# Capacity Expansion and Contraction with Fixed and Quadratic Adjustment Costs*

Makoto Goto
*Department of Industrial and Management Systems Engineering, Waseda University*
Ryuta Takashima
*Department of Nuclear Engineering and Management, University of Tokyo*
Motoh Tsujimura†
*Faculty of Economics, Ryukoku University*

February 17, 2007

**Abstract**

In this paper we examine a firm's capacity expansion and contraction policy with fixed and quadratic adjustment costs. Suppose that the firm's problem is to maximize expected total discounted profit. To solve the firm's problem, we formulate it as an impulse control problem. Then, we show a policy which is derived quasi-variational inequalities is optimal.

*Keywords*: capacity; adjustment costs; impulse control

## 1 Introduction

In the last two decade the theory of investment under uncertainty has made great strides. Dixit and Pindyck (1994) is an excellent textbook of this filed. Many literatures study the case in which the investment is irreversible. The idea behind of the irreversible investment is that the investment expenditure is sunk cost. See Dixit and Pindyck (1994) p. 8 for more detail. In the last decade, as an interesting extension of the theory of investment under uncertainty, reversible investment has been examined. See, for example, Abel and Eberly (1996, 1997) and Dixit and Pindyck (1999).

This paper also examines reversible investment. To be specific, this paper examines a firm's capacity expansion and contraction policy. In this context, we assume that the investment expenditure is not sunk cost. Firms can sell their capacity to the other firms. Thus, we assume that the capacity is not firm-specific capacity. Furthermore, we assume that when the firm expands (contracts) capacity, it incurs fixed and quadratic adjustment costs. Following the line of Abel and Eberly (1997), we assume that a firm produces a single output by using capital and

*The authors thank Masaaki Kijima for helpful comments.
†Corresponding author
Address: 67 Tsukamoto-cho, Fukakusa Fushimi-ku, Kyoto 612-8577 JAPAN
Phone: +81-75-645-8460; E-mail: tsujimura@econ.ryukoku.ac.jp

labor and sells it in competitive market. The output price process is governed by the stochastic differential equation. We also assume that the firm has a Cobb-Douglas production function. The firm wants to maximize the expected total discounted profit. To solve the firm's problem, we formulate the firm's problem as an impulse control problem. Then, we show that a policy which is derived quasi-variational inequalities (QVI) is optimal and that a solution of the QVI is the value function of the firm's problem.

Recently Hartman and Hendrickson (2002), Chiarolla and Haussmann (2005), Guo and Pham (2005), and Merhi and Zervos (2005) also investigate the capacity expansion and contraction problems as related papers. In these papers, firms are assumed to maximize their profits as in this paper. Hartman and Hendrickson (2002) study the firm's capacity expansion and contraction problem as a singular stochastic control problem. Chiarolla and Haussmann (2005) study the firm's capacity expansion problem as a singular stochastic control problem. Guo and Pham (2005) study the entry and the capacity expansion and contraction problems with proportional adjustment costs. To solve the firm's problem, they formulate the firm's problem as a singular stochastic control problem. Their adjustment cost is assumed to be linear and asymmetric. When the firm expands the capacity, it incurs the cost. On the other hand, the firm can contract the capacity at less cost. Chiarolla and Haussmann (2005) and Guo and Pham (2005) assume that the firm's capacity directly evolves the stochastic differential equation. Then, the firm directly controls the capacity. Merhi and Zervos (2005) also study the firm's capacity expansion and contraction problem as a singular stochastic control problem. They solve the firm's problem explicitly. The main difference between these papers and this paper is that this paper treats with the fixed adjustment cost. Then, this paper adopts impulse control method.

The rest of the paper is organized as follows. Section 2 describes the firm's problem. Section 3 shows a capacity expansion and contraction policy (QVI-policy) is optimal. Lastly, Section 4 concludes the paper.

## 2 The Model

We assume that a firm produces a single output by using capital $K_t$ and labor $L_t$ and sells it in competitive market as in Abel and Eberly (1997). The output price process $P = \{P_t\}_{t\geq 0}$ is governed by the following stochastic differential equation:

$$\mathrm{d}P_t = \mu_P P_t \mathrm{d}t + \sigma_P P_t \mathrm{d}W_t, \quad P_{0^-} = p\ (>0), \tag{2.1}$$

where $\mu_P$ $(\in \mathbb{R})$ and $\sigma_P$ $(>0)$ are constants. $W_t$ is a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$, where $\mathcal{F}_t$ is generated by $W_t$ in $\mathbb{R}$, i.e., $\mathcal{F}_t = \sigma(W_s, s \leq t)$. We also assume that the firm's production function $F(L_t, K_t)$ has the form as

$$F(L_t, K_t) = L_t^{\gamma} K_t^{1-\gamma}, \tag{2.2}$$

where $\gamma \in (0,1)$. The firm's operating profit at time $t$ is given by

$$P_t F(L_t, K_t) - wL_t, \tag{2.3}$$

where $w$ is a constant wage. We assume that labor is assumed to be costlessly and instantaneously adjusted. Then, the firm's maximized instantaneous operating profit $\tilde{\pi}(K_t, P_t)$ at time $t$ is derived as

$$\tilde{\pi}(K_t, P_t) = hP_t^{\alpha} K_t, \tag{2.4}$$

where $\alpha = 1/(1-\gamma)$ $(> 1)$ and $h = \alpha^{-\alpha}(\alpha-1)^{\alpha-1}w^{1-\alpha}$ $(> 0)$.

The firm controls the capital stock corresponding to the business environment. That is, capital investment is reversible. When the business environment is good (resp. severe), the firm expands (resp. contracts) the scale of capital stock, i.e., the capacity. If the business environment is strictly severe, the firm sells the remaining capacity and exits the market. In this paper, we assume that if the output price is less than or equal to $\check{p}$, the firm sells the remaining capacity and exits the market. We assume that the amount of the capacity is changed by $\xi_i$ at time $\tau_i$, $i = 0, 1, \cdots$. Then, the dynamics of the capacity is governed by

$$\begin{cases} \mathrm{d}K_t = -\delta K_t \mathrm{d}t, & \tau_i \le t < \tau_{i+1}, \\ K_{\tau_i} = K_{\tau_i^-} + \xi_i, \\ K_{0^-} = k_0 \ (> 0), \end{cases} \tag{2.5}$$

where $\delta$ $(\ge 0)$ is a constant depreciation rate. We put $\tau_0^- = 0$ and $\tau_{i+1}^- = \tau_i$ if $\tau_{i+1} = \tau_i$. These $\xi_i$ and $\tau_i$ correspond to the impulses and the stopping times in impulse control theory, respectively. A capacity expansion and contraction policy (CEC policy) is defined as the following double sequence:

$$\tilde{v} := \{(\tau_i, \xi_i)\}_{i \ge 0}. \tag{2.6}$$

When the firm controls the capacity, it incurs the fixed and proportional adjustment costs. We assume that the convex adjustment costs function $C(\xi_i)$:

$$C(\xi_i) = \begin{cases} c_0^+ + \tilde{c}_1^+ \xi_i + \tilde{c}_2^+ \xi_i^2, & \xi_i \ge 0, \\ c_0^- + \tilde{c}_1^- \xi_i + \tilde{c}_2^- \xi_i^2, & \xi_i < 0, \end{cases} \tag{2.7}$$

where $c_0^{+,-}$ $(> 0)$ is the fixed adjustment cost, $\tilde{c}_1^{+,-}$ $(> 0)$ is the price of purchasing or selling capital, and $\tilde{c}_2^{+,-}$ $(> 0)$ is quadratic adjustment cost parameters[1]. For simplicity, we assume that $c_0 = c_0^{+,-}$ and $\tilde{c}_2 = \tilde{c}_2^{+,-}$. Note that, for $\xi, \xi'$ $(\ge 0)$, $C(\xi_i)$ satisfies the following:

$$C(\xi + \xi') \ge C(\xi) + C(\xi'); \tag{2.8}$$

$$C(\xi) \le C(\xi'), \quad \text{if } \xi \le \xi'. \tag{2.9}$$

Inequality (2.8) represents superadditivity with respect to $\xi$ and implies that reasonable $\{\mathcal{F}_t\}_{t\ge 0}$-stopping times become strictly increasing sequences; i.e., $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_i < \cdots < \infty$.

Let change the variable $P_t^\alpha K_t$ to $X_t$:

$$X_t := P_t^\alpha K_t \tag{2.10}$$

so that we consider the direct control on the state variable. See, for example, Vollert (2003) pp. 61–62 for more detail. Then, it follows from (2.1) and (2.5) that the dynamics of $X_t$ is given by:

$$\begin{aligned} \mathrm{d}X_t &= \left(\alpha\mu_P + \frac{1}{2}\alpha(\alpha-1)\sigma_P^2 - \delta\right) X_t \mathrm{d}t + \alpha\sigma_P X_t \mathrm{d}W_t, \\ &= \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}W_t, \quad X_{0^-} = x \ (> 0), \end{aligned} \tag{2.11}$$

[1] In Abel and Eberly (1997) p. 836, the cost function is formulated as $C(\xi) = \tilde{c}_1^+ \xi_i + \tilde{c}_2^+ \xi_i^n$, $n \in \{2, 4, 6, \cdots\}$. That is, our formulation (2.7) expands the cost function of Abel and Eberly (1997) by adding the fixed adjustment cost and putting $n = 2$.

where $\mu = \alpha\mu_P + \alpha(\alpha-1)\sigma_P^2/2 - \delta$ and $\sigma = \alpha\sigma_P$. Let $\zeta_i$ be defined by

$$\zeta_i = P_{\tau_i}^{\alpha}\xi_i. \tag{2.12}$$

Then, the CEC policy $\tilde{v}$ becomes $v$ defined by

$$v := \{(\tau_i, \zeta_i)\}_{i\geq 0}. \tag{2.13}$$

Thus, the dynamics of the controlled state variable $X_t^v$ denotes

$$\begin{cases} \mathrm{d}X_t^v = \mu X_t^v \mathrm{d}t + \sigma X_t^v \mathrm{d}W_t, & t \in [\tau_{i-1}, \tau_i), \\ X_{\tau_i}^v = X_{\tau_i^-}^v + \zeta_i, \\ X_{0^-}^v = x \ (> 0). \end{cases} \tag{2.14}$$

Furthermore, the profit function $\pi$ and the cost function $C$ become respectively as follows:

$$\pi(X_t^v) = hX_t^v; \tag{2.15}$$

$$C(\zeta_i) = \begin{cases} c_0 + c_1^+\zeta_i + c_2\zeta_i^2, & \zeta_i \geq 0, \\ c_0 + c_1^-\zeta_i + c_2\zeta_i^2, & \zeta_i < 0. \end{cases} \tag{2.16}$$

We define the set of admissible CEC policy as follows:

**Definition 2.1 (Admissible investment policy).** *A CEC policy $v$ is admissible if the followings are satisfied:*

$$0 \leq \tau_i \leq \tau_{i+1}, \quad a.s. \quad i \geq 0; \tag{2.17}$$

$$\tau_i \textit{ is an } \{\mathcal{F}_t\}_{t\geq 0}\textit{-stopping time}, \quad i \geq 0; \tag{2.18}$$

$$\zeta_i \textit{ is } \mathcal{F}_{\tau_i}\textit{-measurable}, \quad i \geq 0; \tag{2.19}$$

$$P\left[\lim_{i\to\infty} \tau_i \leq \hat{T}\right] = 0, \quad \hat{T} \in [0, \infty). \tag{2.20}$$

The condition given by (2.20) means that the CEC policy will only occur finitely before a terminal time, $\hat{T}$. See, for example, Cadenillas and Zapatero (2000). Let $\mathcal{V}$ be the set of admissible CEC policies.

The firm's expected discounted profit $J(x; v)$ is given by:

$$J(x; v) = \mathbb{E}\left[\int_0^\infty e^{-rt}\pi(X_t^v)\mathrm{d}t - \sum_{i=1}^\infty e^{-r\tau_i}C(\zeta_i)1_{\tau_i<\infty}\right], \tag{2.21}$$

where $r$ $(> 0)$ is a discount rate. In this context we assume that

$$r > \mu. \tag{AS.1}$$

Therefore, the firm's problem is to maximize the expected discounted profit over $\mathcal{V}$:

$$V(x) = \sup_{v\in\mathcal{V}} J(x; v) = J(x; v^*), \tag{2.22}$$

where $V$ is the value function and $v^*$ is an optimal CEC policy.

## 3 QVI Policy

In the previous section, we formulated the firm's problem as an impulse control problem. From that formulation, we naturally guess that, under an optimal CEC policy, the firm expands (resp. reduces) the capacity whenever the state process $X^v = \{X_t^v\}_{t\geq 0}$ reaches an upper (resp. lower) threshold. In order to verify this conjecture, we prove that a policy induced by the QVI is an optimal CEC policy for the firm's problem (2.22).

Suppose that $\phi : \mathbb{R}_{++} \to \mathbb{R}_{++}$ is a continuous function. Let $\mathcal{M}$ denote the capacity expansion and contraction operator on the space of functions $\phi$ defined by

$$\mathcal{M}\phi(x) = \sup_{\zeta\in(-x,\infty)} \{\phi(x+\zeta)) - C(\zeta)\}, \tag{3.1}$$

We assume that $\phi$ is a twice continuously differentiable function on $\mathbb{R}$, $C^2$, except on the boundary of the considered region. Let us define an operator $\mathcal{L}$ of the $X$ by

$$\mathcal{L}\phi(x) = \frac{1}{2}\sigma^2 x^2 \phi''(x) + \mu x\phi'(x) - r\phi(x). \tag{3.2}$$

Since $\phi$ is not necessarily $C^2$ in the whole region, we must apply the generalized Dynkin formula for $\phi$. The formula is available if $\phi$ is stochastically $C^2$. That is, we use the generalized Dynkin formula on the set which has a Green measure of $X^v$ zero. Here the Green measure of $X^v$ is the expected total occupation measure $\mathbb{G}(\cdot, x; v)$ defined by

$$\mathbb{G}(\Xi, x; v) = \mathbb{E}\left[\int_0^\infty X_t^v 1_\Xi \mathrm{d}t\right], \tag{3.3}$$

where $1_\Xi$ is the indicator of a Borel set $\Xi \subset \mathbb{R}$. The continuous function $\phi$ is called *stochastically* $C^2$ with respect to $X^v$ if $\mathcal{L}\phi(x)$ is well defined point wise for almost all $x$ with respect to the Green measure $\mathbb{G}(\cdot, x; v)$. Henceforce, we assume that $\phi$ is stochastically $C^2$ with respect to $X^v$. The following equality, which is called the *generalized Dynkin formula*, will be used in the proof of Theorem 3.1.

$$\mathbb{E}[e^{-r\theta-}\phi(X_{\theta-}^v)] = \mathbb{E}[e^{-r\tau_i}\phi(X_{\tau_i}^v)] + \mathbb{E}\left[\int_{\tau_i}^{\theta-} e^{-rt}\mathcal{L}\phi(X_t^v)\mathrm{d}t\right], \tag{3.4}$$

for all $i$, all bounded stopping times $\theta$ such that $\tau_i \leq \theta \leq \tau_{i+1}$. See Brekke and Øksendal (1998) for more details of the Green measure and the generalized Dynkin formula.

We are now in a position to define the quasi-variational inequalities (QVI).

**Definition 3.1 (QVI).** *The following relations are called the QVI for the firm's problem (2.22):*

$$\mathcal{L}\phi(x) + \pi(x) \leq 0, \quad \textit{for a.a. } x \textit{ w.r.t. } \mathbb{G}(\cdot, x; v),\ v \in \mathcal{V}; \tag{3.5}$$

$$\phi(x) \geq \mathcal{M}\phi(x); \tag{3.6}$$

$$[\mathcal{L}\phi(x) + \pi(x)][\phi(x) - \mathcal{M}\phi(x)] = 0, \quad \textit{for a.a. } x \textit{ w.r.t. } \mathbb{G}(\cdot, x; v),\ v \in \mathcal{V}. \tag{3.7}$$

Eq. (3.7) is the complementary condition and is able to be rewritten as

$$\mathcal{L}\phi(x) + \pi(x) = 0, \quad x \in \mathcal{C} \tag{3.8}$$

and

$$\phi(x) - \mathcal{M}\phi(x) = 0, \quad x \in \mathcal{I}, \tag{3.9}$$

where $\mathcal{C}$ is the continuation region defined by

$$\mathcal{C} := \{x; \phi(x) > \mathcal{M}\phi(x) \text{ and } \mathcal{L}\phi(x) + \pi(x) = 0\} \tag{3.10}$$

and $\mathcal{I}$ is the controlled region defined by

$$\mathcal{I} := \{x; \phi(x) = \mathcal{M}\phi(x) \text{ and } \mathcal{L}\phi(x) + \pi(x) < 0\}. \tag{3.11}$$

We define the policy which is derived from the QVI.

**Definition 3.2 (QVI policy).** *Let $\phi$ be a solution of the QVI. Then, the following CEC policy $\hat{v}$ is called a QVI policy:*

$$(\hat{\tau}_0, \hat{\zeta}_0) = (0, 0); \tag{3.12}$$

$$\hat{\tau}_i = \inf\{t \geq \hat{\tau}_{i-1}; X_t^{\hat{v}} \notin \mathcal{C}\}; \tag{3.13}$$

$$\hat{\zeta}_i = \arg\max \left\{ \phi\left( X_{\hat{\tau}_i^-}^{\hat{v}} + \zeta_i \right) - C(\zeta_i) : \zeta_i \right\}. \tag{3.14}$$

*In this context, $\hat{v}$ is defined by $\hat{v} = \{(\hat{\tau}_i, \hat{\zeta}_i)\}_{i \geq 0}$ and $X_t^{\hat{v}}$ is the result of applying the CEC policy $\hat{v}$.*

We can now prove that a QVI policy is an optimal CEC policy. The following theorem is well-known verification theorem. See, for example, Brekke and Øksendal (1998) and Ohnishi and Tsujimura (2002, 2006).

**Theorem 3.1.** *(I) Let $\phi$ be a solution of the QVI and satisfy the following:*

$$\phi \text{ is stochastically } C^2 \text{ w.r.t. } X^v; \tag{3.15}$$

$$\lim_{t \to \infty} e^{-rt}\phi(X_t^v) = 0, \quad a.s. \quad v \in \mathcal{V}; \tag{3.16}$$

$$\text{the family } \{\phi(X_\tau^v)\}_{\tau < \infty} \text{ is uniformly integrable w.r.t. } P \quad v \in \mathcal{V}. \tag{3.17}$$

*Then, we obtain*

$$\phi(x) \geq J(x; v) \quad v \in \mathcal{V}. \tag{3.18}$$

*(II) From (3.5) – (3.7) and (3.12), we have*

$$\mathcal{L}\phi(x) + \pi(x) = 0, \quad x \in H. \tag{3.19}$$

*Suppose $\hat{v} \in \mathcal{V}$, i.e., the CEC policy is admissible. Then, we obtain*

$$\phi(x) = J(x; \hat{v}). \tag{3.20}$$

*Hence, we have*

$$\phi(x) = V(x) = J(x; \hat{v}). \tag{3.21}$$

*Therefore, $\hat{v}$ is optimal.*

*Proof.* (I) Assume that $\phi$ satisfies (3.15) – (3.17). Choose $v \in \mathcal{V}$. Let $\theta_{i+1} := \tau_i \vee (\tau_{i+1} \wedge s)$ for any $s$ ($\geq 0$). Then, by the generalized Dynkin formula, (3.4), we obtain

$$\mathbb{E}[e^{-r\theta_{i+1}-}\phi(X^v_{\theta_{i+1}})] = \mathbb{E}[e^{-r\tau_i}\phi(X^v_{\tau_i})] + \mathbb{E}\left[\int_{\tau_i}^{\theta_{i+1}-} e^{-rt}\mathcal{L}\phi(X^v_t)\mathrm{d}t\right]. \tag{3.22}$$

Hence, from (3.5) we obtain

$$\mathbb{E}[e^{-r\theta_{i+1}-}\phi(X^v_{\theta_{i+1}})] \leq \mathbb{E}[e^{-r\tau_i}\phi(X^v_{\tau_i})] - \mathbb{E}\left[\int_{\tau_i}^{\theta_{i+1}-} e^{-rt}\pi(X^v_t)\mathrm{d}t\right]. \tag{3.23}$$

Taking $\lim_{s\to\infty}$ and using the dominated convergence theorem, we have

$$\mathbb{E}[e^{-r\tau_{i+1}-}\phi(X^v_{\tau_{i+1}-})] \leq \mathbb{E}[e^{-r\tau_i}\phi(X^v_{\tau_i})] - \mathbb{E}\left[\int_{\tau_i}^{\tau_{i+1}} e^{-rt}\pi(X^v_t)\mathrm{d}t\right]. \tag{3.24}$$

Summing from $i = 0$ to $i = m$ yields

$$\begin{aligned}\phi(x) + \sum_{i=1}^{m}\mathbb{E}[e^{-r\tau_i}\phi(X^v_{\tau_i}) - e^{-r\tau_i-}\phi(X^v_{\tau_i-})] \geq& \mathbb{E}[e^{-r\tau_{m+1}-}\phi(X^v_{\tau_{m+1}-})] \\ &+ \mathbb{E}\left[\int_0^{\tau_{m+1}} e^{-rt}\pi(X^v_t)\mathrm{d}t\right].\end{aligned} \tag{3.25}$$

For all $\tau_i < \infty$, following the investment policy, the state process $X^v$ jumps immediately from $X^v_{\tau_i^-}$ to a new state level $X^v_{\tau_i^-} + \zeta_i$. Thus, by (3.1) and $X^v_{\tau_i^-} + \zeta_i = X^v_{\tau_i}$ we obtain

$$\phi(X^v_{\tau_i}) \leq \mathcal{M}\phi(X^v_{\tau_i-}) + C(\zeta_i). \tag{3.26}$$

Thus, we have

$$\begin{aligned}&\phi(x) + \sum_{i=1}^{m}\mathbb{E}\left[\left[e^{-r\tau_i}\mathcal{M}\phi\left(X^v_{\tau_i^-}\right) - e^{-r\tau_i^-}\phi\left(X^v_{\tau_i^-}\right)\right]1_{\{\tau_i<\infty\}}\right] \\ &\quad\geq \mathbb{E}\left[\int_0^{\tau_{m+1}} e^{-rt}\pi(X^v_t)\mathrm{d}t - \sum_{i=1}^{m} e^{-r\tau_i}C(\zeta_i)1_{\{\tau_i<\infty\}} + e^{-r\tau_{m+1}^-}\phi\left(X^v_{\tau_{m+1}^-}\right)\right].\end{aligned} \tag{3.27}$$

It follows from (3.6) that

$$\mathcal{M}\phi\left(X^v_{\tau_i^-}\right) - \phi\left(X^v_{\tau_i^-}\right) \leq 0. \tag{3.28}$$

Hence, we obtain that

$$\phi(x) \geq \mathbb{E}\left[\int_0^{\tau_{m+1}} e^{-rt}\pi(X^v_t)\mathrm{d}t - \sum_{i=1}^{m} e^{-r\tau_i} C(\zeta_i) 1_{\{\tau_i<\infty\}} + e^{-r\tau_{m+1}-}\phi\left(X^v_{\tau^-_{m+1}}\right)\right]. \tag{3.29}$$

Taking $\lim_{m\to\infty}$ and using (3.16), (3.17) and the dominated convergence theorem, we obtain

$$\phi(x) \geq \mathbb{E}\left[\int_0^{\infty} e^{-rt}\pi(X^v_t)\mathrm{d}t - \sum_{i=1}^{\infty} e^{-r\tau_i} C(\zeta_i) 1_{\{\tau_i<\infty\}}\right]. \tag{3.30}$$

Therefore, (3.18) is proved.

(II) Assume that (3.19) holds and $\hat{u}$ is the QVI policy. Then, repeat the argument in part (I) for $v = \hat{v}$. Then, inequalities (3.23) through (3.30) become equalities. Thus, we obtain

$$\phi(x) = \mathbb{E}\left[\int_0^{\infty} e^{-rt}\pi(X^{\hat{v}}_t)\mathrm{d}t - \sum_{i=1}^{\infty} e^{-r\hat{\tau}_i} C(\hat{\zeta}_i) 1_{\{\hat{\tau}_i<\infty\}}\right]. \tag{3.31}$$

Hence, we obtain (3.20). Combining (3.20) with (3.18), we obtain

$$\phi(x) \geq \sup_{v\in\mathcal{V}} J(x;v) \geq J(x;\hat{v}) = \phi(x). \tag{3.32}$$

Therefore $\phi(x) = V(x)$ and $v^* = \hat{v}$ is optimal, i.e., the solution of the QVI is the value function and the QVI policy is optimal. The proof is completed.

□

# 4 Conclusion

In this paper we examined a firm's capacity expansion and contraction policy with fixed and quadratic adjustment costs. Suppose that the firm's problem is to maximize expected total discounted profit. To this end, we formulated it as an impulse control problem. Then, we showed a policy which was derived quasi-variational inequalities is optimal.

Our remaining problem is to show an optimal investment policy.

# References

A. B. Abel and J. C. Eberly, 1996. Optimal Investment with Costly Reversibility, *Review of Economic Studies*, 63, pp. 581–593.

A. B. Abel and J. C. Eberly, 1997. An exact solution for the investment and value of a firm facing uncertainty, adjustment costs, and irreversibility, *Journal of Economic Dynamics and Control*, 21, pp. 831–852.

K. A. Brekke and B. Øksendal, 1998. A Verification Theorem for Combined Stochastic Control and Impulse Control, in: Decreusefond, L. et al. (eds.), *Stochastic Analysis and Related Topics*, Vol.6. Bikhäuser, Basel. pp. 211–220.

A. Cadenillas and F. Zapatero, 2000. Classical and Impulse Stochastic Control of the Exchange Rate Using Interest Rates and Reserves, *Mathematical Finance*, 10, pp. 141–156.

M. B. Chiarolla, and U. G. Haussmann, 2000. Explicit solution of a stochastic, irreversible investment problem and its moving threshold, *Mathematics of Operations Research*, 30, pp. 91–108.

A. Dixit and R. S. Pindyck, 1994. *Invetment under Uncertainty*, Princeton University Press, New Jersey.

A. Dixit and R. S. Pindyck, 1999. Expandability, Reversibility, and Optimal Capacity Choice, in: M. J. Brennan and L. Trigeorgis (eds), *Project Flexibility, Agency, and Competition: New Developments in the Theory and Application of Real Options*, Oxford University Press, New York. pp. 50–70.

X. Guo and H. Pham, 2005. Optimal partially reversible investment with entry decision and general production function, *Stochastic Processes and Their Applications*, 115, pp. 705–736.

R. Hartman and M. Hendrickson, 2002. Optimal partially reversible investment, *Journal of Economic Dynamics and Control*, 26, pp. 483–508.

T. Ø. Kobila, 1991. Partial Investment Under Uncertainty, in: D. Lund and B. Øksendal (eds.), *Stochastic Models and Option Values: Applications to Resources, Environment and Investment Problems*, North-Holland, Amsterdam. pp. 167–186.

A. Merhi and M. Zervos, 2005. A model for reversible investment capacity expansion, submitted.

M. Ohnishi and M. Tsujimura, 2002. Optimal Dividend Policy with Transaction Costs under a Brownian Motion, *Discussion Papers in Economics and Business*, Osaka University, 02-07.

M. Ohnishi and M. Tsujimura, 2006. An Impulse Control of a Geometric Brownian Motion with Quadratic Costs, *European Journal of Operational Research*, 168, pp. 311-321.

B. Øksendal, 1999. Stochastic Control Problems where Small Intervention Costs Have Big Effects, *Applied Mathematics and Optimization*, 40, pp. 355–375.

R. S. Pindyck, 1988. Irreversible investment, capacity choice, and the value of the firm, *American Economic Review*, 79, pp. 969–985.

W. H. Press, S. A. Teukolsky, W. T. Vetterling and B. P. Flannery, 1992. *Numerical Recipes in Fortran: The Art of Scientific Computing*, 2nd edition. Cambridge:Cambridge University Press.

A. Vollert, 2003. *A Stochastic Control Framework for Real Opitions in Strategic Evaluation*, Birkhüser, Boston.

Figure 1: Sample paths of the output price process $P$ and capital stock process $K$.