On the Impulse Control Problem with Outside Jumps*

Makoto Goto†

January 29, 2013

In this paper, we study the impulse control problem with oudside jumps. As represented by (s, S) policies, impulse control problems usually have inside jumps. Namely, when the inventory level goes down and hits a threshold, it jumps up by the order placement. However, in terms of capacity choice problems, firms should install additional capacities when the demand is increasing. That is, the impulse control problem we consider has outside jumps, which is hard to solve through usual approaches as reported in Goto, Takashima and Tsujimura (2006). Our approach is inspired by Guo and Tomecek (2008) who study connections between singular control and optimal switching problems.

The impulse control problem to be solved is given by

$$V(x,y) = \sup_{w \in \mathcal{W}} \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-\rho t}\Pi(X_t, Y_t)\mathrm{d}t - \sum_{i=1}^\infty \mathrm{e}^{-\rho\tau_i}K(\xi_i)\right], \tag{1}$$

where X_t is a diffusion process with $X_0 = x$ (e.g. price, demand), $Y_t = y + \sum_{\tau_i \le t} \xi_i$ is a controlled object (e.g. capacity), $w = (\tau_1, \tau_2, \dots; \xi_1, \xi_2, \dots)$ is an impulse control, $\mathcal{W}$ is a collection of admissible controls, $\Pi(x,y)$ is a profit flow function, $K(\xi)$ is a cost function and $\rho > 0$ is a discount rate. Now we consider the following particular example:

$$\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}W_t, \tag{2}$$

$$\xi_i > 0, \qquad \forall i, \tag{3}$$

$$\Pi(x,y) = y^\gamma x, \qquad \gamma > 2, \tag{4}$$

$$K(\xi) = k_0 + k_1\xi, \qquad k_0, k_1 > 0. \tag{5}$$

Similar to Guo and Tomecek (2008), the impulse control problem (1) can be

---

*The author thanks Xin Guo for the helpful comments and suggestions.

†Graduate School of Economics and Business Administration, Hokkaido University. Department of Industrial Engineering and Operations Research, University of California at Berkeley. E-mail: `goto@econ.hokudai.ac.jp`

transformed to

$$V(x,y) = \int_0^y v_1(x,z)\mathrm{d}z + \int_y^\infty v_0(x,z)\mathrm{d}z, \tag{6}$$

$$v_1(x,y) = \frac{\gamma y^{\gamma-1}x}{\rho-\mu}, \tag{7}$$

$$v_0(x,y) = \begin{cases} Bx^\beta, & x < x^*, \\ \dfrac{\gamma(y^*)^{\gamma-1}x}{\rho-\mu} - k_0 - k_1(y^*-y), & x \geq x^*, \end{cases} \tag{8}$$

$$\beta = \frac{-(2\mu-\sigma^2)+\sqrt{(2\mu-\sigma^2)^2-8\sigma^2\rho}}{2\sigma^2} > 1, \tag{9}$$

where B is an unknown coefficient, x^* is the threshold to expand the capacity and y^* is the expanded capacity.

Thanks to the corresponding boundary conditions:

$$v_0(x^*,y) = v_1(x^*,y^*) - k_0 - k_1(y^*-y), \tag{10}$$

$$\frac{\partial}{\partial x^*}v_0(x^*,y) = \frac{\partial}{\partial x^*}v_1(x^*,y^*), \tag{11}$$

$$\frac{\partial}{\partial y^*}v_1(x^*,y^*) = k_1, \tag{12}$$

we find

$$x^* = \frac{\rho-\mu}{\gamma(\gamma-1)}\left(\frac{\beta(2-\gamma)-1}{\beta(\gamma-1)}\frac{k_1}{k_0-k_1y}\right)^{\gamma-2}k_1, \tag{13}$$

$$y^* = \frac{\beta(\gamma-1)}{\beta(2-\gamma)-1}\frac{k_0-k_1y}{k_1}. \tag{14}$$

Note that both x^* and y^* depend on the initial capacity level y, so we rewrite $F(y) := x^*$ and $G(y) := y^*$. $F(y)$ and $G(y)$ can provide the threshold to expand the capacity and the expanded capacity, respectively, for all capacity levels.

As the remainder, we have to verify the transform from Eq. (1) to Eqs. (6)–(8), and relax Eq. (3) to allow negative controls.