11}^{(0)}$, we expand in λ:

$$\bar{y}_{11}^{(0)} = \frac{r}{D_{11}}I + \left(\frac{r}{D_{11}}I + \frac{\alpha}{r}\right)\sum_{i=1}^{\infty}\lambda^i\left(\frac{D_{21}-D_{11}}{D_{11}r}\right)^i. \tag{B.63}$$

Hence, since $D_{21} - D_{11} > 0$, the introduction of technology adoption leads to a delay in investments up to order zero in σ_y.

To obtain the first-order expansion in σ_y, we can apply the same procedure as before. In particular, we need to solve the ODE in Equation (B.58) for $G^{(0)}(y)+\sigma_y G^{(1)}(y)$, where $G^{(0)}(y)$ is given in (B.61). Then, using the boundary conditions, we can solve for the critical value $\bar{y}_{11}^{(1)}$. Writing

$$\bar{y}_{11}^{(1)} = \bar{y}_{11}^{(0)} + \sigma_y \Delta,$$

we obtain

$$\Delta \quad = \quad -\frac{(D_{21} - D_{11})\,\eta\lambda\rho}{r\,(D_{11}(\lambda + r) - D_{21}\lambda)}. \tag{B.64}$$

Furthermore, an explicit form for $G^{(1)}(y)$ can be derived:

$$G^{(1)}(y) \quad = \quad -\frac{D_{21}\eta\lambda\rho(\lambda+2r) + \frac{r(D_{11}(\lambda+r)-D_{21}\lambda)e^{\frac{(y-1)(\lambda+r)}{\alpha_y}}(\alpha\Delta(\lambda+r)+\eta\rho(\alpha_y-(y-1)(\lambda+r)))}{\alpha_y}}{r^2(\lambda+r)^2}.$$

Performing a series expansion for $\bar{y}_{11}^{(1)}$ gives us

$$\bar{y}_{11}^{(1)} \quad = \quad \bar{y}_{11}^{(0)} - \frac{\rho\sigma_y\eta}{D_{11}r}\sum_{i=1}^{\infty}\lambda^i\left(\frac{D_{21}-D_{11}}{D_{11}r}\right)^i \tag{B.65}$$

$$\quad = \quad \frac{r}{D_{11}}I + \left(\frac{r}{D_{11}}I + \frac{\alpha_y}{r} - \frac{\rho\sigma_y\eta}{D_{11}r}\right)\sum_{i=1}^{\infty}\lambda^i\left(\frac{D_{21}-D_{11}}{D_{11}r}\right)^i \tag{B.66}$$

Hence, in the first-order expansion of the investment threshold, we observe that the correlation ρ critically affects the level of the investment threshold. When this correlation is positive, the threshold is reduced. This observation is clearly different from the case without technological change, where the investment threshold for the first-order expansion is equal to that in the zero-order expansion (see Equation (B.50)). If we set $\rho = 0$, which corresponds to the self-insurance case with no hedging, then we would get $\bar{y}_{11}^{(1)} = \bar{y}_{11}^{(0)}$.

To understand the impact of risk-aversion on the investment threshold, we need to obtain the second-order approximation. We do so by plugging Equation (B.57) into the ODE in Proposition (8).[28] Writing

$$\bar{y}_{11}^{(2)} = \bar{y}_{11}^{(0)} + \sigma_y\Delta + \sigma_y^2\Gamma,$$

we obtain, after solving the ODE under the appropriate boundary conditions,

$$\Gamma = \frac{\alpha_y\gamma D_{11}^2\lambda\left(1-\rho^2\right) + D_{11}r(\lambda+r) - D_{21}\lambda\left(r + \alpha_y\gamma D_{21}\left(1-\rho^2\right)\right)}{2\alpha_y r(D_{11}(\lambda+r) - D_{21}\lambda)}. \tag{B.67}$$

[28]We do not present the explicit form for $G^{(2)}(y)$, but detailed calculations can be obtained from the authors.

A series expansion in λ yields:

$$\bar{y}_{11}^{(2)} = \bar{y}_{11}^{(1)} + \frac{\sigma_y^2}{2\alpha_y} - \sigma_y^2\gamma(1-\rho^2)\frac{D_{11}+D_{21}}{2r}\sum_{i=1}^{\infty}\lambda^i\left(\frac{D_{21}-D_{11}}{D_{11}r}\right)^i. \tag{B.68}$$

When $\lambda \to 0$, we see that the above expression converges to the one in Equation B.52, which is again consistent with the result in Miao and Wang (2007) for an economy without technological innovation. However, if technological innovation is present, we find that the risk-aversion coefficient γ becomes relevant for the second-order approximation of the critical threshold $\bar{y}_{11}$. Indeed, the sign of the last term is negative. Hence, the presence of risk-aversion decreases the investment threshold.

For the third-order expansion, we get an additional term, which does not depend on the risk-aversion parameter γ,

$$\bar{y}_{11}^{(3)} = \bar{y}_{11}^{(2)} + \frac{\eta\rho}{2\alpha_y^2}\sigma_y^3. \tag{B.69}$$

The risk aversion parameter, however, does appear again in the fourth- and higher-order expansions. In particular, for the fourth-order expansion in σ_y, we get

$$\bar{y}_{11}^{(4)} = \bar{y}_{11}^{(3)} + \frac{2\alpha\gamma D_{11}\left(1-\rho^2\right) + 2\eta^2\rho^2 - (r+\lambda)}{4\alpha^3}\sigma_y^4. \tag{B.70}$$

Obviously, the fourth-order term in the threshold $\bar{y}_{11}^{(4)}$ grows with γ. Hence, this term has an increasing effect on $\bar{y}_{11}^{(4)}$. For $\lambda = 0$, the final term in Equation (B.68) vanishes. Consequently, we have the result as in Miao and Wang (2007), i.e., market incompleteness leads to a delayed investment compared to the complete market case. However, if λ increases, the second-order term in (B.68) again decreases the threshold when $\gamma \neq 0$. Therefore, when reaching a critical level of λ, we might indeed observe that the investment decision is accelerated under market incompleteness. When λ is sufficiently large, the precautionary savings effect can dominate the option effect and encourage the agent to exercise the option sooner, unlike the standard real options result and the results in Miao and Wang (2007). Furthermore, the impact not only depends on the level of risk-aversion, but also on the size (and sign) of the correlation and on the frequency of technological change λ. High frequencies will eventually lead to earlier investments:

$$\bar{y}_{11}^{(4)} - \bar{y}_{11,C}^{(4)} = \gamma\sigma_y^2(1-\rho^2)\left(\frac{\sigma_y^2 D_{11}}{2\alpha^2} - \sigma_y^2\frac{D_{11}+D_{21}}{2r}\sum_{i=1}^{\infty}\lambda^i\left(\frac{D_{21}-D_{11}}{D_{11}r}\right)^i.\right)$$

We can also derive higher-order approximations. The fifth- and sixth-order approximations are

$$\bar{y}_{11}^{(5)} = \bar{y}_{11}^{(4)} + \sigma_y^5 \frac{\eta\rho\left(4\alpha_y\gamma D_{11}\left(1-\rho^2\right)+2\eta^2\rho^2-3(\lambda+r)\right)}{4\alpha_y^4} \tag{B.71}$$

and

$$\begin{aligned}\bar{y}_{11}^{(6)} &= \bar{y}_{11}^{(5)} + \sigma_y^6 \frac{2\eta^4\rho^4-6\eta^2\rho^2(\lambda+r)+(\lambda+r)^2}{4\alpha_y^5} \\ &\quad -\gamma\sigma_y^6\frac{\left(1-\rho^2\right)\left(D_{11}(3\lambda-6\eta^2\rho^2)+D_{11}(\lambda+r)-D_{21}\lambda\right)}{4\alpha_y^4}+\gamma^2\sigma_y^6\frac{D_{11}^2\left(1-\rho^2\right)^2}{2\alpha_y^3},\end{aligned} \tag{B.72}$$

respectively. To clarify the effect of the joint presence of market incompleteness and technological innovation, we can calculate the derivatives of $\bar{y}_{11}^{(6)}$ with respect to γ and λ:

$$\frac{\partial^2\bar{y}_{11}^{(6)}}{\partial\gamma\partial\lambda} = -\frac{1}{4}\left(1-\rho^2\right)\sigma_y^2\left(\frac{\sigma_y^4(4D_{11}-D_{21})}{\alpha^4}+\frac{2D_{11}(D_{21}-D_{11})(D_{11}+D_{21})}{(D_{11}(\lambda+r)-D_{21}\lambda)^2}\right). \tag{B.73}$$

To get more intuition about this cross-derivative, we can rewrite it as

$$\begin{aligned}\frac{\partial^2\bar{y}_{11}^{(6)}}{\partial\gamma\partial\lambda} &= -\frac{1}{4}\left(1-\rho^2\right)\sigma_y^2\left(\frac{\sigma_y^4(4D_{11}-D_{21})}{\alpha^4}+\frac{2D_{11}(D_{21}-D_{11})(D_{11}+D_{21})}{D_{11}r^2}\right) \\ &\quad -\frac{1}{2}\left(1-\rho^2\right)\sigma_y^2(D_{11}+D_{21})\sum_{i=1}^{\infty}\lambda^i\frac{(D_{21}-D_{11})^{i+1}(i+1)}{D_{11}^{i+1}r^{i+2}}.\end{aligned} \tag{B.74}$$

Hence, only the zero-order term in λ can become positive. However, since we require for the preemption equilibrium $D_{11}(\lambda+r)-D_{21}\lambda>0$, this would only happen for unrealistically high interest rates.