Market-Entry Sequencing under Uncertainty

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Abstract

At the early stages of industry development, before operating in the market, firms may identify an opportunity to enter yet wait for the market to grow sufficiently (to justify the expense of market entry). We model this problem and show that, in oligopolies, market entry involving lumpy investments takes place in sequence under uncertainty regardless of whether or not firms can observe (and react to) their rivals' moves. Unlike the case in which firms disregard competition (the myopic or open-loop case), entry does not occur in a socially optimal manner when firms condition their moves on competitors' actions. Thus, additional information about rivals' play may lead to social loss.

Keywords: financial economics, investment under uncertainty, real options, strategic investment, option games

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1. Introduction

It is standard practice in finance and strategy to interpret real investment opportunities as being analogous to options. This view is well accepted among academics and practitioners alike and is the core of real options analysis (ROA). It allows one to capture the dynamic nature of decision making because it factors in management's flexibility to revise and adapt strategy in the face of market uncertainty. This approach is summarized in the works of Dixit and Pindyck (1994) and Trigeorgis (1996).

Standard ROA, however, has not adequately addressed strategic interactions. The exercise decisions of a financial option holder have little impact on the underlying asset's value dynamics. Since mainstream ROA draws from contingent-claims analysis (based on financial options), a key feature specific to real assets is often ignored — namely, the interplay or strategic interactions taking place among (real) option holders. This oversight may result in misestimating the value of strategic options and in suboptimal investment policies. The inadequate treatment of competitive dynamics among (real) option holders remains one of the most compelling research gaps (Trigeorgis 1996, p. 376). Bridging this gap requires concurrently taking into account both market and strategic uncertainties via the combined use of ROA and game theory.[1]

In modeling multistage, multiplayer problems, one has to consider the possibility that players will react optimally in the future, conditional on the information they gather over time.[2] In order to analyze strategic interactions in multistage deterministic settings, one often distinguishes two types of information structure. In *closed-loop* strategies, the players can condition their actions (when called upon to play) on the observed sequence of rivals' moves. In contrast, *open-loop* strategies do not depend on rivals' previous play and instead set ex ante a certain investment path to be pursued regardless of the rivals' decisions. If the information structure allows players to condition their actions on previous plays, as in closed-loop strategies, then the subgame-perfect solution concept may yield stronger predictions.[3] Nash equilibria in open-loop strategy profiles often fail to be subgame perfect when firms can observe (and react to) their rivals' moves.[4] The open-loop approach is often employed in economic analysis when many small players cannot condition their play on their opponents' actions. Fudenberg and Levine (1988) have shown that, as the number of players increases, the outcome of a perfect equilibrium in closed-loop strategies converges to a Nash equilibrium in open-loop strategies. Therefore, the Nash equilibrium in open-loop strategies provides a good benchmark and reasonable predictions for large oligopolies when firms formulate their strategy in isolation and their decisions do not materially affect the decision making of rivals.

The distinction between open-loop and closed-loop strategies has not always been clear in the

[1]Smit and Trigeorgis (2004) discuss discrete-time option games, whereas Grenadier (2000) and Huisman (2001) examine a number of continuous-time models. Chevalier-Roignant and Trigeorgis (2010) provide an overview of both types of "option games".

[2]This has different implications depending on whether a single player is faced with deviations from an expected market development or multiple players must predict what will be the outcome of strategic interactions over time. The first case is characterized by Bellman's principle of optimality, whereby the agent acts optimally in every (exogenous) state of the world; the second case is characterized by subgame perfection, whereby players act optimally (as part of a Nash equilibrium) both on and off the equilibrium path.

[3]In most cases, perfect equilibria are best because most dynamic games assume negligible information lags between a player's move and rivals' observations. However, Nash equilibria in open-loop strategies are best when information lags are infinite or firms are precommitted to a certain path of investment. See Fudenberg and Tirole (1991) for further details.

[4]In open-loop models there is a single proper subgame (the game as a whole); this explains why the Nash equilibrium in open-loop strategies is subgame perfect if firms cannot condition their actions on rivals' play.

emerging literature on "option games". Several authors (e.g., Leahy, 1993; Baldursson and Karatzas, 1996; Baldursson, 1998) have examined investment behavior in (perfectly) competitive markets under uncertainty where firms use capital stock as a control variable and may increase that stock by a small amount (infinitely divisible capital).[5] These models assume that the Nash equilibrium (in open-loop strategies) is the appropriate solution concept and do not discuss which information structure is the most descriptive of the dynamic game considered. A new strand in the literature (see, e.g., Back and Paulsen, 2009; Novy-Marx, 2009) considers perfect equilibria in an oligopoly where firms make incremental capital investments. In contrast, several real options models (see, e.g., Smets, 1991; Grenadier, 1996; Huisman and Kort, 1999) have been developed in a duopoly setting and examine lumpy investments under closed-loop strategies.

In this paper we address a discrete or lumpy investment decision in the context of an oligopoly, examining whether firms should enter the market now or later.[6] This problem was previously studied in the field of industrial organization previously by Reinganum (1981a,b); Gilbert and Harris (1984); Fudenberg and Tirole (1985), and Reynolds (1987). However, these authors assume a deterministic market environment. In this paper, we link the two previous approaches by exploring optimal market-entry strategies that allow for uncertainty in market development as modeled by stochastic (Itô) processes. To the best of our knowledge, only Bouis et al. (2009) have addressed oligopoly (more than two firms) and lumpy investment. Our setting and modeling approach differs from theirs which uses numerical analysis to study the investment behaviors in large oligopolies. Here we shall deal concurrently with an exogenous shock and endogenous market-entry decisions.

The paper is organized as follows. Section 2 sets up the problem faced by would-be market entrants and presents the model assumptions. Section 3 examines the investment timing problem in a new market under uncertainty and demonstrates that investment takes place in sequence regardless of whether or not firms can condition their decision on previous rivals' moves. The open-loop

[5]As noted by Pindyck (1988), the assumption that firms can continuously add incremental amounts of capital is an extreme one; in fact, most business problems involve discrete or lumpy investment decisions.

[6]In reality, firms typically do not limit their market-entry strategy solely to determination of the entry time; rather, they determine the staging of entry, the appropriate scale of production, the type of product to manufacture, etc. For simplicity, economists frequently abstract market-entry strategy to focus on the timing issue in isolation.

equilibrium in Section 3.1 describes an oligopoly with a large number of incumbents (i.e., perfect competition); the closed-loop duopoly model is discussed in Section 3.2. Section 4 compares the market-entry sequences with the socially optimal benchmark, confirming that only the open-loop market-entry sequence is necessarily socially optimal; in the closed-loop case, where firms can observe and react to their rivals' previous moves, the market-entry sequence need not be socially optimal. Section 5 concludes.

2. Model Setup and Assumptions

We consider settings to involve complete information on historic market developments and firm investment decisions and in which (option-holding) firms have common priors about the probabilities of (future) exogenous events (homogeneous expectations). Consider the filtered probability space $(\Omega, \mathbb{H}, P)$, where the filtration $\mathbb{H} \equiv \{H_t\}_{t\geq 0}$ is a family of "tribes" that allow perfect recall: $H_s \subseteq H_t$ for all $s = 0, \ldots, t$. The tribe H_t denotes for the possible histories (information set) on which the decision maker bases her decision at time t; the information set keeps track of the exogenous market development and the evolution of industry structure. For tractability, we restrict ourselves to a Markov environment where the latest (i.e., current) state (shock and industry structure) is a sufficient statistic on which firms can condition their decision making.[7] We consider an oligopoly with n identical firms, where each firm has an infinitely lived investment option (modeled as a perpetual American call, as in McDonald and Siegel 1986) but where no firm is active in the marketplace at the outset. When new firms enter the market, they incur a positive lumpy (sunk, exogenous) investment cost I.

Assume the market is subject to an exogenous ($\mathbb{H}$-adapted[8]) shock $\{X_t\}_{t\geq 0}$ that follow a time-homogenous Itô process characterized by the stochastic differential equation

$$\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}z_t, \tag{1}$$

[7] This restriction enables us to reduce the state space and avoid the "curse of dimensionality" by allowing the use of Markov strategies (i.e., strategies based on the latest values of states).

[8] An $\mathbb{H}$-adapted process is H_t-measurable for all $t \in \mathbb{R}_+$. The assumption of adaptedness to the filtration means that no foresight about the future economic development is permitted; this is equivalent to the non–anticipativity constraint in operations research.

where $z = \{z_t\}_{t \geq t_0}$ is a standard Brownian motion and the process starts at $X_0 = x$ almost surely.[9] This process covers a fairly broad family of stochastic processes used in economic analysis, such as arithmetic Brownian motion (where $\mu(X_t) = \mu$ and $\sigma(X_t) = \sigma$ with $\mu, \sigma \in \mathbb{R}$), geometric Brownian motion (where $\mu(X_t) = \mu X_t$ and $\sigma(X_t) = \sigma X_t$ with $\mu, \sigma \in \mathbb{R}$), and the geometric Ornstein–Uhlenbeck process (where $\mu(X_t) = \eta X_t(X_t - \bar{X})$ and $\sigma(X_t) = \sigma X_t$ with $\bar{X}$ the long-term average, η the speed of mean reversion, and $\sigma \in \mathbb{R}$). Assume that the drift process exhibits a long-term positive growth. The state of the industry structure is captured by an integer-valued ($\mathbb{H}$-adapted) nondecreasing process $m = \{m_t\}_{t \geq 0}$ that indicates the number of incumbent firms as of time t.

The revenue function $r(\cdot, \cdot)$ of an incumbent firm at time t depends on the current value of the exogenous shock and also on the number of incumbent firms:

$$r : \mathbb{R} \times \mathbb{N} \to [0, \infty).$$

The (stochastic) revenue function is H_t-measurable, twice continuously differentiable, and nondecreasing in the shock; it is also decreasing in the number of incumbent firms. Denote the cost incurred by an incumbent firm at time t by

$$c : \mathbb{R} \times \mathbb{N} \to [0, \infty).$$

The cost function $c(\cdot, \cdot)$ also is H_t-measurable, twice continuously differentiable, and nondecreasing in the shock. When new entrants arrive, this cost might be reduced owing to spillover effects, best practices, or asset sharing with competitors. On the other hand, the cost might increase in the number of incumbent firms in response to higher demand for inputs.[10] The instantaneous profit flow $\pi(\cdot, \cdot) \equiv r(\cdot, \cdot) - c(\cdot, \cdot)$ to an incumbent is H_t-measurable, twice continuously differentiable, and strictly increasing in the shock. However, it is strictly decreasing in m_t because competitive arrivals are viewed as a negative externality.

We assume that risk-neutral firms face a constant, risk-free rate r that is common to all market

[9] For technical reasons (existence conditions), assume further that the drift and the volatility processes ($\mu : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$, respectively) are $\mathbb{H}$-adapted and have finite variations and, assume there exist constants $C, D \in \mathbb{R}_+$ such that: (i) $|\mu(X_t)| + |\sigma(X_t)| \leq C(1 + |X_t|)$, for all $X_t \in \mathbb{R}$, $t \geq 0$ (linear growth condition); (ii) $\left|\mu(X_t^1) - \mu(X_t^2)\right| + \left|\sigma(X_t^1) - \sigma(X_t^2)\right| \leq D\left|X_t^1 - X_t^2\right|$, for all $X_t^1, X_t^2 \in \mathbb{R}$, $t \geq 0$ (Lipschitz condition).

[10] If $c(\cdot, \cdot)$ were decreasing in m_t, we would assume that the decrease in $c(\cdot, \cdot)$ is strictly less than the decrease in $r(\cdot, \cdot)$.

participants (including the central planner).[11] Assumption 1 reflects the market's incipience, with no firm investing or operating at the outset. Firms that have not yet entered the market are not affected by rivals' investment decisions.

Assumption 1. *No firm is active at the outset* ($m_0 = 0$)*, and none has an incentive to immediately invest:*

$$\mathbb{E}_0\left[\int_0^{\infty} e^{-rs}\pi\left(X_s, 1\right) \mathrm{d}s - I\right] < 0.$$

The operator $\mathbb{E}_t[\cdot]$ is used henceforth as shorthand for $\mathbb{E}\left[\cdot\middle|\tilde{h}_t\right]$, that is, the expectation conditional on the (payoff-relevant) history $\tilde{h}_t \equiv (X_t, m_t) \in H_t$.

3. Market-Entry Sequencing

We now show that regardless of whether or not firms can condition their action on previous moves (closed-loop versus open-loop strategies), the equilibrium market entries are characterized by a time sequencing or ordering of the firms. We first examine a large oligopoly in which firms do not observe their rivals' play (open-loop strategies). We then discuss the preemption that arises when firms observe their rival's move in a duopoly model (closed-loop strategies). This latter approach has been formulated by Dixit and Pindyck (1994) for a multiplicative shock that follows a geometric Brownian motion and (implicit) mixed strategies. We shall derive an analogous result for the general Itô process with related behavioral strategies.

3.1. Market-Entry Sequencing for Open-Loop Strategies

As already noted, the Nash equilibrium in open-loop strategies is a good approximation for large oligopolies. To adapt the game-theoretic notion of open-loop strategies (developed in a deterministic environment) to a setting where market features evolve stochastically, we must refine the notion of open-loop strategies to account for the exogenous shock. In this context, an *investment policy* is a decision rule based solely on the observed resolution of market uncertainty over time but not on the sequence of moves in the industry.[12] This assumption is reasonable in some circumstances, since it

[11] The assumption of risk neutrality can be relaxed if arbitrage opportunities in the market do not exist and the market is complete. Along the lines of Cox and Ross (1976) and Harrison and Kreps (1979), we could replace the drift in the stochastic differential equation (1) by the one that is prevalent in a risk-neutral environment. In this case, z would be a standard Brownian motion under the equivalent martingale measure.

[12] In this sense, an investment policy is open-loop with respect to the filtration of the strategy space but closed-loop with respect to nature's moves.

may be easier to gather information about the prospects of a given market than about rivals' likely competitive moves.

Definition 1. *An investment policy is a decision rule that maps, for every possibly state* $x \in \mathbb{R}$ *of the shock, an action* $a_t^i \equiv a^i(x) \in A^i(h_t) \subseteq \{0,1\}$*. Here,* a_t^i *indicates whether firm* i *operates* (1) *or not* (0)*, and* A_t^i *is the stage action set.*

An investment policy is a pure strategy — that is, a map from the information set (where the payoff-relevant history is the value of the exogenous shock in the Markov environment) to actions (to "enter" or "not"). Investment is assumed to be irreversible, so the firm continues to operate indefinitely once investment takes place. Given the infinite planning horizon, Bellman's principle of optimality prescribes that the existence of a fixed threshold level X_i that partitions the state space for the exogenous shock such that $(-\infty, X_i)$ is the inaction region and $[X_i, \infty)$ is the action region. We can thus simplify the strategy formulation and define it as selecting the investment threshold X_i, rather than specifying the mapping in Definition 1. We associate to this threshold X_i a ($\mathbb{H}$-adapted) stopping time $\tau_i \equiv \inf\{t \geq 0 \mid X_t \geq X_i\}$ at which investment occurs.[13] The assumption of irreversibility makes it possible to relate the stage action a_t^i in Definition 1 to the stopping time τ_i via the relationship

$$a_t^i = \mathbb{1}_{\{t \geq \tau_i\}}, \quad t \geq 0,$$

where $\mathbb{1}$ is the indicator function. The process that indicates the number of incumbent firms at time t is $m = \{m_t\}_{t\geq 0}$, where $m_t = \sum_{i=1}^n a_t^i$. Given Definition 1, we can now define the Nash equilibrium in investment policies for a generic payoff function $F^i(\cdot,\cdot)$. Let X_{-i} denote the strategy profile of all firms *except* firm i.

Definition 2. *The profile of investment policies* $X^* = (X_1^*, \ldots, X_n^*)$ *is a Nash equilibrium if and only if, for each firm* $i = 1, \ldots, n$,

$$F^i\big(X_i^*, X_{-i}^*\big) \geq F^i\big(X_i, X_{-i}^*\big), \quad \forall X_i \in \mathbb{R}.$$

If a firm were to enter the market immediately, then it would receive (at time t)

$$W(X_t, m_t) = \mathbb{E}_t\left[\int_t^\infty e^{-r(s-t)}\pi\left(X_s, m_s\right) \mathrm{d}s - I\right]. \tag{2}$$

[13] The finiteness of the first-hitting time typically involves conditions on the drift of the exogenous shock process $\{X_t\}_{t\geq 0}$. These conditions are specific to the case considered.

3.1.1. Myopic Behavior in Large Oligopolies

The investor's present expected value when firm i decides to enter the market at time τ_i is given by

$$V_0^i(X_i, X_{-i}) = V^i(x; X_i, X_{-i}) \equiv \mathbb{E}_0\left[\int_0^\infty e^{-rs} a_s^i \pi(X_s, m_s)\,\mathrm{d}s - e^{-r\tau_i} I\right]. \tag{3}$$

Here $\{m_s\}_{s\geq 0}$ evolves over time with the arrival of new firms in the marketplace. For a given profile of investment policies by rivals X_{-i}, firm i chooses its investment policy to maximize its payoff $V_0^i(X_i, X_{-i})$. Determining the Nash equilibrium in investment policies corresponds to the multiplayer optimization problem

$$V_0^i\big(X_i^*, X_{-i}^*\big) \equiv \sup_{X_i\in\mathbb{R}} V_0^i\left(X_i, X_{-i}^*\right), \quad \forall i = 1, \ldots, n. \tag{4}$$

Here firm i takes account of the impact of future investments on its value function when formulating its optimal strategy. We then refer to i as a *strategic* firm.

In contrast, a *myopic* firm ignores the investments of other players in its maximizing behavior. In other words, it invests as if no other investment occurs in the industry. The value that myopic investors expect to receive (i.e., their beliefs about payoff) is

$$v_0^i(X_i) = v^i(x; X_i) \equiv \mathbb{E}_0\left[\int_0^\infty e^{-rs} a_s^i \pi(X_s, m_{\tau_i})\,\mathrm{d}s - e^{-r\tau_i} I\right]. \tag{5}$$

The number of incumbents in equation (5) is hypothetized to remain constant (at level m_{τ_i}) after firm i's entry, whereas the strategic firm's maximization problem in equation (3) takes account of future firm arrivals. Because firm arrivals induce negative externalities on the incumbent firm's profit, the myopic firm believes to it is entitled to a higher value than is a strategic firm for the same strategy profile $X = (X_i, X_{-i})$. This may be expressed formally as follows:

$$V_0^i(X_i, X_{-i}) \leq v_0^i(X_i) < \infty, \quad \forall X = (X_i, X_{-i}) \in \mathbb{R}^n.$$

Both $V^i(\cdot;\cdot,\cdot)$ and $v^i(\cdot;\cdot)$ are increasing in x. The myopic firm behaves rationally under the restriction that the effects of rivals' investments on its value can be ignored. In this case, the optimal strategy solves the following problem:

$$v_0^i\left(X_i^*\right) \equiv \sup_{X_i\in\mathbb{R}} v_0^i\left(X_i\right), \quad \forall i = 1, \ldots, n. \tag{6}$$

Proposition 1 establishes that both strategic and myopic firms follow the same Nash equilibrium strategies. The intuition behind this claim involves the strategic firm's taking into account a (negative) value component that is not material for strategy formulation. This property simplifies the underlying problem, allowing one to derive the Nash equilibrium investment policy profiles of strategic firms based on the myopic firm's simpler value functions of equation (5).

Proposition 1. *In a Nash equilibrium in investment policies, a strategic firm formulates the same investment policy as a myopic firm.*

Proof. Define $CE(\cdot,\cdot) \equiv V_0^i(\cdot,\cdot) - v_0^i(\cdot)$, where CE denotes competitive erosion. From equations (3) and (5), it follows that

$$\begin{aligned} CE(X_i, X_{-i}) &= \mathbb{E}_0\left[\int_0^\infty e^{-rs} a_s^i \left\{\pi\left(X_s, m_s\right) - \pi\left(X_s, m_{\tau_i}\right)\right\} \mathrm{d}s\right] \\ &= \mathbb{E}_0\left[\int_{\tau_i}^\infty e^{-rs} \left\{\pi\left(X_s, m_s\right) - \pi\left(X_s, m_{\tau_i}\right)\right\} \mathrm{d}s\right]. \end{aligned}$$

Assume a weak ordering of firms, with firm i investing at $\tau_i < \infty$ (and firm $i+1$ at $\tau_i \leq \tau_{i+1} < \infty$), such that $i = m_{\tau_i}$ firms are operating in the time interval $[\tau_i, \tau_{i+1}]$. Then

$$\begin{aligned} CE(X_i, X_{-i}) = \mathbb{E}_0\Bigg[&\int_{\tau_i}^{\tau_{i+1}} e^{-rs} \left\{\pi(X_s, i) - \pi(X_s, i)\right\} \mathrm{d}s \\ &+ \int_{\tau_{i+1}}^\infty e^{-rs} \left\{\pi(X_s, m_s) - \pi(X_s, i)\right\} \mathrm{d}s\Bigg] \\ = \mathbb{E}_0\Bigg[&\int_{\tau_{i+1}}^\infty e^{-rs} \left\{\pi(X_s, m_s) - \pi(X_s, i)\right\} \mathrm{d}s\Bigg]. \end{aligned} \tag{7}$$

This shows that $CE(X_i, X_{-i})$ depends not on X_i but only on X_{-i}, so $CE(X_i, X_{-i}) = CE(X_{-i})$ and $V_0^i(X_i, X_{-i}) = v_0^i(X_i) + CE(X_{-i})$. Furthermore, since the profit is decreasing in the number of firms, we have $CE(X_{-i}) \leq 0$. The term $CE(X_{-i})$ can thus be interpreted as the competitive value erosion incurred by the (strategic) investor. Following Slade (1994), call $v_0^i(\cdot)$ the *fictitious objective function*. Observe that $CE(X_{-i})$ does not affect the optimizing behavior of the strategic firm i:

$$X_i^* \in \arg\max_{X_i \in \mathbb{R}} V_0^i\left(X_i, X_{-i}^*\right) \Longleftrightarrow X_i^* \in \arg\max_{X_i \in \mathbb{R}} v_0^i(X_i).$$

□

Proposition 1 can now be used to derive sufficient conditions for the Nash equilibrium based on the value function of the myopic firm in equation (5). These conditions are summarized in Proposition 2. For tractability, we omit the dependence of the value functions on the information set and on the investment trigger X_i^*.

Proposition 2. *Firm i's optimal investment policy X_i^* must solve the following system of equations:*

$$rv^i - \mathscr{D}v^i = 0, \tag{8a}$$

$$v^i\big|_{x=X_i^*} = W^i\big|_{x=X_i^*}, \tag{8b}$$

$$v_x^i\big|_{x=X_i^*} = W_x^i\big|_{x=X_i^*}, \tag{8c}$$

$$\lim_{x\to 0} v^i(x; X_i^*) = 0. \tag{8d}$$

The term $\mathscr{D}v^i$ in equation (8a) corresponds to the expected capital gain over an infinitesimally small time interval. It is given by

$$\begin{aligned}\mathscr{D}v^i &\equiv \lim_{h\to 0}\frac{\mathbb{E}_t\left[v^i_{t+h}\right]-v^i_t}{h}\\ &= v^i_x\mu(X_t)+\frac{1}{2}v^i_{xx}\sigma(X_t)^2 .\end{aligned} \tag{9}$$

Here v^i_x and v^i_{xx} indicate respectively the first- and second-order derivatives of v^i_0 with respect to $X_0 = x$ ($\mathscr{D}$ is the infinitesimal generator in stochastic calculus). Equation (8a) is the Hamilton–Jacobi–Bellman (HJB) equation. Equation (8b) is the value-matching condition; it prescribes that, at the time of optimal exercise, the firm is indifferent between investing now (and receiving W) and waiting (and receiving obtaining value v^i). The "smooth pasting" condition represented by equation (8c) ensures that the first-order derivative of the value function is continuous at the optimal threshold. Finally, condition (8d) implies that the value function does not "explode" when the underlying shock has a low value. These conditions are fairly standard is real options analysis.

Proof. The proof is equivalent to that for the Stefan problem in optimal stopping theory as derived by Peskir and Shiryaev (2006), for example. □

3.1.2. Market-Entry Sequencing as a Nash Equilibrium

We have characterized the optimal Nash investment policies formulated by option-holding firms, but we have not yet characterized the timing of when market entry takes place. We next examine whether firms enter simultaneously, sequentially, or sequentially but with clustering effects.

Without loss of generality, we assume a weak ordering of investment with firm i denoting the ith investor. From equation (3), it follows that

$$V^i_0(X_i, X_{-i}) = \mathbb{E}_0\left[\sum_{k=i}^{n}\int_{\tau_k}^{\tau_{k+1}} e^{-rs}\pi\left(X_s, k\right)\mathrm{d}s - e^{-r\tau_i}I\right], \tag{10}$$

where (by convention) $\tau_{n+1} \equiv \infty$.

Proposition 3. *Given a weak ordering of market-entry times, each firm has a unique optimal threshold X^*_i. The thresholds for $i = 1, \ldots, n$ are distinct, and investment takes place in sequence. Formally:*

$$0 < \tau^*_1 < \cdots < \tau^*_n < \infty,$$

where $\tau^*_i = \inf\{t \geq 0 \mid X_t \geq X^*_i\}$.

Proof. We know that $v^i_0(\cdot)$ is strictly increasing in $\left[X^*_{i-1}, X^*_{i+1}\right]$, so the function

$$\mathscr{L}v^i \equiv rv^i - \mathscr{D}v^i \tag{11}$$

is also strictly increasing. Here $\mathscr{D}v^i$ is as defined in equation (9), and by equation (8a) we have $\mathscr{L}v^i\big|_{x=X_i^*} = 0$. To see that $X_i^* \in \left(X_{i-1}^*, X_{i+1}^*\right)$, evaluate (11) at $X_i = X_{i-1}^*$. Since $\mathscr{L}v^i$ is strictly increasing $\mathscr{L}v\big|_{x=X_{i-1}^*} < 0$. Similarly, it follows that $\mathscr{L}v^i\Big|_{x=X_{i+1}^*} > 0$ at $X_i = X_{i+1}^*$. The threshold X_i^* (characterized by the conditions in Proposition 2) is the unique maximum of $v_0^i(\cdot)$ in $\left(X_{i-1}^*, X_{i+1}^*\right)$. □

Proposition 4. *The Nash equilibria in investment policies are characterized by the following properties.*

(i) The set (n-tuple) of market-entry thresholds $X^* = (X_1^*, \ldots, X_n^*)$ where X_i^*, $i = 1, \ldots, n$, solves equations (8a)—(8d)*, is a Nash equilibrium in investment policies.*

(ii) There are in total $n!$ Nash equilibria in market-entry policies, and each is characterized as a permutation of the market-entry thresholds in part (i).

Proof. See Appendix I. □

It cannot be readily predicted which Nash equilibrium is the most likely to occur among the $n!$ equilibria of Proposition 4.[14] Example 1 considers a special case involving a multiplicative shock that follows the geometric Brownian motion and admitting a closed-form solution.[15]

Example 1. *Suppose the shock $\{X_t\}_{t\geq 0}$ follows the geometric Brownian motion*

$$\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}z_t,$$

where $\mu, \sigma \in \mathbb{R}_+$ and $\mu < r$. The shock enters the profit function multiplicatively: $\pi(X_t, m_t) = X_t \pi(m_t)$ for all $X_t \in \mathbb{R}_+$ and all $m_t \in \mathbb{N}$. Then the optimal (Nash equilibrium) threshold X_i^* for the ith market entrant, $i = 1, \ldots, n$, is such that

$$X_i^* \frac{\pi(i)}{\delta I} = \frac{\beta_1}{\beta_1 - 1}.$$

Here $\delta \equiv r - \mu > 0$, and β_1 is the positive root of the "fundamental quadratic" given by

$$\beta_1 \equiv \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + 2\frac{r}{\sigma^2}} \ (> 1).$$

When firms pursue the Nash equilibrium investment policies, the value function for a (strategic) firm $i = 1, \ldots, n$ is

$$V_0^i\left(X_i^*, X_{-i}^*\right) = \left(\frac{X_0}{X_i^*}\right)^{\beta_1} \left(X_i^* \frac{\pi(i)}{\delta} - I\right) + CE\left(X_{-i}^*\right), \tag{12}$$

where the last competitive erosion term is

$$CE\left(X_{-i}^*\right) = \sum_{m=i+1}^{n} \left(\frac{X_0}{X_m^*}\right)^{\beta_1} X_m^* \left(\frac{\pi(m) - \pi(m-1)}{\delta}\right) \ (\leq 0).$$

[14] If firms were asymmetric, then we could reasonably predict that the more likely investment sequence would be the socially optimal one by employing focal-point or commonsense considerations.

[15] Proposition 4 is derived for the fairly general case of when the profit functions are twice continuously differentiable in the shock and are subject to a shock that follows a time-homogeneous Itô process. Existence of an analytical solution to this problem is not ensured generally but only under restrictive assumptions for the process and the profit function. The geometric Brownian motion is standard in economic analysis because it is fairly descriptive of problems faced by economic agents and often yields closed-form solutions.

Proof. In the case of geometric Brownian motion, it follows from equations (8a) and (9) that the solution solves the second-order differential equation

$$rv^i - \mu X_t v_x^i + \frac{1}{2}\sigma^2 X_t^2 v_{xx}^i = 0.$$

This equation has solutions of the form

$$v^i(x; X_i) = Ax^{\beta_1} + Bx^{\beta_2},$$

where A and B are constants to be determined and

$$\beta_2 \equiv \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + 2\frac{r}{\sigma^2}}\ (< 0)\,.$$

From the boundary conditions (8b)—(8d), one can derive the closed-form solutions. According to (8d), $B = 0$. By (8b) and (2), we have $v^i(X_i^*; X_i^*) = A(X_i^*)^{\beta_1} = X_i^* \frac{\pi(i)}{\delta} - I$ and so

$$A = (X_i^*)^{-\beta_1}\left[\frac{\pi(i)}{\delta}X_i^* - I\right]. \tag{13}$$

From (8c), it follows that $v_x^i(X_i^*; X_i^*) = A\beta_1 (X_i^*)^{\beta_1 - 1} = \frac{\pi(i)}{\delta}$. By (13),

$$\beta_1 \left[(X_i^*)^{\beta_1 - 1 - \beta_1 + 1} \times \frac{\pi(i)}{\delta} - (X_i^*)^{\beta_1 - 1 - \beta_1}\, I\right] = \frac{\pi(i)}{\delta}.$$

Therefore, $X_i^* = \frac{\delta I}{\pi(i)}\frac{\beta_1}{\beta_1 - 1}$. The value of the strategic firm is obtained by specializing equations (3) and (7). □

Proposition 5 establishes that early entrants into an emerging market are better-off than later entrants. Given that $\pi(\cdot,\cdot)$ depends on the number of firms operating at time t but not on the ordering of entries, firms cannot secure a sustainable first-mover advantage. It is interesting that, although firms are assumed to be symmetric, they receive asymmetric values in equilibrium (open-loop approach).

Proposition 5. *In a Nash equilibrium in investment policies, firms' equilibrium values decline monotonically with their order of entry: an earlier investor is better-off (on average) than a later market entrant. Hence there exist early-mover advantages* $V_0^i(X^*) > V_0^j(X^*)\,, i < j$.

Proof. For two firms i and j, assume a weak ordering of investments with firm i investing earlier than firm j, so that $X_i^* < X_j^*$ or $\tau_i^* < \tau_j^*$. It then follows from the uniqueness of the maximum X_i^* and the definition of the Nash equilibrium in (4) that

$$V_0^i\big(X_i^*, X_j^*, X_{-i,j}^*\big) > V_0^i\big(X_j^*, X_j^*, X_{-i,j}^*\big)\,.$$

Because firms are symmetric, we have

$$\begin{aligned} V_0^i\big(X_j^*, X_j^*, X_{-i,j}^*\big) &= V_0^j\big(X_j^*, X_j^*, X_{-i,j}^*\big)\,, \\ &\geq V_0^j(X^*)\,. \end{aligned}$$

The last inequality follows because firm j is not worse-off if firm i invests later (at $X_j^* > X_i^*$). □

It may seem counterintuitive that symmetric firms receive asymmetric values in Nash equilibrium (with open-loop investment policies) and that firms do not struggle to become the first investor. These results stem frm the assumption that firms act myopically and do not revise their strategy over time. For closed-loop strategies, as discussed next, results may well differ because firms can react to their opponents' moves.

3.2. Market-Entry Sequencing for Closed-Loop Strategies

In the Section 3.1, we assumed that the investment option was available to a large number of firms, so that potential entrants could not observe their rivals' moves. This assumption was sufficient for the Nash equilibrium in investment policies to be a reasonable solution concept and to provide the most accurate prediction concerning the industry dynamics. The optimal investment triggers were selected myopically in that they were not linked to the investment decisions of future entrants (thus, firms behaved as if they were the last investor to enter the market or like a monopolist with a exclusive right to enter). In this section, we look at the problem where (exogenous) entry barriers exist that prevent more than two firms from entering the market.

Next we briefly summarize what happens when firms devise investment strategies that are closed-loop with respect to the filtration of (both) the strategy and the state space, as opposed to strategies that are open-loop with respect to the strategy space.[16]

3.2.1. Markov Perfection in Investment Strategies

From a game-theoretic viewpoint, investment policies (as formulated in Definition 1) are pure strategies whereby players select an investment path $\{a_t^i\}_{t\geq 0}$ in response to the actual development of a market shock but ignoring the industry structure evolution $\{m_t\}_{t\geq 0}$. Fudenberg and Tirole (1985) have shown that, in a deterministic duopoly setting, closed-loop investment strategies cannot be considered in a similar manner (i.e., via pure strategies). Only randomization allows us to give theoretical explanations for certain effects that are heuristically expected, such as preemption or a war of attrition.

[16]This problem has been discussed heuristically in Dixit and Pindyck (1994, Chap. 9) for the case of a multiplicative shock that follows a geometric Brownian motion. Here, we derive a solution for a general shock — not necessarily multiplicative — that follows a time-homogenous Itô process. For a discussion close to ours but employing mixed strategies, see Thijssen, Huisman, and Kort (2002).

In some circumstances or subgames, the pure-strategy formulation for deriving what to do (a binary decision) in each state of the world fails to result in a unique Nash equilibrium in pure actions; this failure is due to the lack of convexity of the instantaneous discrete action set. Hence, "convexifying" the instantaneous action set may help solve this problem by randomization and the formulation of market-entry strategies as behavioral strategies in continuous time. A behavioral strategy specifies a probability distribution $\Delta A^i\left(\tilde{h}_t\right)$ over pure actions $A^i\left(\tilde{h}_t\right)$ for each history path $\tilde{h}_t = (X_t, m_t)$, where the probability distributions for different histories are independent.[17]

Definition 3. *A behavioral market-entry strategy is a decision rule that maps every possible history $\tilde{h}_t = (X_t, m_t)$ to a mixed action $\alpha_t^i \equiv \alpha^i\left(\tilde{h}_t\right) \in \Delta\, A^i\left(\tilde{h}_t\right) \subseteq [0,1]$.[18] The mixed action α_t^i measures the instantaneous probability of investing in the short time interval $[t, t+h]$ in state $\tilde{h}_t$ as h approaches zero.[19]*

Because investment is irreversible, the mixed action set may change over an industry's evolution. Formally, $\Delta A^i(\tilde{h}_t) \in \{1\}$ after firm i's entry. In light of Definition 3, we can pin down the appropriate solution concept — namely, Markov perfect equilibrium.

Definition 4. *A Markov perfect equilibrium (MPE) is a profile of Markov investment strategies $\alpha^* = (\alpha_t^{i*}, \alpha_t^{j*})_{t\geq 0}$ that form a perfect equilibrium; in other words, $(\alpha_s^{i*}, \alpha_s^{j*})_{s\geq t}$ is a Nash equilibrium for all $\tilde{h}_t,\ t \geq 0$.[20]*

We consider next the values of *strategic* firms. The value of a leader investing at time t is

$$L(X_t) \equiv \mathbb{E}_t\left[\int_t^{\tau_F} e^{-r(s-t)}\pi(X_s, 1)\,\mathrm{d}s + \int_{\tau_F^*}^{\infty} e^{-r(s-t)}\pi(X_s, 2)\,\mathrm{d}s - I\right], \tag{14}$$

where X_F^* is the threshold at which the follower enters and $\tau_F^* = \inf\{t \geq 0 \mid X_t \geq X_F^*\}$ is the follower's entry time. The value of the follower (at time t) is

$$F(X_t) \equiv \begin{cases} \mathbb{E}_t\Big[\int_{\tau_F^*}^{\infty} e^{-r(s-t)}\pi(X_s, 2)\,\mathrm{d}s - e^{-r(\tau_F^*-t)}I\Big] & \text{if } X_t \leq X_F^*, \\ \mathbb{E}_t\Big[\int_t^{\infty} e^{-r(s-t)}\pi(X_s, 2)\,\mathrm{d}s - I\Big] & \text{if } X_t > X_F^*. \end{cases} \tag{15}$$

[17] A *behavioral* strategy differs from a *mixed* strategy in that a mixed strategy determines a probability distribution over pure strategies (i.e., over mappings from information sets to actions). A behavioral strategy randomizes, perhaps degenerately, the action of the player in each state. Kuhn's (1953) theorem establishes an equivalence between these two definitions of randomization (mixed and behavioral strategy) under certain conditions. In both cases, the action sets may differ for different histories $\tilde{h}_t$.

[18] Fudenberg and Tirole (1985) and Thijssen, Huisman, and Kort (2002) provide an alternative formulation using mixed strategies in continuous time rather than behavioral strategies. These authors consider a cumulative distribution function representing the cumulative probability that firm i has invested in state $\tilde{h}_t$ or at any previous time, and α_t^i is a "probability of atoms" — a sort of density function. The equivalence between the definitions of mixed strategy and behavioral strategy in such settings is rigorously shown by Touzi and Vieille (2002).

[19] Here α_t^i is the limit (as the time length h approaches zero) of the discrete-time mixed-action measures in strategic-form games. When $\alpha_t^i > 0$, the firm invests immediately and enters the market.

[20] The equilibrium strategy profile must be adapted to the payoff-relevant history filtration. For a formal definition of MPE, see Maskin and Tirole (2001).

Alternatively, the option-holding firms might decide to invest at the same time, which would result in the following value of simultaneous immediate investment:

$$C(X_t) \equiv \mathbb{E}_t \left[\int_t^{\infty} e^{-r(s-t)} \pi(X_s, 2)\, \mathrm{d}s - I \right]. \tag{16}$$

For $t \geq \tau_F$, we have $C(X_t) = F(X_t)$.

Proposition 6. *Given that no firm invests at the outset (Assumption 1), the profile of symmetric market-entry strategies $\alpha^* = (\alpha_t^{i*}, \alpha_t^{j*})_{t\geq 0}$ written as*

$$\alpha^*(x) = \alpha^{i*}(x) = \alpha^{j*}(x) = \begin{cases} 0 & \text{if} \quad x \in (-\infty, X_P^*) & \text{[don't invest for low values]}, \\ \phi(x) & \text{if} \quad x \in [X_P^*, X_F^*) & \text{[mix for intermediate values]}, \\ 1 & \text{if} \quad x \in (X_F^*, \infty) & \text{[invest for high values]}, \end{cases}$$

where

$$\phi(\cdot) = \frac{L(\cdot) - F(\cdot)}{L(\cdot) - C(\cdot)} \in [0, 1] \tag{17}$$

constitutes a unique Markov perfect equilibrium in investment strategies. The thresholds X_F^* and X_P^* are such that the following statements hold.

(i) X_F^* solves the system

$$\begin{aligned} rF - \mathscr{D}F &= 0, && \text{(18a)} \\ F\big|_{x=X_F^*} &= W\big|_{x=X_F^*}, && \text{(18b)} \\ F_x\big|_{x=X_F^*} &= W_x\big|_{x=X_F^*}, && \text{(18c)} \\ \lim_{x\to 0} F(x) &= 0. && \text{(18d)} \end{aligned}$$

(ii) X_P^* solves the system

$$\begin{aligned} rL - \mathscr{D}L &= 0, && \text{(19a)} \\ L\big|_{x=X_P^*} &= W\big|_{x=X_P^*}, && \text{(19b)} \\ L(X_P^*) &= F(X_P^*), && \text{(19c)} \\ \lim_{x\to 0} L(x) &= 0. && \text{(19d)} \end{aligned}$$

Proof. See Appendix II. □

The equilibrium strategy described in Proposition 6 is interpreted as follows. In the period prior to the preemption time τ_P^* there is no incentive for either firm to invest, so both firms stay out ($\alpha_t^i = \alpha_t^j = 0$). In the period immediately after the preemption time τ_P^*, one firm in the industry will invest first. At the optimal preemption time τ_P^*, each firm is indifferent between being the leader and being the follower (i.e., $L(X_P^*) = F(X_P^*)$) and the probability of investment for each firm at the optimal preemption time τ_p^* is given by $\alpha_{\tau_P^*}^i = \alpha_{\tau_P^*}^j = 0$. For t greater than the follower's optimal time of investment, both firms will operate in the marketplace with the second entrant "following suit". The only possible equilibrium is characterized by a sequence in which one of the firms invests at time τ_P^* and the other at a later time $\tau_F^* > \tau_P^*$. The probability of being the leader in the duopoly market is exactly one half. Proposition 7 summarizes these two properties.

Proposition 7. *The Markov perfect equilibrium established in Proposition 6 has the following properties.*

(i) Investment takes place in sequence:

$$X_P^* < X_F^* \qquad \textit{or} \qquad \tau_P^* < \tau_F^*.$$

(ii) Each (symmetric) firm has a one-half probability of being the leader in the duopoly market — that is, of entering at time τ_P^*.

Proof. See Appendix III. □

The discussion so far has concerned a Markov perfect equilibrium in the case of a duopoly. Bouis et al. (2009) consider a similar problem in the case of a larger number of symmetric oligopolistic firms. The authors consider a multiplicative shock that follows the geometric Brownian motion in the context of reduced-form (deterministic) profits $\pi(n)$ that decrease in the number of incumbent firms n. An equilibrium arises where all firms invest sequentially.[21] The investment trigger of market entrants (except for the last one) is determined by "rent equalization".[22] Therefore, the inclusion of more option-holding firms does not critically affect the sequencial occurrence of investments.

4. Social Optimality of Myopic Market-Entry Sequencing

We have seen that oligopolist firms invest in sequence. Next we examine whether the entry decisions taken by firms acting in their own interest may lead to some form of social optimality. To set a benchmark, we consider the investment timing decisions that a central planner would impose on (decentralized) firms.

[21] Bouis et al. (2009) formalize explicitly the three-firm case and provide numerical analysis for larger oligopolies. The authors show that simultaneous investments may also occur if the starting value of the process is large. In our model, Assumption 1 ensures that such an equilibrium does not arise.

[22] The authors also demonstrate the existence of an additional effect that they call the *accordion* effect. In the three-firm case, if the threshold of the third entrant X_3^* rises, then the second investor has an incentive to invest earlier and thus enjoy duopoly rents longer, thereby setting a lower threshold X_2^*. The first investor then faces earlier entry by the second entrant and so enjoys monopoly profits for a shorter time period; its entry threshold is thus increased. The opposite directions of the change in the "wedges" between X_1^* and between X_2^* and X_2^* and X_3^* is at the core of the accordion effect.

Assume the existence of a given investment threshold choice $X = (X_1, \ldots, X_n)$ in the social planner's admissible strategy set $\mathbb{R}^n$. Then the expected social surplus is given by

$$S(X) = \mathbb{E}_0 \left[\int_0^\infty e^{-rs} \Pi \left(X_s, m_s \right) \mathrm{d}s - \sum_{i=1}^n e^{-r\tau_i} I \right], \tag{20}$$

where $\Pi\left(\cdot,\cdot\right) \equiv R\left(\cdot,\cdot\right) - C\left(\cdot,\cdot\right)$ is the flow of social operating surplus. We have $R\left(\cdot, m_s\right) = \sum_{i=0}^{m_s} r(\cdot, i)$ for the flow of consumer surplus (revenues) and $C\left(\cdot, m_s\right) = \sum_{i=0}^{m_s} c(\cdot, i)$ for the flow of total production costs. The relationship between the (net) social benefit and firms' profits is

$$\Pi(\cdot, m_s) = \sum_{i=0}^{m_s} \left[r(\cdot, i) - c(\cdot, i) \right] = \sum_{i=0}^{m_s} \pi(\cdot, i) = \sum_{i=0}^{n} a_s^i \pi(\cdot, i) \,. \tag{21}$$

The social planner would choose investment thresholds to maximize the expected social surplus $S(X)$. Thus, the social planner is faced with a (stochastic) control problem of the form

$$S(X^*) \equiv \sup_{X \in \mathbb{R}^n} S\left(X\right).$$

We consider three cases: (1) oligopoly, (2) duopoly, and (3) large oligopoly.

4.1. Oligopoly

Proposition 8 asserts that myopic firms — such as those pursuing open-loop investment policies — would invest at the time(s) expected by a social planner. Leahy (1993) uses instantaneous (rather than impulse) control to establish an equivalent result in the context of (infinitely divisible) capacity investment. Here we extend that result to binary (lumpy) market-entry decisions.

Proposition 8. *A Nash equilibrium in investment policies (as obtained in Proposition 4) is socially optimal. That is,* $S(X^*) = \sum_{i=1}^n v_0^i(X_i^*)$.[23]

Proof. See Appendix IV. □

4.2. Duopoly

Following the analysis in Section 3.2, we show that a social loss arises when investment options are available to a small number of firms. Proposition 9 establishes that both firms receive lower payoffs under the Markov perfect equilibrium than under the equivalent Nash equilibrium in investment

[23]Whether the sequence of entry is optimal for society as a whole (i.e., both firms and consumers) or for firms considered jointly is a question that cannot be answered unequivocally here. The answer depends partly on how the revenue function $R(\cdot, m_t)$ is determined. If the optimal choice of revenue is made by a social planner who is looking out for the interest of all market participants, then the investment sequence would benefit society as a whole. However, if the planner is concerned solely with the interest of firms, the resulting Pareto optimality might actually be suboptimal from a social-welfare perspective.

policies for two players.[24] Proposition 10 then asserts that the first market entry takes place too early to be socially optimal.

Proposition 9. *In terms of expectations, no firm is better-off in the region $[X_P^*, X_F^*]$ because the expected value of each firm (including the actual leader) is equal to the value of the follower. In other words, rents are dissipated (on average).*

Proof. See Appendix V. □

If firms cannot commit to sticking to their market-entry strategies, then identical firms receive equal expected values (in equilibrium); this result is some form of "rent dissipation". The intuition behind this result is based on the rent-equalization principle determining the preemption point: If one firm planned to enter as leader at a time t in order to receive a greater value than the follower, then the follower could increase its value by preempting and investing just before t, at $t-\epsilon$ for small ϵ. This strategic interplay would be repeated inducing firms to preempt all the way down to the point X_P^*, beyond which there is no advantage to be gained from preempting.[25]

Proposition 10. *The optimal investment times arising from the MPE are ranked as follows,*

$$\tau_P^* < \tau_L^* < \tau_F^*,$$

where $\tau_L^* = \inf\{t \geq 0 \mid X_t \geq X_L^*\}$ *for* $X_L^* \equiv \arg\max_{X_L \in \mathbb{R}} L_0(X_L, X_F^*)$. *The first entrant in the (duopoly) MPE enters earlier, at time τ_P^*, and faces riskier returns and a higher probability of going bankrupt than is socially optimal.*

Proof. The second inequality follows from Proposition 3 in the two-firm case. To derive the first inequality, suppose by way of contradiction that $\tau_P^* \geq \tau_L^*$. The rent-equalization principle implies that $L(X_P^*) = F(X_P^*)$. Since the follower's value is nondecreasing in the shock, we have $F(X_P^*) \geq F(X_L^*)$. Since $F(x) > L(x)$ for $x \in (-\infty, X_P^*]$, it follows that $F(X_L^*) > L(X_L^*)$. Therefore, $L(X_P^*) > L(X_L^*)$, which contradicts the definition of X_L^*. The uniqueness of $\arg\max_{X_L \in \mathbb{R}} L(X_L, X_F^*)$ was established in Proposition 3. □

In short, because myopic investment policies lead to social optimality (Proposition 8), more knowledge of the competition (closed-loop approach) results in social loss.

5. Conclusion

The choice of an appropriate solution approach (open-loop versus closed-loop) is critical to determining equilibria in real options models that involve competition under uncertainty. When firms

[24] The two-player Nash equilibrium in investment policies described by Joaquin and Butler (2000) is derived in a setting where the exchange-rate process is a multiplicative shock and evolves as a geometric Brownian motion, the investment cost is constant over time, and profit functions are obtained in Cournot quantity competition.

[25] The difference here from the deterministic model in which this rent dissipation was first developed (cf. Fudenberg and Tirole, 1985) is that deviation from the expected market development scenario could lead to positive rents earned by the leader in favorable states and to losses if market development falls short of expectations. This is because entry decisions are now based on *expected* market development scenarios.

can observe and react to rivals' actions, the appropriate solution involves closed-loop strategies and a Markov perfect equilibrium. However, deriving the MPE in closed-loop strategies is usually more involved than deriving the Nash equilibrium for the case of open-loop strategies.

If there are no entry barriers and if a large number of firms can enter a growing market (ignoring rivals' moves), then the sequence of investments is Pareto optimal (an open-loop equilibrium). A similar result was shown previously (e.g., Leahy 1993; Baldursson 1998) in a context where firms could invest incrementally in capital stock but without due reference to the assumed information structure. We have shown that this outcome applies to lumpy market-entry investments as well. When only a few firms are protected from new market entry, the sequence of investment is not necessarily Pareto optimal (closed-loop). In this case, Markov perfection implies investment at an earlier time than the timing a social planner would impose on the firms.

For sufficiently large oligopolistic industries, the optimal market-entry strategy can be determined while ignoring potential rivals. The value obtained, however, represents the effect of competition via an additional (negative) competitive erosion term, as proved in Proposition 1. In the case of a duopoly, firms that jointly share an investment option are subject to adverse effects that alter their incentive to erect entry barriers. Restricted availability of the investment option leads to preemption and rent dissipation (on average). Once both duopoly firms enter, the knowledge that no other firm can arrive ensures fairly high rents in the marketplace (with no competitive erosion term). This trade-off calls for concurrent consideration of both the preemption risk that arises in industries with only a few firms and of the endogenous erection of entry barriers to deter further competitive arrivals.

Our modeling approach has proved useful for explaining dynamic market entry timing decisions and strategic interactions in the case of stochastically growing markets characterized by lumpy investments under uncertainty. We have adopted a fairly general approach that covers a large number of stochastic (Itô) processes and many reduced-form profit functions.

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Appendices

Appendix I: Proof of Proposition 4

We first deal with part (i). From Proposition 3, we know that player i has no incentive to select a threshold in $\left(X_{i-1}^*, X_{i+1}^*\right)$ *other* than the X_i^* characterized in Proposition 2. Suppose now that firm i selects an alternative $X_i \in \left[X_{k-1}^*, X_k^*\right],\ k \leq i$, while seeking profitable deviations from the hypothesized Nash equilibrium. Let $\tau_i \equiv \inf\left\{t \geq 0 \mid X_t \geq X_i\right\}$. We have

$$\begin{aligned}
V_0^i\left(X_i, X_{-i}^*\right) = \mathbb{E}_0\Bigg[& \int_{\tau_i}^{\tau_k^*} e^{-rs}\pi(X_s, i)\,\mathrm{d}s + \sum_{m=k}^{i-2} \int_{\tau_m^*}^{\tau_{m+1}^*} e^{-rs}\pi(X_s, m)\,\mathrm{d}s \\
& + \int_{\tau_{i-1}^*}^{\tau_{i+1}^*} e^{-rs}\pi(X_s, i)\,\mathrm{d}s + \sum_{m=i+1}^{n-1} \int_{\tau_m^*}^{\tau_{m+1}^*} e^{-rs}\pi\left(X_s, m\right)\mathrm{d}s \\
& - e^{-r\tau_i} I \Bigg].
\end{aligned}$$

Selecting X_i to maximize this expression leads to solving the system of equations (8a)–(8d). This corresponds to the previous definition of X_i^*. Hence, the value function $V_0\left(\cdot, X_{-i}^*\right)$ increases monotonically on the interval $\left[X_{k-1}^*, X_k^*\right]$ and reaches a maximum at X_k^*.

Similarly, suppose that firm i deviates and selects $X_i \in \left[X_k^*, X_{k+1}^*\right]$ for $k \geq i$. Then

$$\begin{aligned}
V_0^i\left(X_i, X_{-i}^*\right) = \mathbb{E}_0\Bigg[& \int_{\tau_i}^{\tau_{k+1}^*} e^{-rs}\pi\left(X_s, k\right)\mathrm{d}s \\
& + \sum_{m=k+1}^{n-1} \int_{\tau_m^*}^{\tau_{m+1}^*} e^{-rs}\pi\left(X_s, m\right)\mathrm{d}s - e^{-r\tau_i} I \Bigg].
\end{aligned}$$

Maximizing this program yields the system of equations (8a)–(8d), so $X_i^* = X_k^*$. For $X_i > X_i^*$, we have that $V_0^i\left(\cdot, X_{-i}^*\right)$ is strictly decreasing on each interval of the form $\left[X_k^*, X_{k+1}^*\right]$ and reaches a maximum at the left boundary $\tau_k^*,\ k = 1, \ldots, n$. Because the value function is continuous at X_i^* (owing to the value-matching conditions), X_i^* is the unique maximum. For part (ii), observe that any Nash equilibrium implies a weak sequencing of market entries. Hence, X^* and the other permutations of firm labels in $X^* = (X_1^*, \ldots, X_n^*)$ are the only Nash equilibria in market-entry thresholds ($n!$ permutations).

Appendix II: Proof of Proposition 6

We need to show that no firm has an incentive to deviate from the hypothesized MPE of Proposition 6. We shall consider several cases in turn.

	Invest	Wait
Invest	$C^i(\cdot)$, $C_t^j(\cdot)$	$L^i(\cdot)$, $F^j(\cdot)$
Wait	$F^i(\cdot)$, $L_t^j(\cdot)$	$V^i(\cdot)$, $V^j(\cdot)$

Figure 1: Strategic-Form Representation of the Possible Coordination Problem in $\left(X_P^*, X_F^*\right)$

Case 1: $x < X_P^*$. Here $L(x) < F(x)$, so investing is a strictly dominated strategy.

Case 2: $x \in [X_P^*, X_F^*]$. In this case, a coordination problem may arise. The characterization of α^i as the limit (as the time interval approaches zero) of discrete-time mixed action facilitates depiction of the problem in strategic form, as shown in Figure 1. The value of pursuing the mixed action $\alpha_t^i = \alpha^i(x)$ (while the rival invests with probability $\alpha_t^j = \alpha^j(x)$) is

$$\begin{aligned} V^i\left(\alpha^i, \alpha^j\right) &= \alpha^i\alpha^j C^i + \left(1-\alpha^i\right)\alpha^j F^i + \alpha^i\left(1-\alpha^i\right) L^i \\ &+ \left(1-\alpha^i\right)\left(1-\alpha^j\right) V^i\left(\alpha^i, \alpha^j\right), \end{aligned}$$

resulting in

$$V^i\left(\alpha^i, \alpha^j\right) = \frac{\alpha^i\alpha^j C^i + \left(1-\alpha^i\right)\alpha^j F^i + \alpha^i\left(1-\alpha^j\right) L^i}{\alpha^i + \alpha^j - \alpha^i\alpha^j}$$

for $\alpha^i, \alpha^j \neq 0$. Since this expression is concave in α_t^i (i.e., $\frac{\partial^2 V_t^i}{\partial \alpha_t^{i2}}\left(\alpha_t^i, \alpha_t^j\right) < 0$), the following first-order condition is both sufficient and necessary for an optimal mixed action to obtain:

$$\frac{\partial V^i}{\partial \alpha^i}\left(\alpha^i, \alpha^j\right) = 0 \iff \alpha^i(x) = \phi^i(x),$$

where

$$\phi^i(\cdot) = \frac{L^j(\cdot) - F^j(\cdot)}{L^j(\cdot) - C^j(\cdot)}.$$

For identical firms following symmetric behavioral strategies ($\alpha_t^i = \alpha_t^j = \alpha_t$), the preceding equality simplifies to

$$\phi^i(\cdot) = \frac{L(\cdot) - F(\cdot)}{L(\cdot) - C(\cdot)}.$$

It then follows from equations (15) and (16) that

$$
\begin{aligned}
C\left(X_t\right) &= \mathbb{E}_t\left[\int_t^{\infty} e^{-r(s-t)}\pi(X_s,2)\,\mathrm{d}s - I\right] \\
&\leq \sup_{X_F\in\mathbb{R}} \mathbb{E}_t\left[\int_{\tau_F}^{\infty} e^{-r(s-t)}\pi(X_s,2)\,\mathrm{d}s - e^{-r(\tau_F-t)}I)\right] \\
&\leq F(X_t)\,.
\end{aligned}
$$

Case 3: $x \geq X_F^*$. Here the firm has a dominant strategy to invest immediately and receive $C(x)$. For symmetric firms, the value of the leader and of the follower are equal(ized) for process values higher than X_F^*: $L(x) = C(x) = F(x)$.[26] Firm i can do no better than to pursue the prescribed strategy. Firm i is faced with a decision-theoretic problem. Its optimal entry time is jointly determined by the HJB equation (18a), the "smooth pasting" condition (18c), and equation (18d). In this case, the strategy formulated in Proposition 3 is dominant.

Together these results indicate that the strategy profile given in Proposition 6 is a Markov perfect equilibrium whose uniqueness is shown by Fudenberg and Tirole (1985, Apx. A) in the deterministic context.

Appendix III: Proof of Proposition 7

We suppose the contrary and then derive a contradiction. So let $X_P^* \geq X_F^*$. For $X_P^* > X_F^*$, since $F(X_P^*,\cdot)$ is nondecreasing in X_F it follows that $F(X_P^*, X_P^*) \geq F(X_P^*, X_F^*)$, which means that $F(X_P^*,\cdot)$ is maximized for two distinct trigger values, X_F^* and X_P^*. This contradicts the uniqueness of the threshold as established by conditions (8a)–(8d). If $X_P^* = X_F^*$, then both firms enter at time τ_P^* and receive $C(X_P^*)$. Hence there is an investment trigger $X_i \in (-\infty, X_P^*)$ such that $L(X_i, X_P^*) > C(X_P^*, X_P^*)$ as $\pi(X_s,1) > \pi(X_s,2)$. Since firms are identical, $C\left(X_P^*, X_P^*\right) = F\left(X_P^*, X_P^*\right)$. As a result, there is an incentive to invest before the preemption point X_P^* but this contradicts the rent-equalization principle of equation (19c). Thus we have established part (i) of the proposition.

For part (ii), observe that the probability that one of the firms ends up being the leader the first time X_t is in (X_P^*, X_F^*) is given by $p_L^i = \alpha_t^i\left(1-\alpha_t^j\right) + \left(1-\alpha_t^i\right)\left(1-\alpha_t^j\right)p_L^i$; therefore, $p_L^i = \frac{\alpha_t^i(1-\alpha_t^j)}{\alpha_t^i+\alpha_t^j-\alpha_t^i\alpha_t^j}$ for $\alpha_t^i, \alpha_t^j \neq 0$. Then, in the symmetric case, we have

$$
p_L = p_L^i = p_L^j = \frac{1-\alpha_t}{2-\alpha_t}, \tag{22}
$$

[26] In new-market models, $L(x) = F(x) = C(x)$ for all $x \geq X_F^*$.

which admits a right limit at $\alpha_t = 0$. Part (ii) now follows from equations (17) and (19c).

Appendix IV: Proof of Proposition 8

By equation (21) we have

$$\begin{aligned}
\sum_{i=0}^{n} \int_{\tau_i}^{\infty} e^{-rs}\pi\left(\cdot, m_s\right) \mathrm{d}s &= \sum_{i=0}^{n} \int_{0}^{\infty} e^{-rs} a_s^i \pi(\cdot, m_s) \mathrm{d}s \\
&= \int_{0}^{\infty} e^{-rs} \left[\sum_{i=0}^{n} a_s^i \pi\left(\cdot, m_s\right)\right] \mathrm{d}s \\
&= \int_{0}^{\infty} e^{-rs} \Pi\left(\cdot, m_s\right) \mathrm{d}s \qquad (23)
\end{aligned}$$

because $\Pi\left(\cdot, 0\right) = 0$. It now follows from Fubini's theorem and equations (20), (23), and (5) that

$$\begin{aligned}
S_0\left(X\right) &= \mathbb{E}_0\left[\int_{0}^{\infty} e^{-rs}\Pi\left(X_s, m_s\right) \mathrm{d}s - \sum_{i=0}^{n} e^{-r\tau_i} I\right] \\
&= \sum_{i=0}^{n} \mathbb{E}_0\left[\int_{\tau_i}^{\infty} e^{-rs}\pi\left(X_s, m_s\right) \mathrm{d}s - e^{-r\tau_i} I\right] \\
&= \sum_{i=0}^{n} v_0^i\left(X_i\right).
\end{aligned}$$

The proposition is obtained by taking the supremum X^* over $\mathbb{R}^n$.

Appendix V: Proof of Proposition 9

The proof proceeds in two steps: first we derive the probability of certain scenarios; then we calculate the firm's expected value in the considered region as the weighted average of values obtained in each scenario. Here we examine the symmetric case.

The probability that a firm becomes the leader in the region $[X_P^*, X_F^*]$ was obtained in (22) as $p_L = \frac{1-\alpha_t}{2-\alpha_t}$. The probability of simultaneous investment, should the process X be located for the first time in the preemption region $[X_P^*, X_F^*]$, is equal to $p_C = \frac{\alpha_t^i}{2-\alpha_t^i}$. Hence, the value in this region is

$$\begin{aligned}
V^i\left(\alpha^{i*}, \alpha^{j*}\right) &= p_L\left(L+F\right) + p_C C \\
&= \frac{L + \left(1-\alpha^*\right) F - \alpha_t^*\left(L - C\right)}{2-\alpha^*} \\
&= \frac{L + \left(1-\alpha^*\right) F - L + F}{2-\alpha^*} = F,
\end{aligned}$$

where the second equality follows from the probabilities just described and the third equality from equation (17).

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