Optimal timing of relocation

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Abstract

In this paper we tackle the problem of the optimal relocation policy for a firm that faces two types of uncertainty: one about the moments in which new (and more efficient) sites will become available; and the other regarding the degree of efficiency improvement inherent to each one of these new, yet to be known, potential location places.
In particular, we derive results concerning the expected optimal timing for relocation, the corresponding volatility and the value of the firm under the optimal relocation policy. Impacts on the final results driven by the characteristics of the firm´s original location site, the market environment and the way in which risk is modeled, are studied numerically. The overall results are in line with economic intuition.

Keywords: Globalization, relocation, real options, decision problem, double Poisson process, optimal timing, truncated-exponential distribution, gamma distribution.

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1 Introduction

According to the World Commission on the Social Dimension of Globalization (2004), between 1970 and 1991 the net inflows of foreign direct investment, in percentage of GDP, approximately doubled worldwide. In contrast, between 1991 and 2000, the speed of this process increased dramatically and the corresponding growth was nearly six fold.

Nowadays, globalization tends to be understood as one of the most important questions faced by developed economies. Business attitudes are increasingly dictated by international competitiveness, and perceived differences in terms of location attractiveness have lead to levels of relocation previously unknown.

In effect, one of the main consequences of globalization has been a significant wave of production relocation from technology-rich countries towards low labor cost countries. Historically, the costs associated to business communications were relatively high and the corresponding quality was fairly poor, when compared with today. Consequently, the ability to become fully aware of the competitors' advantages generated by a better choice in terms of location, or by a best set of production practices, and to quickly implement or even to surpass those actions was, on average, not so developed as it is today. The competitive pressure towards the increase in efficiency was naturally less significant and the relatively higher margins earned in a good number of industries provided cushions that could be used to postpone difficult decisions, like those that involve the relocation of a company activities, usually implying job cuts. In addition, lack of information tended to increase perceived uncertainty and this fact was likely to generate further precautions in relation to relocation movements.

Recent increases in the quality of business communications and information in general have been accompanied by a significant decrease in the corresponding costs. Cheap and readily available information imposes the need to make swift decisions and simultaneously reduce uncertainty, creating an environment prone to relocation.

However, in spite of this recent trend, historically, the volume of relocations has apparently followed a slow and gradual course. Regardless of the noticeable appeal, in terms of costs of production, presented by locations in South America and Asia, when compared to the corresponding alternatives in Western Europe and in the USA, for years the pace of relocation was relatively sluggish. Even today, albeit the above mentioned speed increase, the general notion is that a significant number of production sites kept in Europe or in the USA offer disadvantages, when compared to existing alternatives in Eastern Europe or Asia. In fact, the decision process that leads to relocation seems to present significant delays. But, the questions that remain to be answered are: what leads to these delays? What is the dynamic underlying the process of relocation?

Given the natural rational behavior associated to business decision makers, the answers need to be related to the uncertainties facing companies that take this type of decisions. The relocation of a factory or of a big service sector office tends to be carefully analyzed. First of all, because it normally implies high levels of new investment. Additionally, it also tends to create noteworthy costs to layoff current workers and to generate major tribulations (and costs) in terms of public image. However, once the investment is made, if the decision proves to be unwise and the project fails, almost all these costs will be sunk. Consequently, this type of decisions tends to have irreversible characteristics and to be analyzed with prudence. The possibility of new developments capable of making a new location economically unattractive will naturally tend to be seriously considered. The relative appeal of each location is a function of the market environment, the comparative ability and cost of the workforce to handle the production technology in question, the level of political incentives to investment and also of the logistic constraints inherent to the products and markets involved. Or, to put it differently, the decision to relocate is influenced by perceived uncertainties in terms of the potential speed with which new more appealing locations will be made available and by the rate of increase in the corresponding economic efficiency. Both phenomena have stochastic characteristics, and should be modeled properly using an economic model of relocation.

The academic interest in the location of economic facilities has existed since long time ago. However, the focus on relocation has not been much explored, as Sleuwaegen and Pennings (2002) point out. A significant segment of the literature related to location comes from the field of international economics, and has focused mainly on the comparative advantages that might lead companies to prefer a certain location to others available. Special emphasis has been placed in the analysis of effects originated by different labour costs and political barriers, as in Motta and Thisse (1994); Cordella and Grilo (1998); Collie and Vandenbussche (1999). The potential impact of international ventures is also studied in Buckley and Casson (1998); Reuer and Leiblein (2000); Miller and Reuer (1998a,b), but again the focus is not on the rationale for the decision to move from one location to the other, but on the corresponding consequences in terms of risk and return. Conversely, a different approach to the topic has been used in finance. Capital budgeting studies have considered mostly frameworks in which production might shift from place to place inside a business organization, as a function of the evolution of the real exchange rates between different economies (see, e.g., De-Meza and van der Ploeg (1987); Capel (1992); Kogut and Kulatilaka (1994a,b); Botteron et al. (2003)). Underlying these studies is the notion that flexibility regarding the place where to locate production creates value. This flexibility is conceptually regarded as an option, and valued using simple real options frameworks.

However, to our knowledge, in contrast to this paper, none of these studies focuses specifically on the perceived increase in efficiency needed to justify the decision to relocate from one place to another. The rationale underlying the relocation decision is modeled here using a real options framework, in which a perfectly competitive firm adopts an optimal behavior in order to identify a single change in location. In accordance to McDonald and Siegel (1986) and Dixit and Pindyck (1994), the decision maker will need to consider not only the advantages associated to the increased efficiency provided by newly available locations, but also the costs associated to the loss of the option to relocate. In effect, all potential locations for a business venture are already known. However, given political, institutional, geographic (accessibility) and economic constraints, frequently it is not economically rationale for some industries to establish facilities in certain locations. Consequently, we will use the concept of access to a location to mean the economic viability of its rational use. Locations will be identified and distinguished in accordance to their efficiency. Increased levels of efficiency will materialize in the ability to generate more output with the same level of input (i.e., due to increased technical competence by the workforce) or to use lower levels of input to generate the same output (i.e., due to lower wages). In every case, the overall result will present smaller average cost of production. The investment cost will be assumed constant and, as mentioned above, a single change in location will be considered. The modeling approach is akin to that used by Farzin et al. (1998), Huisman (2000) and Huisman and Kort (2003) to tackle to adoption of new technologies. The arrival of new information regarding the availability of new locations with improved levels of efficiency is modeled as a Poisson arrival process. The use of the truncated exponential and the gamma distribution functions to model the Poisson jumps is particularly suitable, given the situation under study. To our knowledge, this is the first time that this type of setting is used to tackle a real options problem.

Along the paper we assume a risk-neutral firm, with a constant discount factor, r, like in Dixit and Pindyck (1994), and we let $\pi(.)$ denote the cash flow of the firm. We analyze a dynamic model with an infinite planning horizon, and we assume that when the firm chooses a new location, it incurs a sunk cost investment, that we denote by I, assumed to remain constant.

The remainder of the paper is organized as follows. Section 2 presents the model of a firm tackling a relocation decision and facing a stochastic environment, where information about new potentially superior locations is modelled according to a stochastic process. Using the classical framework of Dixit and Pindyck (1994), we present, in Section 3, the decision problem associated with the relocalization decision. Taking into consideration uncertainties related to (1) the speed at which new, more efficient locations, will become known and (2) the rate of increase in the corresponding efficiency, in Section 4 we derive the value of the firm, the optimal switching level and some characteristics of the time until relocalization for two particular cases of the probability law that might rule the jumps in the efficiency process, namely the truncated-exponential and the gamma distributions. Section 5 provides the corresponding numerical results and the parallel economic rationale, whereas in Section 6 we compare our analysis with the traditional one. Concluding remarks are provided in Section 7.

A word about notation used in this paper and notably concerning random variables. If X is a random variable, we denote its distribution function by $F_X(.)$ and its density function by $f_X(.)$. Moreover, $E[X] = \int u dF_X(u)$ denotes its expected value, whereas $E[X^2] = \int u^2 dF_U(u)$ is its second order moment and $Var[X]$ its variance, with $Var[X] = \mathbb{E}[X^2] - \mathbb{E}^2[X]$. The indicator function, $\mathbf{1}_a$ is defined as follows: if proposition a is true, then the value is one; otherwise is zero. We use the symbol $\square$ to denote end of a proof of a lemma or a theorem. Finally w.p.1. means *with probability one* and i.i.d. means *independent and identically distributed*.

2 Stochastic Framework

In this section we develop the model of a risk-neutral firm tackling a relocation decision and facing a stochastic environment, where information about new, potentially superior, locations is modelled according to a stochastic process. The approach to capital budgeting decision making is assumed optimal. In the description of the relevant stochastic framework we follow closely the model proposed by Farzin et al. (1998), Huisman (2000), Huisman and Kort (2003) (to tackle to adoption of new technologies), and by Couto (2006) (to tackle to relocation problem).

Nowadays, all potential locations are geographically known. However, this does not mean that every location is available in economic terms. Some potential locations are not effectively accessible due to political and environmental reasons, others because the technology available cannot be rationally used by the workforce on hand, or because they are difficult to access. The economic environment changes continuously at these and other levels that are relevant to the relative economic appeal of the different location sites. With the passing of time, new sites become known and accessible. In order to be competitive they need to be increasingly appealing and gradually more efficient.

In economic terms, neither the increase in efficiency associated to each new location nor the time that will elapse between the moments in which two sequent efficiency optimizing locations become available are known in advance. Thus we are facing a situation with two levels of uncertainty, the first level corresponding to the time at which a new location becomes available, and the second level to the subsequent increase in the efficiency.

Given the uncertainty inherent to the diverse factors capable of affecting the potential economic attractiveness of different locations, and the impending unstable relationships between those factors, it is reasonable to admit that we are not dealing with a learning process with embedded continuous leaps in terms of information arrival. Therefore, it makes sense the use of an exponential distribution (characterized by the referred propriety of absence of memory) to model the time that elapse between the moments in which two consecutive efficiency optimizing locations become available. We note that these assumptions are usually assumed not only in the finance literature (v.g., Merton (1976); Beckers (1981); Ball and Torous (1983); Carr et al. (2002)), but also in some others that involve mathematical applications, as in Benedek and Villars (2000).

We note that if, in addition to the assumption of exponential times, we assume that new locations become available independently of the firm, then the counting process associated with these new locations is a Poisson process, Ross (1996, 2005). Therefore we model the first level of uncertainty using the following setting: we assume that more efficient locations become available according to a Poisson process $N = \{N(t), t \in \Re^+\}$, of rate λ, where $N(t)$ denotes the number of new locations that become available in the time interval $(0, t)$, and we assume that N evolves independently of the firm.

Furthermore, in order to model the uncertainty about the increase of efficiency, we let $\psi(t)$ and $\theta(t)$ denote the efficiency of the firm and the efficiency of the best location available at time t, respectively, with $0 \leq \psi(t) \leq \theta(t)$. Moreover, we assume that $\theta(t)$ is a random variable, and we let $\Theta = \{\theta(t), t \in \Re^+\}$ denote the corresponding stochastic process, such that for $s \leq t$, $\theta(s) \leq \theta(t)$

w.p.1 (i.e., we consider only locations that improve, in some degree, the efficiency). Without loss of generality, we assume that the firm starts its production at time $t = 0$, with $\psi_0 = \psi(0)$ and $\theta_0 = \theta(0)$ denoting its initial efficiency and the efficiency of the best location at time 0, respectively. For simplicity, we drop time-subscripts when no confusion arises.

Note that $\theta(t) > \theta(t^-)$ if and only if at time t an event of N occurs (i.e., if and only if at time t a more efficient location becomes available). Therefore Θ is a jump process, Ross (1996), with strictly increasing jumps.

We let $S = \{S_i, i \in I\!N\}$ denote the sequence of times at which events on N take place. It follows from the definition of Poisson process that S_i is a random variable with Gamma distribution, with parameters i and λ (see Ross (1996)).

Furthermore, we let:

$$U_i = \theta(S_i) - \theta(S_i^-), \quad i \in I\!N$$

and thus U_i is the increase in the efficiency at the ith jump of the process Θ. We assume that $\{U_i, i \in I\!N\}$ is a sequence of positive i.i.d. random variables, identically distributed to the random variable U, and we denote its distribution function by $F_U(.)$. With the previous assumptions, it follows that for any $t \in \Re^+$, $\theta(t)$ can be written as follows:

$$\theta(t) = \theta_0 + \sum_{i=1}^{N(t)} U_i \tag{1}$$

and thus Θ is a double Poisson process, Ross (1996).

Next we introduce the sequential problem associated with the decision of relocation.

3 The decision problem

In this section we describe in more detail the decision problem. We follow closely notation and results presented in Farzin et al. (1998); Huisman (2000); Huisman and Kort (2003); Couto (2006), and references therein. For this reason we skip most technical results and proofs.

It is clear from the description of the problem that the decision to relocate can be stated as a capital budgeting decision problem. Each time a new (and more efficient) location becomes available, the firm has to decide if it stays in the same place (avoiding an investment cost, that we denote by I, but loosing the opportunity to produce more efficiently) or if it changes to the new location. Therefore, at each time S_i the firm has to decide between *continuing* in the present location or *stop*, and move to a new location. This decision strongly depends on the relationship between the current efficiency of the firm and the efficiency that it will achieve in the new location and the investment costs. In order to justify a change in location, the corresponding efficiency gains need to overcompensate the resultant relocation costs.

Thus the problem can be restated as follows: there exists a critical value $\theta^\star$ such that, for every $t \in \Re^+$, if $\theta(t) > \theta^\star$, the firm decides to invest (*stopping action*) in this new location, whereas if

$\theta(t) < \theta^\star$, the optimal decision is to *continue* in its current site, and wait for other locations to become available. Following Dixit and Pindyck (1994), we call the value $\theta^\star$ the *optimal switching level*.

Moreover, let $T^\star$ denote the following (random) variable:

$$T^\star = \inf\{t > 0 : \theta(t) \geq \theta^\star\}. \tag{2}$$

If the firm always acts optimally, then $T^\star$ is simply the time of relocation of the firm. Furthermore, we denote by $N^\star$ the number of locations that become available until the optimal location triggers the relocation of the firm:

$$N^\star = \{n : N(T^\star) = n\}.$$

We note that $T^\star$ and $N^\star$ are related according to the following equation:

$$P(T^\star \leq t) = P(N(t) \geq N^\star) \tag{3}$$

where Equation (3) holds because N is a Poisson process, Ross (1996); moreover $T^\star$ is a *stopping time*, Ross (1996), for the Poisson process N.

Finally, let $V(.)$ denote the value of the firm after relocation (i.e., $V(\theta)$ is the value of the firm when it stays in its current location θ forever), and let $F(., \psi_0)$ denote the value of the firm when its initial efficiency is ψ_0. Whenever possible, we drop ψ_0 from $F(., \psi_0)$, and thus we simply use $F(.)$ (and therefore $F(\psi)$ denotes the value of the firm when its current efficiency is ψ).

In this paper we assume all the necessary conditions on $V(.)$ in order to ensure the existence and uniqueness of $\theta^\star$; in particular we assume that $V(.)$ is a convex function (as in Farzin et al. (1998)). With this assumption, the optimal switching level $\theta^\star$ can be found using the so-called *value matching condition*, which states that the value of the firm is a continuous function on $\theta^\star$, Dixit and Pindyck (1994), i.e.,

$$F(\theta(T^\star)^-) = F(\theta^\star).$$

If the firm decides to change its current location, then it means that for all $\theta > \theta^\star$:

$$F(\theta) = V(\theta) - I \tag{4}$$

which is precisely the termination payoff, as we assume at most one relocation.

On the other hand, if the firm does not change its current location, i.e., if $\theta \leq \theta^\star$, then the value of the firm when the present efficiency is θ, $F(\theta)$, is given by:

$$\frac{\pi(\psi_0)}{r+\lambda} + \frac{\lambda}{r+\lambda}\left[\int_0^{\theta^\star-\theta} F(\theta+u)dF_U(u) + \int_{\theta^\star-\theta}^{\infty} (V(\theta+u) - I)dF_U(u)\right] \tag{5}$$

where the first component corresponds to the present value of the payoffs inherent to the initial location; the first element in the second component represents the value of the real option to relocate production to a place where a higher level of efficiency will be achieved; and the second element in the second component corresponds to the net present value of the firm after moving to the new location - disbursing the investment cost, I, and benefiting from the increase in net cash flows granted by the increased level of efficiency.

In the next section we derive results concerning $\theta^\star$, $F(.)$ and $T^\star$ for two particular classes of random variables, namely the truncated-exponential and the gamma distributions. We note that Huisman (2000), in a different setting, as already presented equivalent results concerning the following density distributions: degenerate, uniform, and exponential.

4 Optimal timings of relocation and corresponding volatilities

The results presented in the previous section allow for the general valuation of firms facing relocation decisions. However, the corresponding values will depend on the levels of efficiency increase (here denoted by U_i) and also on dynamics inherent to the process that governs efficiency evolution (namely through the parameter λ). These dynamics will naturally diverge in accordance to the industry and geographical regions considered. In the present article, the above mentioned dynamics are modeled using different density distribution functions.

Accordingly, in order to study the evolution of the value of the company under an optimal relocation policy for different risk environments and initial locations, in the rest of the paper we consider particular density functions. Namely we consider the following families for U: the truncated-exponential (with parameters M and μ), and the gamma (with parameters 2 and μ). For each one we derive the value of the firm, $F(.)$, the optimal switching level, $\theta^\star$, and the first two order moments for the time of adoption of the new location, $E[T^\star]$ and $\mathbb{E}[(T^\star)^2]$, respectively. In fact, for managerial decision purposes, the period of time that a firm might expect to stay in the current premises prior to relocation is an especially relevant piece of information, since it affects most of the short term operating decisions that will be taken. Something similar might be stated in relation to the corresponding volatility.

We remark that, as $T^\star$ is a non-negative random variable, then it follows that $E[T^\star] = \int_0^\infty u F_{T^\star}(du) = \int_0^\infty (1 - P(T^\star \leq t))\, du$, Ross (1996), where $F_{T^\star}$ denotes the distribution function of $T^\star$. In view of this result and of Equations (1) and (2), it follows that:

$$E[T^\star] = \int_0^\infty (1 - P(T^\star \leq t))\, dt = \int_0^\infty P\left(\theta(t) < \theta^\star\right) dt$$

and therefore:

$$
\begin{aligned}
E[T^\star] &= \int_0^\infty P\left(\theta_0 + \sum_{n=1}^{N(t)} U_n < \theta^\star\right) dt && (6)\\
&= \int_0^\infty \sum_{k=0}^{\infty} P\left(\sum_{n=1}^{k} U_n < \theta^\star - \theta_0\right) P(N(t)=k)dt\\
&= \int_0^\infty \sum_{k=0}^{\infty} P\left(\sum_{n=1}^{k} U_n < \theta^\star - \theta_0\right) \frac{e^{-\lambda t}(\lambda t)^k}{k!}dt\\
&= \int_0^\infty \left[e^{-\lambda t} + \sum_{k=1}^{\infty}\int_0^{\theta^\star-\theta_0} f_{\sum_{n=1}^k U_n}(x)dx\frac{e^{-\lambda t}(\lambda t)^k}{k!}\right] dt\\
&= \frac{1}{\lambda} + \int_0^\infty \left[\sum_{k=1}^{\infty}\left(\int_0^{\theta^\star-\theta_0} f_{\sum_{n=1}^k U_n}(x)dx\right)\frac{e^{-\lambda t}(\lambda t)^k}{k!}\right] dt && (7)
\end{aligned}
$$

where $f_{\sum_{n=1}^k U_n}(.)$ denotes the density function of the sum of independent and identically distributed random variables $U_1, U_2, \ldots, U_k$. Using similar arguments, one can prove that:

$$
E[(T^\star)^2] = \frac{2}{\lambda^2} + \int_0^\infty \left[\sum_{k=1}^{\infty}\left(\int_0^{\theta^\star-\theta_0} f_{\sum_{n=1}^k U_n}(x)dx\right)\frac{e^{-\lambda\sqrt{t}}(\lambda\sqrt{t})^k}{k!}\right] dt. \tag{8}
$$

4.1 Truncated-exponential distribution

Historically, in developed economies, productivity levels have revealed a tendency to grow exponentially. Consequently, economic intuition would lead to the use of the exponential distribution to model increases in efficiency due to changes in location. However, in this paper we decided instead to consider the truncated-exponential distribution. As the density function of the exponential distribution is nonzero for all values in $\Re^+$, if we modeled the increases in efficiency by an exponential distribution, then we would be implicitly assuming an increase in the efficiency that could be arbitrarily large, with non-zero probability. Given the historical background at this level, this possibility does not seem reasonable. It may be appropriate to consider a bounded increase in the efficiency, and therefore the truncated-exponential may be a wise option, as it is bounded w.p.1 by a finite value (that here we denote by M).

We say that U has truncated-exponential distribution with parameters M and μ (and we write $U \sim TExp(M, \mu)$, for short) if its density function is as follows:

$$
f_U(u) = \frac{{}^{-\mu u}}{1 - e^{-\mu M}}, \quad u \in (0, M) \tag{9}
$$

where $M, \mu \in \Re^+$. We note that:

$$
\mathbb{E}[U] = \frac{1}{\mu} - \frac{Me^{-\mu M}}{1 - e^{-\mu M}}, \qquad \mathbb{E}[U^2] = \frac{2}{\mu^2} - \frac{M(2+\mu M)e^{-\mu M}}{\mu(1 - e^{-\mu M})}.
$$

In addition, if we let $M \to \infty$, then U is exponentially distributed, with parameter μ.

In the light of Equation (5), it follows that the value of the firm in the continuation region (i.e., when $\theta < \theta^\star$) is given by the following expression:

$$F(\theta) = \frac{\pi(\psi_0)}{r+\lambda} + \frac{\lambda}{r+\lambda} \int_0^{\min(\theta^\star - \theta, M)} F(u+\theta) \frac{\mu e^{-\mu u}}{1-e^{-\mu M}} du + \\ + \mathbf{1}_{\{\theta^\star - \theta < M\}} \frac{\lambda}{r+\lambda} \int_{\theta^\star - \theta}^{M} (V(u+\theta) - I) \mu \frac{e^{-\mu u}}{1-e^{-\mu M}} du. \tag{10}$$

Therefore the optimal switching level, $\theta^\star$, is the solution of the following equation:

$$\frac{\pi(\psi_0)}{r+\lambda} + \frac{\lambda}{r+\lambda} \int_0^M (V(\theta^\star + u) - I) \mu \frac{e^{-\mu u}}{1-e^{-\mu M}} du = V(\theta^\star) - I. \tag{11}$$

For some particular cases of production functions it is possible to derive the analytical solution of Equation (11). For example, and only for the sake of (a simple) illustration, if we consider a linear production function:

$$\pi(\psi) \propto \psi \tag{12}$$

(where the symbol $\propto$ means *proportional to*) then the value of the firm after relocation is simply:

$$V(\theta) = c\phi \tag{13}$$

where c is a given constant. Then it follows from Equation (11) that $\theta^\star$ is the solution of:

$$\frac{\pi(\psi_0)}{r+\lambda} + \frac{\lambda}{r+\lambda} \int_0^M \left(\frac{\phi(\theta^\star + u)}{r} - I \right) \frac{e^{-\mu u}}{1-e^{-\mu M}} du = c\theta^\star - I$$

whose (unique) solution is:

$$\theta^\star = \frac{M\lambda}{r(1-e^{-M\mu})} + \frac{\lambda}{r\mu} + \pi(\psi_0) + \frac{I}{c}.$$

Next we present one lemma that allows for the computation of a general solution of Equation (10).

Lemma 4.1 Let a_0, a_1, a_2 and a_3 be any non-negative real numbers. Then the solution of the equation:

$$f(x) = a_0 + a_1 \int_0^{a_2 - x} f(x+y) \frac{a_3 e^{-a_3 y}}{1-e^{-Ma_3}} dy \tag{14}$$

is given by:

$$f(x) = e^{a_3 x \left(1 + (-1 + \frac{1}{1-e^{Ma_3}}) a_1\right)} \frac{c(1-e^{-Ma_3})}{a_3} - \frac{(-1+e^{Ma_3}) a_0}{(1+e^{Ma_3}(-1+a_1))^2} \tag{15}$$

where c is a constant determined by initial value.

Moreover, the solution of the equation:

$$f(x) = a_0 + a_1 \left[\int_0^{a_2 - x} f(x+y) \frac{a_3 e^{-a_3 y}}{1 - e^{-M a_3}} dy + \int_{a_2 - x}^{M} g(x+y) \frac{a_3 e^{-a_3 y}}{1 - e^{-M a_3}} dy \right] \tag{16}$$

: is given by

$$\begin{aligned} f(x) = {} & e^{a_3 x \left(1 + (-1 + \frac{1}{1 - e^{M a_3}}) a_1\right)} \frac{1 - e^{-M a_3}}{a_3} \times \\ & \times \left[c_1 - \int_{c_2}^{x} \frac{e^{(-y + M + (1 + \frac{e^{-M a_3}}{1 - 2^{-M a_3}}) y a_1) a_3} \left((-1 + e^{M a_3}) a_0 - g(y+M) a_1 \right) a_3^2}{(1 - e^{M a_3})^2} dy \right] \end{aligned} \tag{17}$$

where c_1 and c_2 are constants determined by initial values.

Proof. Note that Equation (14) can be rewritten as follows:

$$f(x) = a_0 + a_1 e^{a_3 x} \int_x^{a_2} f(y) \frac{a_3 e^{-a_3 y}}{1 - e^{-M a_3}} dy. \tag{18}$$

Let $F'(y) = f(y) \frac{a_3 e^{-a_3 y}}{1 - e^{-M a_3}}$; then Equation (18) is equivalent to:

$$F'(x) = \frac{a_0 a_3}{1 - e^{-M a_3}} e^{-a_3 x} + \frac{a_1 a_3}{1 - e^{-M a_3}} \left[F(a_2) - F(x) \right]. \tag{19}$$

Derivating F once more, we get:

$$F''(x) = -\frac{a_0 a_3^2}{1 - e^{-M a_3}} e^{-a_3 x} - \frac{a_1 a_3}{1 - e^{-M a_3}} F'(x).$$

Let $H(x) = F'(x)$; then the previous equation can be written as:

$$H'(x) = -\frac{a_0 a_3^2}{1 - e^{-M a_3}} e^{-a_3 x} - \frac{a_1 a_3}{1 - e^{-M a_3}} H(x)$$

whose solution is:

$$c e^{-\frac{x a_1 a_3}{1 - e^{-M a_3}}} - \frac{a_0 a_3 e^{(M - x) a_3}}{1 + e^{M a_3}(-1 + a_1)}.$$

Finally Equation (15) follows, as $f(x) = \frac{H(x) e^{a_3 x}(1 - e^{-M a_3})}{a_3}$.

Proceeding similarly, we note that Equation (16) can be rewritten as follows:

$$f(x) = a_0 + a_1 e^{a_3 x} \left[\int_x^{a_2} f(y) \frac{a_3 e^{-a_3 y}}{1 - e^{-M a_3}} dy + \int_{a_2}^{M + x} g(y) \frac{a_3 e^{-a_3 y}}{1 - e^{-M a_3}} dy \right] \tag{20}$$

Let $F'(y) = f(y) \frac{a_3 e^{-a_3 y}}{1 - e^{-M a_3}}$; then Equation (20) is equivalent to:

$$F'(x) = \frac{a_0 a_3}{1 - e^{-M a_3}} e^{-a_3 x} + \frac{a_1 a_3}{1 - e^{-M a_3}} \left[F(a_2) - F(x) + G(x) \right] \tag{21}$$

where $G(x) = \int_{a_2}^{M+x} g(y) \frac{a_3 e^{-a_3 y}}{1-e^{-Ma_3}} dy$. If we derivate F once more, we end up with

$$F''(x) = -\frac{a_0 a_3^2}{1 - e^{-Ma_3}} e^{-a_3 x} + \frac{a_1 a_3}{1 - e^{-Ma_3}} \left[-F'(x) + G'(x) \right].$$

Let $H(x) = F'(x)$; then the previous equation can be written as follows:

$$H'(x) = -\frac{a_0 a_3^2}{1 - e^{-Ma_3}} e^{-a_3 x} + \frac{a_1 a_3}{1 - e^{-Ma_3}} \left[-H(x) + G'(x) \right]$$

whose solution is

$$e^{-x \frac{a_1 a_3}{1-e^{-Ma_3}}} \left[c_1 - \int_{c_2}^{x} \frac{e^{(-y+M+(1+\frac{e^{-Ma_3}}{1-e^{-Ma_3}})ya_1)a_3} \left((-1+e^{Ma_3})a_0 - g(y+M)a_1 \right) a_3^2}{(1-e^{Ma_3})^2} dy \right].$$

Therefore Equation (17) follows after some simple manipulations, using the fact that $f(x) = \frac{H(x)e^{a_3 x}(1-e^{-Ma_3})}{a_3}$. $\square$

In view of Lemma (4.1), if we make the following attributions:

$$a_0 = \frac{\pi(\psi_0)}{r+\lambda}; \quad a_1 = \frac{\lambda}{r+\lambda}; \quad a_3 = \mu; \quad g(y) = V(y)$$

then we have the following result concerning the value of the firm.

Theorem 4.2 The value of the firm when the present efficiency is θ, $F(\theta)$, is given by:

$$\begin{cases} e^{\mu\theta\left(1+(-1+\frac{1}{1-e^{M\mu}})\frac{\lambda}{r+\lambda}\right)} \frac{1-e^{-M\mu}}{\mu} \left[c_1 - \int_{c_2}^{\theta} G(y)dy \right] & \theta < \theta^\star \wedge \theta^\star - \theta < M \\ e^{\mu\theta\left(1+(-1+\frac{1}{1-e^{M\mu}})\frac{\lambda}{r+\lambda}\right)} \frac{c_3(1-e^{-M\mu})}{\mu} - \frac{(-1+e^{M\mu})\frac{\pi(\psi_0)}{r+\lambda}}{(1+e^{M\mu}(-1+\frac{\lambda}{r+\lambda}))^2} & \theta < \theta^\star \wedge \theta^\star - \theta \geq M \\ V(\theta) - I & \theta \geq \theta^\star \end{cases} \tag{22}$$

with:

$$G(y) = \frac{e^{(-y+M+(1+\frac{e^{-M\mu}}{1-2^{-M\mu}})y\frac{\lambda}{r+\lambda})\mu} \left((-1+e^{M\mu})\frac{\pi(\psi_0)}{r+\lambda} - V(y+M)\frac{\lambda}{r+\lambda} \right) \mu^2}{(1-e^{M\mu})^2}$$

where the constants c_1, c_2 and c_3 are the solution of the following system of equations:

$$\begin{cases} c_1 - \int_{c_2}^{0} G(y)dy = 0 \\ e^{\mu\theta^\star\left(1+(-1+\frac{1}{1-e^{M\mu}})\frac{\lambda}{r+\lambda}\right)} \frac{1-e^{-M\mu}}{\mu} \left[c_1 - \int_{c_2}^{\theta^\star} G(y)dy \right] \\ \qquad = \frac{\pi(\psi_0)}{r+\lambda} + \int_0^M (V(\theta^\star+u) - I) \frac{\mu e^{-\mu u}}{1-e^{-\mu M}} du \\ e^{\mu(\theta^\star-M)\left(1+(-1+\frac{1}{1-e^{M\mu}})\frac{\lambda}{r+\lambda}\right)} \frac{1-e^{-M\mu}}{\mu} \left[c_1 - \int_{c_2}^{\theta^\star-M} G(y)dy \right] = \\ \qquad = e^{\mu(\theta^\star-M)\left(1+(-1+\frac{1}{1-e^{M\mu}})\frac{\lambda}{r+\lambda}\right)} \frac{c_3(1-e^{-M\mu})}{\mu} - \frac{(-1+e^{M\mu})\frac{\pi(\psi_0)}{r+\lambda}}{(1+e^{M\mu}(-1+\frac{\lambda}{r+\lambda}))^2} \end{cases}.$$

We stress that Equation (22) and the determination of the constants c_1, c_2 and c_3 is still algebraically much involved, even for simple production functions (as it is the case of the linear production function presented in Equation (22)).

We note that in order to derive the first two order moments of $T^\star$ we need, according to Equations (7) and (8), the distribution of $\sum_i U_i$. As the truncated-exponential is not closed under sums of i.i.d. random variables, and as the distribution of the sum does not have a closed form, we cannot derive closed analytical formulas for $\mathbb{E}[T^\star]$ and $\mathbb{E}[(T^\star)^2]$. Latter on, on Section 5, we show results obtained using numerical simulation.

4.2 Gamma distribution

A gamma distribution with parameters 2 and μ can be seen as the sum of 2 independent and identically distributed Exponential random variables, with parameter μ, Ross (2005). We note that the investment decisions are subject to several risk factors eventually independent. Therefore, in modeling this type of problems, makes sense to introduce distribution functions that enable the analytical treatment of more than one state variable, in order to approximate the modeling exercise to corporate reality.

Assume that U has a gamma distribution, with parameters 2 and μ, so that the density function of U is as follows:

$$f_U(u) = \mu^2 u e^{-\mu u}, \quad u \in \Re^+$$

where $\mu > 0$.

For such a distribution, the two first order moments are given by:

$$\mathbb{E}[U] = \frac{2}{\mu}, \qquad \mathbb{E}[U^2] = \frac{6}{\mu^2}.$$

In the light of Equation (5), it follows that the value of the firm in the continuation region (i.e., when $\theta < \theta^\star$) is given by the following expression:

$$\begin{aligned} F(\theta) = {} & \frac{\pi(\psi_0)}{r+\lambda} + \frac{\lambda}{r+\lambda}\int_0^{\theta^\star-\theta} F(\theta+u)\mu^2 u e^{-\mu u} du + \\ & + \frac{\lambda}{r+\lambda}\int_{\theta^\star-\theta}^{\infty} (V(\theta+u)-I)\mu^2 u e^{-\mu u} du. \end{aligned} \tag{23}$$

Thus the optimal switching level, $\theta^\star$, is the solution of the following equation:

$$\frac{\pi(\psi_0)}{r+\lambda} + \frac{\lambda}{r+\lambda}\int_0^{\infty} (V(\theta^\star+u)-I)\mu^2 u e^{-\mu u} du = V(\theta^\star) - I. \tag{24}$$

For example, if we adopt as value function (12), then $\theta^\star$ is given by:

$$\left(2\lambda + \frac{\sqrt{r^2(\pi(\psi_0)+Ir)\mu^2 + 2\lambda(3r+2\lambda)\phi}}{\sqrt{\phi}}\right)(r\mu)^{-1}.$$

The next lemma provides a result that allows one to compute the solution for the value of the firm.

Lemma 4.3 Let a_0, a_1, a_2 and a_3 be any non-negative real numbers. Then the solution of the equation:

$$f(x) = a_0 + a_1 \left[\int_0^{a_2 - x} f(x+y) a_3^2 y e^{-a_3 y} dy + \int_{a_2 - x}^{\infty} g(x+y) a_3^2 y e^{-a_3 y} dy \right] \tag{25}$$

is given by:

$$f(x) = \frac{a_0}{1 - a_1 a_3} + \frac{c_1}{a_3} \sqrt{a_1} e^{a_3(1+\sqrt{a_1})x} - \frac{c_2}{a_3} \sqrt{a_1} e^{a_3(1-\sqrt{a_1})x}$$

where c_1 and c_2 are such that:

$$\frac{a_0}{1 - a_1 a_3} + \frac{c_1}{a_3} \sqrt{a_1} - \frac{c_2}{a_3} \sqrt{a_1} = a_0 + a_1 \int_{a_2}^{\infty} g(y) a_3^2 y e^{-a_3 y} dy$$

$$\frac{a_0}{1 - a_1 a_3} + \frac{c_1}{a_3} \sqrt{a_1} e^{a_3(1+\sqrt{a_1})a_2} - \frac{c_2}{a_3} \sqrt{a_1} e^{a_3(1-\sqrt{a_1})a_2} =$$

$$= a_0 + a_1 \int_0^{\infty} g(a_2 + y) a_3^2 y e^{-a_3 y} dy.$$

Proof. Equation (25) can be rewritten as follows:

$$\begin{aligned} f(x) = a_0 + a_1 e^{a_3 x} \Bigg[&\int_x^{a_2} f(y) a_3^2 y e^{-a_3 y} dy - x \int_x^{a_2} f(y) a_3^2 e^{-a_3 y} dy \\ &+ \int_{a_2}^{\infty} g(y) a_3^2 y e^{-a_3 y} dy - x \int_{a_2}^{\infty} g(y) a_3^2 e^{-a_3 y} dy \Bigg]. \end{aligned} \tag{26}$$

Let $D'(y) = f(y) a_3^2 e^{-a_3 y}$. Then Equation (26) is equal to:

$$\begin{aligned} D'(x) &= a_0 a_3^2 e^{-a_3 x} + a_1 a_3^2 \Bigg[\int_x^{a_2} y D'(y) dy - x D(a_2) + x D(x) + \\ &\qquad + \int_{a_2}^{\infty} g(y) a_3^2 y e^{-a_3 y} dy - x \int_{a_2}^{\infty} g(y) a_3^2 e^{-a_3 y} dy \Bigg] \\ &= a_0 a_3^2 e^{-a_3 x} + a_1 a_3^2 \Bigg[a_2 D(a_2) - \int_x^{a_2} D(y) dy - x D(a_2) + \\ &\qquad + \int_{a_2}^{\infty} g(y) a_3^2 y e^{-a_3 y} dy - x \int_{a_2}^{\infty} g(y) a_3^2 e^{-a_3 y} dy \Bigg]. \end{aligned}$$

Deriving once more, we get:

$$\begin{aligned} D''(x) &= -a_0 a_3^3 e^{-a_3 x} + a_1 a_3^2 \left[D(x) - D(a_2) - \int_{a_2}^{\infty} g(y) a_3^2 e^{-a_3 y} dy \right] \\ &= -a_0 a_3^3 e^{-a_3 x} + a_1 a_3^2 D(x) - a_4 \end{aligned}$$

where $a_4 = a_1 a_3^2 \left[D(a_2) + \int_{a_2}^{\infty} g(y) a_3^2 e^{-a_3 y} dy\right]$. The solution of the previous equation is

$$D(x) = \frac{a_0 a_3}{a_1 a_3 - 1} e^{-a_3 x} + \frac{a_4}{a_1 a_3^3} + c_1 a_3 \sqrt{a_1} e^{x a_3 \sqrt{a_1}} + c_2 a_3 \sqrt{a_1} e^{-x a_3 \sqrt{a_1}}$$

and therefore:

$$f(x) = D'(x) \frac{e^{a_3 x}}{a_3^2} = \frac{a_0}{1 - a_1 a_3} + \frac{c_1}{a_3} \sqrt{a_1} e^{a_3(1+\sqrt{a_1})x} - \frac{c_2}{a_3} \sqrt{a_1} e^{a_3(1-\sqrt{a_1})x}. \square$$

In view of Lemma (4.3), if we make the following attributions:

$$a_0 = \frac{\pi(\psi_0)}{r+\lambda}; \quad a_1 = \frac{\lambda}{r+\lambda}; \quad a_3 = \mu; \; g(y) = V(y)$$

then we have the following result concerning the value of the firm.

Theorem 4.4 The value of the firm is given by:

$$F(\theta) = \begin{cases} \frac{\pi(\psi_0)}{r+\lambda-\lambda\mu} + \frac{c_1}{\mu} \sqrt{\frac{\lambda}{r+\lambda}} e^{\mu(1+\sqrt{\frac{\lambda}{r+\lambda}})\theta} - \frac{c_2}{\mu} \sqrt{\frac{\lambda}{r+\lambda}} e^{\mu(1-\sqrt{\frac{\lambda}{r+\lambda}})\theta} & \theta > \theta^\star \\ V(\theta) - I & \theta > \theta^\star \end{cases} \tag{27}$$

where c_1 and c_2 are the solutions of the following set of equations:

$$\begin{aligned} \frac{\pi(\psi_0)}{r+\lambda-\lambda\mu} + \frac{c_1}{\mu} \sqrt{\frac{\lambda}{r+\lambda}} - \frac{c_2}{\mu} \sqrt{\frac{\lambda}{r+\lambda}} &= \frac{\pi(\psi_0)}{r+\lambda} + \frac{\lambda}{r+\lambda} \int_{\theta^\star}^{\infty} V(y) \mu^2 y e^{-\mu y} dy \\ \frac{\pi(\psi_0)}{r+\lambda-\lambda\mu} + \frac{c_1}{\mu} \sqrt{\frac{\lambda}{r+\lambda}} e^{\mu(1+\sqrt{\frac{\lambda}{r+\lambda}})\theta^\star} - \frac{c_2}{\mu} \sqrt{\frac{\lambda}{r+\lambda}} e^{\mu(1-\sqrt{\frac{\lambda}{r+\lambda}})\theta^\star} &= \\ = \frac{\pi(\psi_0)}{r+\lambda} + a_1 \int_0^{\infty} V(\theta^\star + y) \mu^2 y e^{-\mu y} dy. \end{aligned}$$

We note that Theorem 4.4, similarly to Theorem 4.2, is still algebraically and numerically quite involved, and very difficult to interpret.

In the next theorem we provide expressions for the computation of the expected value and the volatility of $T^\star$, the time until relocalization.

Theorem 4.5 The first two moments of $T^\star$ are given by:

$$E[T^\star] = \frac{1}{\lambda} + \frac{\mu}{\lambda} \frac{\theta^\star - \theta_0}{2} - \frac{1}{4\lambda} + \frac{e^{-2\mu(\theta^\star - \theta_0)}}{4\lambda} \tag{28}$$

$$\begin{aligned} E[(T^\star)^2] = \frac{2}{\lambda^2} + \frac{1}{2\lambda^2} \Bigg(-\frac{5}{4} + e^{-2\mu(\theta^\star - \theta_0)} (\frac{5}{4} - (\theta^\star - \theta_0)\mu) + \\ + \frac{1}{2} (\theta^\star - \theta_0)(6 + (\theta^\star - \theta_0)\mu) \Bigg). \end{aligned} \tag{29}$$

Proof. Since we assume that $U_i \sim \text{Gamma}(2,\mu)$, it follows that $\sum_{k=1}^{n} U_k \sim \text{Gamma}(2k,\mu)$, as $\{U_i\}$ is a sequence of independent random variables and the Gamma distribution is closed under sums of i.i.d. random variables, Ross (1996). Therefore, in view of Equation (7):

$$
\begin{aligned}
E[T^\star] &= \int_0^\infty \left[e^{-\lambda t} + \sum_{k=1}^{\infty} \int_0^{\theta^\star-\theta_0} f_{\sum_{n=1}^{k} U_n}(x)dx \frac{e^{-\lambda t}(\lambda t)^k}{k!} \right] dt \\
&= \int_0^\infty \left[e^{-\lambda t} + \sum_{k=1}^{\infty} \int_0^{\theta^\star-\theta_0} \frac{\mu^{2k}x^{2k-1}e^{-\mu x}}{(2k-1)!} dx \frac{e^{-\lambda t}(\lambda t)^k}{k!} \right] dt \\
&= \int_0^\infty e^{-\lambda t}dt + \int_0^{\theta^\star-\theta_0} e^{-\mu x} \sum_{k=1}^{\infty} \frac{\mu^{2k}x^{2k-1}}{\lambda(2k-1)!} \left[\int_0^\infty \frac{\lambda^{k+1}t^k}{k!} e^{-\lambda t}dt \right] dx && (30) \\
&= \int_0^\infty e^{-\lambda t}dt + \int_0^{\theta^\star-\theta_0} \frac{e^{-\mu x}}{\lambda} \sum_{k=1}^{\infty} \frac{\mu^{2k}x^{2k-1}}{(2k-1)!} dx && (31) \\
&= \frac{1}{\lambda} + \int_0^{\theta^\star-\theta_0} \frac{e^{-\mu x}}{\lambda} \mu \text{Sinh}(\mu x) dx && (32) \\
&= \frac{1}{\lambda} + \int_0^{\theta^\star-\theta_0} \frac{e^{-\mu x}}{\lambda} \mu \frac{e^{\mu x}-e^{-\mu x}}{2} dx \\
&= \frac{1}{\lambda} + \frac{\mu}{2\lambda} \int_0^{\theta^\star-\theta_0} (1-e^{-2\mu x}) dx \\
&= \frac{1}{\lambda} + \frac{\mu}{\lambda}\frac{\theta^\star-\theta_0}{2} - \frac{1}{4\lambda} + \frac{e^{-2\mu(\theta^\star-\theta_0)}}{4\lambda} && (33)
\end{aligned}
$$

and then Equation (28) follows. We remark that we use in Equation (31) the fact that:

$$
\int_0^\infty \frac{\lambda^{k+1}t^k}{k!} e^{-\lambda t}dt = \int_0^\infty f_{\text{Gamma}(k,\,\lambda)}(t)dt = 1
$$

(as it is the integral all over the support of the density function) and in Equation (32) the fact that $\sum_{k=1}^{\infty} \frac{\mu^{2k}x^{2k-1}}{(2k-1)!} = \mu\text{Sinh}(\mu x)$ (by definition of hyperbolic sinus), where

$$
\text{Sinh}(s) = \frac{e^s - e^{-s}}{2}.
$$

Similarly, it follows from Equation (8) that:

$$
\begin{aligned}
E[(T^\star)^2] &= \int_0^\infty \left[e^{-\lambda\sqrt{t}} + \sum_{k=1}^{\infty} \int_0^{\theta^\star-\theta_0} f_{\sum_{n=1}^{k} U_n}(x)dx \frac{e^{-\lambda\sqrt{t}}(\lambda\sqrt{t})^k}{k!} \right] dt \\
&= \int_0^\infty e^{-\lambda\sqrt{t}}dt + \int_0^\infty \sum_{k=1}^{\infty} \int_0^{\theta^\star-\theta_0} \frac{\mu^{2k}x^{2k-1}e^{-\mu x}}{(2k-1)!} dx \frac{e^{-\lambda\sqrt{t}}(\lambda\sqrt{t})^k}{k!} dt \\
&= \frac{2}{\lambda^2} + \int_0^{\theta^\star-\theta_0} e^{-\mu x} \sum_{k=1}^{\infty} \frac{\mu^{2k}x^{2k-1}}{\lambda(2k-1)!} dx \int_0^\infty \frac{e^{-\lambda\sqrt{t}}\lambda^{k+1}\sqrt{t}^k}{k!} dt
\end{aligned}
$$

If in $\int_0^\infty \frac{e^{-\lambda\sqrt{t}}\lambda^{k+1}\sqrt{t}^k}{k!}dt$ we make the change of variable $u = \sqrt{t}$, then we get:

$$\begin{aligned}\int_0^\infty \frac{e^{-\lambda\sqrt{t}}\lambda^{k+1}\sqrt{t}^k}{k!}dt &= \int_0^\infty \frac{e^{-\lambda u}\lambda^{k+1}u^k}{k!}2udu \\ &= 2\mathbb{E}[\text{Gamma}(\lambda, k)] \\ &= 2\frac{k+1}{\lambda}.\end{aligned}$$

Moreover,

$$\sum_{k=1}^{\infty} \frac{\mu^{2k}x^{2k-1}(k+1)}{(2k-1)!} = \frac{1}{2}\mu^2 x\text{Cosh}(\mu x) + \frac{3}{2}\mu\text{Sinh}(\mu x)$$

where $\text{Cosh}(s) = \frac{e^s+e^{-s}}{2}$. Therefore

$$\begin{aligned}E[(T^\star)^2] &= \frac{2}{\lambda^2} + \int_0^{\theta^\star-\theta_0} e^{-\mu x}\sum_{k=1}^{\infty}\frac{\mu^{2k}x^{2k-1}}{\lambda(2k-1)!}dx\int_0^\infty \frac{e^{-\lambda\sqrt{t}}\lambda^{k+1}\sqrt{t}^k}{k!}dt \\ &= \frac{2}{\lambda^2} + \int_0^{\theta^\star-\theta_0} e^{-\mu x}\sum_{k=1}^{\infty}\frac{\mu^{2k}x^{2k-1}(k+1)}{(2k-1)!}\frac{2(k+1)}{\lambda}dx \\ &= \frac{2}{\lambda^2} + 2\int_0^{\theta^\star-\theta_0} \frac{e^{-\mu x}}{\lambda^2}\sum_{k=1}^{\infty}\frac{\mu^{2k}x^{2k-1}(k+1)}{(2k-1)!}dx \\ &= \frac{2}{\lambda^2} + 2\int_0^{\theta^\star-\theta_0} \frac{e^{-\mu x}}{\lambda^2}(\frac{1}{2}\mu^2 x\text{Cosh}(\mu x) + \frac{3}{2}\mu\text{Sinh}(\mu x))dx \\ &= \frac{2}{\lambda^2} + \frac{\mu}{\lambda^2}\int_0^{\theta^\star-\theta_0} e^{-\mu x}(x\mu\frac{e^{\mu x}+e^{-\mu x}}{2} + 3\frac{e^{\mu x}-e^{-\mu x}}{2})dx \\ &= \frac{2}{\lambda^2} + \frac{1}{2\lambda^2}\left(-\frac{5}{4} + e^{-2\mu(\theta^\star-\theta_0)}(\frac{5}{4} - (\theta^\star-\theta_0)\mu) + \right. \\ &\qquad \left. +\frac{1}{2}(\theta^\star-\theta_0)(6 + (\theta^\star-\theta_0)\mu)\right). \ \square\end{aligned}$$

In the next section we illustrate numerically the results derived in this section for particular instances.

5 Numerical illustrations

In this section we illustrate some of the results derived in the previous section, using particular instances of truncated-exponential and gamma distributions.

We consider the following instances:

We note that we have chosen the parameters of the random variables so that they have the same expected value (but different volatility), i.e., everytime a new location becomes available,

	$\mathbb{E}[U]$	$\sqrt{\text{Var}[U]}$
ExpT(5, 10)	0.1	0.01
Gamma(2, 20)	0.1	0.005

the expected increase in efficiency will be the same in both cases. Therefore we are comparing two situations where the law of the jumps in the efficiency process is different (one is modeled by a truncated-exponential, whereas the other is modeled by a gamma-distribution, with bound 10), but in such a way that on average the jumps in the efficiency are equal. Thus with this simple illustration we can check, at least empirically, if the probability law of the jumps in the efficiency is relevant (and in this case we might find significantly different values of optimal switching levels) or, at the opposite, if the expected value of the increase is the most relevant parameter, regardless the probability law.

Following Farzin et al. (1998) and Couto (2006), we consider a Cobb-Douglas production function with output elasticity equal to 0.5, so that the value of the firm after relocation (see Farzin et al. (1998)) is given by:

$$V(\theta) = \frac{\psi\theta^2}{r}. \tag{34}$$

where $\psi = 0.5(\frac{0.5}{w})p^2$.

Moreover, we assume that the the output price is $p = 1\ 000$, with input price $w = 250$. The firm discount rate is $r = 0.05$, with a sunk cost of investiment in a new location $I = 10\ 000$. Currently, the firm operates in a location with efficiency $\psi_0 = 1$, and the rate at which new locations become available is $\lambda = 0.5$.

Thus Equation (34) takes the following form:

$$V(\theta) = 20\ 000\theta^2 \tag{35}$$

The next table presents numerical results for these situations, concerning:

i) The optimal switching level, $\theta^\star$ - obtained through the solution of Equation (11), for the truncated-exponential case, and of Equation (24), for the gamma case, with $V(\theta)$ given by Equation (34);

ii) The expected value and variance of $T^\star$ for the gamma case, according to Equations (28) and (29), where $V[T^\star] = \mathbb{E}[(T^\star)^2] - \mathbb{E}^2[T^\star]$. We note that for the truncated-exponential case we cannot derive an explicit expression for these two moments (as previously mentioned, the truncated-gamma is not closed under sums of i.i.d. random variables and neither is known its closed form);

iii) Mean and standard deviation of $T^\star$ ($\bar{T}^\star$ and $S_{T^\star}$, respectively), computed using a sample of 1 000 000 simulations[1].

	$\theta^\star$	$\mathbb{E}[T^\star]$	$\sqrt{V[T^\star]}$	$\bar{T}^\star$	$S_{T^\star}$
ExpT(5,10)	2.643	–	–	35.01	11.65
Gamma(2,20)	2.628	34.06	10.05	34.30	10.06

For these two particular situations, the numerical values of both $\theta^\star$ and $T^\star$ are similar, and therefore one may ask if the optimal relocation policy is robust in terms of the distribution of the jumps in the efficiency process. Note that, if one proves that the optimal policy (in terms of optimal switching level and time until adoption of a new location) depends only on the distribution of the jumps in the efficiency through an expected value, then we can simply discard the information concerning the distribution of the jumps, keeping only in mind this expected value.

Next, we present two plots, showing the behavior of $\theta^\star$ as a function of the expected value of U, for different values of arrival rate (λ) of information concerning new locations.

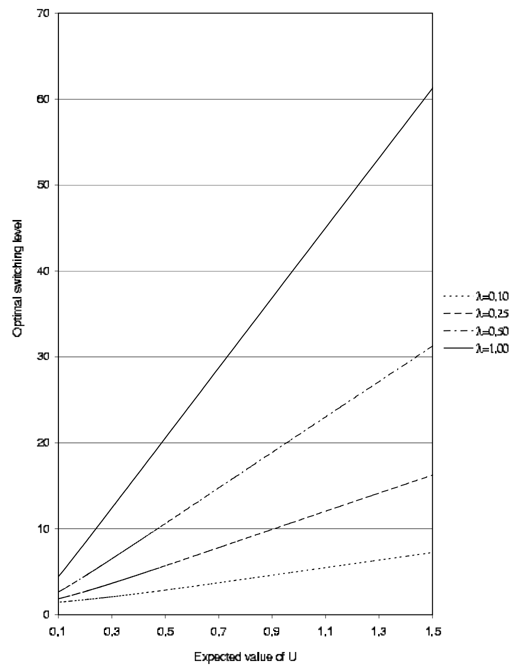

Figure 1: Behaviour of $\theta^\star$ for the truncated-exponential distribution.

The optimal policy seems robust in terms of the particular distribution that we use, as the values of $\theta^\star$ are nearly the same for the truncated-exponential and for the gamma distributions

[1] Simulations were obtained using the Mathematica software.

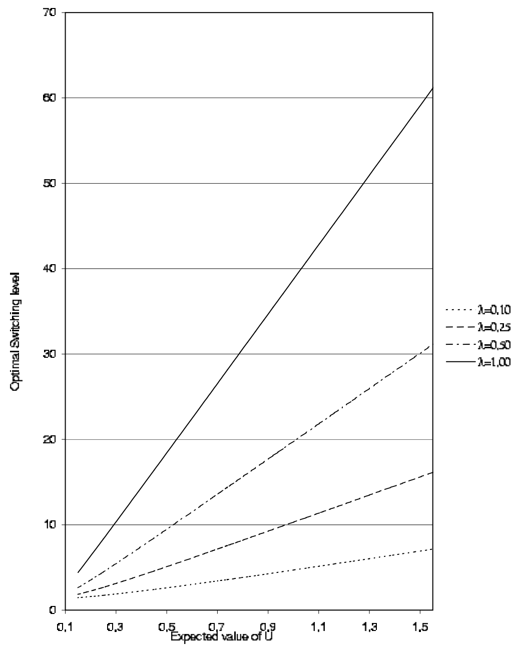

Figure 2: Behaviour of $\theta^{\star}$ for the gamma distribution.

that we have considered. Additionally, the optimal switching level seems to be essentially an increasing linear function of the level of the expected increase in efficiency ($\mathbb{E}[U]$) and an increasing (monotonic but not linear) function of the rate of arrival of information concerning new locations (λ).

Following the above comparative analysis, we also present numerical results concerning the optimal switching level when we impose changes in the above mentioned parameter values. In particular, we analise the effect on the optimal switching level, $\theta^{\star}$ of the following parameters: output price p, input price w, investment cost I, and discount rate r. See Figs. (3-6).

As expected, Fig. (3) shows that initial higher levels of the output prices make the option to delay less valuable leading to smaller optimal switching levels. At the opposite, increases in input prices lead to increases in optimal switching levels (see Fig. (4)). In effect, the level of efficiency that triggers a change in location needs to increase in order to compensate contrary effect induced by the increase in input prices.

The relationship between optimal switching levels and investment costs follows a similar pattern (see Fig. (5)). Increases in investment costs need to be properly compensated by efficiency increases in order to justify changes in location.

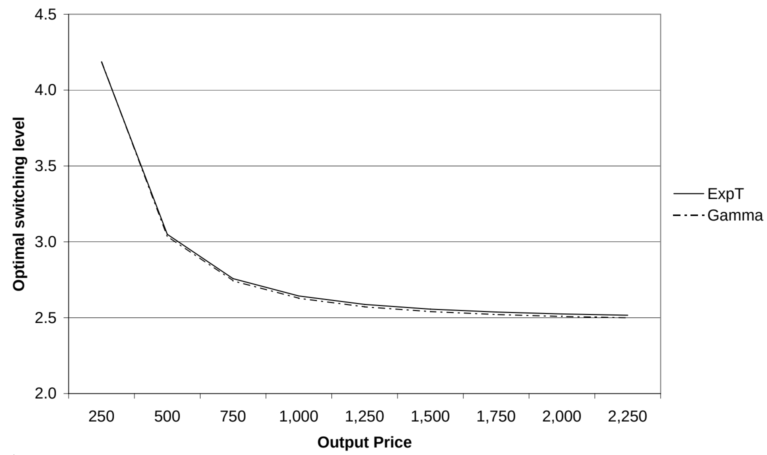

Figure 3: Behaviour of $\theta^\star$ for the truncated-exponential distribution and for the gamma distribution as a function of the output price.

Finally we present a plot concerning the behavior of the optimal switching level as a function of the discount rate (see plot (6)). Reduced discount rate levels mean smaller time value of money and consequently, a small potential loss for postponing the decision to relocate. In contrast, very high discount rate levels imply untenable delaying costs. The convex shape is in accordance with the economic rationale related to the valuation of all interest rate products.

6 Traditional analysis

The traditional capital budgeting decision whose core concepts are Discounted Cash Flow (DCF) and Net Present Value (NPV) suggests that an investment should be implemented when the value of the corresponding operating net cash inflows exceeds the present value of the inherent investment cost, I. It assumes implicitly that the investment is reversible or, if irreversible, that it corresponds to a "now or never" type of opportunity. Dixit and Pindyck (1994) discuss in some detail several investment problems that do not comply with that assumption. Regarding investments related to the choice of location , the ability to reverse a decision depends not only on the purpose of the investment but also on the location itself. For example, an investment in a property aimed commercial real estate development nearby a city centre may be completely reversible, since in case of abandon the recovery of the amount invested will tend to be relatively easy. In contrast, a similar investment in an isolated property in the countryside might be totally irreversible, since, in case of abandon, it might be impossible to find potential buyers. Notwithstanding this obvious difference, and the relevance of its economic implications, traditional capital budgeting decisions based on the application of the NPV rule implicitly assume the reversibility of all investments.

The Real Options Analysis (ROA) explicitly incorporates irreversibility and the possibility of postponing an investment decision. In ROA investment opportunities are seen as options:

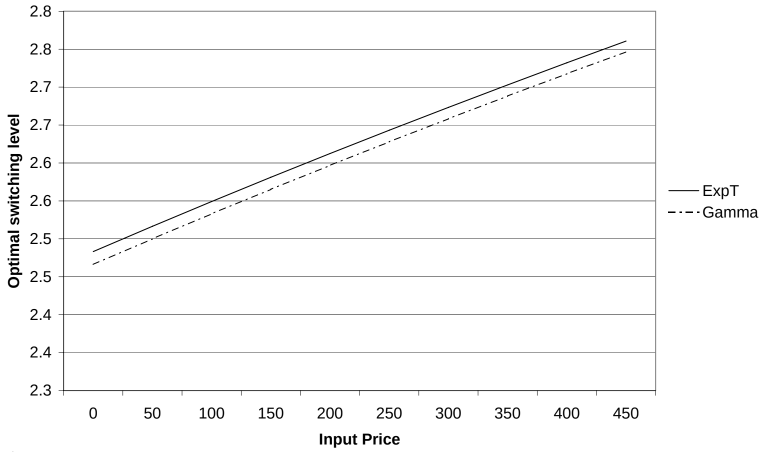

Figure 4: Behaviour of $\theta^\star$ for the truncated-exponential distribution and for the gamma distribution as a function of the input price.

the corresponding holder has the right, but not the obligation of implementing the investment. Investment leads to the "death" of the embedded option, and, as all options are positively valued, the company loses this value whenever invests. Consequently, the loss of the option must be taken into consideration in the investment decision process. Regarding the problem of relocation, the deferral option has value because it exists a positive probability that prior to investing the firm may become aware of a better location capable of allowing the firm to reach efficiency levels impossible to attain in the best site previously available.

According to the traditional capital budgeting analysis based on the concept of NPV, the company adopts the new location with efficiency θ_i if, Huisman (2000):

$$\Psi_i^{\mathrm{NPV}} = \inf\{\theta_j : \theta_j \geq \theta_i^{\mathrm{NPV}}, j \in \mathbb{N}_0\}, \quad i \in \{1, 2, \ldots, n\}$$

where θ_i^{NPV} is the solution of:

$$V(\theta_i^{\mathrm{NPV}}) - I = V(\theta_{i-1}^{\mathrm{NPV}})$$

and $\Psi_0^{\mathrm{NPV}} = \Psi_0$.

Its easy to realize that NPV leads the company to implement the investment too soon, that is, $\Psi_i^{\mathrm{NPV}} < \Psi_i$, for $i \in \{1, 2, \ldots, n\}$.

Determining the efficiency of the first six available locations in our example, according to the traditional NPV method, we verify that $\Psi_i^{\mathrm{NPV}} < \Psi_i$, for $i \in \{1, 2, 3, 4, 5, 6\}$ with $\Psi_i^{\mathrm{NPV}} = 1.3; 1.5; 1.7; 1.9; 2.1;$ and 2.3.

Traditional NPV tends to lead to premature investment, i.e., to non-optimal solutions that are significantly different from those based in a ROA framework.

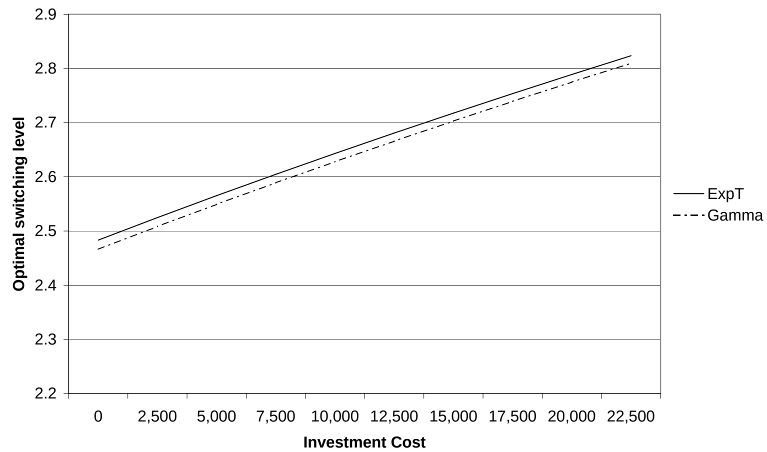

Figure 5: Behaviour of $\theta^\star$ for the truncated-exponential distribution and for the gamma distribution as a function of the investment cost.

7 Concluding remarks

In a period of globalization and market integration, the problem of relocation is especially relevant, given its socio-economic implications. However, in contrast to this article, no other study that we are aware of focuses on the perceived increase in efficiency needed to justify the decision to relocate from one place to another. Using a dynamic programming framework, we have analyzed the problem of the optimal timing for the relocation of a firm that faces a constant and irreversible level of investment expenditure to move its production site. The modeling framework assumes that the availability of new and more efficient location sites evolves according to a Poisson process. The dynamics of the corresponding efficiency increases are modeled using continuous distributions. Given the specific characteristics of the relocation decision, and unlikely any other work that we are aware of in this field, we have used the truncated exponential and the gamma distribution functions for this purpose. According to our research, the optimal timing of relocation is significantly affected by the uncertainties related to both the expected rhythm that characterizes the arrival of information regarding the availability of new, more efficient location sites, and the degree in efficiency improvement from one to the other. Our simulation results suggests the results of the our model are in accordance with economic rationale.

Contrasting relocation decisions taken under the approach proposed in this study with those that would be taken under a traditional capital budgeting framework based on the concept of NPV, it was proved that the latter leads to non-optimal solutions. Rather in line with the intuition usually associated to the decisions to relocate, our framework suggests that the timing of relocation is significantly longer than that inherent to the traditional NPV analysis.

Naturally, a simple framework as the one proposed here has some limits that could and should be overcome in future work. The most relevant are probably those related to the single switching opportunity, the constant level of investment expenditure and the absence of competition.

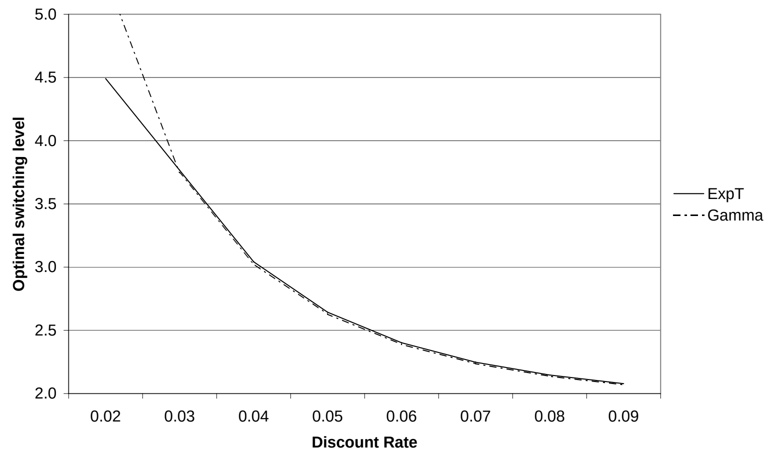

Figure 6: Behaviour of $\theta^\star$ for the truncated-exponential distribution and for the gamma distribution as a function of the discount rate.

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