Pollution reduction policies under uncertainty and their costs

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Abstract In this paper, we investigate pollutant reduction policies under uncertainty. We assume that when an agent reduces quantity of a pollutant, it incurs costs. We consider two kinds of policies distinguished by their costs. One policy incurs proportional reduction cost (Case 1) and the other incurs fixed and proportional reduction costs (Case 2). To solve these problems, we formulate the agent's problems as a singular stochastic control problem in Case 1 and a stochastic impulse control problem in Case 2, respectively. Using this analysis, we show optimal pollutant reduction policies. Furthermore, we present numerical analysis.
Keywords: pollution reduction, proportional and fixed costs, singular stochastic control, stochastic impulse control

1 Introduction

We face many environmental problems such as global warming, acid rain, desertification, soil contamination, and so on. We directly or indirectly suffer damage from these problems. Then, we have to reduce pollutants which cause these problems. Three important characteristics of most environmental problems are the uncertainty, irreversibilities, and the feasibility of postponing decision (Pindyck (2000)). Real options model enables us to solve the environmental problems with these charactaristics.

Arrow and Fisher (1974), Henry (1974) show the value of flexibility of decision-making under uncertainty. Dixit and Pindyck (1994), Conrad (1997), Pindyck (2000, 2002) apply real options model to analyze environmental problems. These research and much of the subsequent literature investigate the timing of environmental policy designed to reduce pollutants once. On the other hand, in this research, the agent reduces a pollutant as often as needed. As well as the timing of reduction, the amount of reduction is endogenously derived.

In this paper, we investigate pollutant reduction policies (PRPs) under uncertainty. We assume that when an agent reduces a pollutant, it incurs costs. We consider two kinds of policies distinguished by their costs. One policy incurs proportional reduction cost (Case 1) and the other incurs fixed and proportional reduction costs (Case 2). To solve these problems, we first formulate the agent's problem as a singular stochastic control problem in Ccase 1. See, for example, Øksendal (1999), Pham (2006), and Yang and Liu (2004) for more detail on singular stochastic control. Next, we formulate it as an impulse control problem in Ccase 2. See, for example, Øksendal (1999), Cadenillas and Zapatero (1999), and Ohnishi and Tsujimura (2006) for more detail on stochastic impulse control. Then, we compare the results of two models.

The rest of the paper is organized as follows. Section 2 describes the setup of the agent's problem. Section 3 examines the case 1 in which the PRP incurs the proportional cost. Section 4 investigates the case 2 in which the PRP incurs the fixed and proportional costs. Next, we present the numerical analysis in Section 5. Section 6 concludes the paper.

2 Setup

Assume that an agent suffers from a pollutant which is emitted by an economic activity. Then the agent have to reduce the pollutant. However it incurs costs. Let Y_t be the stock of the pollutant at time $t \geq 0$. In this paper, we assume that when the agent does not reduce the pollutant, its dynamics is given by:

$$\mathrm{d}Y_t = \mu Y_t \mathrm{d}t + \sigma Y_t \mathrm{d}W_t, \quad Y_0 = y, \tag{2.1}$$

where $\mu > 0$ and $\sigma > 0$. W_t is a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$. Let $D(Y_t)$ be the damage function given by:

$$D(Y_t) = aY_t^b, \tag{2.2}$$

where $a > 0$ and $b > 1$. The damage function D is assumed to satisfy:

$$\mathbb{E}\left[\int_0^\infty \mathrm{e}^{-rt} D(Y_t)\mathrm{d}t\right] < \infty, \tag{2.3}$$

where $r > 0$ is a discount rate. Let z_t be the amount of pollutant reduction at time t. $K(z)$ denotes the reduction cost function. Then, the agent's problem is to choose z to minimize the expected total cost.

3 Pollution Reduction Policy and Proportional Cost

In this section, we consider when the agent reduces the pollutant, it incurs the cost which is proportional to the amount of pollutant reduction (the proportional cost). In this case, the dynamics of pollutant stock (2.1) goes to:

$$\mathrm{d}Y_t = \mu Y_t \mathrm{d}t + \sigma Y_t \mathrm{d}W_t - \mathrm{d}\zeta_t, \quad Y_0 = y, \tag{3.1}$$

where ζ_t is the cumulative amount of pollutant reduction until time t. $\zeta = \{\zeta_t\}_{t \geq 0}$ is assumed to be non-negative, non-decreasing, right-continuous with left-hand limits $\mathcal{F}_t$-adapted process with $\zeta_{0-} = 0$. Furthermore, we assume that:

$$\mathbb{E}\left[\int_0^\infty \mathrm{e}^{-rt}\mathrm{d}\zeta_t\right] < \infty. \tag{3.2}$$

Then the agent's expected total discounted cost function J_{sc} is given by:

$$J_{sc}(y;\zeta) = \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-rt} D(Y_t)\mathrm{d}t + \int_0^\infty \mathrm{e}^{-rt} k_1 \mathrm{d}\zeta_t\right], \tag{3.3}$$

where $k_1 > 0$ is the proportional cost parameter. Therefore, the agent problem is to choose ζ so as to minimize J_{sc}:

$$V_{sc}(y) = \inf_{\zeta \in \mathcal{Z}} J_{sc}(y;\zeta) = J_{sc}(y;\zeta^*), \tag{3.4}$$

where V_{sc} is the value function, $\mathcal{Z}$ is the set of admissible pollutant reduction policies, and ζ^* is an optimal pollutant reduction policy. The agent's problem (3.4) is formulated as an singular stochastic problem.

From the formulation of the agent's problem (3.4), we naturally guess that, under an optimal pollutant reduction policy, the agent reduces the pollutant whenever the pollutant stock reaches a threshold $\overline{y}$. In order to verify this conjecture, we solve the agent's problem (3.4) by using variational inequalities.

The variational inequalities of the agent's problem (3.4) are given as follows:

$$\mathcal{L}V_{sc}(y) + D(y) \geq 0, \tag{3.5}$$

$$V'_{sc}(y) \leq k_1, \tag{3.6}$$

$$[\mathcal{L}V_{sc}(y) + D(y)][k_1 - V'(y)] = 0, \tag{3.7}$$

where $\mathcal{L}$ is the operator defined by:

$$\mathcal{L} \equiv \frac{1}{2}\sigma^2 y^2 \frac{\mathrm{d}^2}{\mathrm{d}y^2} + \mu y \frac{\mathrm{d}}{\mathrm{d}y} - r. \tag{3.8}$$

Let H^{sc} be the continuation region given by:

$$H_{sc} = \{y; y < \overline{y}\}. \tag{3.9}$$

Let $\phi(y) \in C^2$ be a function. For $y < \overline{y}$, the variational inequalities (3.5)-(3.7) lead to the following ordinary differential equation:

$$\frac{1}{2}\sigma^2 y^2 \phi''(y) + \mu y \phi'(y) - r\phi(y) + ay^b = 0. \tag{3.10}$$

If the agent will not reduce the pollutant forever, the expected discounted present value of damage is calculated as follows:

$$\mathbb{E}\left[\int_0^\infty \mathrm{e}^{-rt} D(Y_t)\mathrm{d}t\right] = \frac{ay^b}{\rho}, \tag{3.11}$$

where $\rho = r - \mu b - (1/2)b(b-1)\sigma^2$. It follows from (2.3) that we have $\rho > 0$. Then, the boundary condition $\phi(0) = 0$ yields the solution to (3.10) is:

$$\phi(y) = A_1 y^{\beta_1} + \frac{ay^b}{\rho}, \qquad y < \overline{y}, \tag{3.12}$$

where A_1 is a constant to be determined and $\beta_1 > 1$ is the solution to the following characteristic equation:

$$\frac{1}{2}\sigma^2 \beta(\beta - 1) + \mu\beta - r = 0 \tag{3.13}$$

and is calculated with:

$$\beta_1 = \frac{1}{2} - \frac{\mu}{\sigma^2} + \left[\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}\right]^{\frac{1}{2}} > 1. \tag{3.14}$$

It follows from (3.11) that we have:

$$\phi(y) < \frac{ay^b}{\rho}. \tag{3.15}$$

Then, we obtain $A_1 < 0$. Let ϕ_{sc} be redefined as a candidate function of the value function and be given by:

$$\phi_{sc}(y) = \begin{cases} \psi(y) := A_1 y^{\beta_1} + \frac{ay^b}{\rho}, & y < \overline{y}, \\ k_1 y + C, & y \geq \overline{y}, \end{cases} \tag{3.16}$$

where C is a constant to be determined. Three unknowns A_1, $\overline{y}$, and C are determined by the following simultaneous equations.

$$\psi(\overline{y}) = k_1 \overline{y} + C, \tag{3.17}$$

$$\psi'(\overline{y}) = k_1, \tag{3.18}$$

$$\psi''(\overline{y}) = 0. \tag{3.19}$$

The condition (3.17) is the value-matching condition. The conditions (3.18) and (3.19) are smooth-pasting conditions. From (3.17) we obtain:

$$C = A_1\overline{y}^{\beta_1} + \frac{a\overline{y}^b}{\rho} - k_1\overline{y}. \tag{3.20}$$

Form (3.18) and (3.19) we obtain:

$$A_1 = \frac{k_1(b-1)y^{1-\beta_1}}{\beta_1(b-\beta_1)}, \tag{3.21}$$

$$\overline{y} = \left[\frac{\rho(\beta_1-1)k_1}{ab(\beta_1-b)}\right]^{\frac{1}{b-1}}. \tag{3.22}$$

Notice that from $A_1 < 0$, (3.21), and (3.22), the parameter b must satisfy:

$$1 < b < \beta_1. \tag{3.23}$$

4 Pollution Reduction Policy and Fixed and Proportional Costs

In this section, we investigate the case in which the fixed cost and the cost proportional to reduction are incurred by implementing the policy.

Let ξ_i be the ith amount of pollutant reduction and τ_i be its time. An agent pollutant reduction policy v is defined as the following double sequences:

$$v = \{(\tau_i, \xi_i)\}_{i\geq 0}. \tag{4.1}$$

For all $i \geq 0$, the dynamics of pollutant stock (2.1) changes to:

$$\begin{cases} \mathrm{d}Y_t = \mu Y_t \mathrm{d}t + \sigma Y_t \mathrm{d}W_t, \quad \tau_i \leq t < \tau_{i+1} < \infty, \\ Y_{\tau_i} = Y_{\tau_i -} - \xi_i, \\ Y_{0-} = y. \end{cases} \tag{4.2}$$

We assume that τ_i satisfies:

$$\mathbb{P}\left\{\lim_{i\to\infty} \tau_i \leq \tilde{T}\right\} = 0, \tag{4.3}$$

where $\tilde{T}$ is a terminal time. The condition (4.3) implies that pollutant reduction will only occur finitely before $\tilde{T}$. Let $K(\xi)$ be the cost function given by:

$$K(\xi) = k_0 + k_1\xi, \tag{4.4}$$

where $k_0 > 0$ is the fixed cost. Note that the cost function satisfies subadditibity with respect to ξ:

$$K(\xi + \xi') \leq K(\xi) + K(\xi'). \tag{4.5}$$

Then the agent's expected total discounted cost function J_{im} is given by:

$$\begin{aligned} J_{im}(y; v) = \mathbb{E}\bigg[&\int_0^\infty \mathrm{e}^{-rt} D(Y_t)\mathrm{d}t \\ &+ \sum_{i=0}^{\infty} \mathrm{e}^{-r\tau_i} K(\xi_i)\mathbf{1}_{\{\tau_i<\infty\}}\bigg], \end{aligned} \tag{4.6}$$

Therefore, the agent problem is to choose v so as to minimize J_{im}:

$$V_{im}(y) = \inf_{v\in\mathcal{V}} J_{im}(y; v) = J_{im}(y; v^*), \tag{4.7}$$

V_{im} is the value function, $\mathcal{V}$ is the set of admissible pollutant reduction policies, and v^* is an optimal pollutant reduction policy. The agent's problem (4.7) is formulated as an stochastic impulse problem.

From the formulation of the agent's problem (4.7), we naturally guess that an optimal pollutant reduction policy is in the following from specified by two critical pollutant levels: whenever the pollutant stock reaches a level $\tilde{y}$, the agent reduces the pollutant, so that it instantaneously reduces to another pollutant level $\hat{y}$. In order to verify this conjecture, we solve the agent's problem (4.7) by using quasi-variational inequalities.

Let $\mathcal{M}$ be the pollutant reduction operator defined by:

$$\mathcal{M}V_{im}(y) = \inf_{\xi\in[0,y)} \{V_{im}(y-\xi) + (k_0 + k_1\xi)\}. \tag{4.8}$$

Then, the quasi-variational inequalities (QVI) of the agent's problem (4.7) are given as follows:

$$\mathcal{L}V_{im}(y) + D(y) \geq 0, \tag{4.9}$$

$$V_{im}(y) \leq \mathcal{M}V_{im}(y), \tag{4.10}$$

$$[\mathcal{L}V_{im}(y) + D(y)][\mathcal{M}V_{im}(y) - V_{im}(y)] = 0. \tag{4.11}$$

From the conjecture above, the continuation region H_{im} is given by:

$$H_{im} = \{y; y < \tilde{y}\}. \tag{4.12}$$

Then, an optimal pollutant reduction policy $v^* = (\tau^*, \xi^*)$ characterized by $\tilde{y}$ and $\hat{y}$ with $0 < \hat{y} < \tilde{y} < \infty$ such that

$$\tau_i^* = \inf\{t > \tau_{i-1}^*; X_{t-} \notin H_{im}\}, \tag{4.13}$$

$$\xi_i^* = X_{\tau_i^*-} - X_{\tau_i^*} = \tilde{y} - \hat{y}. \tag{4.14}$$

Let $\phi \in C^2$ be a function. For $y < \tilde{y}$, to the QVI (4.9)-(4.11) lead to the following ordinary differential equation:

$$\frac{1}{2}\sigma^2 y^2 \phi''(y) + \mu y \phi'(y) - r\phi(y) + ay^b = 0. \tag{4.15}$$

As in Section 3, the solution to (4.15) is:

$$\phi(y) = B_1 y^{\beta_1} + \frac{ay^b}{\rho}, \tag{4.16}$$

B_1 is a constant to be determined and β_1 is derived by (3.14). B_1 is negative as well as A_1. Let ϕ_{im} be defined as a candidate function of the value function given by:

$$\phi_{im}(y) = \begin{cases} \varphi(y) := B_1 y^{\beta_1} + \frac{ay^b}{\rho}, & y < \tilde{y}, \\ \varphi(\hat{y}) + k_0 + k_1(y - \hat{y}), & y \geq \tilde{y}, \end{cases} \tag{4.17}$$

Three unknowns B_1, $\tilde{y}$, and $\hat{y}$ are determined by the following simultaneous equations. The first equation is:

$$\varphi(\tilde{y}) = \varphi(\hat{y}) + k_0 + k_1(\tilde{y} - \hat{y}). \tag{4.18}$$

The second one is:

$$\begin{aligned} \varphi'(\tilde{y}) &= \lim_{y \downarrow \tilde{y}} \varphi'(y) \\ &= \lim_{y \downarrow \tilde{y}} \frac{\mathrm{d}}{\mathrm{d}y}[k_0 + k_1(y - \hat{y}) + \phi(\hat{y})] \\ &= k_1. \end{aligned} \tag{4.19}$$

From (4.13) and (4.14), J_{im} is minimized at $\xi^* = \tilde{y} - \hat{y}$:

$$\begin{aligned} \varphi(\tilde{y}) &= k_0 + k_1(\tilde{y} - \hat{y}) + \varphi(\hat{y}) \\ &= \max_{q \in [0, \tilde{y}]} [\varphi(q) + k_0 + k_1(\tilde{y} - q)]. \end{aligned} \tag{4.20}$$

The third one is:

$$\varphi'(\hat{y}) = k_1. \tag{4.21}$$

Unfortunately, as we cannot analytically derive these unknown, we numerically calculate their values in the following section.

5 Numerical Analysis

In this section, we numerically examine the optimal pollutant reduction policies in order to explore their characteristics. The basic parameter values are set out as follows: $r = 0.06$, $\mu = 0.015$, $\sigma = 0.15$, $a = 0.01$, $b = 1.5$, $k_1 = 0.1$, and $k_0 = 0.1$. Then, we obtain $A_1 = -0.4189$, $B_1 = -0.1608$, $C = -0.0021$, $\overline{y} = 0.1177$, $\tilde{y} = 1.4102$, $\hat{y} = 0.0453$.

The comparative static analysis will be left for future research.

6 Final Remarks

In this paper, we examined pollutant reduction policies under uncertainty. When the agent implements the policy, it costs two types of reduction costs. We formulated the agent's problems as the singular stochastic control problem and the stochastic impulse control problem, respectively. Then, we found optimal pollutant reduction policies, respectively. We leave numerical analysis for future research.

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