Strategic Investment Among Asymmetric Firms in Oligopoly [1]

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This paper studies strategic interaction of multiple firms under the assumption of asymmetric sunk cost and profit flow. We estimate the value of each firm and characterize the optimal investment thresholds. We also provide a numerical example on the equilibrium strategies in a three-firm setting. One of main result is that all firms act later than in those under symmetric case. Another important result is that the first investment threshold in an oligopoly market has three kinds of value. This result is strikingly different from the symmetric case where the first investment threshold is always larger than duopoly market.

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1. Introduction

While the orthodox theory in investment concentrates on profit value of certain time in the future, new approach method begins considering option to invest now or later. In perspective of traditional method such as NPV, increase of uncertainty leads decrease of project value. But in viewpoint of new method, also known as real option, we can postpone action to get more information and this can increase profit in uncertainty. Hence those properties -ability to delay investment and uncertainty actually affect decision of firm profoundly with property of irreversibility.

The early real options literature investigates the decision of single firm. However, economies in real market where mergers and acquisitions are prevalent ask for strategic investment decisions between two or more firms. Furthermore, such firms have different investment revenues and costs. Our goal is to demonstrate the interaction between those firms in oligopoly market when they are assumed to be asymmetric on both sunk cost of investment and profit flow. In particular, we investigate the optimal decision of each firm and the effect of asymmetry on competition.

Game theoretical real options have been researched a lot as one of the main interests in real options literature. Fudenberg and Tirole (1985) adopt effects of preemption in games of timing for the first time. They find that the threat of preemption yields a state equilibrium at which the benefit of being a leader equals that of follower. But it needs not do so if there are more than two firms, so does not extend to the general oligopoly game. Dixit and Pindyck (1994) establish a basic model for oligopoly industry by adding overall view of real option. They treat leader and follower in duopoly market, and argue that it is not hard to extend it to n firms even though it is messy in practice. Grenadier (1996) develops an equilibrium framework for solving option exercise strategy. He focuses on a particular example which is to consider the timing of real estate development. The model also explains why some markets experience building booms in the face of declining demand and property values.

There also has been development about competitive interaction under asymmetric structure. Huisman (2001) extends the basic model of two symmetric firms to asymmetric duopoly industry where firms have different sunk costs. He deals with both negative and positive externalites, and finds the condition that gives a lower cost firm's incentive to be a leader. Especially, it is shown that there are three types of equilibrium under negative externalites: only lower cost firm can be a leader, high cost firm has an incentive to be a leader, both firms invest simultaneously. Kijima and Shibata (2002) also investigate the equilibrium of asymmetric two firms, under the assumption that the general volatility depends on state variable. They find that there still exist three types of equilibrium unless it is the strategic complement case. Pawlina and Kort (2006) demonstrate how cost asymmetry cause intensive competition between two firms. They show that the relationship between the firm's value and the cost asymmetry is nonmonotonic and discontinuous. Optimal strategy in asymmetric duopoly industry becomes more specific by Kong and Kwok (2007). They provide a complete characterization of preemptive, dominant and simultaneous equilibriums by analyzing the relative value of leader and follower optimal investment thresholds. And application oligopoly industry make progress on Bouis, R. and Huisman and Kort(2009). In case of three firms change of the wedge between the second and third investment threshold is detected. This leads change of first investment threshold, and can be extended to the n-firm case.

Recent papers related to strategic competition are presented by Mason and Weeds(2010), and Thijssen(2010). Mason and Weeds(2010) investigate the relationship between investment and uncertainty when there may exist preemption. They show that greater uncertainty can lead the leader to invest earlier, while standard results are applied if investments are conducted simultaneously. It is argued that strategic interactions and externalities can have significant qualitative and quantitative effects on that relationship. Thijssen(2010) investigates more about effects of strategic interaction on the option value of waiting. He analyzes game option between two symmetric players each have specific stochastic state variable. It is shown that there exist four types of equilibrium which has qualitatively different properties from those with common state variable.

In this paper, we extend the analysis of Bouis, Huisman and Kort (2009) to the case where three firms are allowed to have asymmetric sunk cost and profit flow. It is obvious that a lower cost firm always becomes a leader when asymmetry is only on sunk cost. But, since we also assume asymmetry on profit flow, higher cost firm can have a chance to preempt. Hence there are some cases where one or two of lower cost firms become dominant or all firms are competitive to be a leader.

To investigate these competitions, we first calculate the value of each firm according to their order of investment and sunk cost. We then characterize the optimal investment thresholds and provide sufficient conditions under which dominant and competitive cases can occur. The optimal investment thresholds for these cases are computed both analytically and numerically. For illustration, we provide extensive examples on the equilibrium strategies in a three-firm setting. We conclude the examples by comparing our results with those under the case of symmetric oligopoly market and asymmetric duopoly market.

Our study on the multiple-firm framework leads to several important implications. We find that all the firms act later than those under the symmetric case. With higher sunk cost, firms would hesitate to enter, despite the assumption of negative externalities. It also affects to the accordion effect which is a remarkable result in the symmetric case. Since higher cost firm is hard to invest, the lowest cost firm will enter sooner to stay longer as a monopolist. It makes the first investment threshold smaller, thus the accordion effect becomes fainter. Besides the asymmetric sunk cost and profit flow, the value of parameters and the difference between the sizes of sunk costs also affect these results. We find that there exist three kinds of first investment threshold: equal to duopoly market, larger than duopoly market, and smaller than duopoly market. This result is strikingly different from the symmetric case where the first investment threshold is always larger than the duopoly counterpart.

The main contribution of our paper is to establish the structure of investment thresholds in asymmetric oligopoly industry. In addition, we demonstrate how the sunk cost and effect of competition influence the act of first investor. In particular, comparing to the symmetric case, it shows distinctive results in the sense of interval between two thresholds. We also derive conditions to predict the investment threshold type classified by our analysis.

The paper is organized as follows. The next section describes model framework and information that we need. In section 3, value of first, second, and third investor and investment threshold is defined. Numerical example is given in section 4. Using numerical analysis, we find out existence and value of investment threshold. We also consider equilibrium strategies, and conclude in section 5. Some proofs of propositions and corollary are given in Appendix.

2. Model

We consider three firms A, B, C, which produce a single, homogeneous good in some oligopolistic industry. Every firm has the option to wait for their optimal entry into the market. The investment opportunity is perpetual and irreversible. Sunk costs are asymmetric between three firms, and there is no variable costs of production after investment. Firms compete with each other to maximize their profits. Uncertainty of each firm's profit is described by state variable $Y(t)$ satisfying geometric Brownian motion given by

$$\begin{aligned} dY(t) &= \mu Y(t)dt + \sigma Y(t)dW(t), \\ Y(0) &= Y, \end{aligned} \tag{1}$$

where μ and σ are constants. Firms are assumed to be a risk neutral with risk-free interest rate $r > \mu$.

When number of n firms are active, profit flow of each active firm is described by

$$Y(t)D_n, \qquad n = 1, 2, 3, \tag{2}$$

where D_n is constant which reflects effect of competition. We assume negative externalities here. In other words, when there are more firms in the market, profit flow of each firm becomes less. So we can express D_n as a strictly decreasing function in n:

$$D_1 > D_2 > D_3 > D_\infty = 0. \tag{3}$$

We also assume asymmetry in sunk cost besides profit flow. Firm A, B, C have the opportunity to invest with distinct sunk cost I_A, I_B, I_C respectively, which satisfies

$$0 < I_A < I_B < I_C. \tag{4}$$

To consider the value function and optimal threshold for nth investor, we denote each of i, j, k as the one of three firms and fix their order. Namely,

$$i, j, k \in \{A, B, C\}, \qquad i \neq j \neq k. \tag{5}$$

Without loss of generality, fix k as a first investor, j as a second investor, and i as a third investor. For example, I_j is sunk cost of firm j which is the second investor.

Finally, we assume the initial value of Y is sufficiently low(i.e,$Y(0) \approx 0$), so that an immediate investment is no optimal.

3. Solution

In this section, we derive value function and optimal threshold for each investor in oligopoly market.

We consider C^n as a value function of nth investor. C^{na} denotes the value of nth investor being active. C^{no} denotes the value of firm which has option to invest as a nth investor but has not invested yet. For example, C^{2a}_{ji} is the value of second investor j, being active. Similarly, C^{1o}_{kji} is the value of first investor k, holding an option.

We also consider Y^n as an optimal threshold of nth investor. Hence Y^1_{kji} is the first investment threshold of firm k, Y^2_{ji} is the second one of firm j, Y^3_i is the third one of firm i.

To compute those values, we apply standard Bellman's optimality argument and use backward approach in time.

3.1. **Investment decision of the third investor.**

First, we analyze the investment decision of third investor. Since two other firms have already invested, we don't have to concern about strategic consideration. So investment problem of the third investor can be regarded as a monopoly situation. We apply our notation to the result of investment problem in monopoly industry well known by Dixit and Pindyck(1994).

Hence the value function of firm i as a third investor is equal to

$$C^3_i(Y) = \max_{\tau^3_i} E^Y \left[\int_{\tau^3_i}^{\infty} e^{-rt} D_3 Y_t dt - e^{-r\tau^3_i} I_i \right], \tag{6}$$

where τ^3_i is optimal stopping time such that

$$\tau^3_i = \inf\{s \geq t : Y_s = Y^3_i\}. \tag{7}$$

Computing the equation(6) leads the formula

$$\begin{aligned} C^{3o}_i(Y) &= \left(\frac{Y}{Y^3_i}\right)^{\beta} \left(\frac{Y^3_i D_3}{r-\mu} - I_i\right), \\ C^{3a}_i(Y) &= \frac{Y D_3}{r-\mu} - I_i, \end{aligned} \tag{8}$$

where β is the positive solution of equation

$$\frac{1}{2}\sigma^2\beta^2 + \left(\mu - \frac{1}{2}\sigma^2\right)\beta - r = 0 \tag{9}$$

By solving value-matching condition, we can get the investment trigger of third investor

$$Y^3_i = \frac{\beta}{\beta-1}\frac{(r-\mu)I_i}{D_3}. \tag{10}$$

Y_i^3 is uniquely defined. And it is decreasing in D_3, increasing in σ.

3.2. **Investment decision of the second investor.**

Secondly, we analyze the investment decision of the second investor. Since one of three firms has already invested, there remains two firms and they are facing duopoly investment game.

Hence the value function of firm j as a second investor is equal to

$$C_{ji}^2(Y) = E^Y\left[\int_{\tau_{ji}^2}^{\tau_i^3} e^{-rt}D_2Y_t dt - e^{-r\tau_{ji}^2}I_j + \int_{\tau_i^3}^{\infty} e^{-rt}D_3Y_t dt\right]. \tag{11}$$

τ_{ji}^2 is optimal stopping time such that

$$\tau_{ji}^2 = \inf\{s \geq t : Y_s = Y_{ji}^2\}, \tag{12}$$

where Y_{ji}^2 denotes optimal threshold of second investor.

Computing the equation(11) leads the formula

$$\begin{aligned} C_{ji}^{2o}(Y) &= \left(\frac{Y}{Y_{ji}^2}\right)^\beta\left(\frac{Y_{ji}^2D_2}{r-\mu} - I_j\right) + \left(\frac{Y}{Y_i^3}\right)^\beta\left(\frac{Y_i^3(D_3-D_2)}{r-\mu}\right), \\ C_{ji}^{2a}(Y) &= \frac{YD_2}{r-\mu} - I_j + \left(\frac{Y}{Y_i^3}\right)^\beta\left(\frac{Y_i^3(D_3-D_2)}{r-\mu}\right). \end{aligned} \tag{13}$$

Detail to derive formula is on Appendix A.

Unlike the trigger of the third investor, trigger of the second investor doesn't always exist because of asymmetrical structure between firms. If sunk cost is too large for the second investor to have incentive to preempt the third investor, trigger of the second investor may not exist. So we should determine whether there exists second investment trigger or not according to size of sunk cost of firm j and i.

Furthermore, even if there exists second investment trigger, it can have different value according to its competition circumstance. We divide it into two cases as in Kong and Kwok(2007). One is case that second investor has its threshold under keen competition versus third investor. The other is second investor has its threshold in dominant position.

If firm j and i are under keen competition, firm j would have benefit to invest when its value of acting as a second investor is larger than that of waiting to be a third investor. We define the investment trigger in this case as Y_{ji}^{21} which satisfies

$$Y_{ji}^{21} = \inf\{Y \in (0, Y_i^3) | C_j^{3o}(Y) \leq C_{ji}^{2a}(Y)\}, \tag{14}$$

and it can be defined again by

$$\left(\frac{Y_{ji}^{21}}{Y_j^3}\right)^{\beta} \left(\frac{Y_j^3 D_3}{r-\mu} - I_j\right) = \frac{Y_{ji}^{21} D_2}{r-\mu} - I_j + \left(\frac{Y_{ji}^{21}}{Y_i^3}\right)^{\beta} \left(\frac{Y_i^3(D_3 - D_2)}{r-\mu}\right). \tag{15}$$

But if firm j has a quite small sunk cost compared to firm i, it will have dominant position and doesn't have to pay attention to act of firm i. So it will invest when its value of acting is larger than that of waiting as a second investor. We define the investment trigger in this case as Y_j^{2*} which satisfies

$$Y_j^{2*} = \inf\{Y \in (0, Y_i^3) | C_{ji}^{2o}(Y) \leq C_{ji}^{2a}(Y)\}, \tag{16}$$

and its exact value is

$$Y_j^{2*} = \frac{\beta}{\beta - 1} \frac{r-\mu}{D_2} I_j. \tag{17}$$

Considering all cases, we have following result for Y_{ji}^2.

Proposition 1. The optimal threshold of second investor, Y_{ji}^2 is determined as below.
(case i) $I_j < I_i$
Y_{ji}^2 always exists and its value is

$$Y_{ji}^2 = \begin{cases} \min(Y_{ij}^{21}, Y_j^{2*}) & \text{if } I_i \in (I_j, I_j^*) \\ Y_j^{2*} & \text{if } I_i \in (I_j^*, \infty) \end{cases} \tag{18}$$

(case ii) $I_j > I_i$
Y_{ji}^2 exists only if $I_j \in (I_i, I_i^*)$ and its value is

$$Y_{ji}^2 = Y_{ji}^{21} \tag{19}$$

where

$$I_{j(i)}^* = \frac{I_{j(i)}}{D_3} \left(\frac{D_2^{\beta} - D_3^{\beta}}{\beta(D_2 - D_3)}\right)^{\frac{1}{\beta-1}}. \tag{20}$$

Equation (20) is a boundary for determining keen or dominant competition. We leave proofs about this boundary and existence of threshold to Appendix B. Here, we investigate the optimal decision of firm j, the second investor.

If firm j has smaller sunk cost compared to firm i, it is not hard to notice that it must pre-empt firm i. Especially if it is in dominant position, it will invest at Y_j^{2*}. But if firm i has also incentive, firm j will invest at Y_{ij}^{21} where firm i can start investing as a second investor. It still can invest at Y_j^{2*} if Y_j^{2*} is smaller than Y_{ij}^{21}.

If firm j has bigger sunk cost compared to firm i, it can invest only if it has incentive to pre-empt firm i. So it cannot be in dominant position, and its optimal decision is Y_{ji}^{21} since it has to invest directly when it has chance.

In fact, we can rewrite equation(18) to $Y_{ji}^2 = \min(Y_{ij}^{21}, Y_j^{2*})$. The reason is that Y_j^{2*} always exists even in case when there is no Y_{ij}^{21} to compare.

We have following corollary about value of $\min(Y_{ij}^{21}, Y_j^{2*})$.

Corollary 2. Let assume $I_j < I_i$ and $I_i \in (I_j, I_j^*)$. Then

$\min(Y_{ij}^{21}, Y_j^{2*}) = Y_j^{2*}$ if and only if

$$\beta - (\beta-1)\frac{I_i}{I_j} \leq \left(\frac{D_3}{D_2}\right)^{\beta} \left\{ \left(\frac{I_j}{I_i}\right)^{\beta-1} - \left(\frac{\beta(D_3-D_2)}{D_3}\right) \right\} < 1 \tag{21}$$

(proof) Appendix C.

3.3. **Investment decision of the first investor.**

Finally, we analyze the investment decision of the first investor. Since the only one firm is left and waiting for investment, the value function of firm k as a first investor is equal to

$$\begin{aligned} C_{kji}^1(Y) = \max_{\tau_{kji}^1} E^Y \Bigg[& \int_{\tau_{kji}^1}^{\tau_{ji}^2} e^{-rt} D_1 Y_t dt - e^{-r\tau_{kji}^1} I_k + \int_{\tau_{ji}^2}^{\tau_i^3} e^{-rt} D_2 Y_t dt \\ & + \int_{\tau_i^3}^{\infty} e^{-rt} D_3 Y_t dt \Bigg] \end{aligned} \tag{22}$$

τ_{kji}^1 is optimal stopping time such that

$$\tau_{kji}^1 = \inf\{s \geq t : Y_s = Y_{kji}^1\}, \tag{23}$$

where Y_{kji}^1 denotes optimal threshold of first investor.

Computing the equation(22) leads the formula

$$\begin{aligned} C^{1o}_{kji}(Y) &= \left(\frac{Y}{Y^1_{kji}}\right)^{\beta}\left(\frac{Y^1_{kji}D_1}{r-\mu} - I_k\right) + \left(\frac{Y}{Y^2_{ji}}\right)^{\beta}\left(\frac{Y^2_{ji}(D_2-D_1)}{r-\mu}\right) \\ &\quad + \left(\frac{Y}{Y^3_{i}}\right)^{\beta}\left(\frac{Y^3_{i}(D_3-D_2)}{r-\mu}\right) \\ C^{1a}_{kji}(Y) &= \frac{YD_1}{r-\mu} - I_k + \left(\frac{Y}{Y^2_{ji}}\right)^{\beta}\left(\frac{Y^2_{ji}(D_2-D_1)}{r-\mu}\right) \\ &\quad + \left(\frac{Y}{Y^3_{i}}\right)^{\beta}\left(\frac{Y^3_{i}(D_3-D_2)}{r-\mu}\right) \end{aligned} \tag{24}$$

Detail to derive formula is on Appendix D.

As it did in second investment trigger, asymmetrical structure between firms affects first investment trigger. If the sunk cost of first investor is too large to have incentive to pre-empt other firms, trigger of the first investor may not exist. Thus we should consider the size of sunk cost of firm k,j, and i.

And we should also consider how keen the competition is between firm k and j. If firm k and j are under keen competition, firm k would have benefit to invest when its value of acting as a first investor is larger than that of waiting to be a second investor. First investment trigger in this case defined as Y^{11}_{kji} which satisfies

$$Y^{11}_{kji} = \inf\{Y \in (0, Y^2_{ji}) | C^{2o}_{ki}(Y) \le C^{1a}_{kji}(Y)\}, \tag{25}$$

and it can be defined again by

$$\left(\frac{Y^{11}_{kji}}{Y^2_{ki}}\right)^{\beta}\left(\frac{Y^2_{ki}D_2}{r-\mu} - I_k\right) = \frac{Y^{11}_{kji}D_1}{r-\mu} - I_k + \left(\frac{Y^{11}_{kji}}{Y^2_{ji}}\right)^{\beta}\left(\frac{Y^2_{ji}(D_2-D_1)}{r-\mu}\right). \tag{26}$$

But if firm k has a quite small sunk cost compared to firm j, it will have dominant position and doesn't have to pay attention to act of firm j. We define the investment trigger in this case as Y^{1*}_k which satisfies

$$Y^{1*}_k = \inf\{Y \in (0, Y^2_{ji}) | C^{1o}_{kji}(Y) \le C^{1a}_{kji}(Y)\}, \tag{27}$$

and its exact value is

$$Y^{1*}_k = \frac{\beta}{\beta-1}\frac{r-\mu}{D_1}I_k. \tag{28}$$

Considering all cases, we have following result for Y^1_{kji}.

Proposition 3. The optimal threshold of first investor, Y^1_{kji} is determined as below.

(case i)$I_k < I_j$
(i-1)$I_k < I_j < I_i$
Y^1_{kji} always exists.

(i-2)$I_k < I_i < I_j$
Y^1_{kji} exists if $I_j \in (I_i, \frac{I_i}{D_3}\left(\frac{D_2^\beta - D_3^\beta}{\beta(D_2-D_3)}\right)^{\frac{1}{\beta-1}})$.

(i-3)$I_i < I_k < I_j$
Y^1_{kji} exists if $I_j \in (I_i, \frac{I_i}{D_3}\left(\frac{D_2^\beta - D_3^\beta}{\beta(D_2-D_3)}\right)^{\frac{1}{\beta-1}})$.

If Y^1_{kji} exists, its value is

$$Y^1_{kji} = \begin{cases} \min(Y^{11}_{jki}, Y^{11}_{jik}, Y^{1*}_k) & \text{if } I_j \in (I_k, I^*_{k,i}) \\ Y^{1*}_k & \text{if } I_j \in (I^*_{k,i}, \infty) \end{cases} . \tag{29}$$

(case ii)$I_k > I_j$
(ii-1)$I_j < I_k < I_i$
Y^1_{kji} exists if $I_k \in (I_j, I^*_{j,i})$ where $I^*_{j,i} = I_k$ satisfies equation (30).

(ii-2)$I_j < I_i < I_k$
Y^1_{kji} exists if $I_k \in (I_i, \frac{I_i}{D_3}\left(\frac{D_2^\beta - D_3^\beta}{\beta(D_2-D_3)}\right)^{\frac{1}{\beta-1}})$
and $I_k \in (I_j, I^*_{j,i})$ where $I^*_{j,i} = I_k$ satisfies equation (30).

(ii-3)$I_i < I_j < I_k$
Y^1_{kji} exists if $I_k \in (I_i, \frac{I_i}{D_3}\left(\frac{D_2^\beta - D_3^\beta}{\beta(D_2-D_3)}\right)^{\frac{1}{\beta-1}})$
and $I_k \in (I_j, I^*_{j,i})$ where $I^*_{j,i} = I_k$ satisfies equation (30).

$$\frac{Y^{1*}_k D_1}{r-\mu} - I_k + \left(\frac{Y^{1*}_k}{Y^2_{ji}}\right)^\beta \left(\frac{Y^2_{ji}(D_2 - D_1)}{r-\mu}\right) - \left(\frac{Y^{1*}_k}{Y^2_{ki}}\right)^\beta \left(\frac{Y^2_{ki} D_2}{r-\mu} - I_k\right) = 0 \tag{30}$$

If Y^1_{kji} exists, its value is

$$Y^1_{kji} = Y^{11}_{kji} \tag{31}$$

(proof) Appendix E.

4. Numerical example

Now we apply argument to real firms, instead of symbolizing them as k, j, i. Then there are six events that can be occurred.

$$\omega \in \{ABC, ACB, BAC, BCA, CAB, CBA\} \tag{32}$$

At here, the order of firms in each case is order to enter the market.

And we also assume some cases according to size of differences between sunk costs. We find investment triggers using numerical analysis and analyze it in sense of existence. Finally, we consider critical point and optimal strategy for each firm.

4.1. **Basic setting.**

Bouis et al.(2006) showed that when there are n firms active, profit flow of each firm is equal to

$$Y(t)D_n, \tag{33}$$

where

$$D_n = \frac{1}{n}\left(\frac{c}{n\gamma - 1}\right)(n\gamma)^{\gamma}, \tag{34}$$

with marginal cost of production c. And $Y(t)$ follows a geometric Brownian motion with drift μ, volatility σ given by

$$\begin{aligned} \mu &= \gamma\mu_x + \frac{1}{2}\gamma(\gamma-1)\sigma_x^2 \\ \sigma &= \gamma\sigma_x \end{aligned} \tag{35}$$

We set μ_x=0.025, σ_x=0.1, c=1, r=1 and consider three difference values for γ, γ=1.25, γ=1.5 and γ=2. Table1 shows how the competition effect would be according to those parameters.

γ=1.25		γ=1.5		γ=2	
effect	parameter	effect	parameter	effect	parameter
D1=0.535	r=0.1	D1=0.385	r=0.1	D1=0.25	r=0.1
D2=0.176	μ=0.033	D2=0.136	μ=0.041	D2=0.094	μ=0.06
D3=0.082	σ=0.125	D3=0.065	σ=0.15	D3=0.046	σ=0.2

Table 1. Parameters for competition effect

To analyze incentive for preempting, we classify some cases according to size of differences between sunk costs. Assume that small difference of sunk cost means difference of their sunk costs are small enough for firm who has bigger sunk cost to preempt the other one. Otherwise, it is big.

Case1 is the case when three firms have small and same amount of differences of sunk costs. It can have three more additional case according to its size of differences. To estimate the sensitive of investing order, we set sunk costs in Case1a to have bigger differences than sunk costs in Case1. But all firms still have incentive to preempt other firms as a second investor. Case1b is similar to Case1a, but differences are bigger for firm C not to preempt firm A as a second investor. Case1c is similar to Case1b, but differences are much bigger so that firm B cannot preempt firm A as a second investor.

In Case2, differences of sunk costs are same and quite large. But in case3 and 4, only the one of two differences is large and the other is small (It is $I_A \ll I_B < I_C$ in case3, and $I_A < I_B \ll I_C$ in case4).

We classify these cases as Table 2 using boundary (20) in Proposition 1.

case1	$\{(I_A, I_B, I_C) \mid I_C \in (I_B, I_B^*), I_B \in (I_A, I_A^*), I_C \in (I_A, I_A^*)\}$
case1a	$\{(I_A, I_B, I_C) \mid I_C \in (I_B, I_B^*), I_B \in (I_A, I_A^*), I_C \in (I_A, I_A^*)\}$
case1b	$\{(I_A, I_B, I_C) \mid I_C \in (I_B, I_B^*), I_B \in (I_A, I_A^*), I_C \in (I_A^*, \infty)\}$
case1c	$\{(I_A, I_B, I_C) \mid I_C \in (I_B, I_B^*), I_B \in (I_A^*, \infty), I_C \in (I_A^*, \infty)\}$
case2	$\{(I_A, I_B, I_C) \mid I_C \in (I_B^*, \infty), I_B \in (I_A^*, \infty), I_C \in (I_A^*, \infty)\}$
case3	$\{(I_A, I_B, I_C) \mid I_C \in (I_B, I_B^*), I_B \in (I_A^*, \infty), I_C \in (I_A^*, \infty)\}$
case4	$\{(I_A, I_B, I_C) \mid I_C \in (I_B^*, \infty), I_B \in (I_A, I_A^*), I_C \in (I_A^*, \infty)\}$

TABLE 2. Classified cases

Set the value of sunk cost for firm A to be 10. And then find other proper values of sunk cost which satisfy Table2. Detailed result for each case are presented in Table3.

case1	case1a	case1b	case1c	case2	case3	case4
I_A=10	I_A=10	I_A=10	I_A=10	I_A=10	I_A=10	I_A=10
I_B=11	I_B=12	I_B=13	I_B=16	I_B=25	I_B=23	I_B=12
I_C=12	I_C=14	I_C=16	I_C=22	I_C=40	I_C=25	I_C =25

TABLE 3. Value of sunk cost for each case

4.2. **Existence of investment trigger.**

Usually firm can rush into the market anytime after the state variable $Y(t)$ exceeds the investment trigger. Since different sunk cost for different firm makes the second investment trigger to be defined as (14), it makes continuous region for firm to have incentive to invest. Namely, firm should also wait to invest till $Y(t)$ reaches in this region, even if the initial value Y is bigger than the investment trigger. Concave property of function ϕ_2 in Appendix B.(b) demonstrates this phenomenon.

Here, we define the continuous region for second investor to have incentive with infimum and supremum of it.

$$\begin{aligned} Y_{ji}^{21} &= \inf\{Y \in (0, Y_i^3) | C_j^{3o}(Y) \le C_{ji}^{2a}(Y)\} \\ Y_{ji}^{22} &= \sup\{Y \in (Y_{ji}^{21}, \infty) | C_j^{3o}(Y) \le C_{ji}^{2a}(Y)\} \end{aligned} \tag{36}$$

Similarly, first investment trigger is defined as (25), so it also makes continuous region for first investor to have incentive. We can prove it using the concave property of function ϕ_1 in Appendix E.(b). We define the continuous region for first investor with infimum and supremum of it.

$$\begin{aligned} Y_{kji}^{11} &= \inf\{Y \in (0, Y_{ji}^2) | C_{ji}^{2o}(Y) \le C_{kji}^{1a}(Y)\} \\ Y_{kji}^{12} &= \sup\{Y \in (Y_{kji}^{11}, \infty) | C_{ji}^{2o}(Y) \le C_{kji}^{1a}(Y)\} \end{aligned} \tag{37}$$

We can get investment triggers and also investment region for each case. Table4 to Table10 show results obtained by numerical analysis when γ=1.25.

order	Y_k^{1*}	Y_{kji}^{11}	Y_{kji}^{12}	Y_j^{2*}	Y_{ji}^{21}	Y_{ji}^{22}	Y_i^3
ABC	2.207	1.443	5.565	7.379	4.900	17.224	17.278
ACB	2.207	1.438	5.653	8.050	5.583	15.782	15.838
BCA	2.428	1.597	5.928	8.050	5.775	14.151	14.398
BAC	2.428	1.604	5.809	6.708	4.378	17.056	17.278
CAB	2.648	1.858	4.992	6.708	4.447	15.779	15.838
CBA	2.648	1.863	4.951	7.379	5.131	14.336	14.398

TABLE 4. Result of case1

We can see that all six cases have values of Y^{11},Y^{21}, and Y^3. Hence we can conclude that every firm has incentive to be a first investor if there are small differences of sunk costs between firms. It also means that every six case of entrance is possible.

order	Y_k^{1*}	Y_{kji}^{11}	Y_{kji}^{12}	Y_j^{2*}	Y_{ji}^{21}	Y_{ji}^{22}	Y_i^3
ABC	2.207	1.399	6.663	8.050	5.279	19.969	20.157
ACB	2.207	1.398	6.707	9.391	6.663	17.072	17.278
BCA	2.648	1.691	7.618	9.391	7.346	13.342	14.398
BAC	2.648	1.729	6.713	6.708	4.282	19.352	20.157
CAB	3.090	2.206	5.505	6.708	4.378	17.056	17.278
CBA	3.090	2.220	5.405	8.050	5.775	14.151	14.398

TABLE 5. Result of case1a

All six cases have values of Y^{11},Y^{21}, and Y^3. Since we set sunk costs to make all firms have incentive to preempt other firms, all firms have second investment trigger in any order. Furthermore, we can check $I_k \in (I_j, I^*_{j,i})$ is satisfied for all cases. Hence there still exist first investment trigger in any case, although we set bigger difference between sunk costs compare to case1.

order	Y^{1*}_k	Y^{11}_{kji}	Y^{12}_{kji}	Y^{2*}_j	Y^{21}_{ji}	Y^{22}_{ji}	Y^3_i
ABC	2.207	1.3709	7.743	8.721	5.668	22.660	23.037
ACB	2.207	1.3708	7.745	10.733	7.783	18.289	18.717
BCA	2.869	·	·	10.733	·	·	14.398
BAC	2.869	1.911	6.600	6.708	4.220	21.360	23.037
CAB	3.531	2.545	6.120	6.708	4.324	18.244	18.717
CBA	3.531	·	·	8.721	6.497	13.832	14.398

TABLE 6. Result of case1b

We set differences of sunk costs to be larger to make firm C cannot preempt firm A. So Y^{21}_{CA} does not exist, and thus Y^{11}_{BCA}, Y^{11}_{CBA} cannot exist according to their definition. But firm C still can be a first investor unless firm B invest next.

order	Y^{1*}_k	Y^{11}_{kji}	Y^{12}_{kji}	Y^{2*}_j	Y^{21}_{ji}	Y^{22}_{ji}	Y^3_i
ABC	2.207	1.333	10.230	10.733	6.867	30.530	31.676
ACB	2.207	1.328	10.721	14.758	11.387	21.575	23.037
BCA	3.531	·	·	14.758	·	·	14.398
BAC	3.531	2.530	6.227	6.708	4.119	26.102	31.676
CAB	4.855	·	·	6.708	4.220	21.360	23.037
CBA	4.855	·	·	10.733	·	·	14.398

TABLE 7. Result of case1c

We get similar result to case1b, but sunk cost of firm B is too big to preempt firm A. So Y^{21}_{CA} and Y^{21}_{BA} do not exist, and thus Y^{11}_{BCA}, Y^{11}_{CBA} cannot exist according to how they defined. And there is no Y^{11}_{CAB} either, since sunk cost of firm C is too big to be a first investor preempting firm A. Hence we can say that the order CAB is the most sensitive one among six orders when differences of sunk costs are same.

But firm B still can preempt firm A as a first investor. And I_C is still in a boundary to overcome I_B, even though their difference is bigger than that of case1b. Hence there exist three first investment triggers in case1c, Y^{11}_{BAC}, Y^{11}_{ABC} and Y^{11}_{ACB}.

order	Y_k^{1*}	Y_{kji}^{11}	Y_{kji}^{12}	Y_j^{2*}	Y_{ji}^{21}	Y_{ji}^{22}	Y_i^3
ABC	2.207	1.305	14.095	16.770	10.550	53.400	57.592
ACB	2.207	·	·	26.833	·	·	35.995
BCA	5.517	·	·	26.833	·	·	14.398
BAC	5.517	·	·	6.708	4.025	33.946	57.592
CAB	8.827	·	·	6.708	4.091	27.933	35.995
CBA	8.827	·	·	16.770	·	·	14.398

TABLE 8. Result of case2

Differences between all sunk costs are quite big, so no firm can preempt other firms if they have larger sunk cost. Hence there exists only one first investment trigger, Y_{ABC}^{11}.

At here, there is no Y_{BAC}^{11} although it is well defined with existence of Y_{BC}^{21} and Y_{AC}^{21}. So we can assume that case BAC is the second sensitive one.

order	Y_k^{1*}	Y_{kji}^{11}	Y_{kji}^{12}	Y_j^{2*}	Y_{ji}^{21}	Y_{ji}^{22}	Y_i^3
ABC	2.207	1.326	10.889	15.429	10.254	35.892	35.995
ACB	2.207	1.326	10.889	16.770	11.618	33.008	33.115
BCA	5.076	·	·	16.770	·	·	14.398
BAC	5.076	·	·	6.708	4.091	27.933	35.995
CAB	5.517	·	·	6.708	4.109	26.745	33.115
CBA	5.517	·	·	15.429	·	·	14.398

TABLE 9. Result of case3

Since difference between I_A and I_B is quite large, there is no Y_{BA}^{21}. Hence it is impossible that there exists Y_{BCA}^{11} or Y_{CBA}^{11}. It means differences between I_A and I_C is also large. So there is no incentive for both firm B and C to preempt firm A. Hence there aren't Y_{CAB}^{11}, Y_{BAC}^{11}. But difference of sunk costs between firm B and C is relatively small. So there exist Y_{CB}^{21}, and also Y_{ACB}^{11}, since firm A always has incentive to preempt any other firm because of its small sunk cost.

After all, there are Y_{ABC}^{11} and Y_{ACB}^{11} in this case. But it is still possible that Y_{ACB}^{11} cannot exist depends on value of parameters, or size of difference between I_B and I_C.

order	Y_k^{1*}	Y_{kji}^{11}	Y_{kji}^{12}	Y_j^{2*}	Y_{ji}^{21}	Y_{ji}^{22}	Y_i^3
ABC	2.207	1.366	7.988	8.050	4.959	30.365	35.995
ACB	2.207	·	·	16.770	·	·	17.278
BCA	2.648	·	·	16.770	·	·	14.398
BAC	2.648	1.732	6.649	6.708	4.091	27.933	35.995
CAB	5.517	·	·	6.708	4.378	17.056	17.278
CBA	5.517	·	·	8.050	5.775	14.151	14.398

Table 10. Result of case4

Difference between I_A, I_C and I_B, I_C are too large to have Y_{CA}^{21} and Y_{CB}^{21}. Hence it is impossible that there exists Y_{BCA}^{11}. Y_{CBA}^{11}, Y_{ACB}^{11} and Y_{CAB}^{11}. But sunk cost of firm B is relatively small to have incentive to preempt firm A. Hence there exists Y_{BAC}^{1}.

After all, there are two investment trigger for first investor, Y_{ABC}^{11} and Y_{BAC}^{11}. But it is still possible that Y_{BAC}^{11} cannot exist depends on value of parameters or size of difference between I_A and I_B.

In case of γ=1.5 and γ=2, results are same to case of γ=1.25 in sense of existence of investment triggers. But there are some definite proofs which show influence of γ when we consider optimal thresholds.

4.3. **Equilibrium.**

In this section, we examine competitive equilibrium in our market. To analyze competitive entrance to the market, we should contemplate both numerical results and graph. We plot graph of value function for each firm and discuss about investment of firms.

Figure 1 to 7 shows value of each firm in every possible case. Note that firm B loses its incentive to be a first investor when state variable $Y(t)$ is over Y_B^{12}. Also firm C loses its incentive to be a first and second investor when state variable $Y(t)$ is over Y_C^{12} and Y_C^{22}, respectively.

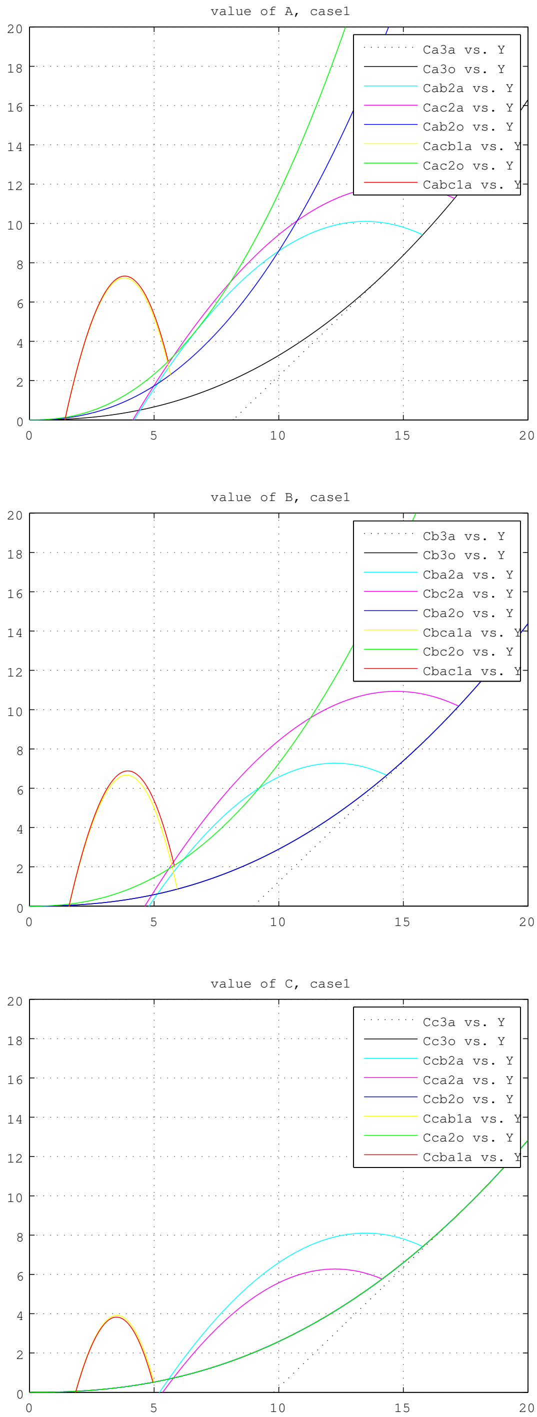

FIGURE 1. Value functions in case1

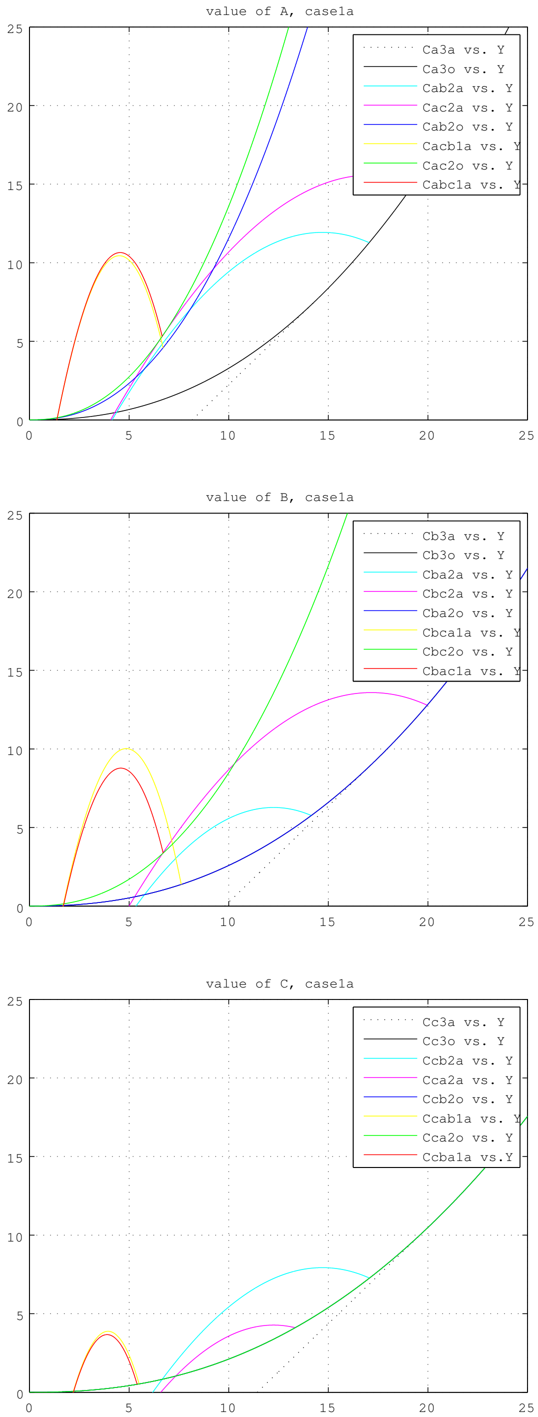

FIGURE 2. Value functions in case1a

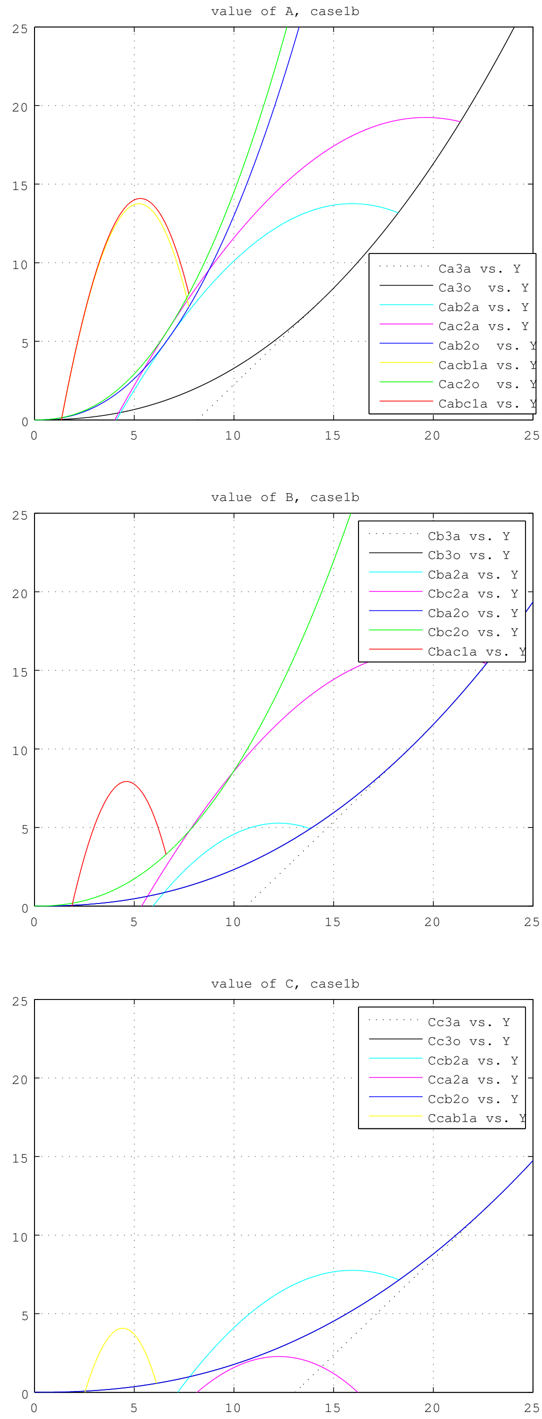

FIGURE 3. Value functions in case1b

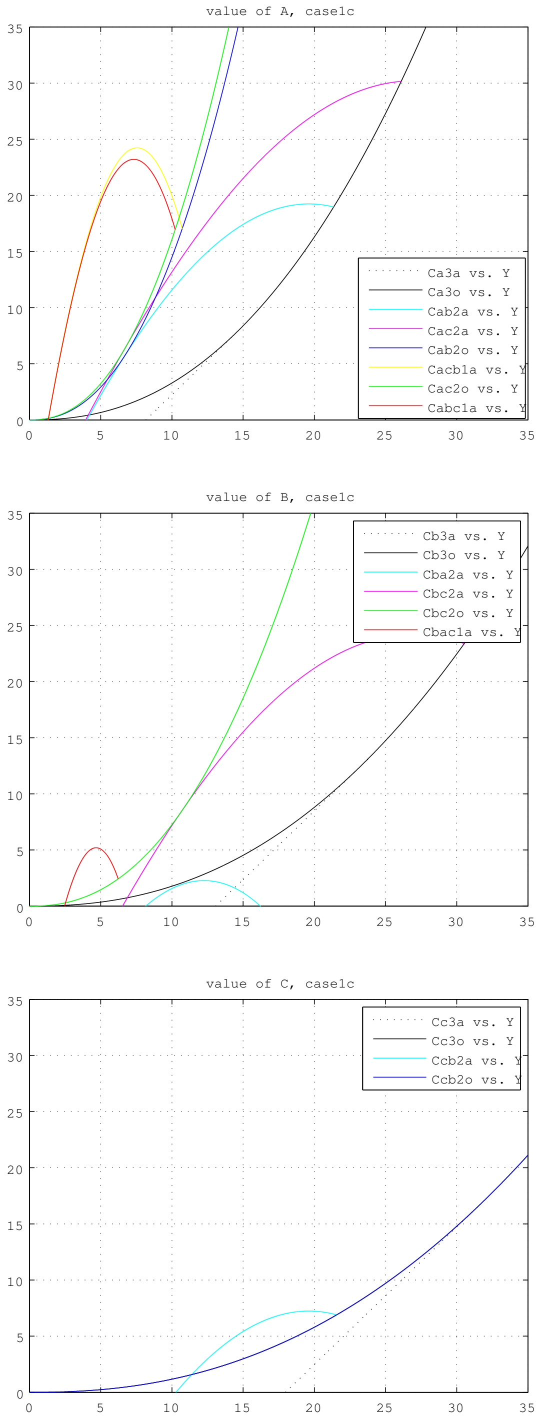

FIGURE 4. Value functions in case1c

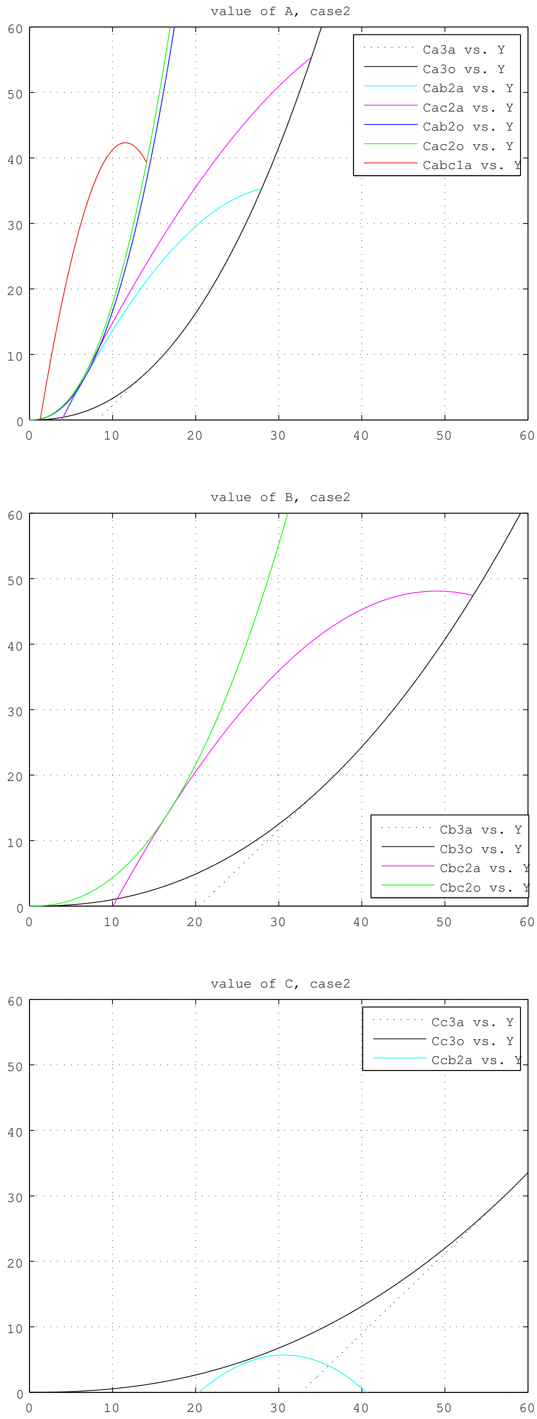

Figure 5. Value functions in case2

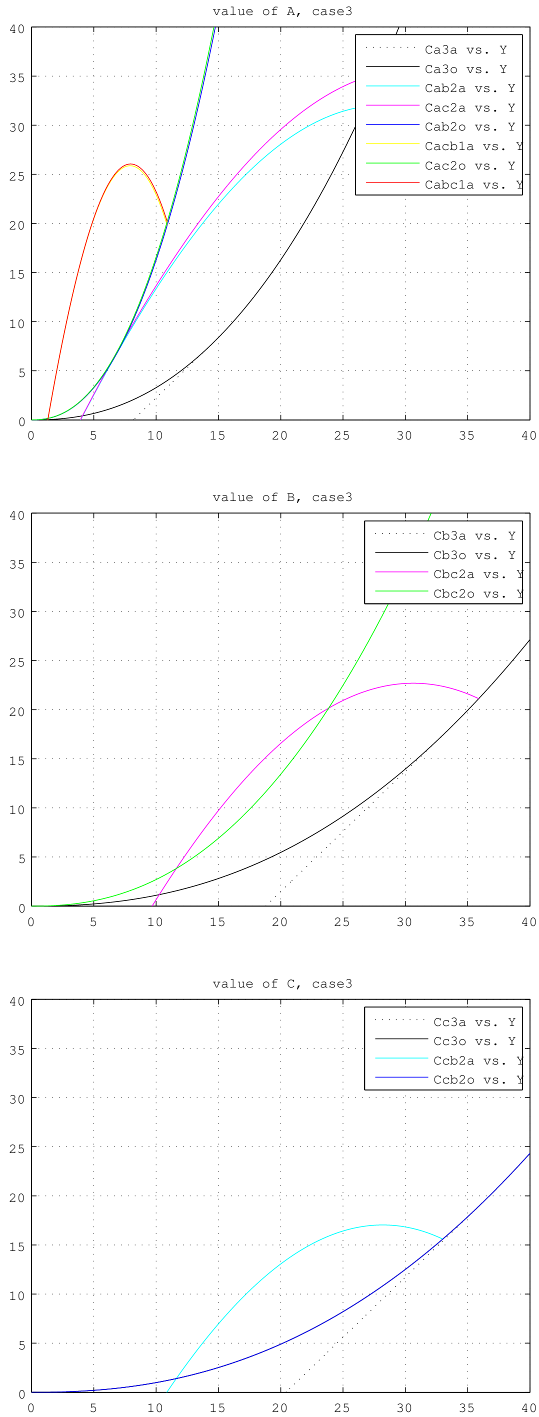

FIGURE 6. Value functions in case3

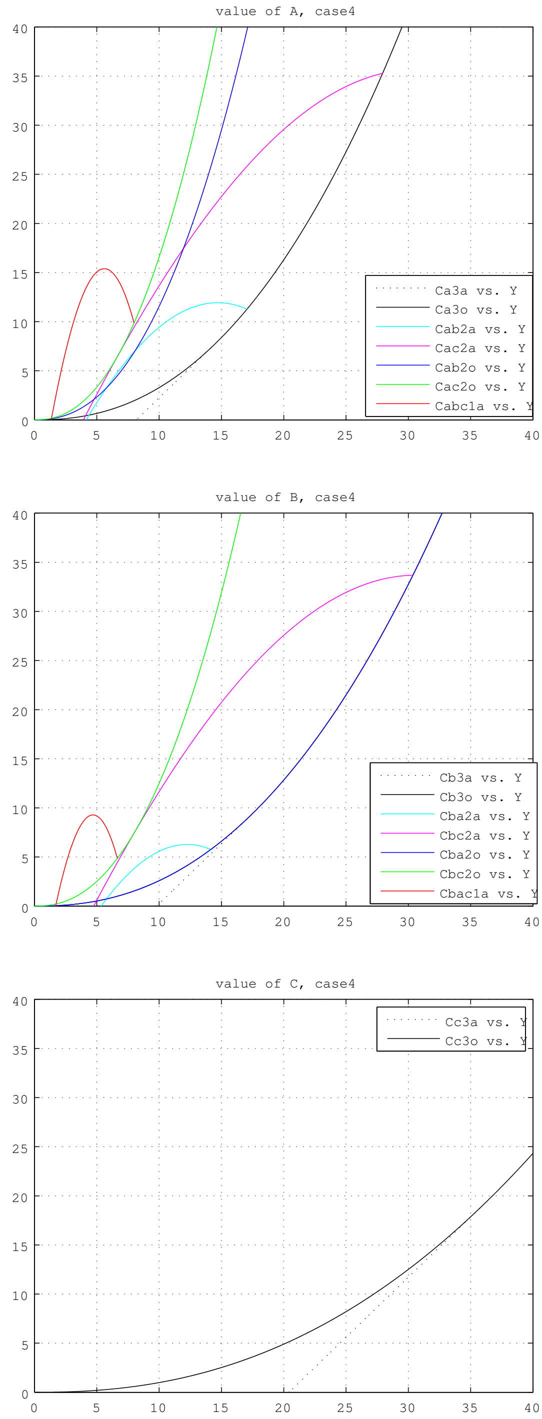

FIGURE 7. Value functions in case4

In case1, A invests at Y^{11}_{BCA}, B does at Y^{21}_{CB}, and C does at Y^3_C. Since we assume that initial value is sufficiently low, only firm A can be a first investor during $Y \in (Y^{11}_{ABC}, Y^{11}_{BCA})$. It is willing to invest whenever after Y^{11}_{ABC}(not Y^{11}_{ACB}, since it can make loss when B comes next and it cannot assure which firm will be the second one). But it delays its investment since it knows that B won't invest till Y^{11}_{BCA}. Similarly, B decides to invest at Y^{21}_{CB} before C invest. And firm C will wait till Y^3_C which is its critical point as a third investor.

In case1a, equilibrium is same to that of case1. Each case has different value and existence of threshold, and value function. But relation of thresholds are same, so they have same optimal strategy.

In case1b, equilibrium is almost same to that of case1 and case1a. But there exist not Y^{11}_{BCA} which was the first chance for firm B to be a first investor. Still there exist Y^{11}_{BAC}, so firm B can have incentive to be a first investor at here. For this reason, firm A should invest at least this point. Firm B and C will follow the strategy of case1 and 1a, because of the same reason in those cases.

In case1c, A invests at Y^{1*}_A, B does at Y^{2*}_B, and C does at Y^3_C. At first, only firm A can be a first investor during $Y \in (Y^{11}_{ABC}, Y^{11}_{BAC})$. But unlike in former cases, Y^{1*}_A is smaller than Y^{11}_{BAC}. It means that the potential of preemptive action of firm B does not affect to action of firm A. Hence firm A will invest at Y^{1*}_A. Furthermore, Y^{2*}_B is smaller than Y^{21}_{CB}. It means potential of preemptive action of firm C does not affect to action of firm B. Hence firm B will invest at Y^{2*}_B.

Case2 has the same result to that of case1c, but the reason is little different. In this case, there does not exist Y^{11}_{BAC}, and also Y^{11}_{BCA}. In fact, only firm A can be a first investor throughout whole region. Hence firm A can act as a monopolist, invests at Y^{1*}_A. And since there is no Y^{21}_{CB}, firm B needs not consider action of firm C. Hence it will invest at Y^{2*}_B.

In case3, A invests at Y^{1*}_A, B does at Y^{21}_{CB}, and C does at Y^3_C. At here, only firm A can be a first investor through whole region. Hence firm A will invest at its critical point which is the point in case of no competition. But firm B and C are under competition. Since B wants to be a second investor before C invests, it invests at Y^{21}_{CB}.

In case4, A invests at Y^{11}_{BAC}, B does at Y^{2*}_B, and C does at Y^3_C. Only firm A can be a first investor during $Y \in (Y^{11}_{ABC}, Y^{11}_{BAC})$, but it delays its investment since it knows that B won't invest till Y^{11}_{BAC}. But there is no competition between firm B and C. Hence firm B will invest at Y^{2*}_B.

Consideration of graphs and applying values of Table4 $\sim$ 10 to Proposition 2 and 3 give us following result for optimal decision of each firm.

	A	B	C
case1	Y^{11}_{BCA}	Y^{21}_{CB}	Y_C
case1a	Y^{11}_{BCA}	Y^{21}_{CB}	Y_C
case1b	Y^{11}_{BAC}	Y^{21}_{CB}	Y_C
case1c	Y^{1*}_{A}	Y^{2*}_{B}	Y_C
case2	Y^{1*}_{A}	Y^{2*}_{B}	Y_C
case3	Y^{1*}_{A}	Y^{21}_{CB}	Y_C
case4	Y^{11}_{BAC}	Y^{2*}_{B}	Y_C

TABLE 11. Optimal decision of each firm

Case1, 2, 3, and 4 shows that firms have obviously different strategy according to their incentive. In case1, firm B has incentive to preempt firm A and firm C has one to preempt firm B. So they concern about firm A or firm B. But they loose that incentive in case2 and don't concern about other firms any more. Only firm B monitors act of firm C in case3, and only firm A does it to firm B in case4.

These differences are based on size of sunk costs, so it becomes weaker when we consider subcases of case1. Firms in case 1a have similar strategy to case1 in sense of investment under competition. They have different existence of threshold if they are forced to enter in certain order assumed before(such as BAC, CBA). But since their basic concept of setting follows that of case1 except the size of sunk cost, they make same decision to case1 while they have same incentive. In case1c, firms loose that incentive and choose same strategy to case2.

Likewise, we can estimate critical points in all cases for each parameter value (when γ=1.25 or 2), and firms choose different strategy under different parameters. When γ is 1.5 or 2, they have different values of investment trigger and some cases show different optimal decision to that of γ=1.25. It means different parameter can also change the existence of optimal threshold, although we have same result after all. Hence we can say parameters affect investment of each firm. The reason seems like the value of μ and σ, since there is only little distinction between value of D_n when we compare parameters in γ.

4.4. **Comparison to symmetric firms.**

Under symmetric firms assumption, Bouis, R. and Huisman, K.J.M. and Kort, P.M.(2009) found distinctive phenomenon among three firms, so called accordion effect. Competition of three firms makes their first investment trigger larger and second investment trigger smaller than those of duopoly firms.

The reason is like this. Since threshold of third firm is decreasing in effect of competition D_3, it is less attempting for third firm to invest. It makes third investor enter later so second firm can have bigger profit rate YD_2 instead of YD_3. To maximize time having that profit rate, second firm decides to enter sooner. This gives less time for first firm to stay as a monopolist, thus it will enter later after all.

We apply our data to symmetric firms. Fix sunk cost for every firm to be $I_A = I_B = I_C = 10$ and get Table12.

	Y_k^1	Y_j^2	Y_i^3
monopoly	2.207		
duopoly	1.397	6.708	
three firms	1.510	4.542	14.398

Table 12. Thresholds of symmetric firms

We also apply our cases to asymmetric duopoly firms to consider accordion effect. In each case of duopoly market, we only use I_A and I_B to compare with oligopoly market.

	Y_A^1	Y_B^2	Y_C^3
monopoly	2.207		
duo1	1.561	7.379	
case1	1.597	5.583	17.278
duo1a	1.732	8.050	
case1a	1.691	6.663	20.157
duo1b	1.913	8.721	
case1b	1.911	7.783	23.037
duo1c	2.207	10.733	
case1c	2.207	10.733	31.676
duo2	2.207	16.770	
case2	2.207	16.770	57.592
duo3	2.207	15.429	
case3	2.207	11.618	35.995
duo4	1.732	8.050	
case4	1.732	8.050	35.995

Table 13. Thresholds of asymmetric firms

Every threshold of asymmetric firms are larger than that of symmetric firms regardless of cases. Under asymmetric structure, a firm which has bigger sunk cost tends to enter later since its bigger sunk cost make it hard to invest. Hence firm C enters later than it was supposed to do if it had same sunk cost to other firms. Firm B also acts like that because of the same reason, and it happens whether firm B or C has incentive to preempt firm A or B.

When we consider accordion effect, threshold of symmetric firms shows it well as expected. But it is quite different in case of asymmetric firms. Analytically, accordion effect occurs among symmetric firms since decrease in D_3 leads increase in first threshold, decrease in second one, and increase in third one. But asymmetric firms choose different second threshold according to their sunk cost size, and it makes different first threshold again. This relation between threshold and sunk cost size distracts unique result in competition effect for asymmetric firms .

In case1, each firm has different but small difference size of sunk cost. This affects firms to compete intensely and have similar result to symmetric firms. Thus we can see accordion effect at here, and it is the only case.

In case1a and 1b, firm B enters sooner to secure its time acting under duopoly market. But firm A also enters sooner even though it is just a little difference. We can interpret it as follows. Bigger sunk cost of firm B makes it harder to invest than case1. This leads firm A to enter sooner since it is attempting to have more profit staying longer as a monopolist.

In case1c, 2, 4, differences of sunk cost size between firms are large enough for firm A and B to act as they are in monopoly or duopoly market. Thus it needs not concern whether accordion effect is found or not. In case3 firm B enters sooner, but we still don't concern accordion effect since firm A act as a monopolist.

In fact, we can find two important phenomenon in oligopoly market compare to duopoly market when asymmetry is assumed. The first one is that the second investment threshold has always smaller value or at least same value compare to duopoly market. It is not hard to explain this when we consider how those thresholds are defined. By Proposition 1, investment threshold of firm B in oligopoly market Y_{BC}^2 has value of $min(Y_{CB}^{21}, Y_2^*) \leq Y_2^*$. On the other hand, investment threshold of firm B in duopoly market denoted by Y_B^{2d} has fixed formula which is equal to Y_2^*.

$$Y_B^{2d} = \frac{\beta}{\beta - 1} \frac{r - \mu}{D_2} I_B \tag{38}$$

Hence $Y_{BC}^2 \leq Y_B^{2d}$ is always satisfied.

The second one is that the first investment threshold in oligopoly market has three kinds of value: same to duopoly market, larger than duopoly market, and smaller than duopoly market. In other words, any exception is possible in oligopoly market. But we can classify those three types and designate conditions that we use to predict. We denote the investment threshold of firm A in duopoly market as Y_{AB}^{1d} for convenience. Since it is the threshold of first investor among two firms, we can define it using same way that we defined the threshold of second investor among three firms. Three kinds of first investment threshold in oligopoly market is described as below. Details are in Appendix F.

(i) $Y^1_{ABC} = Y^{1d}_{AB}$
if

$$Y^{1d}_{AB} = min(Y^{11}_{BA}, Y^{1*}_A) = Y^{1*}_A \tag{39}$$

where Y^{11}_{BA} is defined by

$$\left(\frac{Y^{11}_{BA}}{Y^{2d}_B}\right)^\beta \left(\frac{Y^{2d}_B D_2}{r-\mu} - I_B\right) = \frac{Y^{11}_{BA} D_1}{r-\mu} - I_B + \left(\frac{Y^{11}_{BA}}{Y^{2d}_A}\right)^\beta \left(\frac{Y^{2d}_A (D_2 - D_1)}{r-\mu}\right), \tag{40}$$

or

$$Y^2_{BC} = Y^{2d}_B = Y^{2*}_B \quad \text{and} \quad Y^2_{AC} = Y^{2d}_A = Y^{2*}_A. \tag{41}$$

(ii) $Y^1_{ABC} > Y^{1d}_{AB}$
if

$$Y^{1d}_{AB} = min(Y^{11}_{BA}, Y^{1*}_A) = Y^{11}_{BA} \tag{42}$$

and

$$\begin{aligned}
&\frac{Y^{1d}_{AB} D_1}{r-\mu} - I_B \\
&< \left(\frac{Y^{1d}_{AB}}{Y^2_{BC}}\right)^\beta \left(\frac{Y^2_{BC} D_2}{r-\mu} - I_B\right) - \left(\frac{Y^{1d}_{AB}}{Y^2_{AC}}\right)^\beta \left(\frac{Y^2_{AC}(D_2 - D_1)}{r-\mu}\right) \\
&< \frac{Y^{1d}_{AB} D_1}{\beta(r-\mu)}
\end{aligned} \tag{43}$$

(iii) $Y^1_{ABC} < Y^{1d}_{AB}$ otherwise.

Hence we can conclude that asymmetric structure in oligopoly market interrupts the occurrence of accordion effect and show different interaction between firms.

5. Conclusion

This paper investigates the strategies of firms in an oligopoly market when firms have different sunk costs to enter the market. Using option pricing theory, we calculate the value function of each firm and derive their investment threshold. Asymmetric assumption for both sunk costs and profit flow effects on optimal decision in competition. Especially, existence and value of the first investment threshold is influenced by those of the second investment threshold.

Numerical example also shows some significant results about equilibrium. We have different strategies according to cases what we classified by size of differences between sunk costs. Firms do not affect each other and follow their own optimal strategy when they have large difference in sunk costs. But the degree of that difference is quite sensitive and makes various strategies.

Finally, we analyze our results in sense of comparison to symmetric case. Since we set our sunk costs adding some value to symmetric one, firms become hesitative about entering. Hence all firms act later than those under the symmetric case. Furthermore, the lowest cost firm will enter sooner to stay longer as a monopolist since higher cost firms are hesitative in investment. It makes the first investment threshold smaller, and the accordion effect becomes fainter. Hence we can get three kinds of first investment threshold whereas the first investment threshold in symmetric case is always larger than the duopoly counterpart. We also show how asymmetric structure in oligopoly affect interaction between firms and those three kinds of first investment thresholds.

Appendix A. (Detail to get the value function of second investor)

$$\begin{aligned}
&C_{ji}^2(Y) = E^Y \left[\int_{\tau_{ji}^2}^{\tau_i^3} e^{-rt} D_2 Y_t dt - e^{-r\tau_{ji}^2} I_j \right. \\
&\left. + \int_{\tau_i^3}^{\infty} e^{-rt} D_3 Y_t dt\right] \\
&= E^Y \left[\int_{\tau_{ji}^2}^{\infty} e^{-rt} D_2 Y_t dt - e^{-r\tau_{ji}^2} I_j + \int_{\tau_i^3}^{\infty} e^{-rt} (D_3 - D_2) Y_t dt\right] \\
&= E^Y \left[e^{-r\tau_{ji}^2} \left(\tfrac{Y_{ji}^2 D_2}{r-\mu} - I_j\right) + e^{-r\tau_i^3} \tfrac{Y_i^3}{r-\mu} (D_3 - D_2)\right]
\end{aligned} \tag{44}$$

Let

$$\begin{aligned}
&E^Y \left[e^{-r\tau_{ji}^2} \left(\tfrac{Y_{ji}^2 D_2}{r-\mu} - I_j\right)\right] = K \\
&E^Y \left[e^{-r\tau_i^3} \tfrac{Y_i^3}{r-\mu} (D_3 - D_2)\right] = L
\end{aligned} \tag{45}$$

Since

$$\begin{cases}
K(0) = 0 \\
K(Y_{ji}^2) = \frac{Y_{ji}^2 D_2}{r-\mu} - I_j \\
K'(Y_{ji}^2) = \frac{D_2}{r-\mu}
\end{cases} \tag{46}$$

we can get

$$K = \begin{cases}
\left(\frac{Y}{Y_{ji}^2}\right)^{\beta} \left(\frac{Y_{ji}^2 D_2}{r-\mu} - I_j\right) & \text{if } Y < Y_{ji}^2 \\
\frac{Y D_2}{r-\mu} - I_j & \text{if } Y \geq Y_{ji}^2
\end{cases} \tag{47}$$

And

$$L = E^Y \left[e^{-r\tau_i^3} \frac{Y_i^3}{r-\mu} (D_3 - D_2)\right] = E^Y \left[e^{-r\tau_i^3}\right] \frac{Y_i^3}{r-\mu} (D_3 - D_2) \tag{48}$$

By Laplace transform, $E^Y \left[e^{-r\tau_i^3}\right] = \left(\frac{Y}{Y_i^3}\right)^{\beta}$.
Hence

$$L = \begin{cases}
\left(\frac{Y}{Y_i^3}\right)^{\beta} \left(\frac{Y_i^3 (D_3 - D_2)}{r-\mu}\right) & \text{if } Y < Y_i^3 \\
\frac{Y (D_3 - D_2)}{r-\mu} & \text{if } Y \geq Y_i^3
\end{cases} \tag{49}$$

Finally, we can get value of second investor

$$\begin{aligned}
&C_{ji}^2(Y) = K + L \\
&= \begin{cases}
\left(\frac{Y}{Y_{ji}^2}\right)^{\beta} \left(\frac{Y_{ji}^2 D_2}{r-\mu} - I_j\right) + \left(\frac{Y}{Y_i^3}\right)^{\beta} \left(\frac{Y_i^3 (D_3 - D_2)}{r-\mu}\right) & \text{if } Y < Y_{ji}^2 \\
\frac{Y D_2}{r-\mu} - I_j + \left(\frac{Y}{Y_i^3}\right)^{\beta} \left(\frac{Y_i^3 (D_3 - D_2)}{r-\mu}\right) & \text{if } Y_{ji}^2 \leq Y < Y_i^3 \\
\frac{Y D_3}{r-\mu} - I_j \text{ if } Y \geq Y_i^3
\end{cases}
\end{aligned} \tag{50}$$

Appendix B. (Proof of Proposition 2)

Define the function $\phi_2 : [0, Y_i^3] \longrightarrow \mathbb{R}$ such that $\phi_2(Y) = C_{ji}^{2a} - C_j^{3o}$.Then

$$\phi_2(Y) = \frac{YD_2}{r-\mu} - I_j + \left(\frac{Y}{Y_i^3}\right)^\beta \left(\frac{Y_i^3(D_3 - D_2)}{r-\mu}\right) - \left(\frac{Y}{Y_j^3}\right)^\beta \left(\frac{Y_j^3 D_3}{r-\mu} - I_j\right) \tag{51}$$

(a) $\phi_2(0) = -I_j < 0$
(b) Since $D_3 < D_2$,

$$\begin{aligned} \frac{\partial^2 \phi_2(Y)}{\partial Y^2} &= \beta(\beta-1)Y^{\beta-2}(Y_i^3)^{-\beta}\left(\frac{Y_i^3(D_3 - D_2)}{r-\mu}\right) \\ &\quad -\beta(\beta-1)Y^{\beta-2}(Y_j^3)^{-\beta}\left(\frac{1}{\beta-1}I_j\right) < 0 : \text{concave} \end{aligned} \tag{52}$$

(c)

$$\begin{aligned} \phi_2(Y_j^3) &= \left(\frac{Y_j^3}{Y_i^3}\right)^\beta \left(\frac{Y_i^3(D_3 - D_2)}{r-\mu}\right) - \left(\frac{Y_j^3(D_3 - D_2)}{r-\mu}\right) \\ &= \frac{(D_3 - D_2)}{r-\mu}\left(\left(\frac{Y_j^3}{Y_i^3}\right)^\beta Y_i^3 - Y_j^3\right) \\ &= \frac{(D_3 - D_2)}{r-\mu}Y_j^3\left(\left(\frac{Y_j^3}{Y_i^3}\right)^{\beta-1} - 1\right) \end{aligned} \tag{53}$$

In case of (i),

$$I_j < I_i \Longrightarrow Y_j^3 < Y_i^3 \Longrightarrow \phi(Y_j^3) > 0 \tag{54}$$

Hence there always exist $Y_{ji}^2 \in (0, Y_i^3)$.

But in case of (ii), we cannot assure the sign of $\phi_2(Y_j^3)$. It means we cannot assure the existence of Y_{ji}^2 always in this case.

Intuitively, firm j will have incentive to preempt when its sunk cost is relatively small. We want to find the range of this sunk cost which makes Y_{ji}^2 exists.

Assume that there exist Y_{ji}^2 where (15) is satisfied.

By definition of Y_{ji}^2 , $\phi_2(Y_{ji}^2)$ has to be zero.

$$\begin{aligned} \phi_2(Y_{ji}^2) = \frac{Y_{ji}^2 D_2}{r-\mu} - I_j + \left(\frac{Y_{ji}^2}{Y_i^3}\right)^\beta \left(\frac{Y_i^3(D_3 - D_2)}{r-\mu}\right) \\ - \left(\frac{Y_{ji}^2}{Y_j^3}\right)^\beta \left(\frac{Y_j^3 D_3}{r-\mu} - I_j\right) = 0 \end{aligned} \tag{55}$$

By procedure getting (47) from (46) in Appendix A, we can get the maximum value of Y_{ji}^2 as follow.

$$Y_{ji}^2 = \frac{\beta}{\beta-1}\frac{(r-\mu)I_j}{D_2} \tag{56}$$

Therefore,

$$\begin{aligned}
\frac{\beta}{\beta-1}I_j + \left(\frac{D_3 I_j}{D_2 I_i}\right)^{\beta}\left(\frac{\beta}{\beta-1}\frac{I_i}{D_3}(D_3-D_2)\right) - \left(\frac{D_3}{D_2}\right)^{\beta}\frac{\beta}{\beta-1}I_j = 0 \\
I_j + \left(\frac{D_3 I_j}{D_2 I_i}\right)^{\beta}\left(\frac{I_i}{D_3}(D_3-D_2)\right) - \left(\frac{D_3}{D_2}\right)^{\beta} I_j = 0 \\
\left(\frac{D_3}{D_2}\right)^{\beta}\left(\frac{I_j}{I_i}\right)^{\beta}\left(\frac{I_i}{D_3}(D_3-D_2)\right) = \left\{\left(\frac{D_3}{D_2}\right)^{\beta} - 1\right\} I_j \\
\frac{1}{I_j}(D_3)^{\beta}\left(\frac{I_j}{I_i}\right)^{\beta}\left(\frac{I_i}{D_3}(D_3-D_2)\right) = (D_3)^{\beta} - (D_2)^{\beta} \\
(I_j)^{\beta-1}\left(\frac{D_3}{I_i}\right)^{\beta-1} = \frac{D_3^{\beta} - D_2^{\beta}}{D_3 - D_2} \\
I_j = \frac{I_i}{D_3}\left(\frac{D_2^{\beta} - D_3^{\beta}}{\beta(D_2 - D_3)}\right)^{\frac{1}{\beta-1}}
\end{aligned} \tag{57}$$

Hence in case of (ii), Y_{ji}^2 exists only if

$$I_j < \frac{I_i}{D_3}\left(\frac{D_2^{\beta} - D_3^{\beta}}{\beta(D_2 - D_3)}\right)^{\frac{1}{\beta-1}} \equiv I_i^*(\text{function of } I_i). \tag{58}$$

Getting (58), we know that it is boundary to make firm who has bigger sunk cost to have incentive. Hence we can apply it to case(i) to determine when it is under keen competition. Likewise, it has boundary

$$I_j^* = \frac{I_j}{D_3}\left(\frac{D_2^{\beta} - D_3^{\beta}}{\beta(D_2 - D_3)}\right)^{\frac{1}{\beta-1}}. \tag{59}$$

In fact, we can define this form as a boundary to determine whether the competition is keen or not.

Appendix C. (Proof of Corollary 1)
$Y_j^{2*} \leq Y_{ij}^{21}$ if and only if $\phi_{2i}(Y_j^{2*}) \leq 0$, $\frac{\partial}{\partial Y}\phi_{2i}(Y_j^{2*}) > 0$,
where

$$\begin{aligned}\phi_{2i}(Y) &= C_{ij}^{2a} - C_i^{3o} \\ &= \frac{YD_2}{r-\mu} - I_i + \left(\frac{Y}{Y_j^3}\right)^\beta \left(\frac{Y_j^3(D_3-D_2)}{r-\mu}\right) - \left(\frac{Y}{Y_i^3}\right)^\beta \left(\frac{Y_i^3 D_3}{r-\mu} - I_i\right)\end{aligned} \tag{60}$$

(a)

$$\begin{aligned}\phi_{2i}(Y_j^{2*}) = \frac{\beta}{\beta-1}I_j - I_i + \left(\frac{D_3}{D_2}\right)^\beta \left(\frac{Y_j^3(D_3-D_2)}{r-\mu}\right) \\ - \left(\frac{D_3 I_j}{D_2 I_i}\right)^\beta \left(\frac{\beta}{\beta-1}I_i - I_i\right) \leq 0 \\ \Leftrightarrow \frac{\beta}{\beta-1}I_j - I_i + \left(\frac{D_3}{D_2}\right)^\beta \left(\frac{Y_j^3(D_3-D_2)}{r-\mu}\right) \leq \left(\frac{D_3 I_j}{D_2 I_i}\right)^\beta \left(\frac{I_i}{\beta-1}\right)\end{aligned} \tag{61}$$

(b)

$$\begin{aligned}\frac{\partial}{\partial Y}\phi_{2i}(Y_j^{2*}) = \frac{D_2}{r-\mu} + \frac{\beta}{Y_j^{2*}}\left(\frac{D_3}{D_2}\right)^\beta \left(\frac{Y_j^3(D_3-D_2)}{r-\mu}\right) \\ - \frac{\beta}{Y_j^{2*}}\left(\frac{D_3 I_j}{D_2 I_i}\right)^\beta \left(\frac{I_i}{\beta-1}\right) > 0 \\ \Leftrightarrow \left(\frac{D_3 I_j}{D_2 I_i}\right)^\beta \left(\frac{I_i}{\beta-1}\right) < \frac{I_j}{\beta-1} + \left(\frac{D_3}{D_2}\right)^\beta \left(\frac{Y_j^3(D_3-D_2)}{r-\mu}\right)\end{aligned} \tag{62}$$

By (a), (b), $Y_j^{2*} \leq Y_{ij}^{21}$ if and only if

$$\begin{aligned}\frac{\beta}{\beta-1}I_j - I_i + \left(\frac{D_3}{D_2}\right)^\beta \left(\frac{Y_j^3(D_3-D_2)}{r-\mu}\right) \leq \left(\frac{D_3 I_j}{D_2 I_i}\right)^\beta \left(\frac{I_i}{\beta-1}\right) \\ < \frac{I_j}{\beta-1} + \left(\frac{D_3}{D_2}\right)^\beta \left(\frac{Y_j^3(D_3-D_2)}{r-\mu}\right),\end{aligned} \tag{63}$$

which is equal to

$$\frac{\beta}{\beta-1}I_j - I_i \leq \left(\frac{D_3 I_j}{D_2 I_i}\right)^\beta \left(\frac{I_i}{\beta-1}\right) - \left(\frac{D_3}{D_2}\right)^\beta \left(\frac{Y_j^3(D_3-D_2)}{r-\mu}\right) < \frac{I_j}{\beta-1}, \tag{64}$$

and then finally

$$\beta - (\beta - 1)\frac{I_i}{I_j} \leq \left(\frac{D_3}{D_2}\right)^{\beta} \left\{ \left(\frac{I_j}{I_i}\right)^{\beta - 1} - \left(\frac{\beta(D_3 - D_2)}{D_3}\right) \right\} < 1 \tag{65}$$

Appendix D. (Detail to get the value function of first investor)

$$
\begin{aligned}
C^1_{kji}(Y) = E^Y \Bigg[\int_{\tau^1_{kji}}^{\tau^2_{ji}} e^{-rt} D_2 Y_t dt - e^{-r\tau^1_{kji}} I_k + \int_{\tau^2_{ji}}^{\tau^3_i} e^{-rt} D_2 Y_t dt \\
+ \int_{\tau^3_i}^{\infty} e^{-rt} D_3 Y_t dt \Bigg] \\
= E^Y \Bigg[\int_{\tau^1_{kji}}^{\infty} e^{-rt} D_1 Y_t dt - e^{-r\tau^1_{kji}} I_k + \int_{\tau^2_{ji}}^{\infty} e^{-rt} (D_2 - D_1) Y_t dt \\
+ \int_{\tau^3_i}^{\infty} e^{-rt} (D_3 - D_2) Y_t dt \Bigg] \\
= E^Y \Bigg[e^{-r\tau^1_{kji}} \left(\frac{Y^1_{kji} D_1}{r-\mu} - I_k \right) + e^{-r\tau^2_{ji}} \frac{Y^2_{ji}}{r-\mu} (D_2 - D_1) \\
+ e^{-r\tau^3_i} \frac{Y^3_i}{r-\mu} (D_3 - D_2) \Bigg]
\end{aligned}
\tag{66}
$$

Let

$$
\begin{aligned}
E^Y \left[e^{-r\tau^1_{kji}} \left(\tfrac{Y^1_{kji} D_1}{r-\mu} - I_k \right) \right] = M \\
E^Y \left[e^{-r\tau^2_{ji}} \tfrac{Y^2_{ji}}{r-\mu} (D_2 - D_1) \right] = N
\end{aligned}
\tag{67}
$$

Since

$$
\begin{cases}
M(0) = 0 \\
M(Y^1_{kji}) = \frac{Y^1_{kji} D_1}{r-\mu} - I_k \\
M'(Y^1_{kji}) = \frac{D_1}{r-\mu}
\end{cases}
\tag{68}
$$

we can get

$$
M = \begin{cases}
\left(\frac{Y}{Y^1_{kji}} \right)^{\beta} \left(\frac{Y^1_{kji} D_1}{r-\mu} - I_k \right) & \text{if } Y < Y^1_{kji} \\
\frac{Y D_1}{r-\mu} - I_k \quad \text{if } Y \geq Y < Y^1_{kji}
\end{cases}
\tag{69}
$$

And

$$
N = E^Y \left[e^{-r\tau^2_{ji}} \frac{Y^2_{ji}}{r-\mu} (D_2 - D_1) \right] = E^Y \left[e^{-r\tau^2_{ji}} \right] \frac{Y^2_{ji}}{r-\mu} (D_2 - D_1)
\tag{70}
$$

By Laplace transform, $E^Y \left[e^{-r\tau^2_{ji}} \right] = \left(\frac{Y}{Y^2_{ji}} \right)^{\beta}$

Hence

$$
N = \begin{cases}
\left(\frac{Y}{Y^2_{ji}} \right)^{\beta} \left(\frac{Y^2_{ji}(D_2 - D_1)}{r-\mu} \right) & \text{if } Y < Y^2_{ji} \\
\frac{Y(D_2 - D_1)}{r-\mu} & \text{if } Y \geq Y^2_{ji}
\end{cases}
\tag{71}
$$

Finally, we can get value of first investor

$$
\begin{aligned}
&C^1_{kji}(Y) = M + N + L \\
&= \begin{cases}
\left(\frac{Y}{Y^1_{kji}}\right)^\beta \left(\frac{Y^1_{kji}D_1}{r-\mu} - I_k\right) + \left(\frac{Y}{Y^2_{ji}}\right)^\beta \left(\frac{Y^2_{ji}(D_2-D_1)}{r-\mu}\right) \\
\qquad\qquad + \left(\frac{Y}{Y^3_{i}}\right)^\beta \left(\frac{Y^3_{i}(D_3-D_2)}{r-\mu}\right) \text{ if } Y < Y^1_{kji} \\
\frac{YD_2}{r-\mu} - I_k + \left(\frac{Y}{Y^2_{ji}}\right)^\beta \left(\frac{Y^2_{ji}(D_2-D_1)}{r-\mu}\right) \\
\qquad\qquad + \left(\frac{Y}{Y^3_{i}}\right)^\beta \left(\frac{Y^3_{i}(D_3-D_2)}{r-\mu}\right) \text{ if } Y^1_{kji} \le Y < Y^2_{ji} \\
\frac{YD_2}{r-\mu} - I_k + \left(\frac{Y}{Y^3_{i}}\right)^\beta \left(\frac{Y^3_{i}(D_3-D_2)}{r-\mu}\right) \text{ if } Y^2_{ji} \le Y < Y^3_{i} \\
\frac{YD_3}{r-\mu} - I_k \text{ if } Y \ge Y^3_{i}
\end{cases}
\end{aligned}
\tag{72}
$$

Appendix E. (Proof of Proposition 3)

Define the function $\phi_1 : [0, Y^2_{ji}] \longrightarrow \mathbb{R}$ such that $\phi_1(Y) = C^{1a}_{kji} - C^{2o}_{ki}$.Then

$$\phi_1(Y) = \frac{YD_1}{r-\mu} - I_k + \left(\frac{Y}{Y^2_{ji}}\right)^\beta \left(\frac{Y^2_{ji}(D_2 - D_1)}{r-\mu}\right) - \left(\frac{Y}{Y^2_{ki}}\right)^\beta \left(\frac{Y^2_{ki}D_2}{r-\mu} - I_k\right) \tag{73}$$

(a) $\phi_1(0) = -I_k < 0$

(b) Since $D_2 < D_1$,

$$\begin{aligned}\frac{\partial^2 \phi_1(Y)}{\partial Y^2} &= \beta(\beta-1)Y^{\beta-2}(Y^2_{ji})^{-\beta}\left(\frac{Y^2_{ji}(D_2 - D_1)}{r-\mu}\right) \\ &\quad -\beta(\beta-1)Y^{\beta-2}(Y^2_{ki})^{-\beta}\left(\frac{1}{\beta-1}I_k\right) < 0 : \text{concave}\end{aligned} \tag{74}$$

(c)

$$\begin{aligned}\phi_1(Y^2_{ki}) &= \left(\frac{Y^2_{ki}}{Y^2_{ji}}\right)^\beta \left(\frac{Y^2_{ji}(D_2 - D_1)}{r-\mu}\right) - \left(\frac{Y^2_{ki}(D_2 - D_1)}{r-\mu}\right) \\ &= \frac{(D_2 - D_1)}{r-\mu}\left(\left(\frac{Y^2_{ki}}{Y^2_{ji}}\right)^\beta Y^2_{ji} - Y^2_{ki}\right) \\ &= \frac{(D_2 - D_1)}{r-\mu}Y^2_{ki}\left(\left(\frac{Y^2_{ki}}{Y^2_{ji}}\right)^{\beta-1} - 1\right)\end{aligned} \tag{75}$$

In case of (i),

$$I_k < I_j \Longrightarrow Y^2_{ki} < Y^2_{ji} \Longrightarrow \phi_1(Y^2_{ki}) > 0 \tag{76}$$

Hence there exists $Y^1_{kji} \in (0, Y^2_{ji})$ always, only if Y^2_{ki}, Y^2_{ji} exist.

(i-1)$I_k < I_j < I_i$

Since $I_k < I_i$ and $I_j < I_i$, both $Y^2_{ki}$and Y^2_{ji} always exist by Proposition 2.

Hence Y^1_{kji} always exists.

(i-2)$I_k < I_i < I_j$

Since $I_k < I_i$,Y^2_{ki} always exist. And since $I_j > I_i$, Y^2_{ji} exist if $I_j \in (I_i, \frac{I_i}{D_3}\left(\frac{D_2^\beta - D_3^\beta}{\beta(D_2 - D_3)}\right)^{\frac{1}{\beta-1}})$.

Hence Y^1_{kji} exists if $I_j \in (I_i, \frac{I_i}{D_3}\left(\frac{D_2^\beta - D_3^\beta}{\beta(D_2 - D_3)}\right)^{\frac{1}{\beta-1}})$.

(i-3)$I_i < I_k < I_j$

Since $I_k > I_i$,Y^2_{ki} exist if $I_k \in (I_i, \frac{I_i}{D_3}\left(\frac{D_2^\beta - D_3^\beta}{\beta(D_2-D_3)}\right)^{\frac{1}{\beta-1}})$. And since $I_j > I_i$, Y^2_{ji} exist if $I_j \in (I_i, \frac{I_i}{D_3}\left(\frac{D_2^\beta - D_3^\beta}{\beta(D_2-D_3)}\right)^{\frac{1}{\beta-1}})$. But since $I_k \in (I_i, I_j)$, it is enough for I_j to be inside its region.

Hence Y^1_{kji} exists if $I_j \in (I_i, \frac{I_i}{D_3}\left(\frac{D_2^\beta - D_3^\beta}{\beta(D_2-D_3)}\right)^{\frac{1}{\beta-1}})$.

In case of (ii), we cannot assure the sign of $\phi_1(Y^2_{ki})$. It means that we cannot assure the existence of Y^1_{kji} only with the existence of Y^2_{ki}, Y^2_{ji}. Intuitively, firm k will have incentive to preempt when its sunk cost is relatively small. We want to find the range of this sunk cost which makes Y^1_{kji} exists.

Assume that there exist Y^1_{kji} which satisfies (26). By definition of Y^1_{kji} , $\phi_1(Y^1_{kji})$ has to be zero.

$$\begin{aligned}\phi_1(Y^1_{kji}) = \frac{Y^1_{kji}D_1}{r-\mu} - I_k + \left(\frac{Y^1_{kji}}{Y^2_{ji}}\right)^\beta \left(\frac{Y^2_{ji}(D_2-D_1)}{r-\mu}\right) \\ - \left(\frac{Y^1_{kji}}{Y^2_{ki}}\right)^\beta \left(\frac{Y^2_{ki}D_2}{r-\mu} - I_k\right) = 0\end{aligned} \tag{77}$$

By procedure getting (69) from (68) in Appendix D, we can get the maximum value of Y^1_{kji} as follow.

$$Y^1_{kji} = \frac{\beta}{\beta-1}\frac{(r-\mu)I_k}{D_1} \tag{78}$$

In fact, this is equal to Y^{1*}_k. Hence the boundary is $I^*_{j,i} = I_k$ satisfies

$$\frac{Y^{1*}_k D_1}{r-\mu} - I_k + \left(\frac{Y^{1*}_k}{Y^2_{ji}}\right)^\beta \left(\frac{Y^2_{ji}(D_2-D_1)}{r-\mu}\right) - \left(\frac{Y^{1*}_k}{Y^2_{ki}}\right)^\beta \left(\frac{Y^2_{ki}D_2}{r-\mu} - I_k\right) = 0 \tag{79}$$

But unlike Y^i_3, we cannot fix the value of Y^2_{ji} or Y^2_{ki}. Especially when they are equal to Y^{21}_{ji} or Y^{21}_{ki}, there are no closed forms for those values. That means we cannot find the formula of $I^*_{j,i}$. (We define it to be $I^*_{j,i}$ since it will be represented as a function of I_j and I_i.) But it becomes boundary to determine whether the competition between firm k and j is keen or not.

(ii-1)$I_j < I_k < I_i$

Since $I_k < I_i$ and $I_j < I_i$, both $Y^2_{ki}$and Y^2_{ji} always exist by Proposition 2. And I_k has to be small enough at least to satisfy (77).

Hence Y^1_{kji} exists if $I_k \in (I_j, I^*_{j,i})$ where $I^*_{j,i} = I_k$ satisfies equation (79).

(ii-2)$I_j < I_i < I_k$

Since $I_k > I_i$,Y_{ki}^2 exist if $I_k \in (I_i, \frac{I_i}{D_3}\left(\frac{D_2^\beta - D_3^\beta}{\beta(D_2 - D_3)}\right)^{\frac{1}{\beta-1}})$. And since $I_j < I_i$, Y_{ji}^2 always exist.
At the same time, I_k has to be small enough to satisfy (77).

Hence Y_{kji}^1 exists if $I_k \in (I_i, \frac{I_i}{D_3}\left(\frac{D_2^\beta - D_3^\beta}{\beta(D_2 - D_3)}\right)^{\frac{1}{\beta-1}})$ and $I_k \in (I_j, I_{j,i}^*)$ where $I_{j,i}^* = I_k$ satisfies equation (79).

(ii-3)$I_i < I_j < I_k$

Since $I_k > I_i$,Y_{ki}^2 exist if $I_k \in (I_i, \frac{I_i}{D_3}\left(\frac{D_2^\beta - D_3^\beta}{\beta(D_2 - D_3)}\right)^{\frac{1}{\beta-1}})$. And since $I_j > I_i$, Y_{ji}^2 exist if $I_j \in (I_i, \frac{I_i}{D_3}\left(\frac{D_2^\beta - D_3^\beta}{\beta(D_2 - D_3)}\right)^{\frac{1}{\beta-1}})$. But since $I_j \in (I_i, I_k)$, it is enough for I_k to be inside its region. At the same time, I_k has to be small enough at least to satisfy (77).

Hence Y_{kji}^1 exists if $I_k \in (I_i, \frac{I_i}{D_3}\left(\frac{D_2^\beta - D_3^\beta}{\beta(D_2 - D_3)}\right)^{\frac{1}{\beta-1}})$ and $I_k \in (I_j, I_{j,i}^*)$ where $I_{j,i}^* = I_k$ satisfies equation (79).

Finally, we want to examine the value of Y_{kji}^1 for each case. We apply same logic that we used in Proposition 2 and get similar result.

Appendix F.

(i) We want to check conditions when the first threshold of oligopoly market is same to that of duopoly market. By how it is defined, Y_{AB}^{1d} should be either Y_A^{1*} or Y_{BA}^{11}.

If $Y_{AB}^{1d} = Y_A^{1*}$, Y_{ABC}^1 cannot be Y_{BAC}^{11} or Y_{BCA}^{11}. Since Y_{BAC}^{11} and Y_{BCA}^{11} must be less than Y_A^{1*}, and then it will be contradiction to the assumption $Y_{AB}^{1d} = Y_{ABC}^1$. Hence $Y_{AB}^{1d} = Y_A^{1*}$ becomes one condition to be $Y_{AB}^{1d} = Y_{ABC}^1$

If $Y_{AB}^{1d} = Y_{BA}^{11}$, Y_{ABC}^1 cannot be Y_A^{1*} or Y_{BCA}^{11}. If it is Y_A^{1*}, there is contradiction to the fact $Y_{BA}^{11} < Y_A^{1*}$. If it is Y_{BCA}^{11}, Y_{BA}^2 should be equal to $Y_B^{2d} = Y_B^{2*}$ which is impossible. Hence Y_{ABC}^1 should be Y_{BAC}^{11}, and it leads the condition $Y_{BC}^2 = Y_B^{2d} = Y_B^{2*}, Y_{AC}^2 = Y_A^{2d} = Y_A^{2*}$.

(ii) Following the logic of case (i), we can see that there is no Y_{ABC}^1 when $Y_{AB}^{1d} = Y_A^{1*}$, which satisfies $Y_{AB}^{1d} < Y_{ABC}^1$. Hence $Y_{AB}^{1d} = Y_{BA}^{11}$ should be satisfied at first.

Since $Y_{AB}^{1d} = Y_{BA}^{11} < Y_A^{1*}$, $Y_{ABC}^1 = min(Y_A^{1*}, Y_{BCA}^{11}, Y_{BAC}^{11})$, and $Y_{BCA}^{11} < Y_{BAC}^{11}$ in our results, $Y_{AB}^{1d} < Y_{ABC}^1$ if and only if $\phi_1(Y_{AB}^{1d}) < 0$, $\frac{\partial}{\partial Y}\phi_1(Y_{AB}^{1d}) > 0$,
where

$$\begin{aligned}\phi_1(Y) &= C_{BAC}^{1a} - C_{BC}^{2o} \\ &= \frac{YD_1}{r-\mu} - I_B + \left(\frac{Y}{Y_{AC}^2}\right)^\beta \left(\frac{Y_{AC}^2(D_2 - D_1)}{r-\mu}\right) - \left(\frac{Y}{Y_{BC}^2}\right)^\beta \left(\frac{Y_{BC}^2 D_2}{r-\mu} - I_B\right).\end{aligned} \tag{80}$$

(a)

$$\begin{aligned}\phi_1(Y_{AB}^{1d}) = \frac{Y_{AB}^{1d}D_1}{r-\mu} - I_B &+ \left(\frac{Y_{AB}^{1d}}{Y_{AC}^2}\right)^\beta \left(\frac{Y_{AC}^2(D_2 - D_1)}{r-\mu}\right) \\ &- \left(\frac{Y_{AB}^{1d}}{Y_{BC}^2}\right)^\beta \left(\frac{Y_{BC}^2 D_2}{r-\mu} - I_B\right) < 0 \\ \Leftrightarrow \frac{Y_{AB}^{1d}D_1}{r-\mu} - I_B < \left(\frac{Y_{AB}^{1d}}{Y_{BC}^2}\right)^\beta &\left(\frac{Y_{BC}^2 D_2}{r-\mu} - I_B\right) - \left(\frac{Y_{AB}^{1d}}{Y_{AC}^2}\right)^\beta \left(\frac{Y_{AC}^2(D_2 - D_1)}{r-\mu}\right)\end{aligned} \tag{81}$$

(b)

$$\begin{aligned}\frac{\partial}{\partial Y}\phi_1(Y_{AB}^{1d}) = \frac{D_1}{r-\mu} &+ \beta(Y_{AB}^{1d})^{\beta-1}(Y_{AC}^2)^{-\beta}\left(\frac{Y_{AC}^2(D_2 - D_1)}{r-\mu}\right) \\ &-\beta(Y_{AB}^{1d})^{\beta-1}(Y_{BC}^2)^{-\beta}\left(\frac{Y_{BC}^2 D_2}{r-\mu} - I_B\right) > 0 \\ \Leftrightarrow \left(\frac{Y_{AB}^{1d}}{Y_{BC}^2}\right)^\beta \left(\frac{Y_{BC}^2 D_2}{r-\mu} - I_B\right) &- \left(\frac{Y_{AB}^{1d}}{Y_{AC}^2}\right)^\beta \left(\frac{Y_{AC}^2(D_2 - D_1)}{r-\mu}\right) < \frac{Y_{AB}^{1d}D_1}{\beta(r-\mu)}\end{aligned}$$

(82)

By (a), (b),
$Y_{AB}^{1d} < Y_{ABC}^{1}$ if and only if

$$
\begin{aligned}
&\frac{Y_{AB}^{1d} D_1}{r-\mu} - I_B \\
&< \left(\frac{Y_{AB}^{1d}}{Y_{BC}^{2}}\right)^{\beta} \left(\frac{Y_{BC}^{2} D_2}{r-\mu} - I_B\right) - \left(\frac{Y_{AB}^{1d}}{Y_{AC}^{2}}\right)^{\beta} \left(\frac{Y_{AC}^{2}(D_2 - D_1)}{r-\mu}\right) \\
&< \frac{Y_{AB}^{1d} D_1}{\beta(r-\mu)}
\end{aligned}
\tag{83}
$$

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