Long-Term Economic Relationships and Correlation Structure in Commodity Markets*

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Long-Term Economic Relationships and Correlation Structure in Commodity Markets

Abstract

This paper finds that the long-term co-movement among commodities is driven by economic relations, such as, production, substitution or complementary relationships. These economic linkages imply that expected commodity prices, which are determined by convenience yields and risk premia among other factors, tend to move with each other. This source of co-movement is not captured by traditional commodity pricing models. We build a model where the convenience yield of a certain commodity is determined, among other things, by the prices of related commodities. We test this prediction in a multi-commodity model that disentangles a short-term source of co-movement from the long-term component allowing for a flexible correlation structure. We estimate the model using three commodity pairs: heating oil-crude oil, WTI-Brent crude oil and heating oil-gasoline. We find that long-term relations are pervasive and significant, both, statistically and economically. The correlation structure implied from our model matches the upward sloping patterns observed in the data. The long-term economic relationship considerably reduces the long-term volatility of the spread between commodities which implies lower spread option prices. An out-of-sample test using short-maturity crack spread options data shows that our model considerably reduces the negative bias generated by traditional models.

Keywords: correlation structure, long-term economic relationships, commodity prices, convenience yields, cross-commodity feedback effects, spread options.

JEL Classification: C0, G12, G13, D51, D81, E2.

1 Introduction

Commodity markets have experienced dramatic up-and-down movements in a relatively short period of time. Closest-to-maturity crude-oil futures have increased from almost \$50 per barrel in January 2007 to \$147 per barrel in July 2008, the highest level in history since it is traded in NYMEX. Surprisingly, only 5 months later, the oil price dropped to almost \$30 per barrel. The energy sector, agricultural commodities and industry metals all have experienced similar patterns. While academics and policy makers are still trying to understand the causes of this behavior, the following stylized facts, among others, have been reinforced after the turmoil: 1) commodity prices are volatile, 2) spot and futures prices are mean-reverting, and 3) prices of multiple commodities co-move. These characteristics play a critical role in modeling financial contingent claims on commodities.

Since Keynes (1923), many scholars have studied the stochastic behavior of individual commodities. However, relationships between multiple commodities have received little attention in theoretical modeling and commodity-related contingent-claim pricing. These cross-commodity relationships imply that two or several commodities share an equilibrium that links prices in the long run. Examples of economic long-term relationships between commodities include production relationship where upstream commodity and downstream commodity are tied in a production process, and substitute (or complementary) relationships where two commodities serve as substitute (or complement) in either consumption or production.

Temporary deviation from the long-term relation (because of demand and supply imbalances caused by macro-economic factors and inventory shocks, etc.) will be corrected over the long run. This implies that co-movement exists not only in spot prices, but also in expected prices, which are determined by convenience yield and risk premia, among others. The linkage among expected prices suggests for example, that the convenience yield of one commodity is affected by the spot price of other commodities.

Figure 1 shows the correlation structure of weekly futures returns for the heating - crude oil and for the WTI - Brent crude oil pairs between 1986.07 to 2009.04. These commodity pairs follow a production and a substitute relationship, respectively. The plot shows upward sloping correlation structures for both commodity pairs. Prices are tied by the long-term relationship which translates into higher long-term correlations. Interestingly, traditional commodity pricing models, such as correlated versions of the Gibson and Schwartz (1990) (hereafter GS) and the Casassus and Collin-Dufresne (2005) (hereafter CCD) models, are unable to match this evidence. Moreover, the correlation structure is crucial for the pricing of commodity spread options, which suggest that the option prices implied by the traditional models have strong biases. We propose a reduced-form model that allows for a flexible correlation structure that matches the pattern observed in the data. We find that for long-maturity spread options, the prices implied by our model are lower than the ones predicted by the traditional models, because the higher long-term correlation reduces the volatility of the spread. We show that the opposite occurs for short-maturity options. Moreover, an out-of-sample test using short-maturity crack spread options data shows that our model reduces the negative bias in prices to approximately one half the size of the bias in traditional models.

In econometrics, long-term equilibrium relationships are usually expressed in the format of cointegration or Error Correction Models (ECMs). Engle and Granger (1987) shows that the ECM is identical with a cointegration model if the underlying time series are non-stationary. An ECM predicts that the adjustment in a dependent variable depends not only on the explanatory variables but also on the extent to which an explanatory variable deviates from the equilibrium (refer to Banerjee, Dolado, Galbraith, and Hendry 1993). Many scholars have empirically studied the cointegration and ECM relationships among commodities. Among them, Pindyck and Rotemberg (1990) tests and confirm the existence of a "puzzling" phenomenon - the prices of raw commodities have a persistent tendency to move together. Ai, Chatrath, and Song (2006) documents that the market-level indicators such as inventory and harvest size explain a strikingly large portion of price co-movements. Malliaris and Urrutia (1996) documents a long-term cointegration among prices of agricultural commodity futures contracts from CBOT. Girma and Paulson (1999) finds a cointegration relationship in petroleum futures markets. Recently, Paschke and Prokopczuk (2007) and Cortazar, Milla, and Severino (2008) have studied the statistical relationship among commodities in a multi-commodity affine framework using futures prices. However, none of these models gives an economic foundation about which type of assets and why prices of multiple commodities move together through time. To the best of our knowledge no previous research has looked at the patterns of co-movements among multiple commodities under long-term economic relationships.

Our reduced-form model is part of a growing literature on asset pricing that studies the dynamics of commodity prices. This literature documents the following stylized facts for single commodities: the existence of a stochastic convenience yield (e.g. GS and Brennan 1991), mean-reversion in prices (e.g. Bessembinder, Coughenour, Seguin, and Smoller 1995, and Schwartz 1997), seasonality (e.g. Richter and Sørensen 2002), time-varying risk-premia (e.g. CCD) and stochastic volatility (e.g. Trolle and Schwartz 2009).

Our paper is organized as follows. Section 2 identifies three economic equilibrium relationships and provide examples of such relationships. Section 3 solves an economic model for the case of two commodities that have a production relationship and generates an endogenous cross-commodity feedback effect. Guided by the economic model, section 4 develops an empirical model that captures the co-movement among prices (and price dynamics) in a multi-commodity system. We also show that our model is an extension of the "maximal" affine model to a multi-asset case. Section 5 describes the estimation of the model and shows the estimation results. Section 6 presents the valuation of spread options under our multi-commodity framework and shows a out-of-sample comparison of different pricing models, and section 7 concludes.

2 Long-Term Economic Relationships

The co-movement of commodity prices and the existence of long-term relationships are pervasive in the economy. Examples of the economic relationships between different commodities include, but are not restricted to the following cases:

Production Relationships

One commodity can be produced from another commodity when the former is the output of a production process that uses the other commodity as an input factor. For example, the petroleum refining process "cracks" crude oil into its constituent products, among which heating oil and gasoline are actively traded commodities on the New York Mercantile Exchange along with crude oil. Spread futures and spread options, such as the 3:2:1 crack spreads (the purchase of three crude oil futures with the simultaneous sales of two unleaded gasoline futures and one heating oil future), are widely used by refiners and oil investors to lock in profit margins. A similar production relationship can be found in the soybean complex. Soybeans can be crushed into soybean meal and soybean oil. The three commodities in the complex are traded separately on the Chicago Board of Trade. By analogy to the crack spread, the crush spread is also an actively traded derivative. Not all production-linked commodities have spread derivatives established for trading. Aluminum - Aluminum Alloy and corn - ethanol are other examples of the production-linked relationships without spread trading.

Substitution Relationships

A Substitution relationship exists when two traded commodities are substitutes in consumption. Crude oil and natural gas are commonly viewed as substitute goods. Competition between natural gas and petroleum products occurs principally in the industrial and electric generation sectors. According to the EIA Manufacturing Energy Consumption Survey (Energy Information Administration 2002), approximately 18 percent of natural gas usage can be switched to petroleum products. Other analysts estimate that up to 20 percent of power generation capacity is dual-fired. West Texas Intermediate (WTI), a type of crude oil often referenced in North America, and Brent crude oil from the North Sea, are commonly used as benchmarks in oil pricing and the underlying commodity of NYMEX oil futures contracts. WTI and Brent crude represent an example of a substitute relationship. Recently NYMEX started to trade WTI-Brent spread option. Corn and soybean meal serve as substitute cattle feeds. Ethanol - petroleum products are potentially competitive products. Furthermore crude oil and minerals from industrial metals are generally concentrated in developing countries whose economy relies heavily on commodity exports. Therefore investment and production would be shifted to the commodity that yields the highest profit.

Complementary Relationships

A complementary relationship exists when two commodities share a balanced supply or are complementary in either consumption or production. Let's consider the case of gasoline and heating oil. If the gasoline price increases dramatically, and crude oil is cracked to supply gasoline, this process also produces heating oil and may result in a drop in the price of heating oil. The relationship between these two commodities is one of complementarity. On the other hand, lead, tin, zinc and copper are often smelted from paragenesis mineral deposits. The equilibrium assemblage of mineral phases gives those industrial metals a natural relationship in supply. In addition, industrial metals are seldom used in their pure forms. They find most applications in the form of alloys. For example, the principal alloys of tin are bronze (tin and copper), soft solder (tin and lead), and pewter (75% tin and 25% lead). Two-thirds of nickel stocks are used in stainless steel, an alloy of steel. In 1998, 48% of zinc was applied as zinc coatings, jointly used with aluminium.

The three above-mentioned economic relationships can be present simultaneously among commodities. For example, while complementarity exists between gasoline and heating oil, some substitutability is also at play. In the following section we present a simple structural model for the production relationship and its implication for the prices dynamics. The structural models of substitution and complementary relationships are presented in appendix B.

3 The Economic Model

Commodity prices link two interconnected markets: the cash (or futures) market and the inventory market. Immediate ownership of a physical commodity offers some benefit or convenience that is not provided by futures ownership. This benefit, in terms of a rate, is called the "convenience yield" (see Brennan 1991, and Schwartz 1997). The "Theory of Storage" of Kaldor (1939), Working (1948) and Telser (1958), predicts that the return from purchasing a commodity and selling it for delivery (using futures) equals the interest forgone less the convenience yield net of storage costs. The convenience yield is attributed to the benefit of protecting regular production from temporary shortages of a particular commodity or by taking advantage of a rise in demand and price without resorting to a revision of the production schedule.

The traditional presentation of the Theory of Storage proposes that *a "high" convenience yield is associated with a "high" spot price* (see Pindyck 2001). If we only consider the market for any single commodity, the statement indicates: 1) the convenience yield is an increasing function of the spot price; or 2) there is a positive correlation of *incremental changes* between the spot price and the convenience yield. This paper extends the Theory of Storage by introducing a third interpretation, i.e., 3) a high level of convenience yield of a particular commodity corresponds to a high *price-level difference* between related commodities. The first two interpretations have been studied by several authors. For example, CCD explicitly models the positive dependence of the convenience yield on the spot price and the instantaneous positive correlation between the spot price and the convenience yield. However, the third interpretation has received little attention so far.

To motivate the importance of the third interpretation, let's give an example where interpretation 1) is violated, however it is consistent with interpretation 3). Consider a system of two commodities, heating oil and crude oil, where there is a production relationship in long-term equilibrium. Assume at time 0 that heating and crude oil are \$20 and \$15, respectively, while at time 1 they move to \$22 and \$21, respectively. First, let's consider the convenience yield of heating oil. If we look only at the heating oil market, since the heating oil is more expensive at time 1, we expect to have a greater convenience yield at time 1 than at time 0. However, if we look at both markets –heating oil and crude oil– together, we should expect the convenience yield of heating oil to be smaller at time 1 than at time 0. Indeed, since heating oil is only refined from crude oil, a high spread between heating and crude oil at time 0 (i.e., the high production profit), indicates that the refining capability cannot satisfy the strong demand for heating oil. Thus heating oil is relatively scarce and should have relatively higher convenience yields than at time 1 when the heating oil is very likely in abundance. Thus, the relative prices of heating and crude oil do influence the convenience yield of the commodities. The dependence of the convenience yield of a certain commodity on other commodities is not part of the traditional Theory of Storage.

We provide the intuition with a simple example of a production economy highlighting the economic relationship between crude oil an heating oil. This economy builds on the single commodity equilibrium models of Casassus, Collin-Dufresne, and Routledge (2008) and Routledge, Seppi, and Spatt (2000) and is similar in spirit to the cross-commodity model of Routledge, Seppi, and Spatt (2001).

The economy has a capital sector (K_t) and two storable commodity sectors: crude oil ($Q_{1,t}$) and heating oil ($Q_{2,t}$). An infinitively-lived representative agent derives utility from the following two consumption goods: heating oil and the standard consumption good from the capital sector that is used as the numeraire. The representative agent maximizes expected log utility with respect to consumption of capital ($C_{K,t}$), consumption of heating oil ($C_{2,t}$) and demand for crude oil ($q_{1,t}$):[1]

$$\sup_{\{C_{K,t},C_{2,t},q_{1,t}\}\in\mathcal{A}} \mathbb{E}_0\left[\int_0^\infty e^{-\theta\, t}\left(\phi\log\left(C_{K,t}\right)+(1-\phi)\log\left(C_{2,t}\right)\right)dt\right] \tag{1}$$

where $\mathcal{A}$ is the set of admissible strategies. The optimization problem is subject to the following processes that describe the dynamics of the stocks of capital, crude oil and heating oil, respectively:

$$dK = (\alpha\, K - C_K)dt + \sigma_K\, K\, dW_K \tag{2}$$

$$dQ_1 = -q_1\, dt \tag{3}$$

$$dQ_2 = (\gamma\, \log(q_1) Q_2 - C_2)dt. \tag{4}$$

The production rate of heating oil is an increasing function of the input quantity q_1 that flows from the crude oil stocks. For simplicity, we assume that this rate has a logarithmic form and that crude oil can be used only as an input to the heating oil technology. We assume that the capital sector has a constant return-to-scale technology. Finally, following Cox, Ingersoll Jr., and Ross (1985), we assume that the output of the capital sector is stochastic. Uncertainty in the economy is captured by the Brownian motion W_K and σ_K is the volatility of output returns.

As expected, the representative agent optimally consumes a constant fraction of capital ($C_K = \theta\, K$), a constant fraction of heating oil ($C_2 = \theta\, Q_2$), and demands a constant rate of crude oil ($q_1 = \theta\, Q_1$).[2] The market-clearing prices are determined by marginal utility indifference. The commodity prices correspond to the amount of capital the representative agent is willing to give for an extra unit of commodity (i.e. the shadow price). In this simple economy the equilibrium prices for crude oil (S_1) and heating oil (S_2) are given by:

$$S_1 = \frac{1-\phi}{\phi}\frac{\gamma}{\theta}\frac{K}{Q_1} \quad \text{and} \quad S_2 = \frac{1-\phi}{\phi}\frac{K}{Q_2}. \tag{5}$$

The equilibrium convenience yields are related to the marginal productivity of each commodity in the economy (see Casassus, Collin-Dufresne, and Routledge 2008). A relevant prediction for us is that the convenience yield of heating oil (δ_2) is a time-varying and increasing function of the crude

[1] These variables are all time dependent. Hereafter, we drop this dependance throughout the paper to simplify the notation.

[2] See Appendix A for a sketch of the solution to the representative agent's problem.

oil stocks:[3]

$$\begin{aligned}\delta_2 &= \gamma \log(\theta\, Q_1) \\ &= \gamma \left[\log\left(\frac{1-\phi}{\phi}\gamma K\right) - \log(S_1)\right] \end{aligned} \tag{6}$$

Furthermore, since the crude oil price (S_1) is decreasing in its stock (Q_1), equation (6) shows that the heating oil convenience yield is a decreasing linear function of the (log) crude oil price plus another risk factor that in this case is $\log(K)$. Higher crude oil inventories imply lower crude oil prices and higher heating oil convenience yields. The intuition is the following. Since the production rate of heating oil is increasing in the crude oil inventories, more inventories of crude oil today imply more inventories and lower prices of heating oil in the near future. The heating oil spot price is expected to decrease, which in this model implies lower futures prices.[4]

This cross-commodity relationship exists because crude oil is an input for heating oil production. The model can be extended in several ways, but as long as the production relationship exists, the crude oil price will influence the heating oil price dynamics (through the heating oil convenience yield). Appendices B.1 and B.2 provide structural models for the substitute and complementary relationship respectively, which show a similar phenomenon to the one mentioned above.

In summary, if an economic relationship exists among commodities, the structural model predicts that the dynamics of a certain commodity is partly determined by the behavior of related commodities. In particular, the structural model suggests that the cross-commodity connection is through the convenience yields. In the next section, we propose a reduced-form model with the interdependence of the convenience yield on other commodity prices that is in line with our theoretical prediction.

4 The Empirical Model

This section introduces a reduced-form model that is consistent with the stylized facts from economically related commodities (i.e. upward sloping correlation structure, stochastic convenience

[3]In this simplified economy, the convenience yield of crude oil is zero.

[4]Indeed, in this economy the heating oil risk-premium is constant (σ_K^2). The interest rate is also constant ($r = \alpha - \sigma_K^2$), thus all the action in the expected spot price is given by the time-varying convenience yield.

yields, mean-reversion, etc.). Our multi-commodity model is parsimonious in the sense of "maximal" affine models.[5] We prefer to build a maximal model in order to avoid the risk of model mis-specification. Furthermore, we distinguish two sources of co-movement across commodities: 1) a short-term effect associated to the correlation of commodity prices, and 2) a long-term effect that is a consequence of the economic relationship. The long-term effect manifests in that the dynamics of one commodity is a function of the other commodities in the economy. In particular, we choose a representation of the convenience yield in such a way that the long-term effect is present, because as shown in the previous section the convenience yield of a particular commodity depends on the other commodities.

4.1 The Data-generating Processes

Assume there are n commodities in the system, in which the commodities have an long-term economic relationships. Denote

$$x_i = \log(S_i) \quad \text{for} \quad i = 1, \ldots, n \tag{7}$$

where S_i is the spot price of commodity i. Under the physical measure ($\mathbb{P}$), we assume the (log) spot prices follow Gaussian processes

$$dx_i = (\widetilde{\mu}_i - \delta_i)dt + \sigma_i dW_i \quad \text{for} \quad i = 1, \ldots, n \tag{8}$$

where δ_i is the convenience yield of commodity i, and $\widetilde{\mu}_i$ and σ_i are constants. Here, W_i ($i = 1, \ldots, n$) are correlated Brownian motions. Motivated by our structural framework above, we propose a specification where the convenience yield of commodity i, δ_i, is a function of the spot prices of the n commodities in the economy. Furthermore, there are also n extra latent factors, η_j ($j = 1, \ldots, n$), affecting the n convenience yields. For simplicity, we consider an *affine* relationship

[5] An affine structure is the standard framework for commodity pricing reduced-form models (see for example, GS and Schwartz 1997). See Dai and Singleton (2000) and CCD for the definition of "maximal" in this context.

among the convenience yields and the risk factors. Therefore,

$$\delta_i = -\sum_{j=1}^{n} b_{i,j} x_j + \eta_i - \sum_{j=1, i\neq j}^{n} a_{i,j}\eta_j \tag{9}$$

where $b_{i,j}$ and $a_{i,j}$ are constants. The latent factors η's follow mean-reverting processes of the form,

$$d\eta_i = (\widetilde{\theta}_i(t) - k_i\eta_i)dt + \sigma_{n+i} dW_{n+i} \quad \text{for} \quad i = 1, \ldots, n. \tag{10}$$

Here, $\widetilde{\theta}_i(t) = \widetilde{\chi}_i + \omega_i(t)$, where $\widetilde{\chi}_i$ is a constant and $\omega_i(t)$ is a periodical function on t to capture the seasonality of commodity futures prices (if any). Refer to Richter and Sørensen (2002) and Geman and Nguyen (2005) for a similar setup on the seasonality of the convenience yields. Following Harvey (1991) and Durbin and Koopman (2001), we specify $\omega_i(t)$ as:

$$\omega_i(t) = \sum_{l=1}^{L} (s_i^{c,l} \cos 2\pi\, l\, t + s_i^{s,l} \sin 2\pi\, l\, t). \tag{11}$$

Letting $Y = (x_1, \ldots, x_n, \eta_1, \ldots, \eta_n)^\top$ denote the $2\,n$ factors driving the system of n commodity prices, our model can be rewritten in a vector form,

$$dY = \left(\widetilde{U}(t) + \Psi\, Y\right) dt + d\beta \tag{12}$$

where $\widetilde{U}(t) = (\widetilde{\mu}_1, \ldots, \widetilde{\mu}_n, \widetilde{\theta}_1(t), \ldots, \widetilde{\theta}_n(t))^\top$, and $\Psi = \begin{pmatrix} B & A \\ 0 & \mathcal{K} \end{pmatrix}$ with

$$B = \begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \ddots & b_{2,n} \\ \vdots & \ddots & \ddots & \vdots \\ b_{n,1} & b_{n,2} & \cdots & b_{n,n} \end{pmatrix}, A = \begin{pmatrix} -1 & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & -1 & \ddots & a_{2,n} \\ \vdots & \ddots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & -1 \end{pmatrix}, \mathcal{K} = \begin{pmatrix} -k_1 & 0 & \cdots & 0 \\ 0 & -k_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -k_n \end{pmatrix}$$

In equation (12), $\beta = (\sigma_1 W_1, \ldots, \sigma_{2n} W_{2n})^\top$ is a scaled Brownian motion vector with covariance matrix $\Omega = \{\rho_{i,j}\sigma_i\sigma_j\}$ for $i, j = 1, 2, \ldots, 2n$, where $\rho_{i,j}dt$ is the instantaneous correlation between the Brownian motion increments dW_i and dW_j.

Our model nests several other classical models:

1. If $b_{i,k} = 0$ and $a_{i,k \neq i} = 0$ $(i = 1, \ldots, n; k = 1, \ldots, n)$, our model reduces to correlated GS models on commodities.
2. If $b_{i,k \neq i} = 0$ and $a_{i,k \neq i} = 0$ $(i = 1, \ldots, n; k = 1, \ldots, n)$, our model reduces to correlated CCD models with constant interest rate on commodities.

The correlated GS and CCD models correspond to the GS and CCD models when the spot prices and convenience yields across commodities are correlated. The correlated version of the models are more flexible than the original models and later will be considered as benchmarks for our model.

4.2 Co-movement in Commodity Prices

A natural way of extending the traditional single commodity-pricing models to a multi-asset framework, is to assume that the shocks of the factors are correlated. This is the case for the correlated versions of the GS and CCD models. Indeed, if the objective is to study the valuation of derivatives or the portfolio selection problem in a multi-commodity framework, then correlated factors need to be considered. However, these correlations only generate a short-term source of co-movement in commodity prices. This type of co-movement fails to recognize the long-term effect that exists in the equilibrium relationships.

The proposed empirical model in this paper makes an important distinction between the two components of the co-movement among commodities. In contrast to the short-term effect due to the instantaneous correlation between different commodity prices, the economic relationship generates a longer term effect. This long-term source of co-movement is a feedback effect that is mainly at play through the connection between the expected returns (or the expected prices) of different commodities, i.e. the way a particular commodity impacts the expected return of the other commodities in the economy.[6] This cross-commodity feedback effect corresponds to an error correction or the cointegration between different time series in the discrete-time econometric literature.[7]

[6] The term "feedback effect" has had different interpretations in the econometrics and finance literature. Here, we borrow the concept from the term-structure literature, that refers basically to the non-diagonal terms of the long-run matrix Ψ. See Dai and Singleton (2000) and Duffee (2002) for more details.

[7] For details, please refer to de Boef (2001) and Hamilton (1994).

In the model, the expected return of x_i is

$$\mathbb{E}\left[dx_i\right] = \left(\widetilde{\mu}_i + \sum_{j=1}^{n} b_{i,j}x_j - \eta_i + \sum_{j=1, i\neq j}^{n} a_{i,j}\eta_j\right) dt \tag{13}$$

The $a_{i,j}$'s and the $b_{i,j}$'s (for $j \neq i$) represent the long-term source of co-movement.[8] These parameters relate the expected return of the commodity i with the price and convenience yield of commodity j. The correlated GS and CCD models set these parameters to zero, therefore they completely ignore the cross-commodity feedback effect.

According to the sign of the $b_{i,j}$'s, we classify the co-movement between commodity (log) prices x_i and x_j ($j \neq i$) into three classes. That is, if both $b_{i,j} > 0$ and $b_{j,i} > 0$, a positive increment of x_i tend to feedback a positive increment on x_j, which is in turn likely to strengthen x_i by another positive feedback; hence x_i and x_j move together. Similarly, if $b_{i,j} < 0$ and $b_{j,i} < 0$, x_i and x_j move in opposite directions. Lastly, we have the mixed cases $b_{i,j} > 0$, $b_{j,i} < 0$ and $b_{i,j} < 0$, $b_{j,i} > 0$, where it is not easy to tell the type of co-movement between the commodity prices.

The covariance matrix $\Sigma(t,T)$ for the vector of commodity prices X_T conditional on X_t is

$$\Sigma(t,T) = \int_t^T e^{\Psi(T-u)}\Omega e^{\Psi^\top(T-u)}du. \tag{14}$$

The covariance is stationary as long as all eigenvalues of the long-run matrix Ψ are negative, which is indeed the case for all the commodity pairs studied in the empirical section. From the definition of the conditional covariance we obtain the conditional price correlation (i.e. the correlation structure),

$$\rho(t,T)_{i,j} = \frac{\Sigma(t,T)_{i,j}}{\sqrt{\Sigma(t,T)_{i,i}\Sigma(t,T)_{j,j}}} \quad \text{for} \quad i,j = 1,\ldots,n. \tag{15}$$

It is easy to see that when $T \to t$ the *instantaneous* conditional price correlation is $\rho_{i,j}$ which does not depend on the long-run matrix Ψ, i.e. $\lim_{T\to t}\rho(t,T)_{i,j} = \rho_{i,j}$. This means that in the short run, the correlation among the factors is an important source of co-movement.

For a longer period of time $\tau = T - t > 0$, the conditional price correlation does depend on Ψ,

[8] Note from equation (9) that if $a_{i,j} \neq 0$, then the convenience yield for both commodities i and j share the common factor η_j.

and it is impacted by the relationship among the commodities. If there is a long-term economic relationship, it will appear in the a's and b's, which in turn affects the long-run matrix Ψ. This dynamics creates another source of co-movement that takes effect at relatively longer horizons.

Figures 5 and 9 show that the cross-commodity feedback effect due to the economic relationship, does play an important role in explaining the co-movement of commodity prices. The figure shows that by neglecting the cross-commodity parameters, the GS and CCD models impose strong restrictions on the correlation structure. The cross-commodity feedback effect is important to match the upward sloping correlation structure in the data.

4.3 Futures Pricing

Assuming a constant risk premium for each factor, the risk-neutral process can be expressed as follows:

$$d\beta^{\mathbb{Q}} = \Pi\, dt + d\beta \tag{16}$$

where $\Pi = (\pi_{x,1}, \ldots, \pi_{x,n}, \pi_{\eta,1}, \ldots, \pi_{\eta,n})^\top$ is the risk premium vector. A constant risk premium restricts the long-run behavior (i.e. the Ψ matrix) to be the same under both, risk-neutral and physical measures, but reduces considerably the number of parameters to estimate.

The drift part $U(t)$ under the risk neutral measure can be specified as, $U(t) = \widetilde{U}(t) - \Pi$, hence,

$$dY = (U(t) + \Psi\, Y)dt + d\beta^{\mathbb{Q}} \tag{17}$$

where $\beta^{\mathbb{Q}} = (\sigma_1 W_1^{\mathbb{Q}}, \ldots, \sigma_{2n} W_{2n}^{\mathbb{Q}})^\top$ and $U(t) = (R, L(t))^\top$ with $R = (r^f - \frac{1}{2}\sigma_1^2, \ldots, r^f - \frac{1}{2}\sigma_n^2)^\top$, $L(t) = (\theta_1(t), \ldots, \theta_n(t))^\top$, $\theta_i(t) = \chi_i + \omega_i(t)$ and $\chi_i = \widetilde{\chi}_i - \pi_{\eta,i}$. We assume a constant interest risk-free rate r^f to keep the model simple.[9]

The following proposition shows the futures prices for each commodity i:

Proposition 1 *Let $F_{i,t}(Y_t, T)$ be the i^{th} commodity futures price maturing in $\tau = T - t$ periods.*

[9] It is straightforward to extend our model to consider a stochastic interest rates as in Schwartz (1997).

In the model setup (17), the futures prices are determined by

$$\log(F_{i,t}(Y_t, t+\tau)) = m_i(\tau) + G_i(\tau)Y_t \quad \textit{for} \quad i = 1, \ldots, n \tag{18}$$

where

$$\begin{aligned} m_i(\tau) &= \int_0^\tau \left(G_i(u)\, U + \frac{1}{2} G_i(u)\, \Omega\, G_i(u)^\top \right) du \\ G(\tau) &= \exp(\Psi\, \tau) \end{aligned}$$

where $G_i(\tau)$ denotes the i^{th} row of the $G(\tau)$ matrix.

Proof See Appendix C.1. □

4.4 "Maximal" Affine Model in a Multi-commodity System

Duffie and Kan (1996), Duffie, Pan, and Singleton (2000) and Dai and Singleton (2000) propose a "maximal" canonical form for affine multi-factor model of the form:

$$x_i = \alpha_0^i + \psi_{\widehat{Y}}^i \widehat{Y}, \tag{19}$$

where x_i denotes the (log) value of the i^{th} asset, $\psi_{\widehat{Y}}^i$ is a $1 \times m$ constant row vector and α_0^i is a constant. $\widehat{Y}$ is an $m \times 1$ column vector of latent state variables that follow mean-reverting Gaussian diffusion processes under the risk-neutral measure,

$$d\widehat{Y} = -\Lambda \widehat{Y} dt + dW_{\widehat{Y}}^{\mathbb{Q}} \tag{20}$$

where Λ is a lower triangular matrix and $W_{\widehat{Y}}^{\mathbb{Q}}$ is a vector of independent Brownian motions. The above-mentioned model is "maximal" in the sense that, conditional on observing the *single* asset, the model offers the maximum number of identifiable parameters (*c.f.* Dai and Singleton 2000, and CCD).

In order to use this model into a multi-commodity system, we have to extend it in two ways. First, the above maximal model is only suitable for a single asset, thus we need to extend the model to a canonical affine representation for multiple assets. We hence define the maximal model for multiple assets as follows:

In a system of n assets which are governed by m factors, a model for the system is "maximal" if and only if every single asset in the system is modeled by an m-factor maximal model as defined in Dai and Singleton (2000):

$$X = \psi_0 + \psi_{\widehat{Y}} \widehat{Y}, \tag{21}$$

where $X = (x_1, \ldots, x_n)^\top$ represent the n assets which are governed by $\widehat{Y}$ in equation (20). Here, $\psi_{\widehat{Y}} = (\psi^1_{\widehat{Y}}, \ldots, \psi^n_{\widehat{Y}})^\top$ is an $n \times m$ matrix and $\psi_0 = (\psi^1_0, \ldots, \psi^n_0)^\top$ is an $n \times 1$ vector.

Thus a simple combination of maximal models for single commodities does not necessarily form a maximal model for a multi-commodity system. For example, the CCD model is maximal for single commodities, but is not maximal in a multi-commodity system. The previous section shows that an extended version of the CCD model is nested in our model and hence is not maximal, because this model restricts some parameters in the expected return of the factors to be zero. These constraints considerably influence the joint long-run behavior of the commodities.

Second, the above maximal model only allows a constant ψ_0, however, many commodity prices are subject to seasonal movements. Thus, we need to extend the maximal model by letting ψ_0 be time-varying. The extended model for multiple assets is:

$$\begin{aligned} X &= \psi_0(t) + \psi_{\widehat{Y}} \widehat{Y}, &(22)\\ d\widehat{Y} &= -\Lambda \widehat{Y} dt + dW^{\mathbb{Q}}_{\widehat{Y}} &(23) \end{aligned}$$

where $\psi_0(t) = (\psi^1_0(t), \psi^2_0(t), \ldots, \psi^n_0(t))^\top$ is an $n \times 1$ vector, $\psi^i_0(t) = \alpha^i_0 + \varpi^i_0(t)$, and where $\varpi^i_0(t)$ is a periodical function.

To address the maximal model for multiple assets in an n commodities system governed by $2\,n$ factors, we specify X as the $n \times 1$ vector of (log) spot commodity prices, Λ in (20) as a $2n \times 2n$ lower triangular matrix and $W^{\mathbb{Q}}_{\widehat{Y}}$ as a $2n \times 1$ vector of independent Brownian motions.

Following CCD we now show that for the multi-commodity maximal model, the convenience yield vector $\Delta = (\delta_1, \ldots, \delta_n)^\top$ is an affine function of the state variables $\widehat{Y}$. The absence of arbitrage implies that under the risk-neutral measure ($\mathbb{Q}$) the drift of the spot price of the i^{th} commodity must follow

$$\mathbb{E}_t^{\mathbb{Q}}[dS_i] = (r^f - \delta_i)S_i dt \quad \text{for} \quad i = 1, \dots, n. \tag{24}$$

Applying Itô's lemma, we obtain the following expression for the maximal convenience yield vector Δ implied by our model,

$$\begin{aligned} \Delta &= r^f 1_n - \frac{\mathbb{E}_t^{\mathbb{Q}}[dV] + \frac{1}{2}(\mathbb{V}ar_t^{\mathbb{Q}}[dx_1], \dots, \mathbb{V}ar_t^{\mathbb{Q}}[dx_n])^\top}{dt} \\ &= r^f 1_n + \psi_{\widehat{Y}} \Lambda \widehat{Y} - \frac{1}{2}\text{diag}(\psi_{\widehat{Y}} \psi_{\widehat{Y}}^\top) \end{aligned} \tag{25}$$

where $\mathbb{V}ar_t^{\mathbb{Q}}(.)$ denotes the variance under the risk-neutral measure, and 1_n is an $n \times 1$ column vector with all elements equal to 1.

In order to show that our empirical model from the beginning of this section is indeed maximal, we first introduce an intermediate representation that allows us to show that our model and the one presented in equations (22)-(23) are equivalent. The intermediate representation rotates the state vector $\widehat{Y}$ to state variables that have a better economic meaning: the (log) spot prices and the convenience yields of the n commodities. The canonical form model has m factors, while our empirical model has $2\,n$ factors, therefore, we set $m = 2\,n$. Proposition 2 formalizes the intermediate representation.

Proposition 2 *Assume $2\,n$ factors driving the dynamics of the futures prices of n commodities, as in equations (22)-(23). The maximal model under the risk-neutral measure can be presented equivalently by an affine model where the state variables are the log spot prices x_i and the convenience yields δ_i $(i = 1, \dots, n)$. The dynamics of the new state vector $\overline{Y} = (x_1, \dots, x_n, \delta_1, \dots, \delta_n)^\top$ is:*

$$d\overline{Y} = (\overline{U}(t) + \overline{\Psi}\,\overline{Y})dt + d\beta_{\overline{Y}}^{\mathbb{Q}} \tag{26}$$

where $\overline{U}(t) = \left(\overline{R}, \overline{L}(t)\right)^\top$, $\overline{\Psi} = \begin{pmatrix} 0 & -I_{n\times n} \\ \overline{A} & \overline{B} \end{pmatrix}$ and $\beta_{\overline{Y}}^{\mathbb{Q}}$ is a scaled Brownian motion vector with covariance matrix $\overline{\Omega}$. The $n \times 1$ vectors $\overline{R}$ and $\overline{L}(t)$ and the $n \times n$ matrices $\overline{A}$, $\overline{B}$ and $\overline{\Omega}$ are specified in Appendix C.2.

Proof By writing equations (22) and (25) together, we have

$$\overline{Y} = \begin{pmatrix} X \\ \Delta \end{pmatrix} = \begin{pmatrix} \psi_0(t) \\ \psi_c \end{pmatrix} + \begin{pmatrix} \psi_{\widehat{Y}} \\ \psi_{\widehat{Y}}\Lambda \end{pmatrix} \widehat{Y}, \tag{27}$$

where $\psi_c = r^f 1_n - \frac{1}{2}\mathrm{diag}(\psi_{\widehat{Y}}\psi_{\widehat{Y}}^\top)$. Equation (27) shows that the intermediate representation, $\overline{Y}$, is an invariant transformation of $\widehat{Y}$ (see Dai and Singleton 2000). This transformation rotates the state variables, but all the initial properties of the model are maintained, that is, the resulting model is still a maximal affine $2n$-factor Gaussian model. Furthermore, we apply Itô's lemma to obtain the specific relationships between the model parameters specified in the proposition and those specified in equations (22)-(23). Appendix C.2 shows the derivation in a greater detail. □

An important corollary of Proposition 2 is that, in a maximal model, the *drift* of the convenience yield of a certain commodity depends on other commodity spot prices. This is consistent with the structural model in section 3 (for example, see equation (6)).

Now we are ready to show that our model is maximal. The next proposition formalizes this.

Proposition 3 *The maximal model specified in Proposition 2 is equivalent with our model in (17).*

Proof Equation (9) shows that the convenience yield vector is $\Delta = -B\,X - A\,\eta$, where $\eta = (\eta_1, \ldots, \eta_n)^\top$ is the vector of latent state variables that follow the dynamics in (10). Thus, we find the following invariant transform from $\overline{Y}$ to Y:

$$Y = \begin{pmatrix} X \\ \eta \end{pmatrix} = \begin{pmatrix} I_{n\times n} & 0 \\ -A^{-1}B & -A^{-1} \end{pmatrix} \overline{Y} \tag{28}$$

Similar with Proposition 2, we apply Itô's lemma to compare the parameters in (26) and (28) and show that they are identical. Appendix C.3 shows the derivation in detail. □

Proposition 2 and 3 show that our model belongs to the maximal model of multi-commodity system. Furthermore, it captures the long-term relationship among different commodities. In the following section, we show how to calibrate this model.

5 Estimation

We demonstrate the importance of long-term economic relationships in futures pricing using the heating oil and crude oil production pair. Even though our model can be applied to price a system of n commodities jointly, two commodities are enough to highlight the main characteristics of our model and the intuition behind the results.[10] We also estimate the model for two goods that are substitutes (WTI crude oil and Brent crude oil) and for two commodities that are complement goods (heating oil and gasoline). These estimations are analogous to the heating and crude oil pair, therefore, we leave the details for Appendix D.

5.1 Empirical Method – the Kalman Filter

One of the difficulties of calibrating the model is that the state variables are not directly observable. A useful method for maximum likelihood estimation of the model is addressing the model in a state-space form and to use the Kalman filter methodology to estimate the latent variables.[11] The state-space form consists of a transition equation and a measurement equation. The transition equation shows the data-generating process. The measurement equation relates a multivariate time series of observable variables (in our case, futures prices for different maturities) to an unobservable vector of state variables (in our case, the (log) spot prices x_i and η_i $(i = 1, \ldots, n)$). The measurement equation is obtained using a log version of equation (18) by adding uncorrelated noises to take account of the pricing errors.

Suppose that data are sampled in equally separated times t_k, $k = 1, \ldots, K$. Denote $\Delta t = t_{k+1} - t_k$ as the time interval between two subsequent observations. Let Y_k represent the vector of state variables at time t_k. Thus, we can obtain the transition equation,

$$Y_{k+1} = (\Psi\, \Delta t + I)\, Y_k + \widetilde{U}(t)\, \Delta t + w_k \tag{29}$$

[10]The computational loads increase exponentially for the case of more than two commodities. Furthermore, commodity pairs are building blocks of any commodity system. Any multi-commodity system can be decomposed into multiple commodity pairs, e.g., the system with three commodities can be priced using no more than 3 pairs of commodities.

[11]Hamilton (1994) and Harvey (1991) give a good description of estimation, testing, and model selection of state-space models.

where w_k is a $2n \times 1$ random noise vector following zero-mean normal distributions.

For the measurement equation at time t_k, we consider the vector of the log of futures prices $F_k = (F_{1,k}(\tau_1), \ldots, F_{n,k}(\tau_1), \ldots, F_{1,k}(\tau_M), \ldots, F_{n,k}(\tau_M))^\top$, where τ_j denotes the time to maturities.[12] The log $(n\,M) \times 1$ vector $\mathcal{F}_k$ can be written as,

$$\log(F_k) = \overline{m} + \overline{G}\,Y_k + \varepsilon_k \tag{30}$$

where

$$\begin{aligned}
\overline{m} &= (m_1(\tau_1), \ldots, m_n(\tau_1), \ldots, m_1(\tau_M), \ldots, m_n(\tau_M))^\top, \\
\overline{G} &= (G_1(\tau_1), \ldots, G_n(\tau_1), \ldots, G_1(\tau_M), \ldots, G_n(\tau_M))^\top,
\end{aligned}$$

and ε_k is a $(n\,M) \times 1$ vector representing the model errors with its variance covariance matrix Υ. In order to reduce the number of parameters to estimate, we assume that the standard errors for all contracts are the same. This also reflects the notion that we want our model to price the n commodities and M contracts equally well. Therefore, we define $\Upsilon = \epsilon^2 I_{nM}$, where ϵ is the pricing error of the log of the futures prices and I_{nM} is the $(n\,M) \times (n\,M)$ identity matrix.

5.2 The Data

Our data consist of weekly futures prices of West Texas Intermediate (WTI) crude oil and heating oil. The weekly WTI crude oil and heating oil futures are obtained through the New York Mercantile Exchange (NYMEX) for the period from 1995.01 to 2006.02 (582 observations for each commodity). The time to maturity ranges from 1 month to 17 months for these two commodities. We denote Fn as futures contracts with roughly n months to maturity; e.g., $F0$ denotes the cash spot prices and $F12$ denotes the futures prices with 12 months to maturity. We use five time series —$F1$, $F5$, $F9$, $F13$, $F17$– for WTI, crude oil, and heating oil contracts. Table 1 summarizes the data. Note that, in the calibration, we take the risk-free rate as 0.04, which is the average interest rate during these years.

[12] Since our model has $2n$ factors we need $M \geq 2$.

5.3 Empirical Examination of the Long-term Economic Relationship

As mentioned before, since WTI crude oil and heating oil are the input and output of the oil refinery firm, this commodity pair has a production relationship.

We arbitrarily define crude oil as commodity 1 and heating oil as commodity 2. Figure 2 shows the historical crude and heating oil time series. Crude oil prices do not show seasonality, which is consistent with the literature on oil futures, such as Schwartz (1997). However, heating oil shows quite strong seasonality. This is because in winter, demand for heating oil is typically high, but there are usually not enough facilities existent to store the heating oil; hence, in the winter, heating oil has relatively higher convenience yields. Therefore, winter-maturing futures tend to be higher than those maturing in summer. Since the seasonality of heating oil is in an annual frequency, for simplicity we set $L = 1$ in equation (11), so that

$$\omega_i(t) = s_i^c \cos 2\pi\, t + s_i^s \sin 2\pi\, t. \tag{31}$$

We use the Kalman filter to calibrate our model. Table 2 shows the results. From the model estimation, we see that most parameters are significant. In particular, $b_{1,2}$ and $b_{2,1}$ are highly significant, which is consistent with the our prediction that the convenience yields depend on other commodity prices. The positive signs of $b_{1,2}$ and $b_{2,1}$ are also in line with the prediction of the production relationship. Figure 3 shows the time series of the mean errors (ME) and root mean squared errors (RMSE). The MEs are negligible, and the RMSEs fluctuate between 0.002 to 0.03, which shows that our model performs reasonably well in fitting futures prices. Figure 4 shows the convenience yield for both WTI crude oil and heating oil implied by our model. As is well known, the convenience yields of productive commodities are highly volatile and can be as high as 100% (see CCD).

In order to test whether our model is better than the correlated versions of the GS and CCD models, we run a likelihood ratio test on the three models. Table 3 shows that, in terms of fitting the futures curves, our model is significantly better than the correlated GS model and correlated CCD model. This result suggests that a maximal specification is indispensable when jointly modeling multiple commodities.

Figure 5 shows the correlation structure for correlated GS, correlated CCD and our model. The plot shows that only our model is able to generate the upward sloping correlation curve present in the data. In the short run, we see that the correlation in our model is smaller than the correlated GS and CCD models. This occurs because our model is more flexible when capturing the co-movement between two futures prices, which allows us to disentangle the different sources of co-movement (i.e. the correlation and the long-term effects). Indeed, the correlated versions of GS and CCD, which don't consider long-term relationships, are forced to include some existing mid-term correlation in the short-term component of co-movement. In the long run, our model allows for a greater correlation than the other two models, which is consistent with the significance of the cross-commodity relationship.

In the next section we show that a well-behaved empirical model can guide investors in correctly pricing financial contingent claims.

6 Spread Options Valuation

Spread options are based on the difference between two commodity prices. This difference can be, for example, between the price of an input and the price of the output of a production process (processing spread). NYMEX offers tradable options on the crack spread: the heating oil-crude oil and gasoline-crude oil spread options (introduced in 1994) and the recently announced substitute spread between the WTI and the Brent crude oil. Also, many firms may face "real options" on spreads. For example, manufacturing firms possess an option of transferring the raw material to products at a certain cost, because they can choose not to produce. This option is on the spread between input and output prices and the strike price corresponds to the production cost. The spread option is of great importance for both commodity market participants and real production firms.

Since the spread is determined by the difference of two asset price, it is natural to model the spread by modeling each asset *separately.* This is the main characteristic of the so-called two-price model, where the short-term correlation is the driver for most of the action in the spread (as in the correlated GS and CCD models). Up to now, nearly all researchers use the two-price model for pricing spread options (see Margrabe (1978) and Carmona and Durrleman (2003)). However, as we see from section 4.2, the two-price model ignores the long-term co-movement component implied by our model. Thus, the two-price models might be flawed especially for the long run. Mbanefo (1997) and Dempster, Medova, and Tang (2008), among others have documented that the traditional two-price model suffers a problem of overpricing the spread option. Therefore, spread option pricing can be regarded as an out-of-sample test for our theoretical model.

At current time t, the pricing of call and put spread options, $c_t(T, M)$ and $p_t(T, M)$, with strike K on two commodities with futures prices $F_{1,t}(M)$ and $F_{2,t}(M)$, are specified as:

$$c_t(T, M) = e^{-r^f (T-t)} \mathbb{E}_t^{\mathbb{Q}} \left[\max \left(F_{2,T}(M) - F_{1,T}(M) - K, 0 \right) \right] \tag{32}$$

$$p_t(T, M) = e^{-r^f (T-t)} \mathbb{E}_t^{\mathbb{Q}} \left[\max \left(K - (F_{2,T}(M) - F_{1,T}(M)), 0 \right) \right] \tag{33}$$

where the time to maturity for the spread options is T. To the best of our knowledge, the analytical solution for spread options is not available if $K \neq 0$. Thus, to price the options we use Monte Carlo simulation. In this section, we simulate the futures prices using three models – ours, the correlated CCD, and the correlated GS models. The futures price dynamics under the risk-neutral measure are specified as,

$$\frac{dF_{i,t}(M)}{F_{i,t}(M)} = G_i(M - t)\, d\beta^{\mathbb{Q}}, \quad \text{for} \quad i = 1, 2. \tag{34}$$

We choose two spread options: the crack spread option – spread between heating oil and the WTI crude oil, and the substitute spread option – spread between the WTI crude oil and Brent crude oil. For the crack spread, we assume crude and heating oil prices as $F_{1,t}(M) = 100$ (crude oil) and $F_{2,t}(M) = 105$ (heating oil), respectively; and for Brent and WTI crude oil, we use $F_{1,t}(M) = 100$ (Brent crude) and $F_{2,t}(M) = 102$ (WTI crude), respectively.[13]

We focus on spread options of different maturities to understand the effect of the correlation structure implied by the models. We choose $T = 3$ month for short-maturity options and $T = 5$ years for long-maturity options. Also, for both, crack and substitution spreads, we choose the same maturity on futures and options, which is the convention of the spread option specification on NYMEX. We use the estimates from the crude-heating oil and WTI-Brent oil pairs to conduct our

[13] Note that generally heating oil is about 5 dollars higher than the crude oil, and WTI crude is 1.5 to 2 dollars above Brent crude.

simulations, where 2000 paths are simulated for the three models. In order to make the simulation accurate, we use anti-variate techniques in generating random variables and use the same random seed for all three models. The risk free rate r^f is 0.04 in the simulation.

Tables 4 and 9 show the option values with different strikes for both call and put options of crack spread and substitutive spread, respectively. The tables show that both, short-term and long-term effects, are important determinants of spread option prices. The results indicate that for long-maturity options ($T = 5$ years), our model implies lower call and put spread option prices than the correlated GS and CCD models.[14] Our finding is consistent with the evidence of Mbanefo (1997) that the two-prices models tend to overprice the spread option by ignoring the equilibrium relationship, specially for long-maturity options. This is a consequence of the higher long-term correlations implied by our model. Intuitively, the feedback-effect (positive $b_{1,2}$ and $b_{2,1}$'s) restricts the commodity prices from large deviations from their equilibrium, and thus make the spread of the prices relatively smaller and less volatile than models without this feature. The lower term volatility of the spread traduces into lower options values.

The opposite occurs for short-maturity options ($T = 3$ month). The results suggest that the two-price model may underprice short-maturity option values. The short-term correlation in the CCD and GS models is contaminated because these models are misspecified.[15] Indeed, these models can not capture the long-term source of co-movement, therefore, they tend to accommodate long-term effects in the short-end of the correlation structure. This creates important biases in spread option prices.

Table 5 presents an out-of-sample test for short-maturity heating oil-crude oil (1:1) crack spread options for our model and for the correlated GS and CCD models. The results show that our model does considerably better than the others in matching real data. The three models tend to underprice the price of both call and put options, however, our model reduces the mean pricing error to approximately one half the size of the error in the CCD model. The lower option values are consistent with higher short-term correlation estimates as predicted by our previous analysis.

[14] Note that CCD model has lower option prices than those in the GS model because the CCD model captures the mean-reversion of commodity prices, while GS model does not.

[15] Figures 5 and 9 show that the cross-commodity feedback effect in our model implies a lower short-term correlation and a larger long-term correlation than the correlated GS and CCD models.

The root mean square error columns also show that our model outperforms the benchmark models. Long-maturity options data is not available so we are unable to test the long-term predictions implied by our model.

7 Conclusions

We study the determinants of the co-movement among commodity prices in a multi-asset framework. We find that a long-term source of co-movement is driven by economic relations, such as, production, substitution or complementary relationships. Using a structural model, we show that the economic relation implies a cross-commodity feedback effect that influences the long-term joint dynamics of prices. This effect implies that the convenience yield of a certain commodity depends on the prices of other related commodities. This notion is not presented in the traditional "Theory of Storage".

We propose a maximal affine reduced-form model for a multi-commodity setup which nests the GS and CCD models. We explicitly consider the interdependence of convenience yields on the spot prices of all commodities in the economy. Our model allows us to disentangle the two sources of co-movement and implies a flexible correlation structure that matches the upward sloping shape observed in the data for related commodities. We find that traditional commodity pricing models, such as the GS and CCD models, impose strong restrictions on the correlation structure. These models account only for a short-term source of co-movement, therefore the estimation accommodates this component to match the higher long-term correlation in the data. We estimate the model for three commodity pairs: heating oil-crude oil, WTI-Brent crude oil and heating oil-gasoline. Likelihood-ratio tests show that our model is significantly better than the correlated versions of the GS and CCD models, which proves the importance of modeling cross-commodity relationships.

Our model is then used to price spread options because spread options largely depend on the equilibrium relationship between the two underlying commodities. The flexibility in the correlation structure implied by our model has an important effect on option prices. For long-maturity options, our model predicts lower prices than those from the correlated GS and CCD models. This occurs because our model correctly accounts for an upward sloping correlation structure. The long-run relationship ties both commodity prices, reducing the volatility of the spread and yielding lower spread option values. Our results also show that the short-term correlation is lower than the one in the GS and CCD models. This imply higher prices for short-maturity spread options. An out-of-sample test shows that our model does a much better job fitting short-maturity crack spread options than the benchmark models.

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Appendix

A Derivation of the Economic Model of Production Relationship

First, note that we can extract the convenience yield δ_j for each commodity j using the pricing kernel (ξ) and the price of the commodity (S_j):

$$\mathbb{E}\left[d(\xi\, S_j) + \xi\, \delta_j S_j dt\right] = 0 \tag{A1}$$

which implies that

$$\mathbb{E}\left[\frac{d\chi_j}{\chi_j}\right] = -\delta_j dt \tag{A2}$$

with $\chi_j = \xi\, S_j$. The interpretation of this result is that the convenience yield corresponds to the interest rate in a world that uses the commodity as the numeraire (see Richard and Sundaresan 1981, and Casassus, Collin-Dufresne, and Routledge 2008).

Let us denote by $J(K, Q_1, Q_2) = \sup_{\{C_{K,u}, C_{2,u}, q_{1,u}\}\in\mathcal{A}} \mathbb{E}_t\left[\int_t^\infty e^{-\theta(u-t)} U(C_{K,u}, C_{2,u}) du\right]$ the "current" value function associated with the representative agent's problem. Note that given the set-up, the value function $J(\cdot)$ is not a function of time.

The solution of the our problem is determined by the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\sup_{\{C_K, C_2, q_1\}\in\mathcal{A}} \{U(C_K, C_2) + \mathcal{D}J - \theta J\} = 0 \tag{A3}$$

where $\mathcal{D}$ is the Itô operator

$$\mathcal{D}J = (\alpha\, K - C_K)\,\frac{\partial J}{\partial K} - q_1\,\frac{\partial J}{\partial Q_1} + (\gamma \log(q_1) Q_2 - C_2)\,\frac{\partial J}{\partial Q_2} + \frac{1}{2}\sigma_K^2 K^2 \frac{\partial^2 J}{\partial K^2} \tag{A4}$$

with $\frac{\partial J}{\partial K}$, $\frac{\partial J}{\partial Q_1}$ and $\frac{\partial J}{\partial Q_2}$ representing the marginal value of an additional unit of numeraire good, crude oil and heating oil, respectively. $\frac{\partial^2 J}{\partial K^2}$ is the second derivative with respect to K.

The first-order conditions with respect to consumption of capital, consumption of heating oil and demand for crude oil are $U_{C_K} = \frac{\partial J}{\partial K}$, $U_{C_2} = \frac{\partial J}{\partial Q_2}$ and $\frac{\gamma\, Q_2}{q_1}\frac{\partial J}{\partial Q_2} = \frac{\partial J}{\partial Q_1}$, respectively. Given our logarithmic utility function, these conditions imply that the optimal consumptions are $C_K = \phi\left(\frac{\partial J}{\partial K}\right)^{-1}$ and $C_2 = (1-\phi)\left(\frac{\partial J}{\partial Q_2}\right)^{-1}$. After replacing these controls in the HJB equation we obtain an ordinary differential equation with a closed-form solution that is linear in $\log(K)$, $\log(Q_1)$ and $\log(Q_2)$.

We note that the pricing kernel is $\xi_t \propto e^{-\theta\, t} U_{C_{K,t}}(C_{K,t}, C_{2,t})$ and define commodity prices as the marginal prices that solve $J(K, Q_1, Q_2) = J(K + S_1\,\epsilon, Q_1 - \epsilon, Q_2) = J(K + S_2\,\epsilon, Q_1, Q_2 - \epsilon)$ when $\epsilon \to 0$. These imply that $S_j = \left(\frac{\partial J}{\partial K}\right)^{-1}\frac{\partial J}{\partial Q_j}$ for $j \in \{1, 2\}$. Finally, using the envelope condition above, we obtain the result that commodity j pricing kernel is $\chi_{j,t} \propto e^{-\theta\, t}\frac{\partial J(K_t, Q_{1,t}, Q_{2,t})}{\partial Q_j}$. We apply Itô's to this expression to obtain the convenience yield of commodity j.

B Substitution and Complementary Relationships

B.1 The Economic Model for a Substitution Relationship

Consider now a similar economy to the production case in Section 3, but with two substitute commodities, say, West Texas Intermediate (WTI) crude oil from the North Sea (Q_1) and Brent crude oil (Q_2). There is also a production technology in the capital sector (K) that uses both types of crude oils to produce the consumption good. The representative agent in the economy maximizes the expected log utility with respect to the consumption of capital (C_K), the demand of WTI crude oil (q_1) and the demand of Brent crude oil (q_2):

$$\sup_{\{C_{K,t},q_{1,t},q_{2,t}\}\in\mathcal{A}} \mathbb{E}_0\left[\int_0^\infty e^{-\theta\, t}\log\left(C_{K,t}\right)dt\right] \tag{B1}$$

where $\mathcal{A}$ is a set of admissible strategies. First, consider a simple case where the capital and crude oil stocks evolve in the following way:

$$\begin{aligned} dK &= (\alpha(\log(q_1)+\log(q_2))\,K - C_K)dt + \sigma_K K dW_K & \text{(B2)}\\ dQ_1 &= -q_1 dt + \sigma_1 Q_1\, dW_1 & \text{(B3)}\\ dQ_2 &= -q_2 dt + \sigma_2 Q_2\, dW_2. & \text{(B4)}\end{aligned}$$

The uncertainty is captured by the independent Brownian motions W_i for $i\in\{K,1,2\}$. The stochastic crude oil stocks capture the fact that available barrels of oil are affected by some exogenous factors. As we will note later, this type of uncertainty generates the very appealing feature that the WTI and Brent crude oil prices are less than perfectly correlated.

At this point, given the simplicity of the economy, the two commodities Q_1 and Q_2 are not substitutes. The crude oil demands q_1 and q_2 depend only on their own stock level. Whether the WTI crude oil is cheaper or more expensive than the Brent crude oil does not affect the demand for Brent oil.

A simple way of making these two commodities substitute is by allowing some interaction between the two crude oil stocks. For example, if the agents can move some units from the Brent stock to the WTI stock and vice-versa, then the two commodities will have some degree of substitutability. There are multiple ways of doing this, but only few of them have closed-form solutions. It is important to have analytical expressions in order to understand the economics behind the results.

The case with optimal adjustment from Brent to WTI crude oil and vice-versa at an infinite rate and at no cost can easily be solved, but the model is unrealistic. Without any friction both crude oil prices will be identical. If we consider some degree of irreversibility by including proportional adjustment costs, the problem becomes similar to that of the shipping model of Dumas (1992) which needs to be solved numerically.[16] Including fixed costs as in Casassus, Collin-Dufresne, and Routledge (2008) involves an even more complex solution. If there is a finite upper bound for the rate of adjustment from one stock to the other, the problem has the same flavor as in the bounded investment rate model of Kogan, Livdan, and Yaron (2008). Because of the extra state variable, to the best of our knowledge, there is no closed form solution to this problem, either.

A common characteristic of the endogenous decisions in the three equilibrium models mentioned above is that the optimal adjustment occurs when the level of the target stock is relatively lower than the level of the source stock. We propose an exogenously defined adjustment strategy that captures this feature and allows for closed-form solutions. The strategy involves transporting a time-varying fraction of Brent oil stocks to the WTI sector when the Brent stocks are greater than the WTI stocks, and vice-versa. Doing this at a finite rate captures the irreversibility characteristic embedded in the endogenous decisions of Dumas (1992), Casassus, Collin-Dufresne, and Routledge (2008) and Kogan, Livdan, and Yaron (2008). The modified processes are:

$$\begin{aligned} dQ_1 &= (\omega\, Q_1 - q_1)dt + \sigma_1\, Q_1\, dW_1 & \text{(B5)}\\ dQ_2 &= (-\omega\, Q_2 - q_2)dt + \sigma_2\, Q_2\, dW_2 & \text{(B6)}\\ d\omega &= \kappa\left(\log\left(\frac{Q_2}{Q_1}\right) - \omega\right)dt & \text{(B7)}\end{aligned}$$

The adjustment rate ω can take both signs. It moves continuously towards a time-varying long-term mean that depends on the stocks Q_1 and Q_2. If there is more Brent oil than WTI oil in the economy (i.e. $Q_2 > Q_1$),

[16] Actually, the problem here is more complex, since we have three state variables instead of the two state variables representing the two countries in Dumas (1992).

the rate ω moves towards a positive value until the stocks are balanced. The positive parameter κ is the speed of adjustment from one oil stock to the other and captures the degree of substitutability between the two commodities. The higher the κ the better substitutes are the commodities, because the adjustment occurs at a higher speed. It also captures the persistence of the adjustment rate, and thus the degree of irreversibility in the adjustment decision. A low κ implies greater irreversibility because it will take longer to balance the stocks.

Let us denote by $J(K, Q_1, Q_2, \omega) = \sup_{\{C_{K,u}, q_{1,u}, q_{2,u}\} \in \mathcal{A}} \mathbb{E}_t \left[\int_t^\infty e^{-\theta(u-t)} U(C_{K,u}) du\right]$ the "current" value function for the representative agent's problem in the substitute relationship example.

The solution to our problem is determined by the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\sup_{\{C_K, q_1, q_2\} \in \mathcal{A}} \{U(C_K) + \mathcal{D}J - \theta J\} = 0 \tag{B8}$$

where $\mathcal{D}$ is the standard Itô operator associated to this economy. The first-order conditions with respect to consumption of capital and demands for the two types of crude oil are $C_K = \left(\frac{\partial J}{\partial K}\right)^{-1}$, $\frac{\alpha\, K}{q_1} \frac{\partial J}{\partial K} = \frac{\partial J}{\partial Q_1}$ and $\frac{\alpha\, K}{q_2} \frac{\partial J}{\partial K} = \frac{\partial J}{\partial Q_2}$, respectively. After replacing these controls in the HJB equation we obtain an ordinary differential equation with a closed-form solution that is linear in $\log(K)$, $\log(Q_1)$, $\log(Q_2)$ and ω.

The representative agent problem that maximizes equation (B1) subject to equations (B2) and (B5)-(B7) has an affine solution similar to the one in the production relationship example. The representative agent optimally consumes a constant fraction of capital ($C_K = \theta\, K$) and demands a constant rate of WTI and Brent crude oils ($q_1 = \theta\, Q_1$ and $q_2 = \theta\, Q_2$). Note that as before, the instantaneous crude oil demands are a function only of their own crude oil stocks, but now both crude oil stocks are related because of the adjustment rate ω. This implies that future Brent oil demands will be affected by the current WTI stock level.

In a similar way to the production relationship, the equilibrium crude oil prices are very simple:[17]

$$S_1 = \frac{\alpha}{\theta} \frac{K}{Q_1} \quad \text{and} \quad S_2 = \frac{\alpha}{\theta} \frac{K}{Q_2} \tag{B9}$$

The prices have the same structure, because the problem is symmetric for both commodities.[18] The equilibrium convenience yields are:

$$\delta_1 = \omega - \sigma_1^2 \quad \text{and} \quad \delta_2 = -\omega - \sigma_2^2 \tag{B10}$$

The convenience yields are directly related to the adjustment rate ω. The WTI oil convenience yield is increasing in ω, because a high ω implies more expected WTI oil stocks in the next period.[19] This expected increase in stocks decreases expected prices, thus generating a positive convenience yield (after risk premium adjustments). The WTI oil convenience yield is time varying and has the same dynamics as the adjustment rate. If we also consider that equation (B9) implies that $\frac{Q_2}{Q_1} = \frac{S_1}{S_2}$, then the dynamics of δ_1 is:

$$d\delta_1 = \kappa \left(\log\left(\frac{S_1}{S_2}\right) - \delta_1 \right) dt \tag{B11}$$

A higher differential between S_1 and S_2 implies that the convenience yield δ_1 is more likely to increase in the near future. In this particular case, the convenience yields are conditionally deterministic, because the exogenous adjustment strategy was assumed to have this characteristic. Equation (B12) shows that the convenience yield of WTI oil depends on the price of Brent crude oil,

$$\delta_{1,t} = \delta_{1,t_0} e^{-\kappa(t-t_0)} + \int_{t_0}^{t} \kappa\, e^{-\kappa(t-u)} \log\left(\frac{S_{1,u}}{S_{2,u}}\right) du \tag{B12}$$

[17]Again, see Appendix A for more details on the solution of the model.

[18]Note that because Q_1 and Q_2 are driven by independent Brownian motions, the price of the commodities are only partly correlated. This correlation emerges because they share the common factor K.

[19]The opposite occurs for the Brent convenience yield.

B.2 The Economic Model for a Complementary Relationship

In this appendix we study the joint equilibrium dynamics of two commodities that have a complementary relationship. We consider the case of gasoline (Q_1) and heating oil (Q_2) that share a balanced supply from crude oil (Q_3) crack process and also are complementary in consumption. The representative agent consumes capital (C_K) and a mix of gasoline and heating oil, defined as $C_M = C_1^{\nu} C_2^{1-\nu}$, where ν represents the share of gasoline in the mix. The agent maximizes expected utility of consumption and also chooses how much crude oil to demand (q_3) for production of gasoline and heating oil:

$$\sup_{\{C_{K,t}, C_{M,t}, q_{3,t}\} \in \mathcal{A}} \mathbb{E}_0 \left[\int_0^{\infty} e^{-\theta\, t} \left(\phi \log\left(C_{K,t}\right) + (1-\phi) \log\left(C_{M,t}\right) \right) dt \right]. \tag{B13}$$

The dynamics of the stocks of capital, gasoline, heating oil and crude oil, respectively, are:

$$\begin{aligned} dK &= (\alpha\, K - C_K) dt + \sigma_K K dW_K && \text{(B14)} \\ dQ_1 &= (\gamma\, \log(q_3) - C_1) dt + \sigma_1 Q_1\, dW_1 && \text{(B15)} \\ dQ_2 &= ((1-\gamma)\, \log(q_3) - C_2) dt + \sigma_2 Q_2\, dW_2 && \text{(B16)} \\ dQ_3 &= -q_3 dt. && \text{(B17)} \end{aligned}$$

Where $\gamma \in (0,1)$ represents the relative productivity of gasoline in the crack production process.[20] Let us denote by $J(K, Q_1, Q_2, Q_3) = \sup_{\{C_{K,u}, C_{M,u}, q_{3,u}\} \in \mathcal{A}} \mathbb{E}_t \left[\int_t^{\infty} e^{-\theta (u-t)} U(C_{K,u}, C_{M,u}) du \right]$ the "current" value function for the representative agent's problem in the complementary relationship example.

As before, the solution to our problem is determined by the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\sup_{\{C_K, C_1, C_2, q_3\} \in \mathcal{A}} \left\{ U(C_K, C_1^{\nu} C_2^{1-\nu}) + \mathcal{D}J - \theta J \right\} = 0 \tag{B18}$$

where $\mathcal{D}$ is the standard Itô operator associated to this economy.[21] We proceed as before and obtain the first order conditions with respect to the consumptions and the demand of crude oil. As in the previous cases, the agent optimally consumes a constant fraction of the stocks ($C_K = \theta\, K$, $C_1 = \theta\, Q_1$ and $C_2 = \theta\, Q_2$) and demands a constant fraction of crude oil ($q_3 = \theta\, Q_3$).

The shadow price of a unit of each of the stocks is decreasing on its own stock and increasing on how important they are for the economy:

$$S_1 = \nu \frac{1-\phi}{\phi} \frac{K}{Q_1}, \qquad S_2 = (1-\nu) \frac{1-\phi}{\phi} \frac{K}{Q_2} \quad \text{and} \quad S_3 = \frac{\gamma\, \nu + (1-\gamma)(1-\nu)}{\theta} \frac{1-\phi}{\phi} \frac{K}{Q_3}. \tag{B19}$$

They are also increasing on the capital stock, because this imply that the commodities are relatively scarce in the economy.

The equilibrium convenience yields of gasoline and heating oil are as follows

$$\delta_1 = \gamma\, \log(\theta\, Q_3) - \sigma_1^2 \quad \text{and} \quad \delta_2 = (1-\gamma)\, \log(\theta\, Q_3) - \sigma_2^2 \tag{B20}$$

Both convenience yields are increasing in the crude oil stocks, because this means a higher marginal productivity of gasoline and heating oil. Interestingly, as opposed to the substitute example in the previous section, here the convenience yields of the complementary commodities are positively correlated (in this simple case, they are even perfectly correlated). It is straightforward to show that the dynamics of the gasoline convenience yield depends on the heating oil convenience yield.

[20] The crack product ratio of unleaded gasoline and heating oil usually is $\frac{2}{3} : \frac{1}{3}$. On average, cracking three barrels of crude oil produces 2 barrels (84 gallons) of unleaded gasoline and one barrel heating oil.

[21] Note that since the consumption of the mix (C_M) is increasing in the consumption of gasoline (C_1) and heating oil (C_2), we can assume that the agent maximizes with respect to C_1 and C_2.

The co-movement behavior of heating oil and gasoline prices is more complex than in a pure production relationship and in a pure substitute relationship. In our example above, the heating oil and gasoline co-move with crude oil due to production relationships. The prices of heating oil and gasoline are inversely related to their inventories, which in turn are determined by supply from crack production and demand from complementary consumptions. The consumption intensity of gasoline in the consumption mix is determined by ν and the production share of gasoline from output mix is determined by γ. Unless there is a perfect product-mix where the outputs and consumption are in perfect proportion, the gasoline and heating oil prices will not co-move perfectly.

C Proofs for the Empirical Model

C.1 Proof of Proposition 1

Under the risk-neutral measure, the i^{th} futures prices $F_{i,t}(Y_t, T)$ need to satisfy,

$$F_{i,t}(Y_t, T) = \mathbb{E}_t^{\mathbb{Q}}[S_{i,T}] \quad \text{for} \quad i = 1, \dots, n \tag{C1}$$

Let $\tau = T - t$. The futures price $F_{i,t}(Y_t, t + \tau)$ should satisfy the following vector-based Feynman-Kac equation,

$$-\frac{\partial F_i}{\partial \tau} + \frac{\partial F_i}{\partial Y}^{\top}(U + \Psi Y) + \frac{1}{2}\mathrm{Tr}\left(\frac{\partial^2 F_i}{\partial Y^2}\Omega\right) = 0 \tag{C2}$$

with boundary condition $F_{i,t}(Y_t, t) = \exp(x_{i,t})$.

Assume that

$$\log(F_{i,t}(Y_t, t + \tau)) = m_i(\tau) + G_i(\tau) Y_t \tag{C3}$$

where $m_i(\tau)$ is the i^{th} element of the $m(\tau)$ vector, and $G_i(\tau)$ is the i^{th} row of the $G(\tau)$ matrix. By plugging (C3) into (C2), we have two ordinary differential equations

$$\begin{aligned} -\frac{\partial m_i}{\partial \tau} + G_i U + \frac{1}{2} G_i(\tau)\,\Omega\, G_i(\tau)^{\top} &= 0 \\ \frac{\partial G_i}{\partial \tau} - G_i(\tau)\Psi &= 0 \end{aligned} \tag{C4}$$

with boundary condition

$$\begin{aligned} m_i(0) &= 0 \\ G_{i,i}(\tau) &= 1 \\ G_{j,i}(\tau) &= 0\,(i \neq j) \end{aligned}$$

Thus, the solution for (C2) is

$$\begin{aligned} m_i(\tau) &= \int_0^{\tau} \left(G_i(u)\, U + \frac{1}{2} G_i(u)\,\Omega\, G_i(u)^{\top} \right) du \\ G(\tau) &= \exp(\Psi\, \tau) \end{aligned} \tag{C5}$$

$G_i(\tau)$ denotes the i^{th} row of the $G(\tau)$ matrix. When Ψ is diagnosable,

$$G(\tau) = \Xi\, \mathrm{diag}(\exp(\lambda_1 \tau), \dots, \exp(\lambda_{2n} \tau))\Xi^{-1}$$

where Ξ is the matrix composed of eigenvectors of Ψ and λ_k $(k = 1, \dots, 2n)$ are the eigenvalues of Ψ; otherwise $G(\tau)$ can be calculated by Taylor expansion, i.e. $G(\tau) = I + \frac{1}{2}(\Psi\, \tau)^2 + \frac{1}{6}(\Psi\, \tau)^3 \dots$

Grouping the elements m_i's from equation (C5) yields the solution in Proposition 1.

C.2 Proof of Proposition 2

Equation (27) specifies a unique transformation from the latent variables $\widehat{Y}$ to $\overline{Y}$. Thus the $\overline{Y}$ processes in (27) preserves the maximal specification of the model. Letting $\Gamma_0(t) = \begin{pmatrix} \psi_0(t) \\ \psi_c \end{pmatrix}$, $\Gamma_{\widehat{Y}} = \begin{pmatrix} \psi_{\widehat{Y}} \\ \psi_{\widehat{Y}}\Lambda \end{pmatrix}$ and applying Itô's lemma to (27) we see that

$$d\overline{Y} = \Gamma_{\widehat{Y}}\Lambda\Gamma_{\widehat{Y}}^{-1}(\Gamma_0(t) - \overline{Y})dt + \Gamma_{\widehat{Y}}d\beta_{\widehat{Y}}^{\mathbb{Q}}, \tag{C6}$$

Denoting $\psi_{\widehat{Y}} = \begin{pmatrix} \psi_1 & \psi_2 \end{pmatrix}$ and $\Lambda = \begin{pmatrix} \Lambda_1 & 0 \\ \Lambda_2 & \Lambda_3 \end{pmatrix}$ where $\psi_1, \psi_2, \Lambda_1, \Lambda_2, \Lambda_3$ are all $n \times n$ matrixes and, comparing this with (26), we have,

$$\begin{aligned}
\overline{\Omega} &= \psi_{\widehat{Y}}^{\top}\psi_{\widehat{Y}} + K^{\top}\psi_{\widehat{Y}}^{\top}\psi_{\widehat{Y}}\Lambda \\
\overline{B} &= (\psi_2\Lambda_3\psi_2^{-1}\psi_1 - \psi_1\Lambda_1 - \psi_2\Lambda_2)^{-1}(\psi_1\Lambda_1^2 + \psi_2\Lambda_2\Lambda_1 + \psi_2\Lambda_3\Lambda_2 - \psi_2\Lambda_3^2\psi_2^{-1}\psi_1) \\
\overline{A} &= (\psi_1\Lambda_1 + \psi_2\Lambda_2 - \psi_2\Lambda_3\psi_2^{-1}\psi_1)^{-1}(\psi_1\Lambda_1^2 + \psi_2\Lambda_2\Lambda_1 + \psi_2\Lambda_3\Lambda_2 - \psi_2\Lambda_3^2\psi_2^{-1}\psi_1)\psi_2\Lambda_3\psi_2^{-1} - \psi_2\Lambda_3^2\psi_2^{-1}
\end{aligned} \tag{C7}$$

There is a one-one relationship from $(\Lambda, \psi_{\widehat{Y}})$ to $(\overline{\Omega}, \overline{A}, \overline{B})$. Note that there are, in total, $n + 2n^2$ parameters in Λ and $2n^2$ in $\psi_{\widehat{Y}}$. Also, there are, in total $n + 2n^2$ parameters in $\overline{\Omega}$ and $2n^2$ in $\overline{A}$ and $\overline{B}$.

Given $\overline{B}$ and $\overline{A}$, $\overline{R}$ can be determined easily from $\overline{\Omega}$ and has the form $\overline{R} = (r^f - \frac{1}{2}\overline{\sigma}_1^2, \ldots, r^f - \frac{1}{2}\overline{\sigma}_n^2)^{\top}$. The other mean vector has the form $\overline{L}(t) = (\overline{\theta}_1(t), \ldots, \overline{\theta}_n(t))^{\top}$ with $\overline{\theta}_i(t) = \overline{\chi}_i + \overline{\omega}_i(t)$ and it can be determined by

$$\overline{L}(t) = -(\overline{A}\,\psi_0(t) + \overline{B}\,\psi_c), \tag{C8}$$

Specifically, $\overline{\chi}_i = \sum_{k=1}^{n} \overline{A}_{i,k}\alpha_k + \sum_{k=1}^{n} \overline{B}_{i,k}(\psi_c)_k$, and $\overline{\omega}_i(t) = \sum_{k=1}^{n} \overline{A}_{i,k}\varpi_k(t)$.

Therefore, equation (26) is identical to the maximal specification under the risk-neutral measure.

C.3 Proof of Proposition 3

Equation (28) specifies a unique linear transformation from the latent variables $\overline{Y}$ to Y. Denote $\Gamma_{\overline{Y}} = \begin{pmatrix} I_{n\times n} & 0 \\ -A^{-1}B & -A^{-1} \end{pmatrix}$. Performing Itô's lemma on (28) we have

$$dY = \Gamma_{\overline{Y}}(\overline{U} + \overline{\Psi}\,\Gamma_{\overline{Y}}^{-1}Y)dt + \Gamma_{\overline{Y}}\,d\beta_{\overline{Y}}^{\mathbb{Q}}. \tag{C9}$$

By comparing the parameters in (C9) and those in (17), we find that if the following equations hold, the two models are identical:

$$0 = B^2 - \overline{B}B + \overline{A} \tag{C10}$$

$$\overline{B} - B = A\,\mathcal{K}\,A^{-1} \tag{C11}$$

$$\Omega = (\Gamma_{\overline{Y}})^{\top}\overline{\Omega}\Gamma_{\overline{Y}} \tag{C12}$$

Equation (C10) is a quadratic matrix equation, which has been studied quite often (e.g., Smith, Singh, and Sorensen (1995)). In most of the cases, there is no analytical solution for the quadratic matrix equation, but it can be solved by numerical methods such as the Newton method (*c.f.* Higham and Kim (2001)). After obtaining B, we can solve (C11). Since A and $\mathcal{K}$ can be seen as the Eigenvalue and Eigenmatrix of $(\overline{B} - B)$, we can first obtain $\mathcal{K}$ by calculating the eigenvalues of $(\overline{B} - B)$, then we normalize the i^{th} eigenvector to make its i^{th} element equal to one. A is just the collection of the those eigenvectors. After obtaining A and B, we can easily obtain Ω by equation (C12). Note that there are, in total, $2n^2$ parameters in $\overline{A}$ and $\overline{B}$, and also $2n^2$ parameters in A, B and $\mathcal{K}$. Thus, (C10) and (C11) provide a mapping from $(\overline{A}, \overline{B})$ to $(A, B, \mathcal{K})$. Also, it is easy to show that $R = \overline{R}$, and

$$L = -(A^{-1}B\overline{R} + A^{-1}\overline{L}). \tag{C13}$$

Specifically, $\overline{\chi}_i = \sum_{k=1}^{n}(A^{-1}B)_{i,k}\overline{R}_k + \sum_{k=1}^{n}(A^{-1})_{i,k}(\overline{\theta})_k$, and $\omega_i(t) = \sum_{k=1}^{n}(A^{-1})_{i,k}\overline{\omega}_k(t)$.

D Estimation of the Substitute and Complementary Commodity Pairs

In this section we describe the data and results for the estimation of the WTI-Brent crude oil pair and the heating oil-gasoline pair. The first pair of commodities share a substitute relationship, while in the second case, the commodities are complementary goods.

D.1 The WTI and Brent crude oil pair

The WTI and Brent crude oils have very similar quality and thus a similar usage. The relationship between WTI and Brent crude oil belongs to the substitute relationship.

For the estimation we consider weekly futures prices of West Texas Intermediate (WTI) and Brent crude oils. The weekly WTI and Brent crude oil futures are obtained through the NYMEX and London International Petroleum Exchange for the period from 1995.01 to 2006.02 (582 observations for each commodity). The time to maturity ranges from 1 month to 11 months for these two commodities.[22] We use five time series – $F1$, $F3$, $F6$, $F9$, $F11$ – for WTI and Brent crude oil contracts. Table 6 summarizes the data. As before, we take the risk-free rate as 0.04, which is the average interest rate during these years. Note that, since the quality of WTI oil is slightly better (lighter) than that of Brent oil, on average the futures prices for WTI oil are around 1.5 to 2 dollars higher than those for Brent oil. Figure 6 shows the historical evolution of the WTI and Brent crude oil prices, where we see both of WTI and Brent crude oil do not show seasonal behavior.

In the model estimation, we arbitrarily set WTI crude oil as commodity 1 and Brent crude oil as commodity 2. Since nearly no seasonality is found in the WTI and Brent oil futures prices, we thus set $s_1^c = s_1^s = s_2^c = s_2^s = 0$ in equation (11). Table 7 shows the results for the WTI and Brent crude oil pair.

From the parameter estimation, we see that nearly all the parameters are significant except $a_{1,2}$, $a_{1,2}$ and some correlation estimations. Similar to the case of the previous example, $b_{1,2}$ and $b_{2,1}$ are also significant, i.e. the convenience yield of WTI depends on the price of Brent and vice versa. The positive sign of $b_{1,2}$ and $b_{2,1}$ is consistent with the prediction of the substitute relationship. Figure 7 shows the time series of the ME and RMSE of the pricing errors. The MEs are very small, and the RMSEs fluctuate between 0.002 to 0.02 for WTI crude oil and 0.001 to 0.03 for the Brent crude oil. This again shows that our model performs reasonably well in fitting futures prices. Figure 8 depicts the filtered spot convenience yields. From the likelihood ratio tests in table 8, we again see that our model is significantly better than either the CCD model or the GS model in fitting the futures prices. Figure 9 shows the correlation term-structure for correlated GS, correlated CCD and our model. We find quite similar relationships between correlated GS, CCD and our model as the crude-heating oil example.

[22] The reason for not using longer maturity futures is that the maturity for Brent futures was only up to 1 year in the early sample period.

D.2 The heating oil and unleaded gasoline pair

The heating oil and unleaded gasoline pair is a good example of commodities that has a complementary relationship, because both share a balanced supply from crude oil.

For the estimation we consider weekly futures prices of heating oil and unleaded gasoline. The weekly futures are obtained from Bloomberg for the period from 1995.01 to 2006.02 (582 observations for each commodity). As before, The time to maturity of the contracts ranges from 1 month to 11 months and we use five time series – $F1$, $F3$, $F6$, $F9$, $F11$ – for heating oil and gasoline contracts. Table 10 summarizes the data. Again, we assume that the risk-free rate as 0.04.

In the model estimation, we arbitrarily set heating oil as commodity 1 and gasoline as commodity 2. Table 11 shows the results. From the parameter estimation, we see that nearly all cross-sectional parameters are significant. As in the other estimations, $b_{1,2}$ and $b_{2,1}$ are highly significant, i.e. the convenience yield of the heating oil depends on the price of the gasoline and vice versa. The positive sign of $b_{1,2}$ and $b_{2,1}$ is consistent with the prediction of the complementary relationship. Figure 10 depicts the filtered spot convenience yields. From the likelihood ratio tests in table 12, we again see that our model is better than either the CCD model or the GS model in fitting the futures prices. Figure 11 shows the correlation term-structure for correlated GS, correlated CCD and our model.

Table 1: **Data Summary for heating and WTI crude oil**

The data consist of weekly futures prices of heating oil and West Texas Intermediate (WTI) crude oil from 1995.01 to 2006.02. Fn is denoted as futures contracts with roughly n months to maturity. The mean and standard deviation of returns are in annual terms. The unit for WTI oil and Brent oil futures prices are \$/bbl, \$/bbl respectively, while heating oil futures prices are originally in cents/gallon, we have transferred it to \$/bbl.

Contracts	Mean Price	Std of Price	Mean Return (Annualized)	Std of Price
Panel A: WTI crude oil				
$F1$	28.34	12.79	0.1689	0.3306
$F5$	27.73	13.19	0.1471	0.2637
$F9$	26.82	12.90	0.1404	0.2025
$F13$	26.41	12.93	0.1343	0.1979
$F17$	25.97	12.76	0.1315	0.1842
Panel B: Heating oil				
$F1$	32.77	15.84	0.1815	0.3752
$F5$	32.22	15.91	0.1522	0.2746
$F9$	31.51	15.51	0.1398	0.2246
$F13$	31.03	15.54	0.1304	0.2019
$F17$	30.59	15.27	0.1321	0.1968

Table 2: **Parameter estimation for the heating and WTI crude oil pair**

The data consist of weekly futures prices of heating oil and West Texas Intermediate (WTI) crude oil from 1995.01 to 2006.02. The estimates corresponds to the 4-factor maximal multi-commodity model.

Parameter	Estimate	Std. Err.	Parameter	Estimate	Std. Err.
$b_{1,1}$	-3.523	(0.022)	σ_1	0.456	(0.017)
$b_{1,2}$	3.500	(0.023)	σ_2	0.442	(0.016)
$b_{2,1}$	2.298	(0.136)	σ_3	0.696	(0.034)
$b_{2,2}$	-2.441	(0.152)	σ_4	0.225	(0.013)
$a_{1,2}$	0.018	(0.031)	χ_1	0.494	(0.035)
$a_{2,1}$	-0.455	(0.042)	χ_2	-1.178	(0.079)
k_1	2.066	(0.048)	$\widetilde{\chi}_1$	0.430	(0.106)
k_2	0.664	(0.045)	$\widetilde{\chi}_2$	-1.100	(0.028)
$\rho_{1,2}$	0.821	(0.033)	$\widetilde{\mu}_1$	0.216	(0.139)
$\rho_{1,3}$	0.672	(0.048)	$\widetilde{\mu}_2$	0.135	(0.135)
$\rho_{1,4}$	0.736	(0.062)	s_1^c	1.117	(0.020)
$\rho_{2,3}$	0.802	(0.040)	s_1^s	0.687	(0.018)
$\rho_{2,4}$	0.544	(0.068)	s_2^c	7.956	(0.153)
$\rho_{3,4}$	0.246	(0.092)	s_2^s	4.290	(0.158)
ϵ	0.011	(0.001)			
Log-likelihood	15398				

Table 3: **Likelihood ratio tests for the heating and WTI crude oil pair**

This table compares our maximal model with the correlated CCD and GS models. The parameters used in the calculation are from table 2. Correlated CCD and GS models correspond to the cases $b_{1,2} = b_{2,1} = a_{1,2} = a_{2,1} = 0$ and $b_{1,1} = b_{1,2} = b_{2,1} = b_{2,2} = a_{1,2} = a_{2,1} = 0$ respectively. The 1% significant levels are 13.28, 16.81 and 9.21, respectively for our model vs. correlated CCD, our model vs. correlated GS and correlated CCD vs. correlated GS model. The statistics significant at the 1% level are marked with an asterisk.

	Log-likelihood		LR statistic
Our model	15398	Our model vs. CCD	748 (*)
CCD	15024	Our model vs. GS	1788 (*)
GS	14504	CCD vs. GS	1040 (*)

Table 4: **Values for the heating oil-crude oil crack spread option**

The table shows the crack spread option prices between heating oil and WTI crude oil for different strikes. Panel A presents the call option values, while Panel B presents the put option values. The options and the underlying futures have the same maturity. The parameters used in the calculation are from table 2.

Panel A: Call Options

	Time to maturity = 3 months			Time to maturity = 5 years		
Strike	Our model	CCD	GS	Our model	CCD	GS
0	5.917	5.142	4.654	5.976	6.708	10.051
1	4.659	3.815	3.284	4.819	5.655	8.952
2	3.585	2.729	2.189	3.840	4.725	7.953
3	2.691	1.860	1.377	3.038	3.920	7.042
4	1.968	1.215	0.804	2.394	3.230	6.217

Panel B: Put Options

	Time to maturity = 3 months			Time to maturity = 5 years		
Strike	Our model	CCD	GS	Our model	CCD	GS
0	1.920	1.138	0.680	1.980	3.130	6.461
1	2.662	1.812	1.311	2.670	3.923	7.208
2	3.588	2.726	2.215	3.538	4.839	8.055
3	4.693	3.857	3.404	4.582	5.881	8.990
4	5.970	5.211	4.831	5.784	7.037	10.012

Table 5: **Out-of-sample comparison of heating oil-crude oil crack spread options**

The table shows the results of the out-of-sample tests using short-maturity heating oil-crude oil (1:1) crack spread options data. The market data consists on 244 calls and 367 puts from March 2006 to December 2006 with maturities between 3 and 12 months and moneyness between 0.8 to 1.2 (strike/spot). The parameters used in the calculation are from table 2.

	Mean Pricing Error (calls)	Root Mean Squared Error (calls)	Mean Pricing Error (puts)	Root Mean Squared Error (puts)
Our Model	-0.3582	0.4776	-0.2244	0.4761
CCD	-0.6977	0.7423	-0.5864	0.6904
Schwartz	-0.8856	0.9156	-0.7919	0.8594

Table 6: **Data Summary for WTI and Brent oil**

The weekly WTI and Brent crude oil futures are obtained through the NYMEX and London International Petroleum Exchange for the period from 1995.01 to 2006.02 (582 observations for each commodity). Fn is denoted as futures contracts with roughly n months to maturity. The mean and standard deviation of returns are in annual terms.

Contracts	Mean Price	Std of Price	Mean Return (Annualized)	Std of Price
Panel A: WTI crude oil				
$F1$	28.34	12.79	0.1689	0.3306
$F3$	27.97	12.98	0.1562	0.2787
$F6$	27.36	13.00	0.1455	0.2295
$F9$	26.82	12.90	0.1404	0.2025
$F11$	26.52	12.81	0.1381	0.1895
Panel B: Brent crude oil				
$F1$	26.70	12.49	0.1740	0.3292
$F3$	26.41	12.69	0.1612	0.2770
$F6$	25.89	12.75	0.1516	0.2334
$F9$	25.41	12.67	0.1457	0.2067
$F11$	25.13	12.60	0.1440	0.1950

Table 7: **Parameter estimation for the WTI and Brent oil pair**

The data consist of weekly futures prices of West Texas Intermediate (WTI) crude oil and Brent oil from 1995.01 to 2006.02. The estimates corresponds to the 4-factor maximal multi-commodity model.

Parameter	Estimate	Std. Err.	Parameter	Estimate	Std. Err.
$b_{1,1}$	-9.538	(0.042)	σ_1	0.380	(0.012)
$b_{1,2}$	8.840	(0.054)	σ_2	0.345	(0.011)
$b_{2,1}$	0.145	(0.051)	σ_3	0.317	(0.024)
$b_{2,2}$	-0.296	(0.058)	σ_4	0.312	(0.012)
$a_{1,2}$	-0.304	(0.172)	χ_1	-2.489	(0.210)
$a_{2,1}$	-0.053	(0.045)	χ_2	-0.290	(0.087)
k_1	0.504	(0.005)	$\widetilde{\chi}_1$	-2.836	(0.297)
k_2	1.116	(0.032)	$\widetilde{\chi}_2$	-0.268	(0.126)
$\rho_{1,2}$	0.906	(0.022)	$\widetilde{\mu}_1$	0.202	(0.114)
$\rho_{1,3}$	-0.106	(0.094)	$\widetilde{\mu}_2$	0.207	(0.103)
$\rho_{1,4}$	0.813	(0.034)	s_1^c	0	
$\rho_{2,3}$	0.089	(0.091)	s_1^s	0	
$\rho_{2,4}$	0.817	(0.032)	s_2^c	0	
$\rho_{3,4}$	0.052	(0.157)	s_2^s	0	
ϵ	0.006	(0.001)			
Log-likelihood	18508				

Table 8: **Likelihood ratio tests for the WTI and Brent oil pair**

This table compares our maximal model with the correlated CCD and GS models. The parameters used in the calculation are from table 2. Correlated CCD and GS models correspond to the cases $b_{1,2} = b_{2,1} = a_{1,2} = a_{2,1} = 0$ and $b_{1,1} = b_{1,2} = b_{2,1} = b_{2,2} = a_{1,2} = a_{2,1} = 0$ respectively. The 1% significant levels are 13.28, 16.81 and 9.21, respectively for our model vs. correlated CCD, our model vs. correlated GS and correlated CCD vs. correlated GS model. The statistics significant at the 1% level are marked with an asterisk.

	Log-likelihood		LR statistic
Our model	18508	Our model vs. CCD	1112 (*)
CCD	17952	Our model vs. GS	2986 (*)
GS	17015	CCD vs. GS	1874 (*)

Table 9: **Values for the WTI - Brent oil substitution (or location) spread option**

The table shows the substitution (or location) spread European option prices between WTI and Brent crude oil for different strikes. Panel A presents the call option values, while Panel B presents the put option values. The options and the underlying futures have the same maturity. The parameters used in the calculation are from table 7.

Panel A: Call Options

	Time to maturity = 3 months			Time to maturity = 5 years		
Strike	Our model	CCD	GS	Our model	CCD	GS
0	2.963	2.446	2.414	3.113	3.775	4.101
1	2.330	1.758	1.721	2.583	3.238	3.572
2	1.785	1.198	1.161	2.125	2.763	3.101
3	1.332	0.767	0.731	1.731	2.338	2.683
4	0.963	0.459	0.430	1.399	1.967	2.316

Panel B: Put Options

	Time to maturity = 3 months			Time to maturity = 5 years		
Strike	Our model	CCD	GS	Our model	CCD	GS
0	0.969	0.445	0.413	1.277	1.865	2.220
1	1.335	0.756	0.720	1.670	2.252	2.614
2	1.790	1.197	1.160	2.135	2.699	3.067
3	2.338	1.765	1.730	2.664	3.197	3.572
4	2.969	2.458	2.429	3.256	3.749	4.127

Table 10: **Data Summary for Heating oil and Unleaded Gasoline**

The weekly Heating oil and Unleaded Gasoline futures data considers the period from 1995.01 to 2006.02 (582 observations for each commodity). Fn is denoted as futures contracts with roughly n months to maturity. The mean and standard deviation of returns are in annual terms.

Contracts	Mean Price	Std of Price	Mean Return (Annualized)	Std of Price
Panel A: Heating oil				
$F1$	32.68	15.82	0.1739	0.3484
$F3$	32.40	15.88	0.1566	0.2888
$F6$	31.80	15.53	0.1471	0.2383
$F9$	31.30	15.29	0.1411	0.2126
$F11$	31.06	15.30	0.1378	0.2052
Panel B: Unleaded gasoline				
$F1$	34.95	14.79	0.1772	0.4041
$F3$	34.31	14.73	0.1461	0.3097
$F6$	33.53	14.78	0.1351	0.2493
$F9$	32.92	14.80	0.1287	0.2262
$F11$	32.52	14.61	0.1262	0.2065

Table 11: **Parameter estimation for Heating oil and Unleaded Gasoline pair**

The data consist of weekly futures prices of Heating oil and Unleaded Gasoline from 1995.01 to 2006.02. The estimates corresponds to the 4-factor maximal multi-commodity model.

Parameter	Estimate	Std. Err.	Parameter	Estimate	Std. Err.
$b_{1,1}$	-2.118	(0.173)	σ_1	0.36	(0.012)
$b_{1,2}$	0.444	(0.175)	σ_2	0.444	(0.015)
$b_{2,1}$	2.304	(0.199)	σ_3	0.049	(0.011)
$b_{2,2}$	-3.795	(0.186)	σ_4	0.248	(0.03)
$a_{1,2}$	-1.049	(0.148)	χ_1	-0.028	(0.056)
$a_{2,1}$	4.514	(1.087)	χ_2	-3.701	(0.689)
k_1	0.504	(0.088)	$\widetilde{\chi}_1$	-0.026	(0.065)
k_2	0.059	(0.016)	$\widetilde{\chi}_2$	-5.989	(0.961)
$\rho_{1,2}$	0.728	(0.041)	$\widetilde{\mu}_1$	0.051	(0.094)
$\rho_{1,3}$	-0.336	(0.132)	$\widetilde{\mu}_2$	0.126	(0.12)
$\rho_{1,4}$	-0.69	(0.051)	s_1^c	1.341	(0.322)
$\rho_{2,3}$	-0.301	(0.119)	s_1^s	0.608	(0.14)
$\rho_{2,4}$	-0.622	(0.052)	s_2^c	14.794	(4.236)
$\rho_{3,4}$	0.266	(0.142)	s_2^s	0.055	(1.459)
ϵ	0.015	(0.002)			
Likelihood	14229				

Table 12: **Likelihood ratio tests for the Heating oil and Unleaded Gasoline pair**

This table compares our maximal model with the correlated CCD and GS models. The parameters used in the calculation are from table 2. Correlated CCD and GS models correspond to the cases $b_{1,2} = b_{2,1} = a_{1,2} = a_{2,1} = 0$ and $b_{1,1} = b_{1,2} = b_{2,1} = b_{2,2} = a_{1,2} = a_{2,1} = 0$ respectively. The 1% significant levels are 13.28, 16.81 and 9.21, respectively for our model vs. correlated CCD, our model vs. correlated GS and correlated CCD vs. correlated GS model. The statistics significant at the 1% level are marked with an asterisk.

	log-likelihood		LR statistic
Our model	14229	Our model vs. CCD	240 (*)
CCD	14109	Our model vs. GS	274 (*)
GS	14092	CCD vs. GS	34 (*)

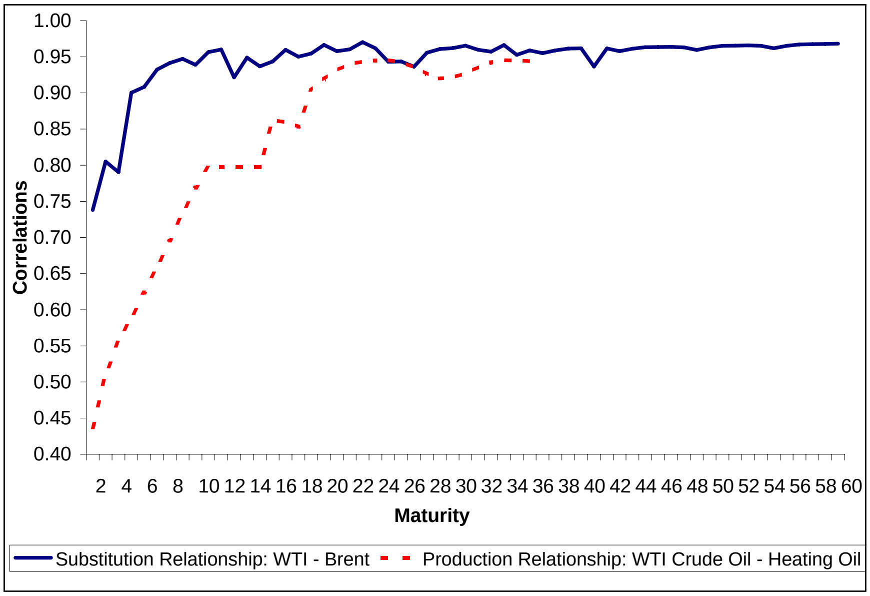

Figure 1: **Correlation structure for heating-crude oil and WTI-Brent crude oil pairs.** The figure plots the correlation between weekly futures returns for different maturities.

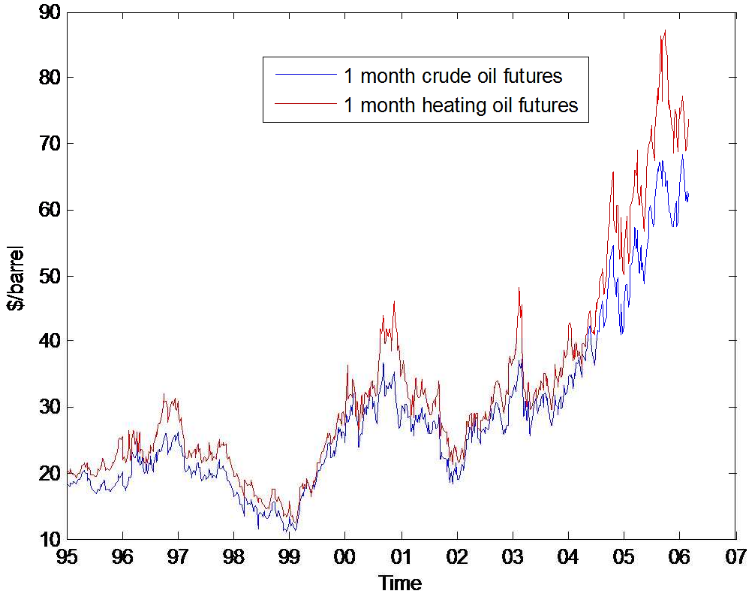

Figure 2: **Historical evolution of heating and crude oil futures.** Weekly data with 1 month futures prices of West Texas Intermediate (WTI) crude oil and heating oil are used.

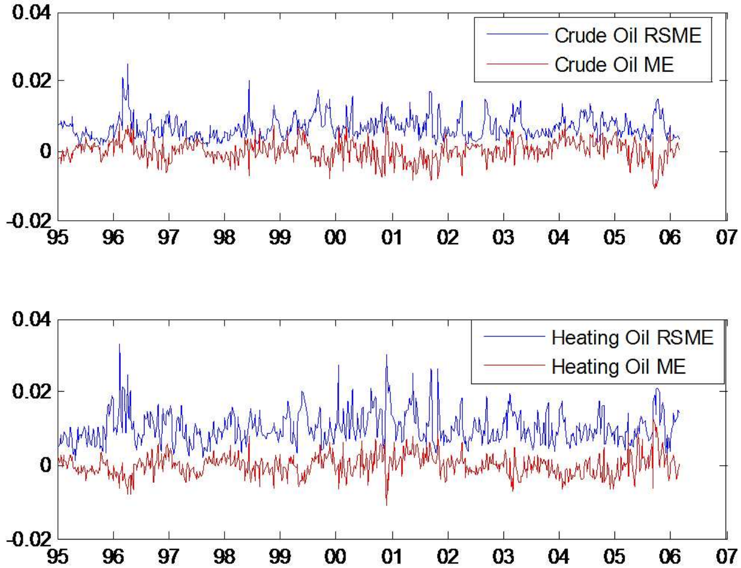

Figure 3: **ME and RMSE for the heating and WTI crude oil futures.** Both ME and RMSE are very small, which show that our model fits the futures prices reasonably well.

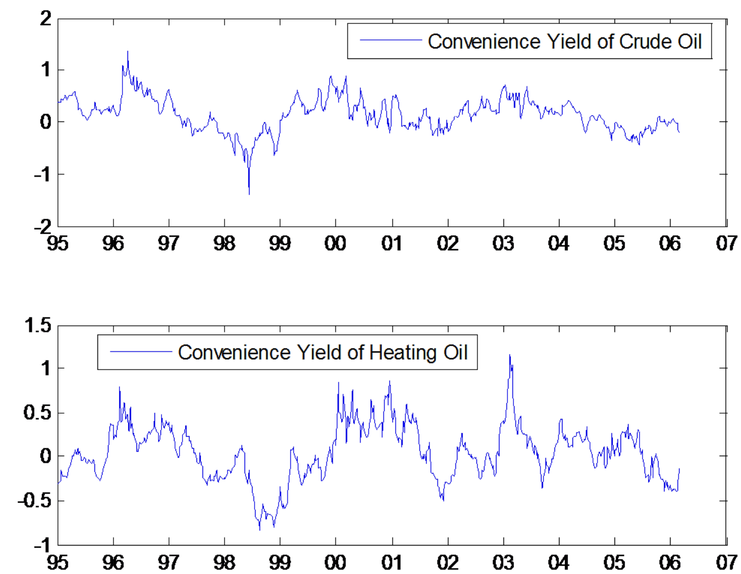

Figure 4: **The implied convenience yield for heating oil and WTI crude oil.** The implied convenience yield are from the 4-factor maximal multi-commodity model with the parameters from table 2.

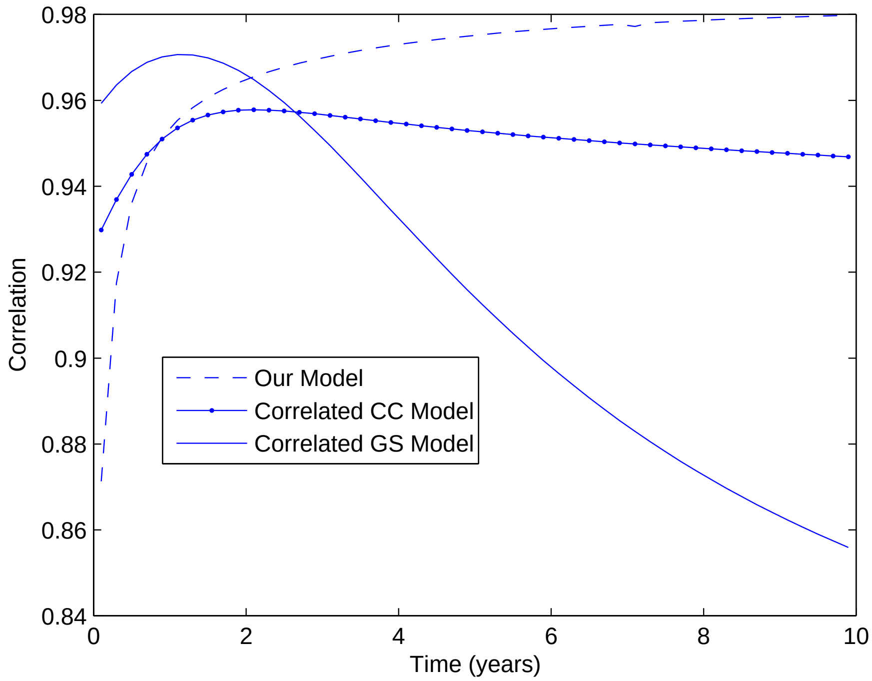

Figure 5: **The correlation term structure for heating and WTI crude oil.** The table shows the correlation term-structure for the correlated GS model, the correlated CCD model and our model. The parameters used in the calculation are from table 2.

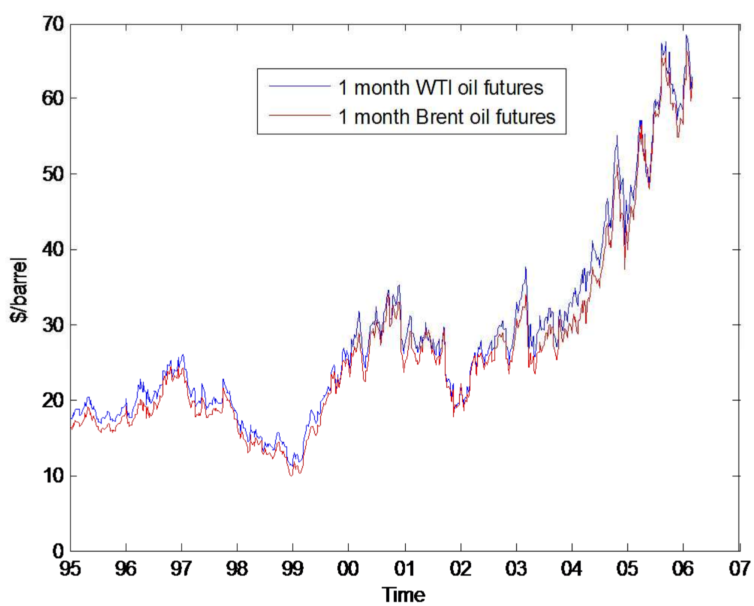

Figure 6: **Historical evolution of WTI and Brent crude oil futures.** Weekly data with 1 month futures prices of West Texas Intermediate (WTI) crude oil and Brent crude oil are used.

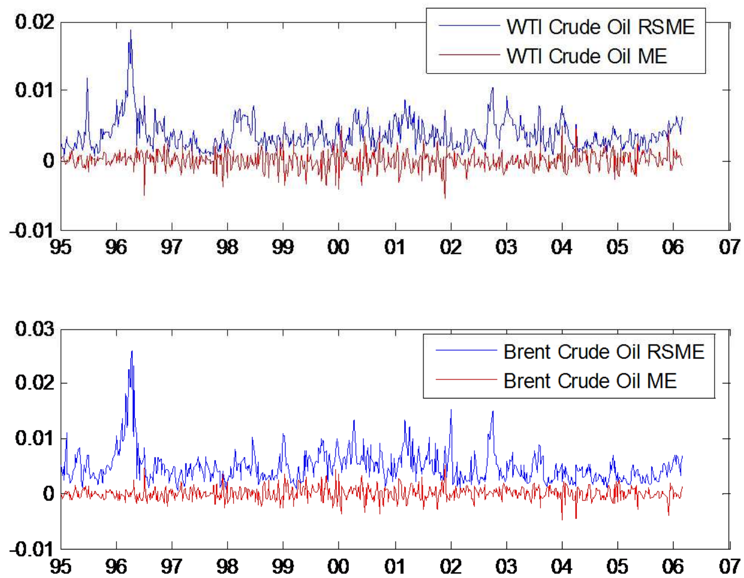

Figure 7: **ME and RMSE for the WTI and Brent crude oil futures.** Both ME and RMSE are very small, which shows our model fits the futures prices reasonably well.

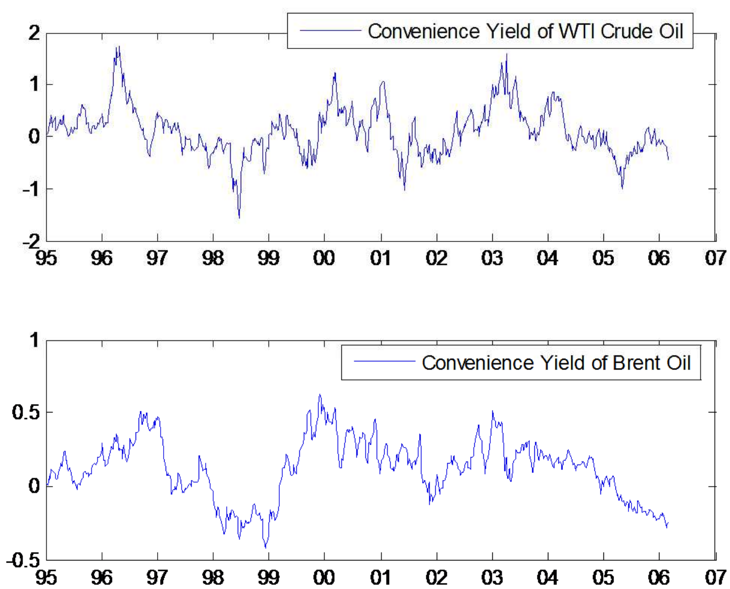

Figure 8: **The implied convenience yield for WTI and Brent crude oil.** The implied convenience yield are from the 4-factor maximal multi-commodity model with the parameters from table 7.

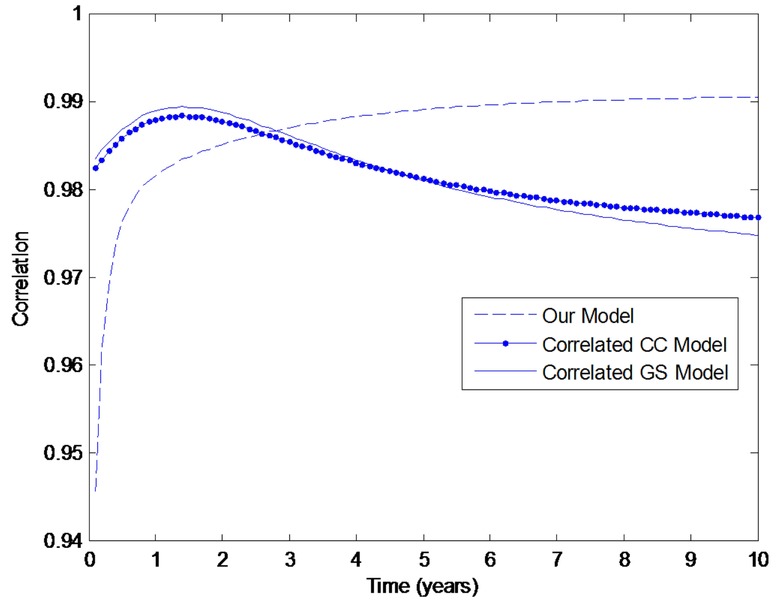

Figure 9: **The correlation term structure for WTI and Brent crude oil.** The table shows the correlation term-structure for the correlated GS model, the correlated CCD model and our model. The parameters used in the calculation are from table 7.

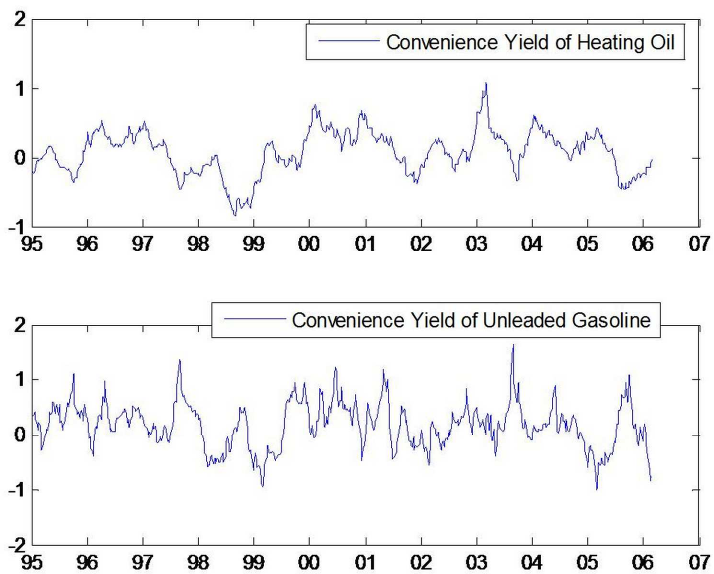

Figure 10: **The implied convenience yield for Heating oil and Unleaded gasoline.** The implied convenience yield are from the 4-factor maximal multi-commodity model with the parameters from table 11.

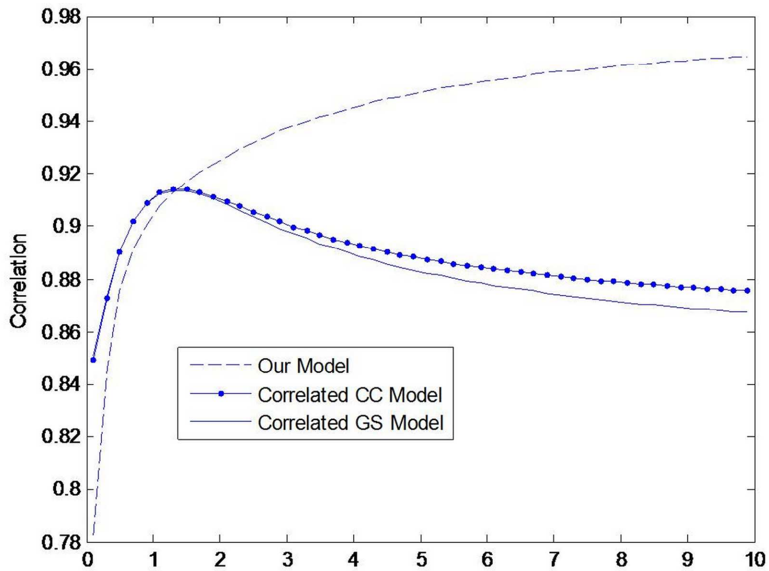

Figure 11: **The correlation term structure for Heating oil and Unleaded gasoline.** The table shows the correlation term-structure for the correlated GS model, the correlated CCD model and our model. The parameters used in the calculation are from table 11.