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Real options and electricity capacity generation

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Abstract

This paper studies capacity expansion for a competitive electricity industry when agents consider investment as an option exercise on a real asset.

The originality of this work is that the electricity price process is endogenously given in an unconventional manner : it solves a multi-technology optimization problem. The direct consequence of this sophistication is that one can not separate the option to invest between technologies. One is forced to evaluate directly the entire expansion plan as a whole. A necessary mathematical tool to work in this direction is singular stochastic control / optimal stopping equivalence often used in real options.

We'll show as one proceeds that the drawback of price internalization is dramatic for interdependent technologies as it prevent for an optimal stopping - stochastic control to hold and, at the same time, makes our ability to prove optimality of myopia less likely.

The addition of a myopia assumption as a remedy to reach an investment criterium is discussed. We motivate the fact that myopia, well known to have been proved optimal in symmetric cases, is likely to be the observed behavior under asymmetries.

A numerical and practical solution grounded on myopia arguments is worked out by combination of analytic treatment and forward Monte Carlo simulations.

Keywords: capacity generation, real options, optimal dispatch

JEL Classification: L11, L94, C61.

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Symbol Glossary

Sets

Ω	:	Set of random events
$\mathbb{L}$	:	Set of demand segments
$\mathbb{K}$	:	Set of technologies
$d(\mathbb{L})$	:	dimension of $\mathbb{L}$
$d(\mathbb{K})$	:	dimension of $\mathbb{K}$

Operation variables

$c(k)$	:	Marginal cost (€/MWh) of technology k, $k \in \mathbb{K}$
$I(k)$	:	Investment cost (€/MW) of technology k, $k \in \mathbb{K}$
$q(k,l)$	:	Production flow (MW) for technology $k \in \mathbb{K}$, demand segment $l \in \mathbb{L}$
$K(k)$	:	Capacity (MW) for technology $k \in \mathbb{K}$
$OMC(k)$	:	Operation and Maintenance cost for technology $k \in \mathbb{K}$
Q	:	Consumption flow (MW) ($Q = \sum_k q_k$)

Economic variables

$Y(t,\omega)$	:	Diffusion process for $\omega \in \Omega$, $\{\Omega, F_t, \mathbb{P}\}$ probability space for $t \in [0,T]$.
μ	:	Drift rate of Y as a GBM
σ	:	Volatility rate of Y as a GBM
$P(Y,Q,l)$	:	Inverse demand function (€/MWh) in section $l \in \mathbb{L}$
ρ	:	annual discount rate

Economic functions

$F(Y(t))$	:	Bellman function (€$_t$) of the expansion plan
$\Psi(Y(t),K)$	:	Welfare flow (€$_t/y$)
$\bar{\Psi}(Y(t),K)$	:	Regression of the welfare flow (€$_t/y$)

1 Introduction

With restructuring of electricity market came competition and higher risks. The generators position, relax and stable once upon a time, became — if not harsh — at least complex : market is no longer protected and there is no more insurance of cost recovery as in the *cost plus system*[1]. The difficulty of exercising in this context is widely increased by uncertainties affecting power market : uncertainty over the demand, over the costs, "subjectives"[2] uncertainties. Having in mind these uncertainties surrounding cash flows of capital in place in the energy sector and regarding irreversibility of those investment, the real options point of view is of considerable interest to determine the construction timing or the technology choice for new power plants.

Various *applied* real options model treat the problem of a single asset investment opportunity, using closed form solutions, binomial lattices or backward Monte Carlo.[3] In all cases is provided the value of an hypothetical additional plant and the optimal policy. They however usually make two critical simplifications :

1. the price process is exogenous : it thus implicitly excludes strategic interaction among firms and impact of new capacities on the price design;

2. assets in place are independent.

While the first assumption may be true for infinitesimal energy maker in a perfectly liquid market — besides the fact that the energy market is nowadays neither competitive nor liquid —, a good price process has at least to take into account the market saturation in capacity at the aggregate level. It has been done in what we will call *stylized* real option literature in capacity expansion models for elastic goods.[4] Concerning the second assumption, it is simply not

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[1] The *cost plus system* was the power pricing regime under regulated monopoly. Producers were protected from external competition, and price regulated to avoid abuses. Price were computed on the basis of incurred costs, and regulators checked that these costs were reasonable.

[2] Uncertainties on environmental policy, impact of dramatic changes of governance, impact of juridical changes, ethical uncertainty and public opinion fluctuations. Though subjective uncertainty may have the worst impact on the value of capital in place in the power sector, it is by definition hardly quantifiable and for that reason always neglected. We will conform to that statement and focus on demand uncertainty.

[3] Closed form solutions are used e.g. by Siddiqui and Fleten[33] and Näsäkkalä and Fleten[13],[14] via calculus on spark spread options.

Binomial lattices are used in Abadie and Chamorro[1],[2]. Tseng and Barz[36] and Gardner and Zhuang[16] showed programming skills in this direction by implementing technical constraints.

Monte Carlo using Longstaff and Schwartz[26] procedure includes Abadie and Chamorro[3].

All these applications consider the option to invest in a precise asset, independently of the pre-existing park. The aim is project valuation, and not a capacity expansion planification, as in stylized real options models like Pindyck[30], He and Pindyck[18], Bertola[7], Leahy[24], Baldursson and Karatzas[5], Grenadier[17] and Aguerrevere[4].

[4] This stream of theoretic real options developed after works of Pindyck[30] and Bertola[7], and connected to game theory and equilibrium under uncertainties with Leahy[24] and Grenadier[17].

true that operating units are independent in value. Operation research in energy networks adhere strongly to — and emerge from — this point of view.

An example will clearly highlight this point. Consider a market composed of 3 units : 1 coal fired power plant, 1 combined cycle gas turbine(CCGT) and 1 open cycle gas turbine(OCGT). Traditionally, the marginal cost(MC) of the coal plant is lower than the marginal cost of the CCGT, itself lower than the MC of the OCGT. If one is considering the option to increase capacity in one of these 3 units, what is the effect of an increase of the MC of the coal plant on the investment policy?

An approximation — widely used in *applied* R.O. literature — is to say that the values of the three plants are independent, and that an increase of the cost of the coal will decrease the value of the coal plant but let the value of the two other plants constant. First, it is dangerous : the conjunction of any policy emerging of such a belief with both construction lags and indivisibilities — two features of real life investment that are sure to play role — will lead to disastrous overbuildings.[5] Second, it precisely does not seem adequate for power : the conjugacy of non storability and inelasticity really make that slight variations of capacity can profoundly affect tightness of a system, along with the probability of price spikes. Adding construction lags and indivisibilities to that approximation will dangerously lead to boom and bust behaviors.

Operation research in energy economics pretends that the values of the three plants are not independent, that an increase in the MC of the coal plant will reduce its value but increase the value of the two other plants. This is a purely intuitive appreciation of the value of the plant based on *load factor* variation for each technologies. The general advantage of an optimization formulation is that it can handle a global view of the system.[6]

Capacity expansion models in *stylized* R.O. literature are grounded on the argument that there is no individual values for plants but just one single value : the value of the project. This value has two components : the capital in place — the three existing assets — and the growth options in these three technologies. The value of the capital in place, the optimal expansion plan and the option value are determined simultaneously as the solution of a — rather complex — optimal control problem. The advantage of this dry mathematic formulation is — however — to allow an endogenous treatment of the price process (see e.g. Leahy[24], Baldursson and Karatzas[5] and Grenadier[17]).

This paper try to bound the classic real option theory, initiated by Myers[29], Bernanke[6], and Kester[23] and mathematically stylized by Brennan and Schwartz[8] and McDonald and Siegel[28] to the more practical case of electricity. It would be very interesting to apply general — rather theoretic — results as the optimality of myopia introduced by Leahy[24] and extended by Grenadier[17] to investment in electric physical assets.

This work will be structured as follows. Section 1 introduces the long run capacity expansion problem faced by the social planner. Section 2 gives the

[5] System dynamic literature on Energy investment pointed out the importance of forward-looking. See Ford[15], Bunn and Larsen[9],[10] and the literature therein.

[6] We will always refer to Stoft[35] when referring to O.R. energy economics arguments.

immediate profit of the social planner as a solution of a linear program. Section 3 establishes the investment trigger and comment the results. Section 4 treats a two technologies example and Section 5 concludes.

2 An industry expansion formulation

The conception of this model followed certain objectives. Precisely, an ambitious capacity expansion model for power would have the three following characteristics :

1. an endogenous price process;
2. demand uncertainty;
3. a direct evaluation of the expansion plan as a whole.

The nature of competitive industry allows us to achieve these three goals simultaneously : the competitive equilibrium in a market submitted to a random Markovian shocks is given by the solution of a unique optimal control problem : the maximization of the cumulated welfare. This result due to Lucas and Prescott[27] is a convenient link between a hardly handleable equilibrium problem and a more convenient dynamic optimization problem.[7]

The main purpose of this section is the formulation of the planner dynamic optimization problem. The benchmark case in this situation is an elastic demand and an homogenous good (see Dixit and Pindyck[11], chapter 9, sections 1.A and 1.B for a complete treatment of this case). Unfortunately, this benchmark is dangerous as power has a subtile nature : it is neither homogeneous nor really elastic.

2.1 Power

Power is a differentiated product. Power at different times is not desired by the same type of customers, and especially not for the same use. Two extremes in this distinction on power is the peak load during certain working hours for the heavy industry, and base load at night for private consumption. In this paper, we indeed choose to differentiate power by load states. The main input of the following analysis is the *load duration curve* of the power industry.

Power is quasi-inelastic. The inverse demand curve associated to each load state is almost vertical.

This point is critical as we are forced — in order to conduct a long term investment analysis — to use a welfare optimization problem i.e. to assume that

[7] Note that to increase the realism of our model, one should improve the technologic description with fixed size investments, uncertain operation costs and other technical constraints. Discrete investment are not amenable to simple investment rule, operation costs uncertainty enter in the course of dimensionality for optimal stopping.

there is a well defined elasticity for power.[8]

Our point of view is that power is nowadays still inelastic : private consumers do not have access to real time prices, and professionals are usually tight in delivery schedule.

Power can be produced via several technologies. Nuclear and coal plants are historically the two main base load generators. Gas showed recent attentions as a middle base generator (CCGT plants) and peak load generator (OCGT plants) due to relative low gas prices at the end of the eighties, an important decrease of the construction and installation costs for theses plants and the practical comfort of their relatively modest sizes. Greenhouse gas emissions constraints reinforced this trend and paved the way to green technologies. Among them, wind turbine is surely nowadays the most popular.

Real options models applied to power generation usually focus on the investment in a fixed size asset of a given technology, assuming an exogenous power price process. In doing so, they assume that the value of this future asset is independent of the value of already existing ones.

These models have the advantage that they provide the financial value of the plant in addition to the optimal strategy. The drawback is that synergies and incompatibilities between different technologies are never taken into account. For instance, it is well known that coupling wind farm and hydro with pump storage increases the value of the wind farm — or any other plant —, as potential wind overproduction can be stored in hydro installations.

As well, the choice of an exogenous price process deforms the fundamental structure of power systems : price spikes do not arise purely randomly. Their frequency reflect the *tightness of the system.* This tightness depends on capital already in place and can be profoundly affected by the addition of a single new asset.

2.2 The model

The ground of our model follows Leahy[24], namely, we consider perpetual options exercise in perfect competition. Investment is irreversible. There is at least three differences with Leahy[24] or Baldursson and Karatzas[5] :

1. Technologies may also differ by their investment costs.
2. We treat power as a differentiated good.[9]
3. The profit flow is not given in closed form but as the solution of a linear program.

[8] There is no accepted consensus on what should be the power demand elasticity. In Lijesen[25], elasticities varies from -0.04 to -0.00001, which makes a huge difference in the conclusion of the long term analyses. This difficulty in calibrating demand for power will highlight in the result sections.

[9] The trick to achieve this goal is a piecewise constant approximation of the load duration curve. It's usage is widespread O.R. energy economics literature. It is linked to the unit commitment problem. See for instance Ehrenmann and Smeers[12] for an application, and Smeers[34] (appendix A.1) for an overview of optimal dispatch and unit commitment models.

Observe that 1 is really an important improvement : assuming a single and same investment cost for all technologies greatly simplifies the problem, even in conjunction with 2 and 3 : one falls back on capacity expansion in competitive industry with technologies differing only by operation costs, a problem treated in Dixit and Pindyck[11] (chapter 9, section 1.B) : one invest in the various technologies in quantities such that marginal cost is equalized across firms. This problem set up is the only multi-technology capacity expansion problem with non additively separable profit that allows an analytical solution. Baldursson and Karatzas[5] treat in full mathematical rigor — in a one technology setting — the optimality of myopia result stated by Leahy[24], and note that the myopia result is easily extended to technology having different operation costs, if they share the same investment cost.

The point 2 is suggested by the nature of power and was discussed earlier.

The point 3 is not an innovation as such : small program were used at least since Brennan and Schwartz[8], Pindyck[30] and He and Pindyck[18]. However, it is the first time that the program complexity[10] does not allow an explicit calculation of the profit flow. Note on that point that the authors were not looking for a mathematical challenge, but a more realistic description of power market.

We first define the complete structure of the model. Then we give more informations on parts that may need enlightenment.

Model Assumptions 1. *We propose the following increasing capacity model.*

1. *Fundamental structure.*
   (a) *power market is competitive;*
   (b) *investment options have infinite lifetime;*
   (c) *there is one single risk factor;*
   (d) *investment is irreversible.*
2. *Demand. Let $\mathbb{L}$ be the set of sub-periods for which the load level is supposed constant.*
   (a) *$\tau(l)$, $l \in \mathbb{L}$ is the duration of these sub-periods in hour s.t. $\sum_{l\in\mathbb{L}} \tau(l) = 8760$.*
   (b) *$P(Y, Q, l)$, $l \in \mathbb{L}$ is the power price in these sub-periods in €/MWh, with Y a stochastic process and*

$$P(Y, Q, l) = Y(t, \omega)A(l) - b(l)Q(l). \qquad (1)$$

3. *Supply. Let $\mathbb{K}$ be the set of available technologies.*[11]

[10] Complexity is however a relative point of view : in convex optimization, the optimal dispatch we use in Problem 1 is a quadratic program, solved relatively easily by interior points methods.

[11] E.g. $\mathbb{K} = \{$nuclear, coal, ccgt, ocgt, wind farm$\}$ for a model with these five technologies.

(a) $K_t(k)$, $k \in \mathbb{K}$ *is the capacity level in MW for plant of type* k *during the year* t.

(b) $q_t(k,l)$, $k \in \mathbb{K}$, $l \in \mathbb{L}$ *is the production in MW for plant of type* k *during period* l *of the year* t *s.t. the dispatch constraints*

$$q_t(k,l) \leq K_t(k) \qquad k \in \mathbb{K}, l \in \mathbb{L} \tag{2}$$

$$\sum_{k\in\mathbb{K}} q(k,l) = Q(l) \qquad l \in \mathbb{L}. \tag{3}$$

(c) $c(k)$, $k \in \mathbb{K}$ *is the operation cost for technology* k *in €/MWh.*

(d) $I(k)$, $k \in \mathbb{K}$ *is the investment cost for technology* k *in €/MW.*

(e) $OMC(k)$, $k \in \mathbb{K}$ *is the operation and maintenance cost for technology* k *in €/MWy.*

The only point that needs explanations at that stage is the demand specification.

Within a year, the system is entirely characterized by installed capacities, costs and the load duration curve. We work on a piecewise constant approximation of the load duration curve to find a set of demand function : call $\mathbb{L}$ the set of sub-periods for which the load level is supposed constant and $\tau(l)$, $l \in \mathbb{L}$ the duration of these sub-periods in hour (Fig. 1). The idea is then to use a different inverse demand curve for each demand segment : it is a simple way to distinguish power by degree of desirability (Fig. 2).

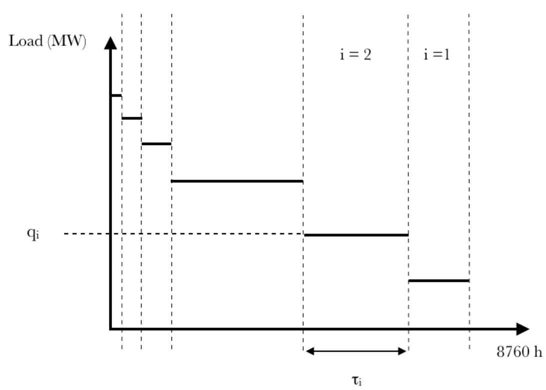

Fig. 1: Load Duration Curve for Card($\mathbb{L}$) = 6.

We choose a linear inverse demand curve where uncertainty is introduced via a geometric Brownian motion $Y_t(\omega)$ starting in 1.

$$P(\omega, Q, l) = Y_t(\omega)A(l) - b(l)Q(l) \tag{4}$$

with $Y_0 = 1$.

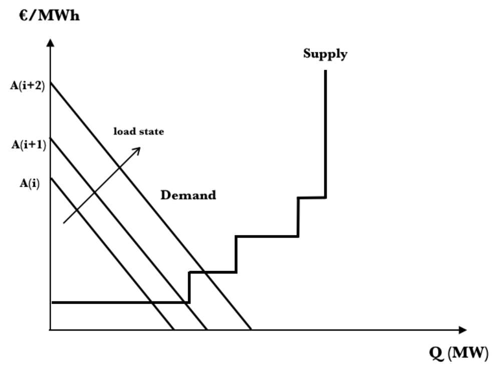

Fig. 2: Supply and demand curves.

For each l, the initial calibration of the demand function needs one observed point $(\bar{P}(l), \bar{Q}(l))$ and the measured elasticity $\bar{E}(l)$. One solves for $A(l)$ and $b(l)$ the simultaneous equations :

$$\left\{ \bar{P}(l) = A(l) - b(l)\bar{Q}(l), \quad \bar{E}(l) = \frac{\bar{P}(l)}{\bar{P}(l) - A(l)} \right\}.$$

After time 0, there is two observed processes : $\bar{P}_t(l)$ and $\bar{Q}_t(l)$ respectively annual average values of prices and loads. The shift parameter $\bar{Y}_t(l)$ for demand segment l is estimated by

$$Y_t(l) = \frac{\bar{P}_t(l) + b(l)\bar{Q}_t(l)}{A(l)} \tag{5}$$

then averaged over l using weights $\tau(l)$.

We are now ready to compute the profit flow of the social planner. Note that since the time unit is the year, the profit flow is the annual welfare, or welfare flow in €/y.

2.3 The profit flow of the social planner

The originality of this paper is the use of a program to compute the profit flow of the social planner. The program we use in the remaining is a welfare optimization version of the unit commitment problem.

Problem 1 (The profit flow of the social planner)**.** *In the following, we call* $\Psi\big(Y, K\big)$ *the social planner profit flow. We choose it to be the solution of the* *program*

$$\Psi\big(Y,K\big) \quad \equiv \quad \max_q \sum_{l\in L} \tau(l)\bigg\{ \int_0^{Q(l)} P(Y,q,l)dq - \sum_k c(k)q(k,l)\bigg\} \\ - \sum_{k\in\mathbb{K}} OMC(k)K(k) \tag{6}$$

$$s.t. \qquad 0 \le q(k,l) \le K(k) \qquad \forall k,\, \forall l \tag{7}$$

$$s.t. \qquad \sum_{k\in K} q(k,l) = Q(l) \qquad \forall l. \tag{8}$$

We check easily that $\Psi(Y,K)$ is a welfare flow in €/y.[12] As there is no explicit time dependence, we will avoid any subscript t from this point.[13]

Note that Y appears in $P(Y,q,l)$ and that K impacts the objective function also through the capacity constraints. When Y and K are given, the program is numerically solved to obtain the value function and the annual strategy. As the demand function is linear, the used software has to handle quadratic programs.[14]

Moreover, note that the Problem 1 is convex. This point is of considerable importance in the choice of the method to solve the long term problem. Precisely, in this case, one can show that the value function $\Psi(Y,K)$ is concave in K.

Proposition 1. *The immediate profit $\Psi(Y,K)$ is a concave function of K.*

Proof. See Appendix B. □

It is important : the function $\Psi(Y,K)$ may be non-differentiable at some points, but it is at least concave in K. Therefore, any regular approximation $\bar{\Psi}$ of Ψ should have the property of being concave in K which is an important property regarding real option theory. In particular, certain conditions on the profit flow function are necessary (but not sufficient) to a *singular stochastic control - optimal stopping equivalence*. These necessary conditions are summarized in Proposition 2.

Proposition 2. *It is necessary for any regression function $\bar{\Psi}(Y,K)$ to be $\mathcal{C}^2$ in Y and $\mathcal{C}^1$ and concave in K for a singular stochastic control - optimal stopping equivalence to hold.*

Proof. See appendix C. □

[12] The time sub periods $\tau(l)$ are in h; $P(Y,q,l)$ and $c(k)$ are in €/MWh; OMC is in €/MWy; q, dq and K are in MW thus $\Psi(Y,K)$ is thus an annual welfare in €, or the welfare flow in €/y.

[13] In general, one may have a profit flow given by $\Psi\big(Y(s),K(s),s\big)$ i.e. explicitly time dependent. Here, as the horizon is infinite, the problem is time homogeneous. There is just an implicit time dependence through processes Y and K; we can therefore simply note $\Psi(Y,K)$.

[14] We use Matlab 7.4 and the function *quadprog*.

Once we have found this "fine enough" fit $\bar{\Psi}$, one can formulate the social planner optimal control problem.

2.4 The social planner's objective

The aim of this section is to determine the optimal behavior of the social planner, based on a fit $\bar{\Psi}$ of the profit flow Ψ.

The unit time of the expansion plan is the year; ρ is the annual discount rate. We call $F(Y)$ the value function (or Bellman function) of the energy expansion plan. $F(Y)$ is the solution of the following optimal control problem.

Problem 2 (The social planner's problem)**.** *Find the value function $F(Y, K)$ and a non-decreasing and left-continuous process $K(v)$, $v \geq s$ such that*

$$F(Y,K) = \max_{K(v), v\geq s} \quad \mathbb{E}\Big[\int_t^{+\infty} \Psi\big(Y(s), K(s)\big) e^{-\rho s} ds - \\ \sum_k \int_t^{+\infty} I(k) e^{-\rho s} dK(k,s) \,\Big|\, Y_t = Y, K_t = K \Big]$$

with $\Psi\big(Y(s), K(s)\big)$ the solution of the optimal dispatch problem given by Problem (1).

This formulation is general, that is, one may assume Ψ in closed form or numeric. We are in the latter case in our model : we only have a collection of points of the welfare.

The reader should moreover note that the stochastic control problem we face is non-trivial. The control process K_s is constrained to be increasing. The mathematical field related to such problems is *singular stochastic control* where the term *singular* comes from the fact that optimal control of such problems is usually of a *bang bang* type. Direct determination of the solution is difficult, but the bang bang nature of the control suggests a link to *optimal stopping*. Usually, mathematicians begin by working on a related optimal stopping problem, then prove that the solution of this optimal stopping problem can be integrated to find the value function and the optimal control of the singular stochastic control problem. This link is of considerable interest as related optimal stopping problems are usually easier to solve.[15] It is however important to realize that this equivalence has been proved only for particular cases, each involving one control variable and one uncertainty.[16] There is therefore in this field a curse of dimensionality that makes the problem hard to deal with.

We do not elaborate on the number of uncertainties as our model includes a single shock process. We rather discuss on the number of control variables.

With a single technology, we can find a solution in closed form : the equivalent optimal stopping problem we face is integrated to find the value of the

[15] Moreover, optimal stopping theory is more topologized than stochastic control theory, and then more amenable to existence proofs.

[16] See Karatzas and Shreve[20],[21] and El Karoui and Karatzas[22].

project (see Bertola[7], Pindyck[30],[31] and the Karatzas and Shreve[20] *monotone follower* problem in related stochastic control literature).

With two technologies, there is no mathematical results defining a certain singular stochastic control - optimal stopping equivalence. The only way out is the distinction of two types of problems :

1. Problems with additively separable[17] profits : they are easily handled as one can separate the project in option exercise in each technology. Mathematically, one use an optimal stopping - singular stochastic control equivalence for each control variable.

2. Problems with non additively separable profit : we have no theory to rely on.

The distinction separable *vs.* non separable profit has to be made when computing a regression $\bar{\Psi}$ of the welfare Ψ. If we allow the fit to be non additively separable, one can not find a trigger unless a myopia additional assumption.

3 Elaboration of a solution

He and Pindyck[18] note that if the profit flow is separable one can write the investment opportunity as a sum of investment opportunities on each technology. We thus have two options :

1. Either we find the better fit $\bar{\Psi}$ for the function Ψ. A better fit implies cross terms and is not additively separable. Then we can not solve the investment problem except with an additional myopia assumption.

2. Or we find an additively separable fit $\bar{\Psi}$. Then we can solve analytically the investment problem for each technology.

In this part, we discuss these two options. We assume that the Ito diffusion Y driven the demand is a geometric Brownian motion

$$dY(t,\omega) = \mu Y dt + \sigma Y dB_t(\omega) \tag{9}$$

to achieve analytical results. A perpetual american call option on an underlying described by this dynamic will have the form AY^{β_1} with β_1 the only positive root of the quadratic[18] :

$$\mathcal{Q}(\beta) = \frac{1}{2}\sigma^2\beta(\beta-1) + \mu\beta - \rho. \tag{10}$$

It is however not a restriction : the same procedure applies to other diffusions or to jumps.

[17] One can formally identify the profit of each technology in the expression of the profit flow.

[18] The positive root is moreover greater than 1 and tends to 1 as σ increases. See Mc Donald and Siegel[28].

3.1 A separable welfare flow

In this part, we assume an additively separable fit $\bar{\Psi}$ of the welfare function. Assuming a geometric Brownian shock affecting the demand, the singular stochastic control - optimal stopping equivalence takes the following form.

Definition 1 (The social planner's optimal stopping problem)**.** *Assume $Y(t,\omega)$ a geometric Brownian motion described by the stochastic differential equation (9). Given the current capacities $K(k)$ ($k = 1\dots d(\mathbb{K})$), find the value of the project $F(Y,K)$ and the $d(\mathbb{K})$ triggers $Y_k^\star(K)$ ($k = 1\dots d(\mathbb{K})$) such that*

$$\mu Y\frac{\partial F}{\partial Y}(Y,K) + \frac{1}{2}\sigma^2 Y^2\frac{\partial^2 F}{\partial Y^2}(Y,K) + \bar{\Psi}(Y,K) - \rho F(Y,K) = 0$$
$$\textit{for} \qquad Y \leq Y_k^\star(K),\ \forall k \in \mathbb{K}$$

and

$$\frac{\partial F}{\partial K_k}\Big(Y_k^\star(K), K\Big) = I_k \qquad \forall k \in \mathbb{K} \tag{11}$$

$$\frac{\partial^2 F}{\partial K_k \partial Y}\Big(Y_k^\star(K), K\Big) = 0 \qquad \forall k \in \mathbb{K}. \tag{12}$$

Proof. See He and Pindyck[18] page 584, equation (14) with the assumptions that there is a single uncertainty and that the profit flow is additively separable. □

On the base of this separable formulation, we propose the following regression model.

Definition 2 (An additively separable regression model)**.** *We propose an interpolation of the profit flow Ψ by the additively separable function $\bar{\Psi}$ given by*

$$\bar{\Psi}(Y,K) = \sum_{k=1}^{d(\mathbb{K})} \sum_{i,j=1}^{d(\gamma),\,d(\alpha)} b_{k,ij} Y^{\gamma_i} K_k^{\alpha_j} - \sum_{k=1}^{d(\mathbb{K})} OMC(k)K(k) \tag{13}$$

with γ and α respectively positive base vectors of dimension $d(\gamma)$ and $d(\alpha)$ such that $\forall i,\ 0 < \gamma_i < \beta_1$; $\forall j,\ 0 < \alpha_j \leq 1$; $b \geq 0$ and β_1 the positive root of the fundamental quadratic $\mathcal{Q}(\beta)$ defined in (10).

The motivation of this formulation is straightforward : one can find the particular integral of each term of this interpolation. The homogeneous solution of the differential equation (18) is well known : $F_h(Y,K) = A(K)Y^{\beta_1}$ with β_1 the positive root of $\mathcal{Q}(\beta)$. It is also well known in the literature that if $\gamma_i \geq 0$ satisfies the condition $\rho - \mu\gamma_i - \frac{1}{2}\sigma^2\gamma_i(\gamma_i - 1) > 0$ — which turns out to say that $\mathcal{Q}(\gamma_i) < 0$ i.e. that $0 \leq \gamma_i \leq \beta_1$ for all i — , then if $\Psi(Y,K) = b_{t,ij}Y^{\gamma_i}K_t^{\alpha_j}$ (resp. $c_{ut,ijk}Y^{\gamma_i}K_t^{\alpha_j}K_u^{\alpha_k}$), the particular integral of

(18) is $F_p(Y,K) = \bar{b}_{t,ij}(\gamma_i) Y^{\gamma_i} K_t^{\alpha_j}$ (resp. $\bar{c}_{ut,ijk} Y^{\gamma_i} K_t^{\alpha_j} K_u^{\alpha_k}$) with

$$\bar{b}_{t,ij}(\gamma_i) = \frac{b_{t,ij}}{\rho - \mu\gamma_i - \frac{1}{2}\sigma^2\gamma_i(\gamma_i - 1)} \tag{14}$$

$$\left(\text{resp.} \quad \bar{c}_{ut,ijk}(\gamma_i) = \frac{c_{ut,ijk}}{\rho - \mu\gamma_i - \frac{1}{2}\sigma^2\gamma_i(\gamma_i - 1)}\right). \tag{15}$$

Note that it is compulsory that every power of Y is lower than β_1.

The solution of the Bellman partial differential equation of the generation plan is :

$$F(Y,K) = A(K)Y^{\beta_1} + \sum_{k=1}^{d(\mathbb{K})} \sum_{i,j=1}^{d(\gamma),\, d(\alpha)} \bar{b}_{k,ij}(\gamma_i) Y^{\gamma_i} K_k^{\alpha_j} - \sum_{k=1}^{d(\mathbb{K})} \frac{OMC(k)K(k)}{\rho}$$

and as the welfare flow is now additively separable, one can write :

$$F(Y,K) = \sum_{k=1}^{d(\mathbb{K})} F_k(Y, K_k)$$

with

$$F_k(Y,K_k) = A_k(K_k)Y^{\beta_1} + \sum_{i,j=1}^{d(\gamma),\, d(\alpha)} \bar{b}_{k,ij}(\gamma_i) Y^{\gamma_i} K_k^{\alpha_j} - \frac{OMC(k)K(k)}{\rho}$$

where $F_k(Y,K_k)$ is interpreted as the value of the project to invest in technology k. The first term $A_k(K_k)Y^{\beta_1}$ is the option value of capacity expansion in technology k and the second term $\sum_{i,j=1}^{d(\gamma),\, d(\alpha)} \bar{b}_{k,ij}(\gamma_i) Y^{\gamma_i} K_k^{\alpha_j} - OMC(k)K(k)/\rho$ is the value of installed capital for the same technology. One can solve the problem independently for each technology (for each $k \in \mathbb{K}$) : it is just the simplest case treated by Bertola[7] and Pindyck[30].

Proposition 3 (Investment trigger for the separable case)**.** *The investment trigger for technology k is given by :*

$$\sum_{i,j=1}^{d(\gamma),\, d(\alpha)} \left\{ \alpha_j \bar{b}_{k,ij}(\gamma_i) Y^{\gamma_i} K_k^{\alpha_j - 1} \left(\frac{\beta_1 - \gamma_i}{\beta_1} \right) \right\} = I_k + \frac{OMC(k)}{\rho} \tag{16}$$

Proof. See Appendix D. □

Appendix D shows that $Y_k^\star(K_k)$ defined by (16) is uniquely defined as an increasing function of K_k. It is sufficient to our purpose : one is not able to find a closed form solution for $Y_k^\star(K_k)$, but in real time one can observe $Y(t)$ and check if (16) holds or not. The same Appendix provides additional conditions under which $A(K_k)$ is well defined.

One should remark for sake of completeness that though solution procedure offered in Appendix D seems the more direct, it is not the quickest way to achieve the result. A quite sophisticate result due to Baldursson and Karatzas[5] ensures *myopia* in competitive equilibrium. This result applied to investment opportunity in each technology would have lead directly to this solution.[19]

3.2 A non separable welfare flow

As a solution of a mathematical program, the profit flow is not separable : the growth option of the entire expansion plan is not the sum of individual growth option on each technology. The exercise of an investment option in a given technology actually depreciate investment options on other technologies. There is therefore a high likelihood that a precise fit of the profit flow requires a regression model with cross(interactions) terms.

Here arises a theoretical complication. If the profit flow of the singular stochastic control problem faced by the social planner is not additively separable, one has no theory to rely on. It is likely — this intuition is based on the work of Baldursson and Karatzas[5] — that a singular stochastic control/optimal stopping equivalence can not emerge from this twisted case. And since myopia rely on this equivalence, it is *a fortiori* improbable to find out a myopia result in this context.

However, we think that practitioners use myopia as a proxy for the true optimal behavior under uncertainty, first because it can be solved in any situation, and second because it is unquestionably true in benchmark — symmetric — cases i.e. perfect competition (Leahy[24] and Baldursson and Karatzas[5]) and symmetric oligopolies (Grenadier[17]). For these two reasons, equilibrium under myopic behavior worth's to be interested in.[20]

Having in mind the use of *myopia* as an assumption and using known particular solution of the Bellman equation, we propose the following regression model.

Definition 3 (A non additively separable regression model)**.** *We propose an interpolation of the profit flow* Ψ *by the non additively separable function* $\bar{\Psi}$

[19] The reader can check that by using Appendix E with $\bar{b}_{kt,ijl} = 0$ for all k, t, i, j, l.

[20] Myopic actions as proved to be the solution in competitive industry in symmetric settings. Leahy[24] proved in a single technology model that myopic behavior is optimal in perfect competition under uncertainty. In Dixit and Pindyck[11] and Baldursson and Karatzas[5], this result is extended to several technologies with the condition that they share a same investment cost. To our knowledge, such a result does not exist in a multi-technology setting with different investment cost.

given by

$$\bar{\Psi}(Y,K) = \sum_{t=1}^{d(\mathbb{K})} \sum_{i,j=1}^{d(\gamma),d(\alpha)} b_{t,ij} Y^{\gamma_i} K_t^{\alpha_j}$$

$$+ \sum_{\substack{t,u=1 \\ u \neq t}}^{d(\mathbb{K})} \sum_{i,j,k=1}^{d(\gamma),d(\lambda),d(\lambda)} c_{ut,ijk} Y^{\gamma_i} K_t^{\lambda_j} K_u^{\lambda_k} - \sum_{t=1}^{d(\mathbb{K})} OMC(t)K(t) \qquad (17)$$

with γ, α and λ respectively positive base vectors of dimension $d(\gamma)$, $d(\alpha)$ and $d(\lambda)$ such that $\forall i,\ 0 < \gamma_i < \beta_1$; $\forall j,\ 0 < \alpha_j \leq 1$; $\forall k,\ 0 < \lambda_k < 1$; with the restriction that any interaction terms should pick 2 powers λ_1 and λ_2 such that $\lambda_1 + \lambda_2 \leq 1$; $b \geq 0$; and β_1 the positive root of the fundamental quadratic $\mathcal{Q}(\beta)$ defined in (10).

The motivation of this formulation is given in motivation of Definition 2. At this stage, we make the assumption that competitive agents are myopic i.e. that they assume no future entry after their own investment. They therefore solve an optimal stopping problem on the only Y variable, assuming the industry output is fixed forever at the current level. This assumption simplifies the problem : as the future capacity is assumed static, one can find the value of the next unit as a function of the shock price process only, and find a so called *myopic* strategy. Each agent solves an American perpetual call on a next marginal unit, assuming that the termination value of this unit is computable — since it is assumed that no further control will be exercised in any direction — as the expectation of its future cash flows.

Definition 4 (Myopic behavior of each agent)**.** *The optimal stopping problem solved by the myopic agent is : given the current capacities in place $K(k)$ ($k = 1 \dots d(\mathbb{K})$), find for each $k \in \mathbb{K}$ the values of marginal unit sized next units $f_k(Y)$ and triggers $Y_k^\star$ such that*

$$\mu Y \frac{\partial f_k}{\partial Y}(Y) + \frac{1}{2}\sigma^2 Y^2 \frac{\partial^2 f_k}{\partial Y^2}(Y) - \rho f_k(Y) = 0 \qquad \forall\, Y \leq Y_k^\star, \qquad (18)$$

and

$$f(Y_k^\star) = m_k(Y_k^\star) - I_k \qquad (19)$$

$$\frac{\partial f}{\partial Y}(Y_k^\star) = \frac{\partial m_k}{\partial Y}(Y_k^\star) \qquad (20)$$

$$m_k(Y) = \mathbb{E}\left[\int_0^{+\infty} \frac{\partial \bar{\Psi}}{\partial K_k}(Y_s, K) e^{-\rho s} ds \middle| Y_0 = Y \right]. \qquad (21)$$

The myopia assumption is mathematically expressed by the very definition of the function $m_k(Y)$ in (21) : it is stated as a function of the sole shock Y because K is assumed forever constant in the computation of the right hand integral. In this expression K is indeed considered as a parameter, not as a variable. Proposition 4 gives optimal investment triggers under myopia assumptions.

Proposition 4. *The myopic investment trigger $Y_k^\star(K)$ for technology $k \in \mathbb{K}$ when capacities in places are $K(k)$ is given by*

$$\sum_{i=1}^{d(\gamma)} Y^{\gamma_i}\left(\frac{\beta_1-\gamma_i}{\beta_1}\right)\left\{\sum_{j=1}^{d(\alpha)} \alpha_j \bar{b}_{k,ij}(\gamma_i) K_k^{\alpha_j-1} + \sum_{\substack{t=1\\ t\neq k}}^{d(\mathbb{K})} \sum_{j,l=1}^{d(\lambda),\, d(\lambda)} \lambda_k \bar{b}_{kt,ijl}(\gamma_i) K_t^{\lambda_j} K_k^{\lambda_l-1}\right\} = I_k + \frac{OMC(k)}{\rho}. \quad (22)$$

Proof. See Appendix E. □

One needs to discuss whether the choice of an additively separable welfare flow is adequate. The welfare flow solves an optimization problem. As we said earlier, a fine approximation of this welfare surely demands interaction terms, as technologies having different characteristics are not just substitutable : they may interact positively or negatively. Seen this way, it does not appear judicious to occult interaction of the formulation.

However, the separable approximation is a projection of the welfare on the space of additively separable functions. The relative error of this projection indicates how bad this approximation can be.

It is thus important to compare relative errors of both separable and non separable fits of the welfare flow : pronounced differences will show that technologies strongly interact in values, while slight differences will indicate that technologies are approximatively independent.

To give a summary of what is done so far : one can choose either to operate an additively separable regression of the welfare flow, with the disadvantage that the investment trigger one obtains for a particular technology does not depend on level of capital stock in other technologies. Or, one can look for a non additively separable regression of the welfare and assume myopia to solve the problem.

The latter supposes computation of interaction coefficients in the regression model. By increasing the number of technologies, we increase strongly the size of the constrained mean square program one needs to solve to accomplish the regression procedure. As we precisely want a model able to handle several technologies, this method will rapidly lead to a dead end.

We now propose a method to tackle this problem. Recall that myopia is optimal if the regression model is separable; and used as an assumption (motivated by the absence of decision rule, and optimality in benchmark cases) otherwise. This leads to an heuristic based on myopia and forward Monte Carlo.

4 Myopia and Monte Carlo : an effective algorithm

This section presents an alternative determination of the trigger by forward Monte-Carlo simulations. We first use the myopia simplification of fixing forever capacity variables. Second, we use the fact that a natural proxy of the marginal profit $\frac{\partial \Psi(Y,K)}{\partial K_k}$ is the sum $\lambda_k(Y,K)$ of Lagrange multipliers $\lambda_{k,l}(Y,K)$, $l \in \mathbb{L}$ of capacity constraints (7) on K_k : this is noted

$$\lambda_k(Y,K) \equiv \sum_{l \in \mathbb{L}} \lambda_{k,l}(Y,K) = \widehat{\frac{\partial \Psi}{\partial K_k}}(Y,K). \tag{23}$$

("λ_k is used as an estimator of $\frac{\partial \Psi}{\partial K_k}(Y,K)$ "). Note that $\lambda_k(Y,K)$ is in €/MWy.

We explain the procedure in a few words (and without mathematical embarrassment).

1. Starting from initial capacity K and shock level Y, we generate a large number of scenarios.

   For each scenario ω, one computes the value of the marginal next unit of technology k as a numerical integral of $\lambda_k(Y,K)$. One obtains the marginal value $\widehat{M_k}(Y,\omega)$.

   Averaging on Ω brings us the value of the marginal unit $\widehat{m_k}(Y)$.

2. One then regress $\widehat{m_k}(Y)$ by power functions of Y, $\overline{\widehat{m_k}}(Y) = \sum_{i=1}^{d(\gamma)} c_i Y^{\gamma_i}$ and solve by analytical treatment the resulting optimal stopping problem on Y.

This intuition is formalized in the following.

Method 1. *1. Let $\{Y_s\}_{0 \le s \le T(\epsilon,\omega)} : \Omega \times \mathbb{R}^+ \to \mathbb{R}$, $Y_0 = Y$, the stochastic process affecting demand. Define the random variable $M_k(Y) : \Omega \to \mathbb{R}$ by :*

$$\begin{aligned} \forall\, \omega \in \Omega, \qquad M_k(Y)(\omega) \quad &\equiv \quad \sum_{s=0}^{T(\epsilon,\omega)} \lambda_k(Y_s(\omega),K)e^{-\rho s} \\ \Bigg(\quad &\approx \quad \int_0^{\infty} \frac{\partial \Psi}{\partial K_k}(Y_s,K)e^{-\rho s} ds \Bigg) \end{aligned} \tag{24}$$

with $T(\epsilon,\omega)$ choosen so that $\sum_{t=T(\epsilon,\omega)}^{T(\epsilon,\omega)+1} |\lambda_k(Y_s(\omega),K)|e^{-\rho s} \le \epsilon$.

2. Generate N paths $\omega_1, \omega_2, \ldots, \omega_N$ and note the corresponding random vector $(Y_s(\omega_1), \ldots, Y_s(\omega_N))_{0 \le s \le T(\epsilon)}$ with $T(\epsilon) = \max(T(\epsilon,\omega_1), \ldots, T(\epsilon,\omega_N))$. The $M_k(Y)(\omega_i)$ form an i.i.d. sequence. The law of large numbers ensures that :

$$\begin{aligned} \widehat{m_k}(Y) \equiv \mathbb{E}\Big[M_k(Y)(\omega)\Big] \quad &= \quad \lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} M_k(Y)(\omega_j) \\ \Bigg(\quad &\approx \quad \mathbb{E}\Big[\int_0^{+\infty} \frac{\partial \Psi}{\partial K_k}(Y_s,K)e^{-\rho s} ds \Big| Y_0\Big]\Bigg). \end{aligned} \tag{25}$$

3. Computes $\widehat{m_k}(Y)$ for different Y and regress by

$$\overline{\widehat{m_k}}(Y) = \sum_{i=1}^{d(\gamma)} c_i Y^{\gamma_i} \tag{26}$$

with γ positive base vectors of dimension $d(\gamma)$ such that $\forall i$, $0 < \gamma_i < \beta_1$ and β_1 the positive root of $\mathcal{Q}(\beta)$ defined in (10). The investment trigger is the value of Y such that :

$$\sum_{i=1}^{d(\gamma)} c_i Y^{\gamma_i} \left(\frac{\beta_1 - \gamma_i}{\beta_1} \right) = I_k + \frac{OMC(k)}{\rho}. \tag{27}$$

The finding of the last trigger only consist in using Appendix E from equation (55) to the end, having replaced (55) by $m_k(Y) = \sum_{i=1}^{d(\gamma)} c_i Y^{\gamma_i}$.

Again, myopia is emphasized by our notations : $M_k(Y)$, $\widehat{m_k}(Y)$ and $\overline{\widehat{m_k}}(Y)$ are noted as sole functions of Y while (obviously) λ_k is a function of both Y and K. Industry capacity expansion is driven by a sequence of myopic optimal stopping problem : capital addition takes place precisely when myopic trigger for the current capacity is reached; then follows an (upward) update of capacity level, and it's a new optimal stopping problem, to be solved again under myopia.

Before we move on to an example and compare the two methods, we discuss some advantages of our new procedure.

First, instead of performing a regression on the *uncertainty* $\otimes$ *technology* space, one just needs to regress on the *uncertainty* space : it is of course both mathematically and computationally slighter. We expect a sharp decrease of the regression error.

Second, note that Monte Carlo is here used to evaluate the bequest function (or terminal payoff) of the optimal stopping problem in order to apply analytic method to solve the latter. This method thus mix forward[21] Monte Carlo to analytics. Besides the fact that their work only applies to option valuation in finite horizon (and not capacity expansion), this method is completely disconnected to Longstaff and Schwartz[26]'s (backward) Monte Carlo method which uses entire paths matrices and compute backward directly the option value using polynomial regression as a way to estimate conditional expectation.

Third (and lastly), the procedure is very efficient by using directly Lagrange multipliers of the capacity constraints (Lagrange multipliers are provided in quasi any solver).

5 An example : 2 technologies

To fix ideas, we consider a capacity expansion problem with two technologies i.e. $K \in \mathbb{R}^2$. We choose to compare nuclear and coal; and call K_1 the nuclear capacity and K_2 the coal capacity. Details on technologies in electricity generations are given in Appendix A.

[21] Forward Monte Carlo allows more effective parallel computations. Elaborate on this point.

The demand is splitted in 6 segments ($d(\mathbb{L}) = 6$) for which we give the load level (see Table 1). Calibration is done for each load state : for state l, we use the load $\bar{Q}(l)$, corresponding price $P(l, \bar{Q}(l))$ and elasticity $\gamma(l)$ to find the coefficient $A(l)$ and the slope $b(l)$ of the demand function.

The elasticity is allowed to depend on the load state. However, for interpretation simplicity, we'll assume the same elasticity value for each load state, that is, $\gamma(l) = \gamma$ for all l. Our base case scenario assumes $\gamma = -0.5$. We will observe how investment trigger reacts to elasticity variations by comparing this base case to $\gamma = -0.35$ (low elasticity) and $\gamma = -0.75$ (high elasticity).

Demand Segment l	1	2	3	4	5	6
$\tau(l)$(kh)	0.01	0.04	0.31	4.4	3	1
$\bar{Q}(l)(MW)$	86000	83000	80000	60000	40000	20000
$P(l, \bar{Q}(l))$ (€/MWh)	300	60	55	40	30	20
$\gamma(l)$	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5
$A(l)$ for $Y = 1$	900	180	165	120	90	60
$b(l)$	0.0070	0.0014	0.0014	0.0013	0.0015	0.0020

Table 1: Demand calibration data (base case).

5.1 Solution by interpolation on the $Y \otimes K$ space

5.1.1 The welfare flow

The defined optimal dispatch problem is solved[22] for different values of the capacities K_1 and K_2 and different values of the demand shift Y. Values of the welfare for each value of K_1, K_2 and Y constitute a surface in the 3 dimensional euclidian space : the welfare flow $\Psi(Y, K_1, K_2)$ in €/y.

The Figure 3 represent the welfare Ψ for $\gamma = -0.5$ as a set of 2D surfaces parametrized by the Y value. It shows the welfare as a strictly increasing function of Y, as successive surfaces do not meet and are higher for higher demand indices. It also shows that Ψ is concave in K_1 and K_2. Note however that Ψ is decreasing for large values of the capacities. This last observation is a consequence of presence of Operation and Maintenance Costs (OMC) in the optimal dispatch. Each held capacity incur at least OMCs to keep the capacity alive. For that reason, even with a zero investment cost, it would not be optimal to hold an infinite capital stock.

[22]We use Matlab 7.4.0 and the function *quadprog*.

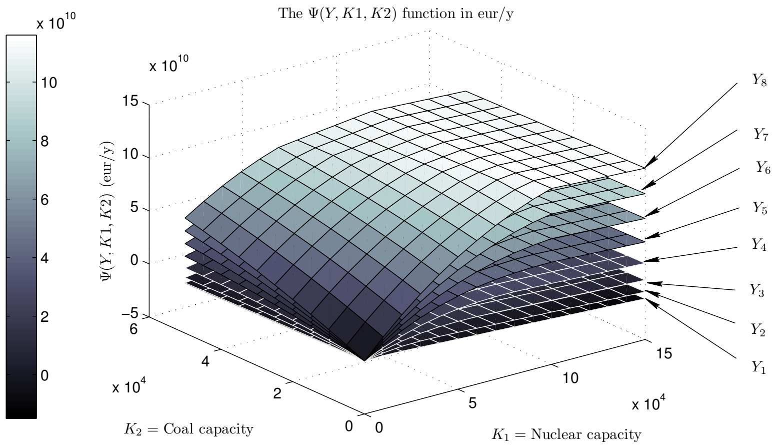

Figure 3: **The welfare function** $\Psi(Y, K_1, K_2)$ **for** $(Y_1, Y_2, \ldots, Y_8) = (0.25, 0.5, \ldots 2)$**,** $K_1 \in [0; 150000]$ **(MW) and** $K_2 \in [0; 60000]$ **(MW).**

5.1.2 Regression of the welfare flow and triggers

We now turn to the regression problem and look for an approximation of $\Psi(Y, K_1, K_2)$ by a function $\bar{\Psi}(Y, K_1, K_2)$ which is $\mathcal{C}^2$ in Y, and $\mathcal{C}^1$ and concave in K.

Before choosing regression bases, one need to specify the demand growth rate. $Y(t, \omega)$ is assumed to be a geometric Brownian motion i.e. to solve a stochastic differential equation $dY = \mu_Y Y dt + \sigma_Y Y dB_t$ having parameters $\mu_Y = 0.2$ and $\sigma_Y = 0.03$. We fix the discount rate to $\rho = 0.1$. With this set up, the positive root of the fundamental quadratic (10) is easily computed : one finds $\beta_1 = 4.62$.

We use the additive regression model given by equation (13) and set $\gamma = (1, 2, 3, 4, 4.5) \in \mathbb{R}^5$ (the only constraint on γ being that it's maximum is under β_1), $\alpha = (0.25, 0.5, 0.75, 1) \in \mathbb{R}^4$ and $\lambda = (0.1, 0.2, 0.3, 0.4, 0.5) \in \mathbb{R}^5$. The regression is achieved using a least square constrained program : regression coefficients are restricted to be positive to ensure that the fit $\bar{\Psi}$ is a concave function.[23]

After the regression, it remains to solve numerically the nonlinear one dimensional trigger equation.[24]

The two steps of generating the welfare Ψ and performing the regression have been done for $\gamma = 0.5$ (base case) and $\gamma = 0.35, 0.75$. Results are presented in the next subsection.

[23]We use Matlab 7.4.0 and the function *lsqnonneg*.
[24]On Matlab 7.4.0, one uses *fzero*.

	$\gamma = 0.5$	$\gamma = 0.35$	$\gamma = 0.75$
Rel. Error (separable)	11.61%	12.23%	10.98%

Table 2: Relative Errors for separable and non separable cases.

5.1.3 Results

Note that the relative error obtained for the separable regression model — available in Table 2, giving all output data for this part on interpolation on the $Y \otimes K$ space — is at best 10%. This passable result suggests either to affine the base, or to move from a non separable to a separable regression model.

Figure 4 displays investment triggers for additively separable regression model. Separable regression assumes that technologies are independent. For that reason, one can plot investment trigger for nuclear (resp. coal) as a only function of level of installed nuclear (resp. coal) units. It is visible on this graph that, for the three elasticities scenarios, nuclear will be the preferred technology (investment triggers for nuclear are lower). One also verifies that the trigger is an increasing function of the elasticity level : for a fixed demand level, the more consumption is sensitive to price increases, the less will the marginal value of the next unit be important; it implies later investment.

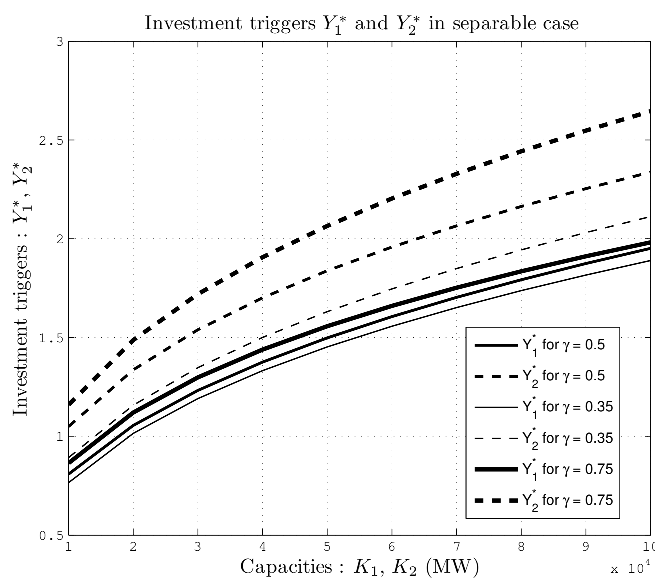

Fig. 4: Investment triggers for the separable regression. Nuclear is the preferred technology, and higher elasticities implies a higher value of the option to wait.

		K_2(MW)		
		10000	20000	30000
	10000	0.0211	0.0170	0.0164
	20000	0.0335	0.0279	0.0220
	30000	0.0395	0.0255	0.0295
K_1 (MW)	40000	0.0413	0.0377	0.0334
	50000	0.0462	0.0507	0.0390
	60000	0.0477	0.0474	0.0354

Table 3: **Relative errors in the regression $\widehat{\widehat{m_1}}(Y)$ of $\widehat{m_1}(Y)$.**

		K_2(MW)		
		10000	20000	30000
	10000	0.0678	0.0662	0.0649
	20000	0.0681	0.0693	0.0628
K_1	30000	0.0656	0.0550	0.0634
(MW)	40000	0.0659	0.0757	0.0699
	50000	0.0643	0.0616	0.0526
	60000	0.0479	0.0545	0.0625

Table 4: **Relative errors in the regression $\widehat{\widehat{m_2}}(Y)$ of $\widehat{m_2}(Y)$.**

5.2 Regression on the Y space only. Analytics - Monte Carlo.

We now treat the capacity expansion problem using the alternative numerical method described in Section 4.

For given initial capacities and initial demand shock $Y_0 = Y = 1$, we conducted $N = 30$ (See (25)) forward simulations, using $\epsilon = 0.01$ as stopping condition in the valuation of $\widehat{M_k}$ (See (24)). We then regressed the marginal unit $\widehat{m_k}$ using the base $\gamma = (0.25, 0.5, \ldots 3)$ in (26).

Table 3 and Table 4 show respectively relative errors in regressions of $\widehat{\widehat{m_1}}(Y)$ and $\widehat{\widehat{m_2}}(Y)$. A comparison with Table 2 show that — compare to global regression —, our method allow a better precision at the regression level. It is normal : the latter only needs to regress on the Y variable.[25]

Figure 5 and Figure 6 give investment triggers $Y_1^*(K_1, K_2)$ and $Y_2^*(K_1, K_2)$ for respectively nuclear and coal investment. For each technology, the investment trigger is increasing in all type of installed capacity : not only it is in-

[25] One may think that a proper analyze should compare the relative error of global regression with the combined error of forward Monte Carlo with regression on the single uncertainty axis. It would not be true, as — in theory, by letting N goes to infinity — Monte Carlo gives us the expectancy with an arbitrary big precision (the sample average converges almost surely to the expected value). In practice, Monte Carlo gives the mean value with a good precision relatively fast ($N = 100$), so our method will remain the most accurate.

creasing in capital stock of his own type, but also in capital stock for other technologies. Note that this simple — and intuitive — characteristic of investment does not appear with additively separable regression and cannot appear in a model considering the price process as exogenous.

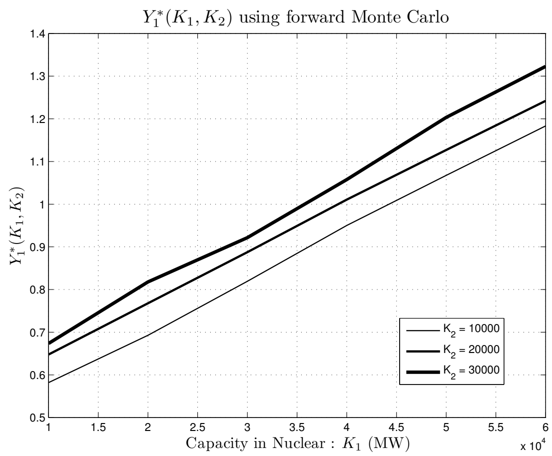

Fig. 5: Investment triggers for *nuclear* combining analytics and forward Monte Carlo.

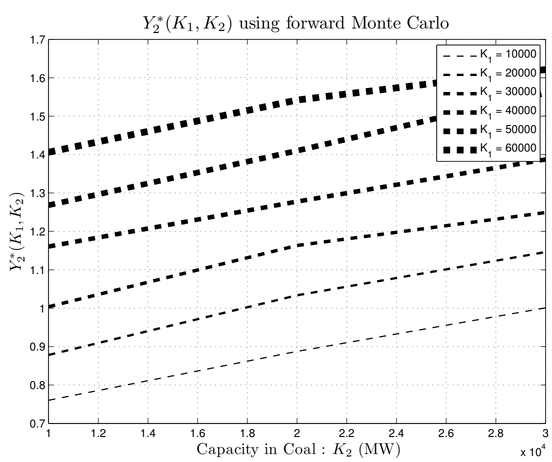

Fig. 6: Investment triggers for *coal* combining analytics and forward Monte Carlo.

6 Conclusion

This paper proposes a capacity expansion model for competitive power market. Considering uncertainties and important capital intensiveness in this sector, real options theory is a necessary tool in the determination of appropriate investment behavior. We use a stylized model to make the price process endogenous : such capacity expansion models are scarce regarding investment in power assets. This paper tries to fill this gap while recent restructuring brought serious fear of boom and bust cycles in construction of power capacities.

Besides being original for the reason we just mentioned, our model differ at least in two ways of the closest work it is related with (Leahy[24]) : first, one uses optimal dispatch to compute the welfare flow of the social planner; second, our model uses several technologies that may also differ by their investment cost.

This next point forces us to fit our welfare flow by convenient analytical forms. At this stage, a solution can be obtained by assuming a separable regression model or by resorting on an assumption of myopia. This second option is discussed and motivated : myopia is in general the observed behavior.

We moreover provide a faster and more accurate method to apply myopia numerically : we combine forward Monte Carlo simulations and free boundary problems to get investment criteria more efficiently. This method is promising : it can be used anytime we use myopia on a capacity generation plan; or for pricing perpetual American options on any complex physical asset, with underlyings given by any stochastic process one can think of for which standard real options theory provides a trigger.

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Appendices

A The 4 main technologies

We use 4 technologies in our model. Main features on these technologies are given in Table 5.[26]

Technology	Nuclear	Coal	CCGT	OCGT
Indice	1	2	3	4
O. & M. Cost (€/MWh)	10	8	5	5
Fuel Cost (€/MWh)	5	25.714	37.241	55.385
Emission Cost (€/MWh)	0	18.729	7.994	14.012
Marginal Cost (€/MWh)	15	52.443	50.236	74.397
Fix Cost (€/MWh)	18.62	10.65	5.74	3.28

Table 5: Generation technologies

O.&M. is for Operation and Maintenance cost. These prices are based on input costs of respectively 3 and 6 €/GJ for the coal and the gas; 1.85 €/MWh for the nuclear input; and finally 23 €/t of CO_2. The real discount rate we use is 5 %.

B Proof of Proposition 1

Indeed, rewriting Problem 1 as a minimization, one can write

$$\Psi\big(Y,K\big) + \sum_{k\in\mathbb{K}} OMC(k)K(k) \equiv -\min_{q} \sum_{l\in L} \tau(l)\Big\{\sum_{k\in\mathbb{K}} c(k)q(k,l) - \int_0^{Q(l)} P(Y,q,l)dq\Big\} \quad (28)$$

$$\text{s.t.} \quad 0 \le q(k,l) \le K(k) \qquad \forall k,\, \forall l \quad (29)$$

$$\text{s.t.} \quad \sum_{k\in\mathbb{K}} q(k,l) = Q(l) \qquad \forall l. \quad (30)$$

We note that (28) is the minimization of a convex function subject to two convex constraints (29) and (30). Standard optimization theory guarantees that the objective cost function at the optimum is convex with respect to the constraint vector K i.e. that $\Psi(Y,K)$ is concave in K.

[26] Costs are given in Risto and Aija[32]. One may also consult Joskow[19] and Stoft[35] for the computation of the fix costs.

C Proof of Proposition 2

That $\bar{\Psi}(Y,K)$ is $\mathcal{C}^2$ in Y is a necessary condition for the function $F(Y,K))$ to be $\mathcal{C}^2$ in Y in order to apply the Ito formula.

That $\bar{\Psi}(Y,K)$ is $\mathcal{C}^1$ in K is a necessary condition for the function $F(Y,K)$ to be $\mathcal{C}^1$ in K in order to solve the value matching (11).

For this expansion planning problem to make sense, the Bellman function $F(Y,K)$ has to be concave in K for all $Y \in \mathbb{R}^+$. It is the case (see e.g. Bertola[7]) if the immediate welfare flow $\bar{\Psi}(Y,K)$ is a concave function of K.

We now turn to the conditions under which the problem of maximizing the total surplus is equivalent to the research of the competitive equilibrium. These conditions are stated in Leahy[24].

1. The investment projects are infinitely divisible.
2. The cost function do not offer increasing return to scale.
3. The welfare flow $\Psi(Y,K)$ is continuous with respect to the output K.

We assume that Condition 1 holds. Condition 2 clearly holds, as shown in Fig. 2. Condition 3 will hold if $\bar{\Psi}(Y,K)$ is $\mathcal{C}^1$.

D Proof of proposition 3

One has to solve

$$F_k(Y,K_k) = A_k(K_k)Y^{\beta_1} + \sum_{i,j} \bar{b}_{k,ij}(\gamma_i) Y^{\gamma_i} K_k^{\alpha_j} - \frac{OMC(k)K_k}{\rho} \tag{31}$$

$$\frac{\partial F_k}{\partial K_k}(Y_k^*(K_k), K_k) = I_k \tag{32}$$

$$\frac{\partial^2 F_k}{\partial K_k Y}(Y_k^*(K_k), K_k) = 0. \tag{33}$$

The following decomposition is insightful.

$$F_k(Y,K_k) = \underbrace{A_k(K_k)Y^{\beta_1}}_{O_k(Y,K_k)} + \underbrace{\sum_{i,j} \bar{b}_{k,ij}(\gamma_i) Y^{\gamma_i} K_k^{\alpha_j} - \frac{OMC(k).K_k}{\rho}}_{V_k(Y,K_k)} \tag{34}$$

The term $O_k(Y,K_k)$ is the expansion option in technology k for a given Y. It is decreasing in K_k. $V_k(Y,K_k)$ is value of installed capital K_k for drift Y. A necessary(but not sufficient) condition to increase capacity is $\frac{\partial V_k}{\partial K_k}(Y,K_k) > 0$.

Preliminary study : the function $V_k(Y,K_k)$.

$$V_k(Y,K_k) \equiv \sum_{i,j} \bar{b}_{k,ij}(\gamma_i) Y^{\gamma_i} K_k^{\alpha_j} - \frac{OMC(k)K_k}{\rho}; \tag{35}$$

$$V_k(Y,0) = 0; \tag{36}$$

$$V_k(0,K_k) = -OMC(k)K_k; \tag{37}$$

$$\lim_{K_k \to +\infty} V_k(Y,K_k) = -\infty. \tag{38}$$

We compute the partial derivatives of the function V_k.

$$\frac{\partial V_k}{\partial K_k}(Y, K_k) = \sum_{i,j} \alpha_j \bar{b}_{k,ij}(\gamma_i) Y^{\gamma_i} K_k^{\alpha_j - 1} - \frac{OMC(k}{\rho} \tag{39}$$

$$\frac{\partial^2 V_k}{\partial K_k^2}(Y, K_k) = \sum_{i,j} \underbrace{\alpha_j}_{>0} \underbrace{(\alpha_j - 1)}_{<0} \underbrace{\bar{b}_{k,ij}(\gamma_i)}_{>0} \underbrace{Y^{\gamma_i} K_k^{\alpha_j - 2}}_{>0} < 0 \tag{40}$$

$$\rightarrow V_k(Y, K_k) \text{ is a concave function of } K_k.$$

$$\frac{\partial^2 V_k}{\partial K_k \partial Y}(Y, K_k) = \sum_{i,j} \underbrace{\alpha_j \gamma_i}_{>0} \underbrace{\bar{b}_{k,ij}(\gamma_i)}_{>0} \underbrace{Y^{\gamma_i - 1}}_{>0} \underbrace{K_k^{\alpha_j - 1}}_{>0, \searrow K_k} > 0 \tag{41}$$

$$\rightarrow \frac{\partial^2 V_k}{\partial K_k \partial Y} \text{ is positive and decreasing in } K_k. \tag{42}$$

One also compute

$$\frac{\partial F_k}{\partial K_k}(Y, K_k) = A_k'(K_k) Y^{\beta_1} + \frac{\partial V_k}{\partial K_k}(Y, K_k) \tag{43}$$

$$\frac{\partial^2 F_k}{\partial K_k Y}(Y_k, K_k) = \beta_1 A_k'(K_k) Y^{\beta_1 - 1} + \frac{\partial^2 V_k}{\partial K_k \partial Y}(Y, K_k) \tag{44}$$

Resolution of the free boundary problem and the trigger.
The smooth pasting (33) gives (using (41) and (44))

$$A_k'(K_k) = \frac{-\sum_{i,j} \alpha_j \gamma_i \bar{b}_{k,ij}(\gamma_i) Y^{\gamma_i - 1} K_k^{\alpha_j - 1}}{\beta_1 Y^{*(\beta_1 - 1)}} < 0 \tag{45}$$

As said earlier , the option value decreases along with K_k.
The value matching (32) gives (introducing (45) and (39) in (43))

$$\frac{-\sum_{i,j} \alpha_j \gamma_i \bar{b}_{k,ij}(\gamma_i) Y^{\gamma_i - 1} K_k^{\alpha_j - 1}}{\beta_1 Y^{*(\beta_1 - 1)}} Y^{*\beta_1} + \sum_{i,j} \alpha_j \bar{b}_{k,ij}(\gamma_i) Y^{*\gamma_i} K_k^{\alpha_j - 1} - \frac{OMC(k}{\rho} = I_k$$

we finally obtain the trigger :

$$\sum_{i,j} \alpha_j \bar{b}_{k,ij}(\gamma_i) Y^{*\gamma_i} K_k^{\alpha_j - 1} \left(\frac{\beta_1 - \gamma_i}{\beta_1} \right) = I_k + \frac{OMC(k)}{\rho}. \tag{46}$$

The trigger is uniquely defined for each K_k and is increasing in K_k.
One can define

$$T(Y, K_k) \equiv \sum_{i,j} \underbrace{\alpha_j \bar{b}_{k,ij}(\gamma_i)}_{>0} \underbrace{Y^{*\gamma_i}}_{>0} \underbrace{K_k^{\alpha_j - 1}}_{>0, \searrow K_k} \underbrace{\left(\frac{\beta_1 - \gamma_i}{\beta_1} \right)}_{>0} \tag{47}$$

so that the optimal investment level $Y^*(K)$ is given by the zero of the function

$$G(Y, K_k) \equiv I_k + \frac{OMC(k)}{\rho} - T(Y, K_k) \qquad \forall K_k. \tag{48}$$

We note that

$$G(0,K) = I_k + \frac{OMC(k)}{\rho}; \tag{49}$$

$$G(+\infty,K) \equiv \lim_{Y\to-\infty} G(Y,K) = -\infty; \tag{50}$$

moreover G is strictly continue and decreasing in Y for all K. The Bolzano's theorem therefore ensures that $G(.,K)$ as only one zero for all K, noted $Y^\star(K)$.

Moreover, as the function $G(Y,K)$ is strictly increasing in K, so will be the trigger $Y^*(K)$.

E Proof of Proposition 4

One has to solve

$$\mathcal{L}f_k(Y) - \rho f_k(Y) = 0 \qquad \forall\, Y \leq Y_k^\star, \tag{51}$$

and

$$f(Y_k^\star) = m_k(Y_k^\star) - I_k \tag{52}$$

$$\frac{\partial f}{\partial Y}(Y_k^\star) = \frac{\partial m_k}{\partial Y}(Y_k^\star) \tag{53}$$

$$m_k(Y) = \mathbb{E}\Big[\int_0^{+\infty} \frac{\partial\bar\Psi}{\partial K_k}(Y_s,K)e^{-\rho s}ds\Big|Y_0 = Y\Big]. \tag{54}$$

Our first step will be to compute $m_k(Y)$. Using $\Psi(Y,K)$ given by equation (3), one computes

$$\begin{aligned}\frac{\partial\bar\Psi}{\partial K_k}\ (Y_s,K) \quad &= \sum_{i,j=1}^{d(\gamma),d(\alpha)} \alpha_j b_{k,ij} Y_s^{\gamma_i} K_k^{\alpha_j-1} \\ &+ \sum_{\substack{t=1\\t\neq k}}^{d(\mathbb{K})} \sum_{i,j,l=1}^{d(\gamma),d(\lambda),d(\lambda)} \lambda_l c_{kt,ijl} Y_s^{\gamma_i} K_t^{\lambda_j} K_k^{\lambda_l-1} - OMC(k)\end{aligned}$$

but

$$\mathbb{E}\Big[\int_0^{+\infty} \frac{\partial\bar\Psi}{\partial K_k}(Y_s,K)e^{-\rho s}ds\Big|Y_0 = Y\Big] = \int_0^{+\infty} \mathbb{E}\Big[\frac{\partial\bar\Psi}{\partial K_k}(Y_s,K)\Big|Y_0 = Y\Big]e^{-\rho s}ds$$

so we just need to evaluate

$$\begin{aligned}\mathbb{E}\Big[Y_s^{\gamma_i}|Y_0 = Y\Big] &= \mathbb{E}\Big[Ye^{\gamma_i(\mu-\frac{1}{2}\sigma^2)s+\gamma_i\sigma B_s(\omega)}\Big] \\ &= Y\mathbb{E}\Big[e^{(\mu\gamma_i-\frac{1}{2}\gamma_i^2\sigma^2)s+(\frac{1}{2}\gamma_i^2\sigma^2-\frac{1}{2}\gamma_i\sigma^2)s+\gamma_i\sigma B_s(\omega)}\Big] \\ &= Ye^{\frac{1}{2}\gamma_i\sigma^2(\gamma_i-1)s}e^{\gamma_i\mu s}.\end{aligned}$$

So

$$\int_0^{+\infty} \mathbb{E}[Y_s^{\gamma_i}|Y_0 = Y]e^{-\rho s}ds = Y^{\gamma_i}\int_0^{+\infty} e^{-(\rho-\gamma_i\mu-\frac{1}{2}\gamma_i\sigma^2(\gamma_i-1))s}ds$$

and this integral converges if $\rho-\gamma_i\mu-\frac{1}{2}\gamma_i\sigma^2(\gamma_i-1)>0$, taking value $Y^{\gamma_i}/(\rho-\gamma_i\mu-\frac{1}{2}\sigma^2\gamma_i(\gamma_i-1)$.

Remembering of notations (14) and (15), one then can note :

$$\begin{aligned} m_k(Y) &= \sum_{i,j=1}^{d(\gamma),d(\alpha)} \alpha_j \bar{b}_{k,ij}(\gamma_i) Y^{\gamma_i} K_k^{\alpha_j-1} \\ &+ \sum_{\substack{t=1\\t\neq k}}^{d(\mathbb{K})} \sum_{i,j,l=1}^{d(\gamma),d(\lambda),d(\lambda)} \lambda_l \bar{c}_{kt,ijl}(\gamma_i) Y^{\gamma_i} K_t^{\lambda_j} K_k^{\lambda_l-1} - \frac{OMC(k)}{\rho} \end{aligned}$$

The general solution of (51) is $f_k(Y)=AY^{\beta_1}$. The value matching and smooth pasting conditions take the form :

$$\begin{aligned} f_k(Y_k^*) &= AY_k^{*\beta_1} = m_k(Y_k^*) - I_k \\ \frac{\partial f_k}{\partial Y}(Y_k^*) &= \beta_1 A Y_k^{*\beta_1-1} = \frac{\partial m_k}{\partial Y}(Y_k^*). \end{aligned} \tag{55}$$

As usual, we start by exploiting the smooth pasting conditions; one obtains

$$A = \frac{\frac{\partial m_k}{\partial Y}(Y_k^*(K_k))}{\beta_1 Y_k^{*\beta_1-1}(K_k)} \tag{56}$$

with

$$\begin{aligned} \frac{\partial m_k}{\partial Y}(Y) &= \sum_{i,j=1}^{d(\gamma),d(\alpha)} \gamma_i\, \alpha_j\, \bar{b}_{k,ij}(\gamma_i)\, Y^{\gamma_i-1} K_k^{\alpha_j-1} \\ &+ \sum_{\substack{t=1\\t\neq k}}^{d(\mathbb{K})} \sum_{i,j,l=1}^{d(\gamma),d(\lambda),d(\lambda)} \gamma_i\, \lambda_l\, \bar{c}_{kt,ijl}(\gamma_i)\, Y^{\gamma_i-1} K_t^{\lambda_j} K_k^{\lambda_l-1}. \end{aligned}$$

Using (56) into the value matching (52), one gets

$$\frac{\frac{\partial m_k}{\partial Y}(Y_k^*(K_k))}{\beta_1} Y_k^*(K_k) = m_k(Y_k^\star(K_k)) - I_k$$

Introducing the expressions of $m_k(Y)$ and $\frac{\partial m_k}{\partial Y}(Y)$ lead directly to the expression :

$$\begin{aligned} \sum_{i=1}^{d(\gamma)} \underbrace{Y^{\gamma_i}}_{>0,\nearrow Y} \underbrace{\left(\frac{\beta_1-\gamma_i}{\beta_1}\right)}_{>0} \Bigg\{ & \sum_{j=1}^{d(\alpha)} \underbrace{\alpha_j\, \bar{b}_{k,ij}(\gamma_i)}_{>0} \underbrace{K_k^{\alpha_j-1}}_{>0,\searrow K_k} \\ & + \sum_{\substack{t=1\\t\neq k}}^{d(\mathbb{K})} \sum_{j,l=1}^{d(\lambda),\,d(\lambda)} \underbrace{\lambda_k\, \bar{b}_{kt,ijl}(\gamma_i)}_{>0} \underbrace{K_t^{\lambda_j}}_{>0} \underbrace{K_k^{\lambda_l-1}}_{>0,\searrow K_k} \Bigg\} = I_k + \frac{OMC(k)}{\rho}. \end{aligned}$$

The left hand expression is positive, increasing in Y and decreasing in K_k for any values K_t, $t\neq k$. The Bolzano's theorem ensures that this equation has a unique zero $Y_k^*(K_k)$ for any given K_k. And because this left hand expression is decreasing in K_k, the trigger is increasing in K_k.