The Role of Uncertainty in Real Options Analysis

Mao-wei Hung* and Leh-chyan So**

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ABSTRACT

An adjusted Black-Scholes pricing formula is derived in this paper. By separating risk and uncertainty through the robust control technique, we find that uncertainty as well as risk raises the management's subjective evaluation of real options. We suggest a simple method to filter the risk of the project and to acquire a more reliable value of real options without the influence of uncertainty. Besides, we propose that one investment opportunity may be postponed inappropriately, since under uncertainty the exercise of investment may be delayed by the project manager.

Keywords: Defer option; Investment opportunity; Uncertainty; Black-Scholes pricing formula; Volatility

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* Hung is from the College of Management, National Taiwan University, Taipei, Taiwan.

** So is from the Department of Quantitative Finance, National Tsing Hua University, Hsinchu, Taiwan.

Address correspondence to: Assistant Professor Leh-chyan So, Department of Quantitative Finance, National Tsing Hua University, 101, Section 2 Kuang Fu Road, Hsinchu, Taiwan; e-mail: lcso@mx.nthu.edu.tw

Introduction

There have been a lot of approaches proposed to evaluate an investment opportunity. Among them, traditional discounted-cash-flow (DCF) approach, such as the standard net-present-value (NPV), is easy to apply but is criticized for its neglect of management's flexibility to adjust later decisions as uncertainty reveals (e.g., Trigeorgis, 1996). On the contrary, real options have enjoyed great popularity these days (e.g., Dixit and Pindyck, 1994; Trigeorgis, 1996; Amram and Kulatilaka, 1999). Since in the real world, the management has the right to undertake the investment opportunity and realizes the positive profit. This flexibility protects the management against the downside risk but provides unlimited upside potential. Hence, in the light of real options, the investment opportunity should be worth more when its volatility is high. Real options tend to extend financial options into investment opportunity analysis of real assets and often assign higher value to the investment opportunity because of time value.

Since the management has the flexibility to retract his initial planning, it is risk and uncertainty in the future that make him to upgrade the investment opportunity. Although the terms risk and uncertainty are often used interchangeably in the literature, they do have different meanings. Risk, an objective term, represents a probability distribution of the potential outcomes. When you sense risk, you have an ability to quantify the potential outcomes. This is the reason why we change probability distributions to make a risk-averse investor risk-neutral. Uncertainty, however, is a subjective term and represents a lack of confidence about probability estimates. Recent studies have found that they have different influences on decision makers. Alessandri's (2003) empirical findings show that managers treat risk and uncertainty separately and that use different decision rules to response to each. Alessandri et al. (2004) also emphasize the importance of identification of the risks and uncertainties inherent in the decision-making process. They further suggest that qualitative approaches should be used in place of quantitative ones to evaluate capital projects with higher uncertainty. Nishimura and Ozaki (2006) showed that uncertainty and risk have different effects on the value of investment opportunity. Miao and Wang (2009) emphasized the effect of distinguishing risk from uncertainty in the situation of option exercise or in the optimal exit problem. Trojanowska and Kort (2010) focused their attention on how uncertainty on the investment timing. They claimed that uncertainty aversion makes a firm to consider the project more risky and tends to overprice the risk under uncertainty. They predicted that the probability of investment is monotonically decreasing as the level of uncertainty in long-term projects. Roubaud, Lapied, and Kast (2010) suggested that the pro- and con-attitude toward ambiguity of a decision maker may influence his exercise of the option to invest by using the Choquet-Brownian motions to describe uncertainty.

In practice, it is the neat Black-Scholes pricing formula that most commonly-used when applying real options analysis. Many studies and reports have been done on the influence and estimation of the six factors affecting the price of an option (for example, Leslie and Michaels, 1997; Davis, 1998; Fernandez, 2002). Among the six key factors, volatility is especially notorious for its difficulties in estimation (for example, Lander and Pinches, 1998). And it is known that option prices are very sensitive to the estimation of the volatility of the underlying assets. As noted by Trigeorgis (1990), a 50% increase in the volatility raises the option value by about 40%. However, according to the classification mentioned above, uncertainty is not taken into consideration in the Black-Scholes model. Since an option can protect the risk-aversor against the downside risk, it is risk along with an attitude of risk aversion that makes the option more valuable in the Black-Scholes world. The parameter, σ, in the Black-Scholes pricing formula refers only to risk not to uncertainty.

Nevertheless, the terms risk and uncertainty are often used interchangeably in the literature. This leads us to suspect that, disguised as risk, uncertainty could be substituted into the Black-Scholes pricing formula when real options analysis is applied in practice. Shouldn't we purely consider risk in the uncertainty-absent Black-Scholes world if we are unanimous in the objectivity of the Black-Scholes model? Therefore, the purpose of this paper is to separate uncertainty from risk in the commonly-used Black-Scholes pricing formula and to see how uncertainty affects option prices. We intend to verify that the value of real options obtained by the Black-Scholes pricing formula may be not real if the concept of risk and uncertainty is vague. Besides, we want to show that how uncertainty could affect the timing of investment.

Before going on with our analysis, we describe the foundation of our framework. Among the approaches of addressing uncertainty, our work is most closely associated with the approach called the robust control approach, which depicts model uncertainty through a set of priors, and introduces the penalty function to a general utility function in order to capture investors' uncertainty (for example, Anderson, Hansen, and Sargent, 1999; Maenhout, 1999; Kogan and Wang, 2002; Boyle, Uppal, and Wang, 2003; Uppal and Wang, 2003). If the investor is under uncertainty, he worries about some worse-case scenario. And he will choose alternative models that are further away from the reference model. Hence, the robust control approach assigns a lower penalty to further-away perturbations. If the level of the investor's uncertainty is low, he will choose alternative models that are very similar to the reference model. Hence, the robust control approach assigns a higher penalty to further-away perturbations. The penalty is inversely relative to the investor's uncertainty.

The organization of this paper is as follows: the model and theoretical results are described in Section II; a numerical example is described in Section III, and the conclusions of the paper are presented in the final section.

Model

Throughout the paper, we denote the risk-free interest rate by a constant r. We assume that the gross project value, S, follows a stochastic process of the form

$$\frac{dS}{S} = v dt + \sigma dZ \tag{1}$$

where dZ is a Wiener process. The drift term, v, and the diffusion term, σ, described above are all constants. Taking the nontraded property of the underlying asset, we make an assumption that the below-equilibrium return shortfall is q (for example, McDonald and Siegel, 1986). Hence, the dynamic process of the gross project value is adjusted to

$$\frac{dS}{S} = (v - q) dt + \sigma dZ \tag{2}$$

In the literature, q functions as a dividend yield. The cum-dividend expected return of S is v.

We first consider the case where the manager knows the true probability law of asset returns, given the probability measure P. The budget constraint of the management is:

$$\frac{dW}{W} = (r + \pi(v - r) - \frac{C}{W})dt + \pi\sigma dZ, \tag{3}$$

where π represents the proportion of wealth allocated to the project. The expected utility of the management in the continuous time is:

$V_t = E^P\left[\int_t^T e^{-\rho s}u(C_s)ds + e^{-\rho(T-t)}V_T\right]$, where $u(C_s)$ takes the form: $u(C_s)=\dfrac{C_s^{1-\gamma}}{1-\gamma}$.

The Bellman equation is:

$$0=\max_{C,\pi}\left\{u(c)-\rho V_t + WV_w\left[r+\pi(v-r)-\frac{C}{W}\right]+\frac{1}{2}W^2V_{ww}\pi^2\sigma^2\right\} \tag{4}$$

After taking derivatives of (4) with respect to C and π, we can obtain the optimal consumption and portfolio choice implied by the first-order conditions:

$$u_c = V_W, \tag{5}$$

$$\pi = -\frac{V_w}{WV_{ww}\sigma^2}(v-r) \tag{6}$$

Under the assumption of utility function $u(C_s)=\dfrac{C_s^{1-\gamma}}{1-\gamma}$, we guess the value function

takes the form $V(W)=\kappa_0\dfrac{W^{1-\gamma}}{1-\gamma}$, (7)

where κ_0 is a constant depending on the parameter of the economy.

Substituting (7) into (6), we obtain that the optimal investment into the investment opportunity for the management is $\pi = \dfrac{(v-r)}{\gamma\sigma^2}$.

However, when we attempt to address his subjective evaluation of the real option, the above discussion would not hold. We should consider the misspecification problem. When the management is not sure of the true probability law of asset returns, he/she considers alternative model Q^{ξ} other than the reference model P. And the Weiner processes of his budget constraint should be changed to Q^{ξ}-measured ones. Following Uppal and Wang (2003), the Bellman equation should be adjusted to:

$$0=\sup_{C,\pi}\inf_{v_R}\left\{\begin{aligned}&u(C)-\rho V_t + WV_w\left[r+\pi(v-r)-\frac{C}{W}\right]+\frac{1}{2}W^2V_{ww}\pi^2\sigma^2\\&+V_wW\pi\sigma^2 v_R+\frac{\Psi(V)}{2}v_R{}^2\phi\sigma^2\end{aligned}\right\} \tag{8}$$

where the first line of the brace is the same as (4), the former term of the second line reflects the adjusted drift term resulting from the change of measure from P to Q^{ξ}. ν_R, in part, reflects the difference between the adjusted drift term and the original one. The latter term of the second line is the penalty function, where $\Psi(V)$ converts penalty to units of the utility, and ϕ denotes the management's subjective measure of confidence about the reference model. Hence, the reciprocal of ϕ can be treated as the level of the management's uncertainty.

The optimal consumption can be obtained by solving for the first-order condition $u_c = V_W$. After taking derivatives of (8) with respect to π, we can obtain

$$\pi = -\frac{V_w}{WV_{ww}\sigma^2}(\nu - r) - \frac{V_w \nu_R}{WV_{ww}} \tag{9}$$

Differentiating (8) with respect to ν_R, we obtain that the optimal

$$\nu_R = -\frac{V_w W\pi}{\Psi(V)\phi}. \tag{10}$$

Substituting (10) into (9) leads:

$$\pi = -\frac{V_w}{WV_{ww}\sigma^2}(\nu - r) + \frac{(V_w)^2\pi}{V_{ww}\Psi(V)\phi} \tag{11}$$

Since $V(W) = \kappa_0 \frac{W^{1-\gamma}}{1-\gamma}$, we get $V_w = \kappa_0 W^{-\gamma}$. Following Maenhout (1999) and Uppal and Wang (2003), we assume that $\Psi(V) = \frac{1-\gamma}{\gamma}V(W) = \frac{W^{1-\gamma}}{\gamma}\kappa_0$. (11) can be rewritten as $\pi = \frac{\nu - r}{\gamma\sigma^2} - \frac{\pi}{\phi}$. That is, the optimal investment into the investment opportunity for the uncertainty-averse management is

$$\pi^* = \frac{1}{(1+\frac{1}{\phi})}\frac{(\nu - r)}{\gamma\sigma^2} = \frac{\hat{p}}{\gamma\sigma^2}, \tag{12}$$

where $\hat{p} \equiv \left(\frac{\nu - r}{1+\frac{1}{\phi}} \right)$

(12) can be explained in this way. $\hat{p}$ in (12) could be treated as the management's subjective estimation of the risk premium for the nontraded asset. The management's uncertainty makes him underestimate the risk premium for market portfolio; hence,

his optimal investment in the investment opportunity would be less than if he knew exactly the data generating process of asset returns.

Instead of the commonly-used objective stochastic discount factor, we have to compute the management's marginal utility function which will serve as the subjective stochastic discount factor for the subjective evaluation of the real option.

Applying Ito's formula, we can write

$$
\begin{aligned}
dV_w &= \frac{\partial V_w}{\partial t}dt + \frac{\partial V_w}{\partial W}dW + \frac{1}{2}\frac{\partial^2 V_w}{\partial W^2}(dW)^2 \\
&= \kappa_0(-\gamma)W^{-\gamma-1}\left\{W\left[r+\pi^*(v-r)+(\pi^*)^2\sigma^2 v_R^* - \frac{C^*}{W}\right]dt + W\pi^*\sigma dZ^\xi\right\} \\
&\quad + \frac{1}{2}\kappa_0(-\gamma)(-1-\gamma)W^{-\gamma-2}W^2(\pi^*)^2\sigma^2 dt \\
&= (-\gamma)\kappa_0 W^{-\gamma}\left\{\left[r+\pi^*(v-r)+(\pi^*)^2\sigma^2 v_R^* - \frac{C^*}{W}\right]dt + \pi^*\sigma dZ^\xi\right\} \\
&\quad + \frac{1}{2}\kappa_0(-\gamma)(-1-\gamma)W^{-\gamma}(\pi^*)^2\sigma^2 dt \\
&= \kappa_0 W^{-\gamma}\left\{(-\gamma)\left[r+\pi^*(v-r)+(\pi^*)^2\sigma^2 v_R^* - \frac{C^*}{W}\right] + \frac{1}{2}\gamma(1+\gamma)(\pi^*)^2\sigma^2\right\}dt \\
&\quad + (-\gamma)\kappa_0 W^{-\gamma}\pi^*\sigma dZ^\xi \qquad (13)
\end{aligned}
$$

Since $V(W) = \kappa_0\frac{W^{1-\gamma}}{1-\gamma}$, we get $V_w = \kappa_0 W^{-\gamma}$. (14)

Substitution of (14) into (13) leads:

$$
\begin{aligned}
dV_w &= V_w\left\{(-\gamma)\left[r+\pi^*(v-r)+\pi^*\sigma^2 v_R^* - \frac{C^*}{W}\right] + \frac{1}{2}\gamma(1+\gamma)(\pi^*)^2\sigma^2\right\}dt \\
&\quad + V_w(-\gamma)\pi^*\sigma dZ^\xi
\end{aligned} \qquad (15)
$$

From (15), we know the dynamic process of the management's marginal utility function is:

$$
\begin{aligned}
\frac{dV_w}{V_w} &= \left\{(-\gamma)\left[r+\pi^*(v-r)+\pi^*\sigma^2 v_R^* - \frac{C^*}{W}\right] + \frac{1}{2}\gamma(1+\gamma)(\pi^*)^2\sigma^2\right\}dt \\
&\quad - \gamma\pi^*\sigma dZ^\xi
\end{aligned} \qquad (16)
$$

Since first-order condition shows that $u_c = V_W$, we get $u_c = C^{-\gamma} = V_w = \kappa_0 W^{-\gamma}$.

That is, $\frac{C^*}{W} = \kappa_0^{\frac{-1}{\gamma}}$. (17)

Substituting (17) into (16), we then obtain

$$\frac{dV_w}{V_w}=\left\{(-\gamma)\left[r+\pi^*(v-r)+\pi^*\sigma^2 v_R^*+\gamma\kappa_0^{\frac{-1}{\gamma}}\right]+\frac{1}{2}\gamma(1+\gamma)(\pi^*)^2\sigma^2\right\}dt \tag{18}$$
$$-\gamma\pi^*\sigma dZ^{\xi}$$

On the other side, using $V(W)=\kappa_0\dfrac{W^{1-\gamma}}{1-\gamma}$ and substituting the optimal values into (8), we get the following useful relationship:

$$u(C^*)+\kappa_0 W^{-\gamma}C^*+\kappa_0 W^{1-\gamma}\left[r+\pi^*(v-r)\right]+\frac{1}{2}(-\gamma)\kappa_0 W^{1-\gamma}(\pi^*)^2\sigma^2$$
$$+\kappa_0 W^{1-\gamma}\pi^*\sigma^2 v_R^*+\frac{\Psi(V)}{2}(v_R^*)^2\phi\sigma^2=0 \tag{19}$$

Using (17) and doing some rearrangement, we obtain:

$$-\gamma\kappa_0^{-\frac{1}{\gamma}}+(1-\gamma)\left[r+\pi^*(v-r)\right]+\frac{1}{2}(1-\gamma)(-\gamma)(\pi^*)^2\sigma^2$$
$$+(1-\gamma)\pi^*\sigma^2 v_R^*+\frac{\Psi(V)}{2}(1-\gamma)\frac{(v_R^*)^2\phi\sigma^2}{W^{1-\gamma}\kappa_0}=0 \tag{20}$$

Using (20), (18) can be simplified as:

$$\frac{dV_w}{V_w}=\left\{\begin{matrix}-r-\left[\pi^*(v-r)\right]+\gamma(\pi^*)^2\sigma^2\\ -\pi^*\sigma^2 v_R^*-\dfrac{\Psi(V)}{2}(1-\gamma)\dfrac{(v_R^*)^2}{W^{1-\gamma}\kappa_0}\end{matrix}\right\}dt-\gamma\pi^*\sigma dZ.^{\xi} \tag{21}$$

Set $\Psi(V)=\dfrac{1-\gamma}{\gamma}V(W)=\dfrac{W^{1-\gamma}}{\gamma}\kappa_0$. Substituting this assumption and

$v_R=-\dfrac{V_w W\pi}{\Psi(V)\phi}=\dfrac{-\gamma\pi}{\phi}$ into (21), we get

$$\frac{dV_w}{V_w}=\left\{-r-\pi^*(v-r)+\frac{\gamma(1+2\phi+\gamma)}{2\phi}(\pi^*)^2\sigma^2\right\}dt-\gamma\pi^*\sigma dZ^{\xi}. \tag{22}$$

We then substitute (12) into (22) and get

$$\frac{dV_w}{V_w}=\left\{-r-\frac{(v-r)^2\phi(1-\gamma)}{2(1+\phi)^2\gamma\sigma^2}\right\}dt-\frac{(v-r)}{(1+\frac{1}{\phi})\sigma}dZ^{\xi}\equiv-\hat{r}dt-\frac{(v-r)}{(1+\frac{1}{\phi})\sigma}dZ^{\xi}, \tag{23}$$

where $\hat{r}\equiv r+\dfrac{(v-r)^2\phi(1-\gamma)}{2(1+\phi)^2\gamma\sigma^2}$.

In the extreme case where ϕ approaches infinity, our result would be $\hat{r}=r$, which means that when the management knows the true probability law of asset returns, his subjective risk-free interest rate is the same as that in the real world.

However, since the management under consideration is risk-averse with $\gamma > 1$, $\hat{r} = r + \frac{(v-r)^2 \phi(1-\gamma)}{2(1+\phi)^2 \gamma\sigma^2} < r$ in general. It implies that the management's uncertainty about the true probability law of asset returns may make him consider that one dollar invested today has a lower future value than its market value. The intuition behind this is quite clear. Since the management's uncertainty would make him underestimate the risk premium of the nontraded asset, his investment in the investment opportunity decreases. That is, he invests too much in the riskfree asset, which provide lower payoff than the nontraded asset, than he would if he knows the true probability law of asset returns. Hence, this suboptimal allocation would make him subjectively perceive that one dollar invested today has a lower future value than its market value. The subjective risk-free interest rate, $\hat{r}$, decreases as the subjective measure of confidence, ϕ, decreases.

We can then compute the subjective value of the real option for the management under model misspecification. Let $F(S,t)$ be the subjective price of the real option with the nontraded asset as its underlying asset. The martingale approach implies:

$$
\begin{aligned}
0 &= E^{Q^{\xi}}\left[d(V_w F(S,t))\right] \\
&= E^{Q^{\xi}}\left[V_w dF(S,t) + F(S,t)dV_w + dV_w dF(S,t)\right] \\
&= V_w E^{Q^{\xi}}\left[F_s dS + F_t dt + \frac{1}{2}F_{ss}(dS)^2 + F\frac{dV_w}{V_w} + \frac{dV_w}{V_w}F_s dS\right]
\end{aligned} \tag{24}
$$

Using (23), we can rewrite (24) as:

$$
0 = V_w \left[F_s S(v-q) + F_t + \frac{1}{2}F_{ss}S^2\sigma^2 - F\hat{r} - F_s S\frac{(v-r)}{1+\frac{1}{\phi}} \right] dt. \tag{25}
$$

Therefore, we get

$$
\begin{aligned}
0 &= F_s S(v-q) + F_t + \frac{1}{2}F_{ss}S^2\sigma^2 - F\hat{r} - F_s S\frac{(v-r)}{1+\frac{1}{\phi}} \\
&= \frac{1}{2}F_{ss}S^2\sigma^2 + SF_s(\hat{r}-\hat{q}) - \hat{r}F + F_t,
\end{aligned} \tag{26}
$$

where $\hat{r} = r + \dfrac{(v-r)^2 \phi(1-\gamma)}{2(1+\phi)^2 \gamma\sigma^2}$ and $\hat{q} = q + \left(\hat{r} - \dfrac{r}{1+\dfrac{1}{\phi}} \right) - \dfrac{v}{1+\phi}$.

When ϕ approaches infinity, $\hat{r} = r$. Hence, $\hat{q}$ degenerates to q.

Because (26) can be treated as the typical Black-Scholes partial differential equation with the subjective required rate of return $\hat{r}$ and subjective dividend yield $\hat{q}$, the subjective value of the real option on the nontraded asset can be obtained immediately.

$$\hat{C} = Se^{-\hat{q}(T-t)} N(d_1) - Ie^{-\hat{r}(T-t)} N(d_2), \tag{27}$$

where I denotes that investment cost,

$$\hat{r} = r + \frac{(v-r)^2 \phi(1-\gamma)}{2(1+\phi)^2 \gamma\sigma^2}, \quad \hat{q} = q + \left(\hat{r} - \frac{r}{1+\dfrac{1}{\phi}} \right) - \frac{v}{1+\phi},$$

$$d_1 = \frac{\ln(\dfrac{S}{I}) + (\hat{r} - \hat{q} + \dfrac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}, \text{ and } d_2 = d_1 - \sigma\sqrt{T-t}.$$

When ϕ approaches infinity, $\hat{r} = r$ and $\hat{q}$ degenerates to q. It is the result of the Black-Scholes pricing formula. When we take the effect of the management's uncertainty into consideration and let ϕ get smaller, we find $\hat{r} < r$ and $\hat{q} < q$. Since the management's uncertainty lower the subjective dividend yield more than it lower the subjective interest rate, the management's subjective value of the real option raises, compared with the situation that he is aware of the true probability law of asset returns. We claim that uncertainty as well as risk has a positive effect on the value of real options.

We now consider the optimal timing to exercise the option. Since (26) can be treated as the typical Black-Scholes partial differential equation with the subjective required rate of return $\hat{r}$ and subjective dividend yield $\hat{q}$, we know immediately the optimal timing to exercise by applying McDonald and Siegel's (1986) results. The project value must be as large as S^* before the manager decides to invest.

$$S^* = \frac{\beta_1}{\beta_1 - 1} I > I, \tag{28}$$

where I denotes that investment cost,

$$\beta_1 = \left(\tfrac{1}{2} - (\hat{r} - \hat{q})/\sigma^2\right) + \sqrt{\left((\hat{r} - \hat{q})/\sigma^2 - \tfrac{1}{2}\right)^2 + 2\hat{r}/\sigma^2} \ ,$$

$$\hat{r} = r + \frac{(v - r)^2 \phi (1 - \gamma)}{2(1 + \phi)^2 \gamma \sigma^2}\ , \ \ \hat{q} = q + \left(\hat{r} - \frac{r}{1 + \dfrac{1}{\phi}} \right) - \frac{v}{1 + \phi} .$$

Numerical Example

Table 1 displays the values of all parameters used in calibration: the risk aversion coefficient (γ), the rate of return of the investment opportunity (v), the riskless interest rate (r), the dividend yield (shortfall) (q), the time to maturity ($T - t$),the volatility of the investment opportunity (σ), and the subjective measure of confidence about the probability law of asset returns (ϕ). Because stocks have volatility between 0.2 and 0.5 in general, we take this range when calibration. Without more information, we make an assumption that $r = q = 0.04$ (for example, Dixit and Pindyck, 1994). Although the subjective parameter, ϕ, is not easy to specify, Maenhout (1999) provided a suggestion to choose appropriate ϕ. According to him, ϕ should be chosen to make the difference between the objective and subjective risk premium for market portfolio less than 3% under 95% confidence interval. We eliminate unreasonable ϕ less than 6 and let the values of ϕ rise from 6 up to infinity to see how the subjective measure of confidence about the probability law of asset returns, or uncertainty, affects the evaluation of the real option.

Table 2 summarizes the values of real options after separating the effects of risk and uncertainty. The results of neglect of uncertainty are shown in the column of $\phi = \infty$. We find that the management's uncertainty, alike risk, raises his subjective value of real options. Base on these findings, if we mistake uncertainty for risk and overestimate the value of parameter, σ, we would make a wrong conclusion that the value of the real option is high.

And what is the more reliable value of the real option? We now provide a simple method to filter the risk of the project and to acquire a more reliable value of real options without the influence of uncertainty. Since the commonly-used Black-Scholes pricing formula does not take uncertainty into consideration, oftentimes under the guise of risk, uncertainty stows away into the Black-Scholes pricing formula. Suppose that the estimated volatility is 0.4, which could contain information both of risk and uncertainty. Taking this value into the commonly-used pricing formula, we find that the value of the real option is 31.75. However, we know that the management is not exactly aware of the situation and that he is indeed under high uncertainty, say ϕ=6. We would interpolate between 25.10 and 32.34 in Table 3 to get an implied volatility of 0.292. After filtering the "true" risk through our model, we get the value of the real option is merely 23.84. This more reliable value is indeed much less than the 31.75, which is obtained by the commonly-used Black-Scholes model.

Table 3 displays the critical values to invest as the subjective measure of confidence about the probability law of asset returns (ϕ) and the volatility of the project (σ) vary. The results of neglect of uncertainty are shown in the column of $\phi = \infty$. We find that management's uncertainty, alike risk, raises the critical value to invest. An increase in uncertainty, or a decrease in his confidence about the probability law, will increase S^* and hence tend to postpone the investment project.

Conclusion

In this paper, we setup a framework, which separates risk and uncertainty, to evaluate real options. We find that besides risk, uncertainty raises the value of real options as well. We claim that interchangeably using risk and uncertainty would overestimate the value of real options. We cannot trust the value of real options unless we clarify and identify risk and uncertainty. Our theoretical model responds well to Alessandri's (2003) empirical findings. Although the Black-Scholes pricing formula is neat and user-friendly, it still has drawbacks when applied to evaluate capital projects with higher uncertainty.

We have a caution for the external users of the results from real options analysis. Once risk and uncertainty are unidentifiable, there would be space for the management to manipulate the parameter, σ, in the Black-Scholes pricing formula to exaggerate the value of an investment opportunity. Besides, it is possible that one investment opportunity may be postponed inappropriately.

Finally, we claim that our model has an advantage that it maintains the neat property of the Black-Scholes pricing formula, and depicts uncertainty and uncertainty-aversion through one single parameter. As a result, it could be applied practically much more easily than other approaches, and is helpful to find a more reliable value of real options and a more optimal timing to invest.

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Table 1: Parameter Values Used in Calibration

The parameters are the risk aversion coefficient, the rate of return of the investment opportunity, the riskless interest rate, the dividend yield (shortfall), the time to maturity, the volatility of the investment opportunity, and the subjective measure of confidence about the probability law of asset returns, respectively. We let the values of σ and ϕ include in the ranges to see their influences on the evaluation of real options.

Parameters	S_0	I	γ	v	r	q	$T-t$	σ	ϕ
Values	100	100	5	0.13	0.04	0.04	10	0.2~0.5	6~∞

Table 2: Subjective Values of Real Options

This table displays the subjective values of real options as the subjective measure of confidence about the probability law of asset returns (ϕ) and the volatility of the project (σ) vary. Other parameter values used are displayed in Table 1.

	ϕ=6	ϕ=8	ϕ=10	ϕ=12	ϕ=14	ϕ=16	ϕ=18	ϕ=20	ϕ=100	$\phi=\infty$
σ =0.2	25.10	23.02	21.75	20.90	20.29	19.84	19.48	19.20	17.15	16.69
σ =0.3	32.34	30.46	29.30	28.51	27.94	27.52	27.18	26.91	24.95	24.50
σ =0.4	39.60	37.73	36.58	35.79	35.23	34.80	34.46	34.19	32.21	31.75
σ =0.5	46.32	44.43	43.25	42.46	41.88	41.44	41.09	40.82	38.78	38.31

Table 3: Critical Values of Project Value

This table displays the critical values of the project value as the subjective measure of confidence about the probability law of asset returns (ϕ) and the volatility of the project (σ) vary. The manager has to defer the investment project until the project value beyond the critical value. Other parameter values used are displayed in Table 1.

	$\phi=6$	$\phi=8$	$\phi=10$	$\phi=12$	$\phi=14$	$\phi=16$	$\phi=18$	$\phi=20$	$\phi=100$	$\phi=\infty$
$\sigma=0.2$	339.28	285.40	261.47	247.99	239.34	233.33	228.91	225.52	204.48	200.44
$\sigma=0.3$	416.94	370.82	347.44	333.32	323.87	317.11	312.02	308.07	282.15	276.88
$\sigma=0.4$	548.78	493.95	465.20	447.50	435.51	426.85	420.30	415.18	381.05	373.98
$\sigma=0.5$	718.51	649.29	612.48	589.64	574.09	562.81	554.27	547.57	502.60	493.21