Ambiguity Aversion in Real Options

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Extended Abstract

Real option valuation has traditionally been concerned with investment under project value uncertainty while assuming the agent has perfect confidence in a specific model. However, agents do not generally have perfect confidence in their model and this *model uncertainty* affects their decisions. In this work, we introduce a simple model for real option valuation to account for the agent's aversion to model ambiguity through the notation of *robust indifference prices*. We derive analytical results for the perpetual option to invest and the linear complementarity problem that the finite time problem satisfies.

Key-words: Real Options; Ambiguity Aversion; Risk Aversion

Quantitative methods to analyze the option to invest in a project enjoy a long and distinguished history. The classical work of McDonald and Siegel (1986) (see also Dixit and Pindyck (1994)) investigates the problem from the point of view of derivative pricing and assigns the value of the option to invest as

$$\text{value} = \sup_{\tau \in \mathscr{T}} e^{-r\tau} \mathbb{E}\left[(P_\tau - I)_+\right]. \tag{1}$$

The expected value is taken under an appropriate risk-adjusted measure, I is the cost of investing in the project, P_t is the value of the project at time t and $\mathscr{T}$ denotes the family of allowed stopping times in $[0, T]$. In the European case, the agent may invest in the project only at maturity, in the Bermudan case, the agent may invest at a set of specific times (e.g. monthly), and in the American case, the agent may invest at any time. As such, the problem is in general a free boundary problem in which the optimal strategy is computed simultaneously with the option's value.

Traditionally, the project value is assumed to be a geometric Brownian motion (GBM) and the investment amount is constant or deterministic, as in the pioneering work of Tourinho (1979). Henderson and Hobson (2002) and Henderson (2007) investigate how an agent's risk aversion affects the valuation of perpetual real options and in particular consider the project value as only partially spanned by a tradable asset. In this work, the European, finite time horizon American and perpetual American versions of the problem will be considered in an incomplete market setting similar to Henderson (2007). To value the real option to invest, we will utilize the concept of *robust indifference pricing*, first introduced by Jaimungal and Sigloch (2009) in the context of credit markets, to account both for the agent's risk aversion and ambiguity aversion (also known as *model uncertainty*).

Our key result is that due to the unhedgable risk in the project value process, the agent's ambiguity can significantly affect the optimal exercise strategy and therefore the value of the real option. The effect of ambiguity aversion is similar, but quite distinct from risk aversion. Indeed, it is economically plausible that an agent is risk-neutral but is severely ambiguity averse. We demonstrate that for such agents, ambiguity plays a crucial role in determining exercise policies as well as the value of the option to invest. Furthermore, the marginal price (Davis 1997) is determined under an ambiguity adjusted minimal entropy martingale measure (MEMM). Some numerical computations illustrate the effect of ambiguity for risk-averse agents.

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Let P_t denote the project value, which we assume for simplicity to be a geometric Brownian motion (GBM)

$$\frac{dP_t}{P_t} = \nu\, dt + \eta\, dW_t^{(1)}. \tag{2}$$

The agent cannot, in general, trade this project value; however, we assume they can trade in a strongly correlated asset denoted S_t also modeled as a GBM,

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, \left(\rho\, dW_t^{(1)} + \sqrt{1-\rho^2}\, dW_t^{(2)}\right). \tag{3}$$

In equations (2) and (3), $W_t^{(1)}$ and $W_t^{(2)}$ denote two uncorrelated $\mathbb{P}$-Wiener processes.

To value the option to invest in the project, we invoke the concept of certainty equivalence (or indifference pricing) which requires solving two optimal investment problems (i) the investment problem in the absent of the option to invest (ii) the investment problem in the presence of the option to invest. However, to account for ambiguity aversion, we allow the agent to consider a set of *candidate measures* $\mathbb{Q} \in \mathcal{Q}$ which are equivalent to the *reference measure* $\mathbb{P}$. Furthermore, the agent is assumed to have preferences invoked by the *robust optimization problem* (Anderson, Hansen, and Sargent (1999), Uppal and Wang (2003), and Maenhout (2004))

$$U(x, P, S) = \sup_{\pi\in\mathcal{A}} \inf_{\pi\in\mathcal{Q}} \mathbb{E}^{\mathbb{Q}}_{x,P,S} \left[u(X_T^\pi) + \frac{1}{\varepsilon}\, h(\mathbb{Q}|\mathbb{P})\right]. \tag{4}$$

The function $u(x)$ is concave and represents the agent's utility. The function h penalizes candidate measures $\mathbb{Q}$ which are very far from the reference measure $\mathbb{P}$ and $\varepsilon > 0$ acts as the penalization strength.

A popular choice for the penalty function h (e.g., Anderson, Hansen, and Sargent (1999)) is the entropic penalty function $h(\mathbb{Q}|\mathbb{P}) = \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \ln \frac{d\mathbb{Q}}{d\mathbb{P}}\right]$. As $\varepsilon \downarrow 0$, the candidate measure is pinned to the reference measure and the robust portfolio optimization problem reduces to the usual portfolio optimization problem. As $\varepsilon \to +\infty$, all candidate measures are considered equally viable. Consequently, ε acts as a measure of the agent's level of ambiguity aversion.

The entropic penalty problem is not solvable in general. However, Maenhout (2004) suggests a modification of the HJB equation which leads to tractable solutions for the complete market case and shows that ambiguity aversion can be absorbed by modifying the agent's risk aversion. In an incomplete market model for credit risk, Jaimungal and Sigloch (2009) demonstrate that ambiguity aversion and risk aversion are quite distinct – it is the presence of a non-traded index, similar to the project value model considered here, which induces the distinction.

Motivated by the robust optimization problem introduced by Jaimungal and Sigloch (2009), here we study the robust portfolio optimization problem

$$\begin{aligned} U^a(t, x, P, S) = \sup_{\tau\in\mathscr{T}_t} \sup_{\pi\in\mathcal{A}} \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{t,x,P,S} \Bigg[& V(\widehat{\tau}, X^\pi_{\widehat{\tau}} + a(P_{\widehat{\tau}} - I)_+, P_{\widehat{\tau}}, S_{\widehat{\tau}}) \\ & - \frac{1}{\epsilon} \int_t^{\widehat{\tau}} U^a(s, X_s^\pi, P_s, S_s) \mathbf{v}_s^{\mathbb{Q}\prime} \Sigma^{-1} \mathbf{v}_s^{\mathbb{Q}}\, ds \Bigg], \end{aligned} \tag{5}$$

where, $\widehat{\tau} = \tau \wedge T$ and

$$V(t, x, P, S) = \sup_{\pi\in\mathcal{A}} \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{t,x,P,S} \left[u(X_T^\pi) - \frac{1}{\epsilon} \int_t^T V(s, X_s^\pi, P_s, S_s) \mathbf{v}_s^{\mathbb{Q}\prime} \Sigma^{-1} \mathbf{v}_s^{\mathbb{Q}}\, ds \right]. \tag{6}$$

The penalty terms in equations (5) and (6) represent scaled versions of the entropic penalty, as in Maenhout (2004). Note the sign flip on the penalty term due to our use of exponential utility function: $u(x) = -\frac{1}{\gamma}e^{-\gamma x}$. The vector $\mathbf{v}_t^{\mathbb{Q}} = (v_t^P, v_t^S)$ represents the excess drifts of the project value P_t and tradable asset S_t under the candidate measure $\mathbb{Q}$. The value function $V(t,x,P,S)$ represents the agents robust utility *post exercise* at which time the agent has no exposure to the option's risk. The value function $U(t,x,P,S)$ represents the agent's robust utility *prior to exercise*, which upon exercise reduces to $V(t,x,P,S)$.

We define the *robust indifference price* p_t of the option to invest as the solution to $U^a(t,x-p_t,P,S) = U^0(t,x,P,S)$. As such, the robust indifference price p_t can be interpreted as the amount of wealth the agent is willing to give up right now in exchange for receiving the value of the option at exercise, without altering their robust utility.

Lemma 1 Post Exercise Value Function. *The post-exercise value function $V(t,x,P,S)$ is independent of P and S and is given explicitly as*

$$V(t,x,P,S) = u\Big(xe^{r(T-t)}\Big)\,\exp\left\{-\frac{1}{2}\lambda^2(T-t)\right\}, \qquad \textit{with} \quad \lambda^2 = \frac{1}{1+\varepsilon}\left(\frac{\mu-r}{\sigma}\right)^2. \tag{7}$$

Furthermore, the optimal investment and drift adjustments are

$$\left(\pi_t^*,\quad v_t^{S*},\quad v_t^{P*}\right) = \left(\tfrac{1}{\gamma(1+\varepsilon)}\tfrac{\mu-r}{\sigma^2}e^{-r(T-t)},\quad -\tfrac{\varepsilon}{1+\varepsilon}(\mu-r),\quad -\tfrac{\varepsilon}{1+\varepsilon}\rho\tfrac{\eta}{\sigma}(\mu-r)\right). \tag{8}$$

The ambiguity aversion parameter ε enters into all expressions. In the limit in which the agent is fully confident in their model (i.e., $\varepsilon \downarrow 0$) the solution reduces to the usual Merton solution. Moreover, there are no drift corrections to the spanning asset or project value in this limit. However, as the agent becomes severely ambiguity averse (i.e, $\epsilon \to +\infty$), the agent no longer invests in the spanning asset, the optimal measure becomes the MEMM under which the spanning asset's drift is equal to the risk-free rate and the project value's drift is $\nu - \rho\frac{\eta}{\sigma}(\mu - r)$ – the tradable spanning asset becomes risk-neutral and orthogonal Brownian motions are unchanged under the MEMM.

Due to the early exercise constraint, the pre-exercise value function $M(t,x,P,S)$ is not solvable exactly; however, we demonstrate that it does solve a linear complementarity problem recorded in the following Theorem.

Theorem 1 Pre-Exercise Value Function. *The pre-exercise value function $M(t,x,P,S)$ is independent of S and factors as $U^a(t,x,P,S) = u\left(xe^{r(T-t)}\right)e^{-\frac{1}{2}\lambda^2(T-t)}G^{\beta}(t,\ln P)$ where $G(t,y)$ solves the following linear complementarity problem*

$$\left\{\begin{aligned} \partial_t G + \mathcal{L}G &\leq 0, \\ \ln G(t,y) &\geq a\tfrac{\gamma}{\beta}(e^y - K)_+ e^{r(T-t)}, \\ (\partial_t G + \mathcal{L}G)\cdot\left(\ln G(t,x) - \tfrac{\gamma}{\beta}(e^y-K)_+ e^{r(T-t)}\right) &= 0, \end{aligned}\right. \tag{9}$$

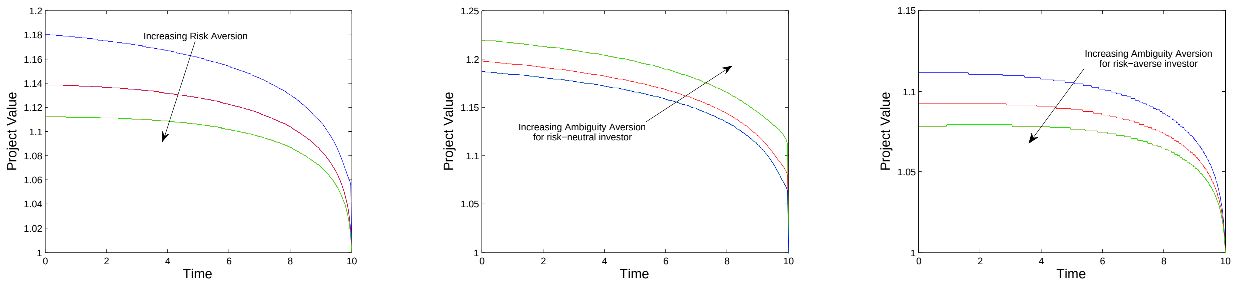
$\mathcal{L}G = \widetilde{\nu}\partial_y G + \frac{1}{2}\eta^2\partial_{yy}G$ and $\widetilde{\nu}$ and β are explicit functions of ε, σ, η, ρ, ν, μ and r.

As a consequence of the above Theorem, we have the following.

Corollary 2 Robust Indifference Price. *The robust indifference price $p(t,y)$ of the real option satisfies the non-linear complementarity problem*

$$\left\{\begin{aligned} \partial_t p + \mathcal{L}p - \tfrac{1}{2}\eta^2\tfrac{\gamma}{\beta}e^{r(T-t)}\left(\partial_y p\right)^2 &\leq r\,p, \\ p(t,y) &\geq (e^y-K)_+, \\ \left(\partial_t p + \mathcal{L}p - \tfrac{1}{2}\eta^2\tfrac{\gamma}{\beta}e^{r(T-t)}\left(\partial_y p\right)^2 - r\,p\right)\cdot\left(p(t,x) - (e^y-K)_+\right) &= 0. \end{aligned}\right. \tag{10}$$

As $\gamma \to 0$ the non-linear term disappears, and $p(t, y)$ reduces to the marginal (Davis 1997) price, albeit with an ambiguity adjusted drift of $\widetilde{\nu}$. As $\varepsilon \downarrow 0$, $\widetilde{\nu}$ reduces to the usual MEMM drift and we obtain the finite-time horizon analog of Henderson (2007). However, with non-zero ε the results may differ considerably. The following figures provide a few examples of how the early exercise boundaries are affected by risk aversion and ambiguity for a finite time horizon problem.



(a) Risk aversion

(b) Ambiguity aversion with $\gamma = 0$

(c) Ambiguity Aversion with $\gamma = 0.2$

Figure 1: The effects of risk aversion and ambiguity aversion on the optimal exercise boundary.

We also derive analytical results for the perpetual case and prove the relevant verification theorems.

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