THE AFTER-TAX REPLACEMENT DECISION UNDER COST AND SALVAGE PRICE UNCERTAINTY

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Abstract

We present an analytical solution for the after-tax replacement investment decision for an asset subject to deteriorating operating costs and salvage value. The advantages of solving the three-factor replacement model analytically are that it extends the scope to include salvage value and the depreciation tax shield, while yielding a solution more efficiently than purely numerical methods. We show that the incremental value rendered by the replacement has to exceed the net re-investment cost, a finding that mirrors the standard result for one-factor models. The replacement policy is shown to vary with asset age, with younger assets being replaced at a lower operating cost threshold than older assets, and to vary with the salvage value, which is inversely related to the threshold. In line with expectations, an operating cost volatility increase raises the threshold, but an increase in the salvage value volatility or a decrease in the correlation between the operating cost and salvage value lowers the threshold.

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We present an analytical method for solving the after-tax optimal timing boundary for a replacement model, which characterizes a productive asset with stochastic operating cost and salvage value that both deteriorate with age. At replacement, the disposal of the incumbent renders a salvage value, which is used to partly offset the fixed re-investment cost for installing the replica. Therefore, in this two-factor model, the replacement policy reflects a trade-off between the operating cost under continued usage and the salvage value obtained under disposal. Since continuance with the incumbent involves an uncertain future stream of potentially deteriorating operating cash flows as well as the opportunity cost of a likely unfavorable change in the salvage value, replacement entails sacrificing the incumbent net value in exchange for a replica with an improved performance, but only after expending the re-investment cost. The presence of salvage value in the formulation makes this model particularly relevant to those assets, such as vehicles, earth moving equipment and aircraft, that have an extensive second hand market, as well as to those, like ships, that are disposed of for their scrap value. Additionally, the presence of a salvage value creates a plausible mechanism for terminating the infinite replacement chain. If the investment in the replica becomes no longer economically justified owing to adverse operating conditions, the chain is terminated whenever the salvage value exactly balances the net worth of the incumbent. Finally, we show the versatility of the analytical method by extending the model to a three-factor formulation through the inclusion of a declining balance depreciation schedule.

Abandonment value, as a source of cash flow, has the potential to alter the replacement policy. According to the simulation model of Robichek and Horne (1967), which is based on a NPV framework, the consequence of a significant abandonment value is to raise the project value since the mobility of funds for a project that turns sour has a flexibility value. Ignoring abandonment is to exclude this option and to under-estimate the project value. However, since their analysis overlooks the opportunity to abandon at a subsequent date prior to project expiration, or the timing of the abandonment decision, Dyl and Long (1969), see also Robichek and Van Horne (1969), their assessment of the value created by abandonment is actually an under-estimate. The formulation of Dyl and Long (1969) is extended by Gaumitz and Emery (1980), who examine the consequences of a like-for-like replacement when the incumbent asset is abandoned, and show that the presence or absence of asset replacement has a significant impact on the abandonment decision. A more comprehensive model of replacement and abandonment is produced by Howe and McCabe (1983), who consider the three cases of pure abandonment, an infinite cycle model with an abandonment value at each replacement, and a finite cycle model.

A stochastic model of replacement and abandonment is built by Bonini (1977) on a dynamic programming framework. Although numerical methods are essential for determining the optimal replacement-abandonment policy, this formulation has the advantages that the two factors are represented by random variables and their treatment in the model is explicit. In contrast, the subsequent real option models of replacement that embrace abandonment only do so implicitly. As a restraint on model dimensionality, Mauer and Ott (1995) represent both the salvage value and the depreciation charge as functions of the operating cost, while Dobbs (2004) deduces from a one-factor replacement model, the salvage value from the operating cost threshold. Because there is no explicit recognition of the salvage value, these formulations ignore the possibility of a trade-off between the operating cost and the salvage value in the replacement policy as well as the co-movements in their stochastic behavior, so questions on the interaction of the two factors and how the properties of the salvage value influence the optimal policy remain unanswered. Abandonment also appears in real option models characterizing other forms of opportunity. McDonald and Siegel (1985) re-interpret their investment opportunity model to consider the abandonment for an ongoing project, but their analysis rests on the property of homogeneity degree-one, which is unavailable for replacement models with reversionary levels and a fixed re-investment cost. Similarly, Myers and Majd (1990) apply a ratio transformation for reducing the dimensionality of their two-factor abandonment model, or use numerical methods when the changes in project value are non-linear.

The major contribution of our study is the development of a quasi-analytical solution to the after-tax optimal timing boundary for a real option replacement-abandonment model. Because both operating cost and salvage value are explicitly and distinctly represented in the formulation, we are able to ascertain the direction and magnitude of the effect produced by including the salvage value on the replacement policy. Although this effect is quantified for deterministic models, no similar measure exists for the stochastic model. Further, it is possible to assess the variations in the trade-off between the thresholds for the two factors along the timing boundary due to changes in properties of the factors as well as their interaction. One merit of using the quasi-analytical method is the ease in forming the solution for an increase in the number of model factors. We develop the solution to the optimal timing boundary for the enlarged model containing a deterministic, declining balance depreciation charge as an additional factor, and show how the combined inclusion of the salvage value and the depreciation charge influences the replacement policy. Finally, the model has the advantage that the implied infinite chain of replacements can be terminated whenever it is economically justified to do so. Because the model explicitly recognizes the salvage value, abandonment occurs whenever the incumbent asset value is exactly balanced by its salvage value. Because termination is formally admissible within the formulation, then according to Preinreich (1940), this model with its explicit recognition of the salvage value provides a more realistic representation than previously designed replacement models.

Our model is founded on the assumption that the operating cost and salvage value are well described by separate, but dependent, geometric Brownian motion processes, while the depreciation charge follows a similar process but with a zero volatility. When the model is enlarged to include depreciation, it contains three factors since all three variables are dynamic, and hence, they all appear as attributes in the asset valuation function. Replacement is economically justified whenever the factors jointly attain their respective threshold levels. At replacement, the levels for the operating cost, salvage value and depreciation charge for the replica re-adjust to their respective reversionary levels, which represent an improvement over their levels just before replacement. Following a replacement, the factor levels begin to deteriorate as the asset ages, so the operating cost typically inflates while the salvage value and depreciation charge deflate, and consequently, the threshold for operating cost is typically higher than its reversionary level, while the thresholds for the salvage value and depreciation charge are typically lower than their respective reversionary levels. Replacement is economically justified when the incumbent asset value is exactly balanced by the replica asset value less the net re-investment cost. In this context, the net re-investment cost is specified by the expenditure in acquiring a replica less the after-tax benefits of disposing of the incumbent. Even though abandonment value is the net value obtainable from asset disposal, collectively from its sale, any recoverable working capital or from the release of resources, in either cash or cash savings, we confine our attention to only the salvage value, the proceeds from the asset sale, because this is more likely to described by the given stochastic process.

The study is organized in the following way. The next section describes the replacement problem when a stochastic salvage value is present, explains the method for obtaining the optimal timing boundary, and provides a preliminary discussion of some of the findings. This is followed by a numerical exploration of the timing boundary behavior and illustrations of how the boundary responds to changes in the parameters. The basic replacement-salvage model is enlarged to a three-factor representation through including the depreciation charge and we demonstrate how the analytical method is adapted to obtain the timing boundary for a three-factor model. The following section provides a variety of numerical illustrations to show the effects of changing parameters on the timing boundary. The final section is a conclusion.

For convenience, the constituent equations for each of the models are assembled into two tables. Tables 1 and 6 present respectively the equations representing the various replacement models with salvage and those for the various replacement models with salvage and depreciation.

Valuing the Replacement Opportunity with Salvage

Valuation Function

Solving the asset replacement problem with an uncertain operating cost and salvage value involves maximizing the after-tax expected present value of the net cash flow stream over all possible replacement policies. The optimal timing boundary for this two-factor model is represented by a function defined over a two-dimensional space. At any time, the operating cost and salvage value for the asset under study are denoted by C and S respectively. The tax rate τ is applicable to all cash flows, both positive and negative, and whether they represent income or capital gains. At replacement, the operating cost and salvage value for the newly installed replica revert to their known initial levels of C_I and S_I respectively. As the asset efficiency deteriorates with usage, we assume that the expected change in operating costs is $\alpha_C > 0$, measured as an annualized continuous rate; correspondingly, its salvage value declines with an expected change rate of $\alpha_S < 0$. The replacement re-investment cost is a known constant K. To avoid round-tripping, $S_I < K$. Asset re-investment is treated here as partly irreversible, since the firm recovers only a fraction of the original outlay if the asset is divested. We assume that the asset revenue remains at a constant known level, denoted by P_I; since asset revenue is common for all replacement but not abandonment opportunities, this quantity is largely excluded from the analysis.

We assume that the two uncertain factors follow distinct geometric Brownian motion processes with drift. For $X \in \{C, S\}$:

$$\mathrm{d}X = \alpha_X X\mathrm{d}t + \sigma_X X\mathrm{d}z_X \tag{1}$$

Where α_X is the instantaneous drift rate, σ_X the instantaneous volatility rate, and $\mathrm{d}z_X$ is the increment of the standard Wiener process. Dependence between the two factors is described by the instantaneous covariance term $\rho\sigma_C\sigma_S$, $\mathrm{Cov}\left[\mathrm{d}C, \mathrm{d}S\right] = \rho\sigma_C\sigma_S CS\mathrm{d}t$ and $|\rho| \le 1$.

Since all other flexibilities are assumed to be absent, the value of the asset with its inherent replacement option is denoted by F_1. The value of F_1 depends on the prevailing operating cost and salvage value levels, so $F_1 = F_1(C, S)$. By assuming complete markets, standard contingent claims analysis can be applied to the asset value to determine its risk neutral valuation relationship, Constantinides (1978), Mason and Merton (1985). This is expressed by the partial differential equation:

$$\begin{gathered}\tfrac{1}{2}\sigma_C^2 C^2 \frac{\partial^2 F_1}{\partial C^2} + \rho\sigma_C\sigma_S CS \frac{\partial^2 F_1}{\partial C \partial S} + \tfrac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 F_1}{\partial S^2} \\ + \theta_C C \frac{\partial F_1}{\partial C} + \theta_S S \frac{\partial F_1}{\partial S} - rF_1 + \left(P_I - C\right)\left(1 - \tau\right) = 0,\end{gathered} \tag{2}$$

where $r > 0$ is the constant risk-free rate of interest, and θ_X are the respective risk-neutral drift rates. We assume that $r - \theta_X > 0$. The simplest kind of generic function satisfying (2) takes the form:

$$F_1 = A_1 C^{\eta_1} S^{\gamma_1} + \frac{P_I\left(1-\tau\right)}{r} - \frac{C\left(1-\tau\right)}{r - \theta_C}, \tag{3}$$

where, generically, A_1 is an unknown coefficient, and η_1 and γ_1 are unknown parameters of the product power function. In (3), the term $A_1 C^{\eta_1} S^{\gamma_1} > 0$ represents the replacement option value, while

$$\frac{P_I(1-\tau)}{r} - \frac{C(1-\tau)}{r-\theta_C}$$

represents the asset value in the absence of any replacement opportunity.

Substituting (3) in (2) yields the characteristic root equation:

$$Q_1(\eta_1,\gamma_1) = \tfrac{1}{2}\sigma_C^2\eta_1(\eta_1-1) + \rho\sigma_C\sigma_S\eta_1\gamma_1 + \tfrac{1}{2}\sigma_S^2\gamma_1(\gamma_1-1) + \theta_C\eta_1 + \theta_S\gamma_1 - r = 0. \tag{4}$$

The function Q_1 represents an ellipse. This ellipse has a presence in all four quadrants of the two-dimensional space, defined by η_1 and γ_1, because $Q_1(\eta_1,0)=0$ and $Q_1(0,\gamma_1)=0$ both have a positive and a negative root. If we envisage a function $H_1(\eta_1,\gamma_1)=0$, which is distilled from the value matching and associated smooth pasting conditions, then $H_1=0$ has to intersect $Q_1(\eta_1,\gamma_1)=0$ for a feasible solution to exist. It follows that the roots of η_1 and γ_1 can belong to any of the four quadrants:

I:	$\{\eta_{11},\gamma_{11}\}$	$\eta_{11}\ge 0, \gamma_{11}\ge 0$
II:	$\{\eta_{12},\gamma_{12}\}$	$\eta_{12}\ge 0, \gamma_{12}\le 0$
III:	$\{\eta_{13},\gamma_{13}\}$	$\eta_{13}\le 0, \gamma_{13}\le 0$
IV:	$\{\eta_{14},\gamma_{14}\}$	$\eta_{14}\le 0, \gamma_{14}\ge 0$

This suggests that (3) takes the specific form:

$$F_1 = A_{11}C^{\eta_{11}}S^{\gamma_{11}} + A_{12}C^{\eta_{12}}S^{\gamma_{12}} + A_{13}C^{\eta_{13}}S^{\gamma_{13}} + A_{14}C^{\eta_{14}}S^{\gamma_{14}} + \frac{P_I(1-\tau)}{r} - \frac{C(1-\tau)}{r-\theta_C}. \tag{5}$$

Boundary Conditions

We first invoke the limiting boundary conditions for the operating cost C and salvage value S in order to constrain the form of (5). As C becomes increasingly large and unfavorable, the pressure to replace the incumbent intensifies and the replacement option value becomes correspondingly large. In contrast, there is little economic justification for replacing the incumbent when C tends to zero and is favorable. This suggests that the operating cost parameter η_1 can only be positive, which means that the eligible solution space does not belong to quadrants III or IV and that $A_{13} = A_{14} = 0$.

We now consider how the salvage value level impacts on the replacement option value. At replacement, the asset owner is required to pay the re-investment cost K for installing the replica, but receives the after-tax salvage value $(1-\tau)S$ on disposing the incumbent, and so, the net acquisition cost for the replica is $K-(1-\tau)S$. Since the salvage value reduces the net re-investment cost, the owner is incentivized to replace the incumbent for high rather than low salvage value levels and the replacement option value varies positively with the salvage value. This suggests that the salvage value parameter γ_1 can only be positive, which means that the eligible solution space does not belong to quadrants II or III and that $A_{12} = A_{13} = 0$.

Collectively, the limiting boundary conditions imply that $A_{12} = A_{13} = A_{14} = 0$, so (5) becomes:

$$F_1 = A_{11}C^{\eta_{11}}S^{\gamma_{11}} + \frac{P_I(1-\tau)}{r} - \frac{C(1-\tau)}{r-\theta_C}. \tag{6}$$

Assuming that replacement is economically viable, the economic boundary conditions characterizing an optimal replacement are represented by the value conservation conditions and the optimality conditions. Our interpretation of economically viable is described below, under abandonment. The threshold levels for the operating cost and salvage value, which signal an optimal replacement, are denoted by $\hat{C}_1$ and $\hat{S}_1$ respectively. Since replacement is perceived as a rational response to asset deterioration, we expect the operating cost threshold to be at least its initial level, $\hat{C}_1 \geq C_I$, and the salvage value threshold to be no more than its initial level, $\hat{S}_1 \leq S_I$. At replacement, the incumbent and replica asset values are given by $F_1\left(\hat{C}_1, \hat{S}_1\right)$ and $F_1\left(C_I, S_I\right)$ respectively. To ensure value conservation, the incumbent asset value must balance the replica asset value less the net re-investment cost:

$$F_1\left(\hat{C}_1, \hat{S}_1\right) = F_1\left(C_I, S_I\right) - K + (1-\tau)\hat{S}_1,$$

Mauer and Ott (1995). This value matching relationship can be explicitly expressed as:

$$A_{11}\hat{C}_1^{\eta_{11}}\hat{S}_1^{\gamma_{11}} - \frac{\hat{C}_1(1-\tau)}{r-\theta_C} = A_{11}C_I^{\eta_{11}}S_I^{\gamma_{11}} - \frac{C_I(1-\tau)}{r-\theta_C} - K + (1-\tau)\hat{S}_1. \tag{7}$$

The two smooth pasting conditions associated with (7), for the two factors C and S, can be expressed as:

$$A_{11}\hat{C}_1^{\eta_{11}}\hat{S}_1^{\gamma_{11}} = \frac{\hat{C}_1(1-\tau)}{\eta_{11}(r-\theta_C)} = \frac{\hat{S}_1(1-\tau)}{\gamma_{11}} > 0. \tag{8}$$

This demonstrates that the option value at replacement $A_{11}\hat{C}_1^{\eta_{11}}\hat{S}_1^{\gamma_{11}}$ is always positive since η_{11} and γ_{11} are both positive. Using (8) to eliminate A_{11} from (7) yields the reduced form value matching relationship:

$$\frac{\hat{C}_1(1-\tau)}{r-\theta_C} - \frac{C_I(1-\tau)}{r-\theta_C} = K - (1-\tau)\hat{S}_1 + \frac{\hat{C}_1(1-\tau)}{\eta_{11}(r-\theta_C)}\left[1 - \frac{C_I^{\eta_{11}}S_I^{\gamma_{11}}}{\hat{C}_1^{\eta_{11}}\hat{S}_1^{\gamma_{11}}}\right]. \tag{9}$$

If we conjecture that γ_{11} is sufficiently small, then $C_I^{\eta_{11}}S_I^{\gamma_{11}}$ would be less than $\hat{C}_1^{\eta_{11}}\hat{S}_1^{\gamma_{11}}$ since $\hat{C}_1 > C_I$ and $\hat{S}_1 < S_I$. It now follows that the optimal replacement relationship (9) now implies that the after-tax value improvement in the operating cost exceeds the net re-investment cost by a positive amount:

$$\frac{\hat{C}_1(1-\tau)}{r-\theta_C} - \frac{C_I(1-\tau)}{r-\theta_C} > K - (1-\tau)\hat{S}_1. \tag{10}$$

This finding for the replacement opportunity with salvage re-investment model is the two-factor equivalent of that for the standard one-factor investment opportunity model. We can also express (9) as:

$$\frac{\hat{C}_1(1-\tau)}{\eta_{11}(r-\theta_C)}\left[\eta_{11} + \gamma_{11} - 1 + \frac{C_I^{\eta_{11}}S_I^{\gamma_{11}}}{\hat{C}_1^{\eta_{11}}\hat{S}_1^{\gamma_{11}}}\right] = K + \frac{C_I(1-\tau)}{r-\theta_C} > 0. \tag{11}$$

The replacement model with a salvage opportunity is represented by three simultaneous equations: (i) the reduced form value matching relationship (9), (ii) the reduced form smooth pasting condition (8), and (iii) the characteristic root equation $Q_1(\eta_{11}, \gamma_{11}) = 0$, (4). The replacement timing boundary can be constructed from calculating the solutions of $\hat{C}_1$, η_{11} and

γ_{11} to the replacement model for a pre-specified $\hat{S}_1$, and then repeating the process for varying $\hat{S}_1$.

Single Replacement Opportunity

The asset owner may wish to pursue a single replacement policy when the operating conditions show signs of becoming adverse. Although the valuation function remains intact, the value matching relationship for the multiple replacement model (7) has to be amended to exclude the replica replacement option value. The revised value matching relationship becomes:

$$A_{11s}\hat{C}_{1s}^{\eta_{11s}}\hat{S}_{1s}^{\gamma_{11s}} - \frac{\hat{C}_{1s}\left(1-\tau\right)}{r-\theta_C} = -\frac{C_I\left(1-\tau\right)}{r-\theta_C} - K + \left(1-\tau\right)\hat{S}_{1s} \tag{12}$$

where the subscript s refers to the single replacement opportunity. Since the two smooth pasting conditions can be expressed in a form identical to (8) except for the inclusion of the subscript s, then by eliminating A_{11s} (12), the reduced form value matching relationship becomes:

$$\frac{\hat{C}_{1s}\left(1-\tau\right)}{\eta_{11s}\left(r-\theta_C\right)}\left[\eta_{11s}+\gamma_{11s}-1\right] = \frac{C_I\left(1-\tau\right)}{r-\theta_C} + K\ . \tag{13}$$

For a single replacement to be economically justified, then from (13), the after-tax value of the operating cost threshold has to exceed the sum of the after-tax value of the operating cost for the replica and the re-investment cost, adjusted by a mark-up factor exceeding one. Since

$$\frac{C_I^{\eta_{11}}S_I^{\gamma_{11}}}{\hat{C}_1^{\eta_{11}}\hat{S}_1^{\gamma_{11}}} > 0\ ,$$

then $\hat{C}_1 < \hat{C}_{1s}$. For any salvage value threshold, the operating cost threshold for the multiple replacement model is always less than that for the single replacement model because its re-investment cost can be recouped over multiple replacements instead of only one.

The single replacement policy is found from solving the three simultaneous equations: (i) the reduced from value matching relationship (13), (ii) the reduced form smooth pasting condition, modified (8), and (iii) the characteristic root equation $Q_1\left(\eta_{11s},\gamma_{11s}\right)=0$, (4).

Zero Salvage Value

When the salvage value is set to equal zero, the multiple replacement model simplifies to the one-factor version presented by Dobbs (2004). When the salvage value is excluded from the formulation, the asset value including the embedded replacement option $F_2 = F_2\left(C\right)$ depends on only the prevailing operating cost, so $F_2 = F_2\left(C\right)$, where:

$$F_2 = A_2 C^{\eta_2} + \frac{P_I\left(1-\tau\right)}{r} - \frac{C\left(1-\tau\right)}{r-\theta_C}, \tag{14}$$

where generically A_2 is an unknown coefficient, and η_1 is an unknown parameter. A comparison of (3) with (14) reveals that $F_2\left(C\right) = F_1\left(C,S\right)$ for $\gamma_1 = 0$ and $S = 0$. If we denote the operating cost threshold by $\hat{C}_2$, then the value matching relationship is specified by $F_2\left(\hat{C}_2\right) = F_2\left(C_I\right) - K$, or more explicitly by:

$$A_{21}\hat{C}_2^{\eta_{21}} - \frac{\hat{C}_2\left(1-\tau\right)}{r-\theta_C} = A_{21}C_I^{\eta_{21}} - \frac{C_I\left(1-\tau\right)}{r-\theta_C} - K. \tag{15}$$

The associated smooth condition can be expressed as:

$$A_{21}\hat{C}_2^{\eta_{21}} = \frac{\hat{C}_2\left(1-\tau\right)}{\eta_{21}\left(r-\theta_C\right)} \tag{16}$$

where $\eta_{21} > 1$ is the positive root of the characteristic equation $Q\left(\eta_{21},0\right) = 0$, (4). The coefficient A_{21} is eliminated from (15) by substituting (16) to yield:

$$\frac{\hat{C}_2\left(1-\tau\right)}{\eta_{21}\left(r-\theta_C\right)}\left[\eta_{21} - 1 + \frac{C_I^{\eta_{21}}}{\hat{C}_2^{\eta_{21}}}\right] = \frac{C_I\left(1-\tau\right)}{r-\theta_C} + K. \tag{17}$$

Alternatively, this can be expressed as:

$$\frac{\hat{C}_2\left(1-\tau\right)}{r-\theta_C} - \frac{C_I\left(1-\tau\right)}{r-\theta_C} = K + \frac{\hat{C}_2\left(1-\tau\right)}{\eta_{21}\left(r-\theta_C\right)}\left[1 - \frac{C_I^{\eta_{21}}}{\hat{C}_2^{\eta_{21}}}\right] \geq K$$

A strategy of multiple replacement is economically justified provided that the after-tax value improvement in operating cost exceeds the re-investment cost.

Since $S_I^{\gamma_{11}} \ge \hat{S}_1^{\gamma_{11}}$ for $\gamma_{11} \ge 0$, a comparison of (11) with (17) reveals that $\hat{C}_2 \ge \hat{C}_1$. The effect of including the salvage value in the formulation is to reduce the operating cost threshold since the salvage value reduces the net re-investment cost.

The one-factor operating cost replacement model is represented by the reduced form value matching relationship (17) and the characteristic root equation $Q_1\left(\eta_{21}, 0\right) = 0$, (4), which is the after-tax version of the solution developed by Dobbs (2004). The reduced forms (11) and (17) are identical when γ_1 is set to equal zero, and when $\hat{S}_1$ is then set to equal zero.

If only a single replacement opportunity is available, then the replacement option value for the replica is omitted from (15). The operating cost threshold is now denoted by $\hat{C}_{2s}$, and it is straightforward to show that the threshold level is given by:

$$\frac{\hat{C}_{2s}\left(1-\tau\right)\left(\eta_{21s}-1\right)}{\eta_{21s}\left(r-\theta_C\right)} = \frac{C_I\left(1-\tau\right)}{r-\theta_C} + K. \tag{18}$$

Clearly, $\hat{C}_{2s} \ge \hat{C}_2$. This shows that the operating cost threshold is higher for a single replacement than for multiple replacements since the re-investment cost can only be recovered within a single occasion.

Abandonment

The multiple replacement model with salvage is founded on an infinite sequence of replacement opportunities and excludes the possibility of abandonment. However, the sequence of replacement opportunities can be interrupted whenever the replacement is not viable, which occurs when the replica asset value fails to compensate for the re-investment cost. For such a failure to occur, there would have to have been a deterioration in the initial properties of the replica asset, and although any of these properties, K, P_I, C_I or S_I, could be selected, we focus exclusively on P_I since the other properties are inter-dependent.

The asset owner is indifferent between replacing the incumbent and abandoning the project of a sequence of multiple replacements whenever $F_1\left(C_I, S_I\right) = K$. This implies that $F_1\left(\hat{C}_3, \hat{S}_3\right) = \left(1-\tau\right)\hat{S}_3$, and the value matching relationship can be more explicitly expressed as:

$$A_{31}\hat{C}_3^{\eta_{31}}\hat{S}_3^{\gamma_{31}} + \frac{\hat{P}_{3I}(1-\tau)}{r} - \frac{\hat{C}_3(1-\tau)}{r-\theta_C} = (1-\tau)\hat{S}_3, \tag{19}$$

where the subscript 3 refers to the abandonment variant of the multiple replacement model, and $\hat{P}_{3I}$ denotes the reversionary revenue level required for abandonment.

The smooth pasting conditions associated with (19) can be expressed in a form identical to (8) except for the change in subscript. By eliminating A_{31} from (19), the reduced form value matching relationship becomes:

$$\frac{\hat{P}_{3I}}{r} = \frac{\hat{C}_3(\eta_{31}+\gamma_{31}-1)}{\eta_{31}(r-\theta_C)} = \frac{\hat{S}_3(\eta_{31}+\gamma_{31}-1)}{\gamma_{31}}. \tag{20}$$

The abandonment variant of the replacement model with salvage is specified by three equations: (i) and (ii) the two reduced forms of the value matching relationship (20), and (iii) the characteristic root equation $Q_1(\eta_{31},\gamma_{31}) = 0$, (4).

Illustrative Results for the Replacement with Salvage Model

The discriminatory boundary for the various versions of the replacement with salvage model is evaluated using the quasi-analytical solution. The responsiveness of the boundary due to specified changes in parametric values is examined through sensitivity analysis. We also compare this replacement policy with those prescribed by Mauer and Ott (1995) and Dobbs (2004). The initial analysis is performed on the data set presented in Table 2. Although this ignores the revenue level since its value has no bearing on the replacement boundary, later we consider its level in conjunction with abandonment.

Replacement Boundary

Figure 1 illustrates the replacement boundary, which is evaluated as the solution to the three simultaneous equations: (i) reduced form value matching relationship (11), (ii) the reduced form smooth pasting condition (8), and the characteristic root equation $Q_1(\eta_{11},\gamma_{11}) = 0$, (4), for a pre-specified $\hat{S}_1$ value. The boundary for the representative set $0 \le \hat{S}_1 \le S_I$ is depicted by the line AB, which separates the continuance region below AB from the replacement region above AB.

The optimal decision is to replace the incumbent whenever the prevailing operating cost and salvage value belong to the replacement region, and to continue with the incumbent if otherwise.

The slope of the line AB is non-linear and negative. The non-linearity implies that $\hat{C}_1$ and $\hat{S}_1$ are not proportionate and that the homogeneity degree-one property is unavailable. The downward sloping replacement boundary means that there is a trade-off between the operating cost and the salvage value. For a certain pair of prevailing operating cost and salvage value levels belonging to the continuance region, there is possibly no regime change for a simultaneous operating cost increase and a salvage price decrease, while simultaneous increases in both the operating cost and salvage value are likely to change the regime from continuance to replacement, particularly if the pair of levels initially lie on the replacement boundary. There is a trade-off between the two factors exists due to the way that the salvage value, through reductions in the net re-investment cost, supplies compensates for increases in the operating cost. Further, the boundary is slightly convex to the origin; its slope becomes increasingly negative for decreases in the salvage value threshold and increases in the operating cost threshold. To promote a prevailing pair of operating cost and salvage value along the boundary into the replacement region requires for a unit decrement in the salvage value, a lower increment in the operating cost for higher salvage values. The significance of the salvage value in tipping the regime from continuance to replacement is greater for higher salvage values and its importance wanes as the salvage value approaches zero.

Representative values of AB are presented in Table 3. This table shows that as the salvage value threshold declines, the operating cost threshold increases, while the parameters η_{11} and γ_{11} both decline. When the salvage value threshold reaches zero, $\gamma_{11} = 0$ implying that the salvage value no longer influences the asset value, and η_{11} is specified by the positive root of $Q_1(\eta_{11}, 0) = 0$, (4). Previously, we conjecture that for the after-tax value improvement in the operating cost to exceed the net re-investment cost, (10), γ_{11} has to be sufficiently small. We observe from the table that γ_{11} is small, but moreover that its value and the salvage value threshold both decline towards zero such that $S_I^{\gamma_{11}} / \hat{S}_1^{\gamma_{11}}$ remains close to one.

Figure 1 also illustrates the replacement boundary for a zero salvage value, depicted by AC. This boundary is evaluated from the reduced form value matching relationship (17) and from the positive root of $Q\left(\eta_{21},0\right)=0$, (4). It is identical to the after-tax version of the solution proposed by Dobbs (2004). Since AC is located at least above AB, the two depicted boundaries endorse the finding that the effect of including the salvage value in the formulation is to lower the operating cost threshold. Ignoring the salvage value produces a sub-optimal replacement policy and the magnitude of the discrepancy grows with the extent of the salvage value at replacement.

The replacement boundary for the single replacement opportunity model is also shown in Figure 1 and Table 4. The boundary, depicted by DE in Figure 1, is evaluated from the reduced form value matching relationship (13) and the characteristic root equation $Q\left(\eta_{11s},\gamma_{11s}\right)=0$, (4). Like AB, the boundary DE is also downward sloping, but it always lies above AB, as predicted by the analysis. The relative positions of the boundaries suggest that the single replacement policy is more conservative than the multiple replacement policy, since the re-investment cost under a single replacement can only be compensated on one occasion. For any trajectory of the operating cost and salvage value starting from their reversionary levels, the asset owner should always exercise multiple replacements as the preferred policy, if possible. Figure 1 also presents the single replacement boundary for a zero salvage value, depicted by DF. This is evaluated from the reduced form value matching relationship (18) and the positive root of $Q\left(\eta_{21s},0\right)=0$, (4). Its boundary DF always lies above the single replacement with salvage boundary DE.

Assets are not forever replaced because of increasingly unfavorable economic conditions. If the reversionary levels at the next replacement become sufficiently adverse, replacement is no longer supported and the multiple replacement chain should be abandoned. Although the abandonment opportunity is not explicitly structured in the model, the asset owner becomes indifferent between the strategies of multiple replacement and abandonment when the value of the replica exactly balances the re-investment cost, or when the incumbent value exactly balances the after-tax salvage value including its depreciation shield. The abandonment boundary is then constructed from the two reduced form value matching relationships (20) and $Q_1\left(\eta_{31},\gamma_{31}\right)=0$, (4) for a particular reversionary revenue level P_I. An alternative interpretation is to compute the implied revenue reversionary level along the multiple replacement boundary that makes abandonment viable. This requires substituting the multiple replacement boundary solutions, the operating cost threshold $\hat{C}_1$, the salvage value threshold $\hat{S}_1$, and the parameters η_{11} and γ_{11}, for $\hat{C}_3$, $\hat{S}_3$, η_{31} and γ_{31} in (20) to obtain the value of the revenue reversionary level. These implied values of P_I are presented in Table 3. This reveals that there is a slight concavity in the implied P_I approximately around salvage value threshold level of 20. However, the changes in P_I along the boundary are very modest and we can almost conclude that P_I is invariant for all pairs of operating cost and salvage value thresholds. This means that when the implied P_I is translated as an abandonment policy, abandonment is not exercised for specific operating cost and salvage value thresholds, but when the revenue at the next replacement fails to achieve a minimum constant level.

Our numerical illustrations have endorsed the importance of including the salvage value in forming the replacement decision. The replacement boundary with salvage always lies below that without salvage, so its exclusion will produce sub-optimal decisions and assets economically ready for replacement will continue to be used in service. Further, the presence of the salvage value in the replacement model means that the infinite chain of replacements can be terminated with a justifiable cause. This means that the replacement with salvage model supplies a more realistic representation of the replacement decision in practice.

Sensitivity Analysis

The separate effects of parametric changes on the operating cost threshold $\hat{C}_1$ are presented in Table 5 for a representative set of salvage value thresholds $0 \le \hat{S}_1 \le 60$. It contains ten panels and each panel is devoted to the specific parameter according to the list in Table 5. For $\hat{S}_1 = 0$, the replacement with salvage model is identical to the basic replacement model, so the resulting operating cost threshold does not depend on the salvage value properties. Because of this, it is sometimes more informative to consider the operating cost threshold at $\hat{S}_1 = 60$ as well as the boundary location and shape.

The sensitivity analysis for the re-investment cost K, the initial operating cost C_I and the initial salvage value S_I are presented in panels A, B and E respectively of Table 3. The results are unsurprising. A positive change in K or C_I produces an increase in the operating cost threshold for all salvage value thresholds since any deterioration in the re-investment cost or the initial operating cost would have to be compensated by a higher operating cost threshold. Changes in the initial salvage value have the opposite effect on the boundary. An increase in S_I produces a fall in the operating cost threshold since its improvement entails a lower net re-investment cost. However, of the three, an initial salvage value change has the smallest impact on the boundary; in fact, the effect produced by a change in S_I is almost insignificant. This suggests that on replacement, asset owners should negotiate with suppliers to obtain improved terms for the asset investment price and its initial operating cost level, but should not to be overly concerned about the salvage value.

Panels C and D of Table 5 display the effects of changes in the two stochastic properties of the operating cost factor, the risk neutral drift rate θ_C and the volatility σ_C, respectively. Again, the results are unsurprising. A deterioration in θ_C produces a higher operating cost threshold, but the effect is not very significant. An operating cost threshold increases is also produced by an increase in σ_C, but its effect is more significant and pronounced for $\hat{S}_1 = 0$. This suggests that replacement asset owners should pay greater attention to its operating cost volatility than the drift rate, and, since volatility is an indicator of reliability, should seek improvements in the asset reliability.

There are more interesting results regarding the effects on the boundary arising from changes in the salvage value stochastic properties. Panels F and G of Table 5 present these effects for the salvage value risk neutral drift rate θ_S and volatility σ_S. Although its effect is hardly significant, a θ_S decrease lowers the operating cost threshold, while a σ_S increase lowers the threshold but its effect is more significant. For each case, a deterioration in these salvage value parameters reduces the operating cost threshold. This suggests that as these parameters deteriorate, the operating cost threshold is being lowered as a way of protecting the asset owner from losses given a likely fall in the salvage value. This protection effect is also visible, and amplified, for the correlation coefficient ρ between the two factors, the operating cost and the salvage value. Panel H of Table 5 presents the effect of changes in ρ on the boundary. The two factors form a natural hedge for a positive ρ since to some extent, any deterioration in operating cost is compensated by a salvage value improvement. However, a more realistic representation is a negative correlation, because an operating cost increase is likely to be reflected in a salvage value decrease. The effect of a fall in ρ is to lower the operating cost threshold, which suggests that the threshold is acting to avert the misfortune of a greater salvage value loss. It is interesting to observe that while volatility in the standard investment model normally raises the threshold, a volatility increase lowers the threshold, and this feature is replicated for the correlation coefficient.

The last two panels of Table 5, I and J, show the respective effects on the boundary of changes in the risk-free rate and the tax rate. The risk-free rate effect, although not very significant, is consistent with expectations and any risk-free rate increase is reflected in a higher operating cost threshold in order to compensate the lower discount factor. Similarly, any deterioration in the tax rate also creates a rise in the operating cost threshold because of its negative effect on the after-tax cash flow.

The deterministic variant is a special case of the stochastic replacement model with zero volatilities. In Appendix A, we formulate the deterministic variant, determine the optimal cycle time and show that the solutions for the stochastic and deterministic variants are identical when $\sigma_C = \sigma_S = 0$. Using the tilde symbol to denote a deterministic solution value, the optimal cycle time is $\tilde{T}_1 = 22.225$, with $\tilde{\eta}_{11} = 1.8049$, $\tilde{\gamma}_{11} 0.04340$, $\tilde{C}_1 = 24.327$ and $\tilde{S}_1 = 19.749$, for the parametric data presented in Table 2. When we ignore the salvage value threshold, the operating cost threshold for the deterministic variant lies entirely below the boundary for the stochastic variant, AB in Figure 1. This suggests that applying the deterministic rule in a context of uncertain operating costs and salvage value is sub-optimal., and the error is magnified as the salvage value declines towards zero.

Valuing the Replacement Opportunity with Salvage and Depreciation

Valuation Function

Identifying the timing boundary for the replacement model with salvage and depreciation is developed in exactly the same way as for the model not including depreciation. Consequently, we only present an abbreviated analytical derivation and include only those aspects that are materially different. Specifically, the derivation concentrates on the specification of (i) the depreciation schedule, (ii) the risk neutral valuation relationship, (iii) the valuation function, (iv) the value matching relationship, and (v) the smooth pasting conditions.

The selected depreciation schedule is DB (declining balance) since among the three alternatives, DB, SL (straight line) and SYD (sum-of-year's-digits), this form is the most tractable. The DB depreciation level, denoted by D, is described by the deterministic geometric process:

$$\mathrm{d}D = -\theta_D \mathrm{d}t \tag{21}$$

where $0 < \theta_D < 1$ is a known constant proportional depreciation rate. Being time dependent, the time elapsed since the last replacement, or the age of the incumbent, can be deduced directly from the value of D. The principal difference between the evolutionary forms of C and S compared with D is the absence of the volatility term in (21). When the incumbent is replaced by the replica, the depreciation level reverts to its initial level D_I. If the re-investment cost K is fully depreciable for tax purposes, then $D_I = \theta_D K$. However, we leave this matter open since it is straightforward to accommodate this provision in the model solution.

The value of the asset with its inherent replacement option, which is denoted by F_4, depends on the prevailing levels of the operating cost, salvage value and depreciation, so $F_4 = F_4\left(C, S, D\right)$. The risk neutral valuation relationship, expressed as a partial differential equation, is given by:

$$\begin{aligned}&\tfrac{1}{2}\sigma_C^2 C^2 \frac{\partial^2 F_4}{\partial C^2} + \rho\sigma_C\sigma_S CS\frac{\partial^2 F_4}{\partial C \partial S} + \tfrac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 F_4}{\partial S^2} \\ &+ \theta_C C\frac{\partial F_4}{\partial C} + \theta_S S\frac{\partial F_4}{\partial S} - \theta_D D\frac{\partial F_4}{\partial D} - rF_1 + \left(P_I - C\right)\left(1-\tau\right) + D\tau = 0.\end{aligned} \tag{22}$$

The simplest kind of function satisfying (22) takes the generic form:

$$F_4 = A_4 C^{\eta_4} S^{\gamma_4} D^{\lambda_4} + \frac{P_I(1-\tau)}{r} - \frac{C(1-\tau)}{r-\theta_C} + \frac{D\tau}{r+\theta_D} \tag{23}$$

where A_4 denotes an unknown coefficient. The term $A_4 C^{\eta_4} S^{\gamma_4} D^{\lambda_4} > 0$ represents the replacement option value, while

$$\frac{P_I(1-\tau)}{r} - \frac{C(1-\tau)}{r-\theta_C} + \frac{D\tau}{r+\theta_D}$$

is the asset value in the absence of any replacement opportunities.

Substituting (23) in (22) yields the characteristic root equation:

$$\begin{aligned} Q_4(\eta_4, \gamma_4, \lambda_4) = \tfrac{1}{2}\sigma_C^2 \eta_4(\eta_4 - 1) + \rho\sigma_C\sigma_S\eta_4\gamma_4 + \tfrac{1}{2}\sigma_S^2\gamma_4(\gamma_4 - 1) \\ + \theta_C\eta_4 + \theta_S\gamma_4 - \theta_D\lambda_4 - r = 0. \end{aligned} \tag{24}$$

The function Q_4 represents a hyper-parabola defined over the three-dimensional space $\{\eta_4, \gamma_4, \lambda_4\}$. It should be conceived as the extension to the third dimension of the Q function parabola for the replacement with DB depreciation model, and it therefore has a presence in all of the eight quadrants of this three-dimensional space. The solution to $\{\eta_4, \gamma_4, \lambda_4\}$ can possibly belong to any of the following eight quadrants:

I:	$\{\eta_{41}, \gamma_{41}, \lambda_{41}\}$	$\eta_{41} \geq 0, \gamma_{41} \geq 0, \lambda_{41} \geq 0$
II:	$\{\eta_{42}, \gamma_{42}, \lambda_{42}\}$	$\eta_{42} \geq 0, \gamma_{42} \geq 0, \lambda_{42} \leq 0$
III:	$\{\eta_{43}, \gamma_{43}, \lambda_{43}\}$	$\eta_{43} \geq 0, \gamma_{43} \leq 0, \lambda_{43} \geq 0$
IV:	$\{\eta_{44}, \gamma_{44}, \lambda_{44}\}$	$\eta_{44} \geq 0, \gamma_{44} \leq 0, \lambda_{44} \leq 0$
V:	$\{\eta_{45}, \gamma_{45}, \lambda_{45}\}$	$\eta_{45} \leq 0, \gamma_{45} \geq 0, \lambda_{45} \geq 0$
VI:	$\{\eta_{46}, \gamma_{46}, \lambda_{46}\}$	$\eta_{46} \leq 0, \gamma_{46} \geq 0, \lambda_{46} \leq 0$
VII:	$\{\eta_{47}, \gamma_{47}, \lambda_{47}\}$	$\eta_{47} \leq 0, \gamma_{47} \leq 0, \lambda_{47} \geq 0$
VIII:	$\{\eta_{48}, \gamma_{48}, \lambda_{48}\}$	$\eta_{48} \leq 0, \gamma_{48} \leq 0, \lambda_{48} \leq 0$

This suggests that (23) takes the specific form:

$$F_4 = \sum_{m=1}^{8} A_{4m} C^{\eta_{4m}} S^{\gamma_{4m}} D^{\lambda_{4m}} + \frac{P_I(1-\tau)}{r} - \frac{C(1-\tau)}{r-\theta_C} + \frac{D\tau}{r+\theta_D}\,. \tag{25}$$

Boundary Conditions

We first constrain the form of (25) by imposing the limiting boundary conditions. As previously argued, since the justification for an economic replacement becomes weightier as the operating cost increases, the operating cost parameter η_4 has to be positive. This means that the eligible $\{\eta_4, \gamma_4, \lambda_4\}$ solution space excludes quadrants V – VIII and $A_{45} = A_{46} = A_{47} = A_{48} = 0$.

The effects of variations in the salvage value and depreciation on the replacement option value are connected to the net re-investment cost. At replacement, the re-investment cost is partly mitigated by the after-tax cash flow arising from the disposal of the incumbent. If at replacement, the levels of the salvage value and depreciation for the incumbent are denoted by S and D respectively, then the remaining cumulative depreciation is D/θ_D since D declines geometrically, the after-tax capital gain on the incumbent is $(S - D/\theta_D)(1-\tau)$, so the after-tax cash flow arising from the disposal is $S(1-\tau) + D\tau/\theta_D$. Equivalently, this can be interpreted as the after-tax sale revenue from the disposal plus the depreciation tax shield. Now, the owner has an incentive to replace the incumbent when the level of the salvage price or depreciation is high, and this is correspondingly reflected in the replacement option value. Therefore, the salvage value and depreciation parameters, γ_4 and λ_4, have to be positive, which means that the eligible $\{\eta_4, \gamma_4, \lambda_4\}$ solution space excludes quadrants II – IV and VI – VIII, and $A_{42} = A_{43} = A_{44} = A_{46} = A_{47} = A_{48} = 0$.

Collectively, the limiting boundary conditions imply that (25) simplifies to:

$$F_4 = A_{41} C^{\eta_{41}} S^{\gamma_{41}} D^{\lambda_{41}} + \frac{P_I(1-\tau)}{r} - \frac{C(1-\tau)}{r-\theta_C} + \frac{D\tau}{r+\theta_D}. \tag{26}$$

For an economically viable replacement, the economic boundary conditions for the replacement with salvage and depreciation are constituted by the value matching relationship and the associated smooth pasting conditions. The threshold levels triggering an optimal replacement are denoted by $\hat{C}_4$, $\hat{S}_4$ and $\hat{D}_4$ for the operating cost, the salvage value and depreciation respectively, where $\hat{C}_4 \geq C_I$, $\hat{S}_4 \leq S_I$ and $\hat{D}_4 \leq D_I$. The values of the incumbent and replica assets at replacement are defined by $F_4(\hat{C}_4, \hat{S}_4, \hat{D}_4)$ and $F_4(C_I, S_I, D_I)$ respectively, while the net re-investment cost is $K-\hat{S}_4(1-\tau)-\hat{D}_4\tau/\theta_D$. Following Mauer and Ott (1995), value conservation at replacement demands that:

$$F_4\left(\hat{C}_4,\hat{S}_4,\hat{D}_4\right)=F_4\left(C_I,S_I,D_I\right)+\hat{S}_4(1-\tau)+\hat{D}_4\tau/\theta_D-K,$$

so explicitly, the value matching relationship is:

$$\begin{aligned}&A_{41}\hat{C}_4^{\eta_{41}}\hat{S}_4^{\gamma_{41}}\hat{D}_4^{\lambda_{41}}-\frac{\hat{C}_4(1-\tau)}{r-\theta_C}+\frac{\hat{D}_4\tau}{r+\theta_D}\\&=A_{41}C_I^{\eta_{41}}S_I^{\gamma_{41}}D_I^{\lambda_{41}}-\frac{C_I(1-\tau)}{r-\theta_C}+\frac{D_I\tau}{r+\theta_D}+\hat{S}_4(1-\tau)+\hat{D}_4\tau/\theta_D-K.\end{aligned}\tag{27}$$

The three smooth pasting conditions associated with (27), for each of the three factors C, S and D, can be expressed as:

$$A_{41}\hat{C}_4^{\eta_{41}}\hat{S}_4^{\gamma_{41}}\hat{D}_4^{\lambda_{41}}=\frac{\hat{C}_4(1-\tau)}{\eta_{41}(r-\theta_C)}=\frac{\hat{S}_4(1-\tau)}{\gamma_{41}}=\frac{\hat{D}_4\tau r}{\lambda_{41}\theta_D(r+\theta_D)}>0,\tag{28}$$

which proves that the replacement option value is always positive since η_{41}, γ_{41} and λ_{41} are all positive. Using (28) to eliminate A_{41} from (27) yields:

$$\begin{aligned}&\frac{\left(\hat{C}_4-C_I\right)(1-\tau)}{r-\theta_C}+\frac{\left(D_I-\hat{D}_4\right)\tau}{r+\theta_D}\\&=K-\hat{S}_4(1-\tau)-\hat{D}_4\tau/\theta_D+\frac{\hat{C}_4(1-\tau)}{\eta_{41}(r-\theta_C)}\left[1-\frac{C_I^{\eta_{41}}S_I^{\gamma_{41}}D_I^{\lambda_{41}}}{\hat{C}_4^{\eta_{41}}\hat{S}_4^{\gamma_{41}}\hat{D}_4^{\lambda_{41}}}\right].\end{aligned}\tag{29}$$

By conjecturing that γ_{41} and λ_{41} are both small, then $C_I^{\eta_{41}}S_I^{\gamma_{41}}D_I^{\lambda_{41}}<\hat{C}_4^{\eta_{41}}\hat{S}_4^{\gamma_{41}}\hat{D}_4^{\lambda_{41}}$, and it follows that:

$$\frac{\left(\hat{C}_4-C_I\right)(1-\tau)}{r-\theta_C}+\frac{\left(D_I-\hat{D}_4\right)\tau}{r+\theta_D}>K-\hat{S}_4(1-\tau)-\hat{D}_4\tau/\theta_D.$$

This implies that for an optimal replacement to occur, the value improvements in the operating cost and the depreciation tax shield has to exceed the net re-investment cost. This finding for the replacement opportunity with salvage and depreciation re-investment model is the three-factor equivalent of that for the standard one-factor investment opportunity model.

Alternatively, (29) can be expressed as:

$$\frac{\hat{C}_4\left(1-\tau\right)}{\eta_{41}\left(r-\theta_C\right)}\left[\eta_{41}+\gamma_{41}+\lambda_{41}-1+\frac{C_I^{\eta_{41}}S_I^{\gamma_{41}}D_I^{\lambda_{41}}}{\hat{C}_4^{\eta_{41}}\hat{S}_4^{\gamma_{41}}\hat{D}_4^{\lambda_{41}}}\right]=K+\frac{C_I\left(1-\tau\right)}{r-\theta_C}-\frac{D_I\tau}{r+\theta_D} \tag{30}$$

The replacement and salvage with depreciation model is represented by four equations: (i) the reduced form value matching relationship (30), (ii) and (iii) two reduced from smooth pasting conditions (28), and (iv) the characteristic root equation $Q_4\left(\eta_{41},\gamma_{41},\lambda_{41}\right)=0$ (24).

Single Replacement Opportunity

If there exists only one further remaining replacement opportunity, the value matching relationship excludes the replica option value, so (27) becomes:

$$A_{41s}\hat{C}_{4s}^{\eta_{41s}}\hat{S}_{4s}^{\gamma_{41s}}\hat{D}_{4s}^{\lambda_{41s}}-\frac{\hat{C}_{4s}\left(1-\tau\right)}{r-\theta_C}+\frac{\hat{D}_{4s}\tau}{r+\theta_D}=-\frac{C_I\left(1-\tau\right)}{r-\theta_C}+\frac{D_I\tau}{r+\theta_D}+\hat{S}_{4s}\left(1-\tau\right)+\hat{D}_{4s}\tau/\theta_D-K. \tag{31}$$

Since the reduced form smooth pasting conditions are identical to (28) except for the inclusion of the subscript s, then by eliminating A_{41s}, the reduced form value matching relationship becomes:

$$\frac{\hat{C}_{4s}\left(1-\tau\right)}{\eta_{41s}\left(r-\theta_C\right)}\left[\eta_{41s}+\gamma_{41s}+\lambda_{41s}-1\right]=K+\frac{C_I\left(1-\tau\right)}{r-\theta_C}-\frac{D_I\tau}{r+\theta_D}. \tag{32}$$

For a single replacement to be economically justified, then from (32), the after-tax operating cost threshold has to exceed the re-investment cost plus the after-tax operating cost value for the replica less its depreciation tax shield value. By comparing (32) with (30), then $\hat{C}_{4s}\geq\hat{C}_4$. For any trajectory starting from the reversionary levels, the multiple replacement policy is always exercised before the single replacement policy, since the net re-investment cost is discounted over multiple instead of a single replacement.

The single replacement policy is determined from four equations: (i) the reduced form value matching relationship (32), (ii) and (iii) two reduced form smooth pasting conditions, amended (28), and (iv) the characteristic root equation $Q_4\left(\eta_{41s},\gamma_{41s},\lambda_{41s}\right)=0$ (24).

Abandonment

The possibility of abandonment emerges when the owner becomes indifferent between the replacing and abandoning the asset, which occurs whenever replica asset value exactly balances the re-investment cost, $F_4\left(C_I, S_I, D_I\right) = K$, or the incumbent value balances the after-tax salvage value including the value of the remaining tax shield, $F_4\left(\hat{C}_5, \hat{S}_5, \hat{D}_5\right) = \hat{S}_5\left(1-\tau\right) + \hat{D}_5\tau / \theta_D$, where the subscript 5 refers to the abandonment possibility. From (27) this implies that:

$$A_{51}\hat{C}_5^{\eta_{51}}\hat{S}_5^{\gamma_{51}}\hat{D}_5^{\lambda_{51}} + \frac{\hat{P}_{5I}\left(1-\tau\right)}{r} - \frac{\hat{C}_5\left(1-\tau\right)}{r-\theta_C} + \frac{\hat{D}_5\tau}{r+\theta_D} = \hat{S}_5\left(1-\tau\right) + \hat{D}_5\tau / \theta_D, \tag{33}$$

where $\hat{P}_{5I}$ denotes the reversionary revenue level required for abandonment. The smooth pasting conditions associated with (33) can be expressed in the same form as (28) except for the change in subscript. The reduced form value matching relationships are obtained by eliminating A_{51}:

$$\begin{aligned}\frac{\hat{P}_{5I}}{r} &= \frac{\hat{C}_5}{\eta_{51}\left(r-\theta_C\right)}\left[\eta_{51}+\gamma_{51}+\lambda_{51}-1\right] \\ &= \frac{\hat{S}_4}{\gamma_{41}}\left[\eta_{51}+\gamma_{51}+\lambda_{51}-1\right] \\ &= \frac{\tau}{1-\tau}\frac{\hat{D}_5 r}{\lambda_{51}\theta_D\left(r+\theta_D\right)}\left[\eta_{51}+\gamma_{51}+\lambda_{51}-1\right].\end{aligned} \tag{34}$$

The abandonment variant of the replacement with salvage and depreciation model is composed of four simultaneous equations: (i), (ii) and (iii) three reduced form value matching relationship (34), and (iv) the characteristic root equation $Q_4\left(\eta_{51s}, \gamma_{51s}, \lambda_{51s}\right) = 0$ (24).

Illustrative Results for the Replacement with Salvage and Depreciation

Even if one[1] of the factors is deterministic, the optimal timing boundary for the three-factor real option model occupies a three dimensional space since every factor is dynamic. While the timing boundary for a one-factor model is describable by a one dimensional space, and for a two-factor model by a two dimensional space, the boundary for the three-factor model is describable by a three dimensional space. As the number of dynamic factors grows, the space spanned by the resulting timing boundary grows commensurately. This also implies that the

[1] If two of the factors in a three-factor model were deterministic, their threshold levels would be related through a time variable and the three dimensional optimal timing boundary could be fully represented by a two dimensional space.

extent of model indeterminacy increases likewise. While a one-factor model yields a point solution boundary and there is full model determinacy, and a two-factor model yields a two dimensional solution boundary and is one equation short of model determinacy, for the three-factor model, the solution boundary occupies a three dimensional space and is two equations short of model determinacy. So, although the optimal timing boundary for the replacement model with operating costs, salvage value and depreciation are fully represented by four simultaneous equations, the reduced form value matching relationship (30), two reduced from smooth pasting conditions (28), and the characteristic root equation $Q_4\left(\eta_{41},\gamma_{41},\lambda_{41}\right)=0$ (24), there are in fact six unknown quantities, $\hat{C}_4$, $\hat{S}_4$, $\hat{D}_4$, η_{41}, γ_{41} and λ_{41}.

While there is an element of choice in displaying the three dimensional optimal timing boundary, we depict the boundary as a representative set within a two-dimensional space. Specifically, a salvage value threshold is initially pre-specified, and then for this fixed threshold level, the optimal timing boundary is found for the two remaining factors, operating costs and depreciation by varying the depreciation threshold level. This procedure is repeated several times for alternative pre-specified salvage value thresholds. In this way, a representative set of optimal timing boundaries can be constructed. Since depreciation is a deterministic variable, its threshold level implies a certain timing level or asset age, $\hat{T}_4$, that is:

$$\hat{T}_4 = \frac{1}{\theta_D}\ln\left(\frac{D_I}{\hat{D}_4}\right)$$

Because time seems to be perceptibly a more natural quantity than depreciation, the optimal timing boundaries are expressed in a two dimensional space of operating costs and time.

The numerical illustrations are computed using the data set and the supplementary presented respectively in Tables 2 and 7. The supplementary data provides the depreciation reversionary level D_I and the declining balance rate θ_D. We assume that the whole amount of the re-investment cost K is allowable for depreciation, so $D_I = \theta_D K$. The merit of using this condition is that changes in either the declining balance rate or the re-investment cost are automatically cascaded into the depreciation reversionary level.

Replacement Boundary

The replacement boundary for the three-factor model is illustrated in Figure 2 for pre-specified threshold levels for the salvage value and from the parametric values in Tables 2 and 7. It is evaluated from the reduced form value matching relationship (30), two reduced from smooth pasting conditions (28), and the characteristic root equation $Q_4\left(\eta_{41}, \gamma_{41}, \lambda_{41}\right)=0$ (24). The set of representative boundaries are presented for salvage value threshold levels from a minimum of 0, to 20, 40 and until 60, which is its maximum reversionary level. Each boundary is labeled by AB with a subscript denoting the salvage value threshold level. When the salvage value threshold is zero, a comparison of the optimal timing boundaries for the current replacement model and the replacement model with depreciation, Chapter 4 Figure 2 (The effect of tax policy on the stochastic replacement decision), reveals that they are identical, as predicted by the theory. When the depreciation threshold is zero, or $\hat{T}=\infty$, the operating cost and salvage value thresholds is lower for the current model than for the replacement with salvage model, Figure 1 and Table 3, because of the inclusion of the replica depreciation tax-shield value:

$$\frac{D_I \tau}{r+\theta_D}$$

in the value matching relationship for the current model, (27). The boundary AB separates the continuance decision region below the line from the replacement decision region above the line. When a prevailing salvage value equals the salvage value threshold level, the optimal decision is to replace the incumbent whenever the prevailing operating cost and implied time value belong to the replacement region, and to continue with the incumbent if otherwise.

The slopes for the optimal timing boundaries are all positive but non-linear. For any fixed prevailing salvage value, the operating cost threshold increases as the asset ages and younger assets are replaced at lower operating cost thresholds than older assets. Again, as the asset ages, a smaller and smaller operating cost increment is required to promote the prevailing operating cost and time levels from on the optimal timing boundary into the replacement region. On considering a change in the salvage value threshold, we observe from Figure 2 that the distance between adjacent optimal timing boundaries, say for $\hat{S}_4$ equal to 60 and 40, is almost invariant. Irrespective of the asset age, a drop in the salvage value from 60 to 40 is going to produce an almost identical rise in the operating cost threshold. However, the distance between the two optimal timing boundaries grows as the salvage value threshold decreases. A unit drop in the salvage value threshold has to be compensated by greater increases in the operating cost threshold as the salvage value threshold declines towards zero.

Representative values along the optimal timing boundaries are presented in Table 8. This reveals that a decrease in either the salvage value threshold or the depreciation threshold (or an increase in the age threshold) produces an increase in the operating cost threshold and decreases in the parameters η_{41}, γ_{41} and λ_{41}. Since the net re-investment cost is defined as the re-investment cost less any gains from disposing of the incumbent together with the depreciation tax shield, there is an increase in the net re-investment cost as the levels for the depreciation and salvage value thresholds decrease, and this increase has to be compensated by a raised operating cost threshold level.

Collectively, Figure 2 and Table 8 illustrate that the effect of including any of the salvage value or depreciation in the formulation is to lower the operating cost threshold. Figure 2 reveals that the optimal timing boundaries for the replacement model are vertically stacked for differing salvage value thresholds, and that the position of the boundary for a lower salvage value threshold always lies above that for a higher salvage value threshold. Table 8 endorses this finding. Also, it can be observed from Table 8, that for any salvage value threshold, the operating cost threshold increases for falls in the depreciation threshold. Any trajectory of prevailing levels for the operating cost, salvage value and depreciation, starting from their reversionary levels, is almost bound to hit the optimal timing boundary for the operating cost replacement model with salvage value and depreciation before hitting that for the operating cost replacement model. The sole exception arises when the trajectory hits both boundaries simultaneously, which can only occur if the threshold levels for the salvage value and depreciation are both zero. When the levels of salvage value and depreciation are relevant, a replacement policy based only on uncertain operating costs is sub-optimal and is going to be too conservative. Under this sub-optimal replacement policy, assets are allowed to continue to be in use when it is economically justifiable to have replaced them.

Whenever the underlying economic conditions become unfavorable, and the reversionary levels at the next replacement become adverse, there is a likelihood that the multiple replacement chain remains no longer viable. The asset owner is indifferent between sustaining and discontinuing the chain when the replica asset value and the re-investment cost are in strict balance, or when the incumbent asset value exactly equals the after-tax salvage value including any depreciation tax shield. Again, we consider the magnitude of the revenue reversionary level in keeping with the owner's indifference between continuing and discontinuing the multiple replacement chain. The required level of reversionary revenue is determined from (34), and these are presented in Table 8 for the representative values along the optimal timing boundary. If the reversionary revenue level slips below the level for indifference, it is economically justified to accept the salvage value and to discontinue the multiple replacement chain. Table 8 reveals that the reversionary revenue levels for indifference are slightly concave around a salvage value threshold of 30 and an age threshold of 10. However, the revenue reversionary level for indifference is almost invariant. Further, this almost constant amount is significantly lower than that for the replacement model with salvage value, Table 3. This shows that by increasing the available net cash flow, the inclusion of the depreciation charge produces a fall in the revenue reversionary level for indifference, and the condition for discontinuing the multiple replacement chain is less severe when depreciation is included in the formulation.

The optimal timing boundary for the single replacement opportunity model is illustrated in Table 9. The shape for the single replacement boundary, Table 9, is very similar to that for the multiple replacement boundary, Table 8; however, their locations are very different. The two boundaries are vertically stacked, with the single replacement boundary always having a greater operating cost threshold for any salvage value and depreciation threshold levels. This means that any trajectory of prevailing values for operating cost, salvage value and depreciation will hit the multiple replacement boundary before hitting the single replacement boundary. A multiple replacement strategy is always preferred to a single replacement strategy, because the re-investment cost in the multiple replacement model can be recouped over several future replacements, while it can only be recouped on one occasion for the single replacement model. Although a replacement model with decremental re-investment opportunities can be constructed, a more satisfying approach for terminating the multiple replacement chain is to introduce conditions specifying when the chain should be discontinued, as we show above.

Sensitivity Analysis

Table 10 illustrates the optimal timing boundary for the multiple replacement model with salvage and depreciation in response to changes in the parameters. Each entry in the main body of the table records the operating cost threshold for a representative set of salvage value thresholds $0 \le \hat{S}_4 \le 60$ and time thresholds $0 \le \hat{T}_4 \le \infty$. The table is organized into 11 panels, and each panel is devoted to a separate parameter.

Panels A and B of Table 10 present, respectively, the operating cost threshold for variations in the re-investment cost and the reversionary operating cost. The results are unsurprising. In response to an increase in either the re-investment cost or the operating cost revisionary level, there is an accompanying rise in the operating cost threshold, since the additional sacrifice has to be compensated by an increased threshold level. The arrays of illustrated operating cost thresholds reveal that the threshold attains its highest level at $K = 120$ or $C_I = 12$ for the very oldest asset with a zero salvage value, but its lowest level at $K = 80$ or $C_I = 8$ for a newly installed asset with a salvage value at the top of its range. Clearly, a fall in the operating cost threshold at replacement is being compensated by a rise in either the salvage value or the residual depreciation tax shield. The extent of the compensation varies, though, over the ranges of the thresholds. As an illustration, for any age threshold, the operating cost threshold increase due to an operating cost reversionary level increase declines as the salvage price rises. This suggests that as the salvage value declines towards zero, it gains in relative importance when the operating cost reversionary level increases. In contrast, the operating cost threshold increase due to a re-investment cost increase mainly decreases as the salvage value threshold declines towards zero, and so, loses in relative importance.

The response of the timing boundary to variations in the reversionary salvage value is illustrated in Panel C of Table 10. The effect is invariant for a zero salvage value threshold, since for $\hat{S}_4 = 0$, $\gamma_{41} = 0$, so the salvage value plays absolutely no role in the value matching relationship (27). For a salvage value threshold exceeding zero, the effect of an increase in the reversionary salvage value is to reduce the operating cost threshold for all reported entries. This occurs because such an increase effectively lowers the net re-investment cost, which then leverages a lower operating cost threshold. Although the response of the timing boundary to a re-investment cost decrease and a reversionary salvage value increase is similar, their magnitudes are significantly different. A negative change in the re-investment cost creates a greater relative impact on the replacement policy than a like positive change in the reversionary salvage value, because of the distinct natures of the re-investment cost and the salvage value. At the next replacement, while the re-investment cost is known, the salvage value is uncertain and unknown, but most likely to be less than its reversionary level. If we compare the effect of a change in the reversionary salvage value on the operating cost threshold, this effect is greatest for the youngest asset having a salvage value threshold at the reversionary level, and least for old assets with a low salvage value threshold. It follows that in the selection of the right asset, prospective owners should devote more attention to obtaining improved terms of sale, and should not be unduly concerned with its salvage value.

Panel D of Table 10 illustrates the timing boundary behavior in response to variations in the operating cost volatility. For any salvage value and age threshold, the effect of a volatility increase is to raise the operating cost threshold, as expected. An increase in the operating cost uncertainty produces a more conservative policy on replacing the incumbent and suggests the adoption of a more cautious stance towards replacement. The extent of the timing boundary adjustment varies with the threshold levels for the salvage value and asset age. As the incumbent grows increasingly old and its salvage value dwindles to zero, the extent of the adjustment increases, so that the greatest adjustment occurs for an infinitely aged asset having a zero salvage value. In contrast, the effect of a volatility increase is least for a newly installed asset having a salvage value at the top of its range. The response of a volatility rise on the timing boundary is less for young assets with high salvage values than for old assets with low salvage values.

The model characteristics change radically when one of the uncertain factors is assigned to have a zero volatility. If the volatility of the operating cost is zero, then both the operating cost and the depreciation charge are describable by a time dependent function, and consequently, their threshold levels are related through this dependency on time. As their threshold levels are related to the age threshold $\hat{T}_4$ through:

$$\hat{C}_4 = C_I \mathrm{e}^{\theta_C \hat{T}_4} \text{ and } \hat{D}_4 = D_I \mathrm{e}^{-\theta_D \hat{T}_4}$$

then:

$$\left(\frac{\hat{C}_4}{C_I}\right)^{\theta_C} = \left(\frac{D_I}{\hat{D}_4}\right)^{\theta_D} . \tag{35}$$

For $\sigma_C = 0$, (35) represents an additional equation to be included in the model, which reduces the extent of the model indeterminacy by one. This revised model is relevant for assets under study whose operating cost can be legitimately treated as certain, but whose salvage value is uncertain, such as electric fork-lift trucks. The optimal timing boundary for the revised model is illustrated in Table 11. This reveals that the salvage value threshold declines with asset age. If we assume that the salvage value can never exceed its reversionary level, then the asset has a minimum asset lifetime of approximately 12.4 years before any replacement can take place. Further, the asset also has a maximum lifetime, which occurs when the salvage value threshold equals zero, at approximately 22.4 years. Replacements are confined to an age range between 12.4 and 22.4 years. This analysis demonstrates a key benefit of the quasi-analytical method, which stems from the ease of obtaining a solution to a particular problem from the solution to a general problem.

The timing boundary behavior in response to variations in the salvage value volatility is illustrated in Panel E of Table 10. When the salvage value threshold is zero, its parameter γ_{14} is also zero, and so changes in its volatility can have no impact on the timing boundary. For salvage value thresholds exceeding zero, a rise in the salvage value volatility produces a fall in the operating cost threshold for all asset ages. This contrasts with the finding concerning the response of the boundary to variations in the operating cost volatility. It implies that a volatility increase leads to a more liberal replacement policy. The extent of the fall in the operating cost threshold is greatest for a salvage value at the top of its range and for the oldest assets. Compared with similar changes in the operating cost volatility, the extent of the variations in the operating cost threshold due to a salvage value volatility change is, however, relatively modest. Even so, the finding may be interpreted as a form of protection policy. If the prevailing salvage value for the incumbent is relatively high and the asset is relatively old, increasing uncertainty concerning the salvage value produces a more liberal replacement policy because of the possible, significant future decline in the salvage value, particularly for older assets.

When the salvage value volatility is zero, $\sigma_S = 0$, the thresholds for the salvage value and the depreciation charge are related through time, and we can apply an expression similar to (35) for solving the timing boundary:

$$\left(\frac{\hat{S}_4}{S_I}\right)^{\theta_S} = \left(\frac{D_I}{\hat{D}_4}\right)^{\theta_D} \tag{36}$$

The resulting timing boundary is illustrated in Table 12. This reveals that the operating cost threshold increases with age, but at a decreasing rate, until it attains its maximum level when the thresholds for the salvage value and the depreciation charge are both zero, which occurs at time infinity. This representation is appropriate for assets under study that experience a volatile operating cost behavior but a time dependent, known salvage value. This would be relevant to problems where the owner acquires the asset from a vendor, but he has the right to return it to the vendor at a price according to a deterministic time schedule.

When both volatilities are set equal to zero, $\sigma_C = 0$ and $\sigma_S = 0$, the model becomes deterministic. This entails determinacy since the model is supplemented by (35) and (36), so the number of equations and unknowns is identical. The optimal timing boundary is presented in Table 13. It is interesting to note that the operating cost threshold for the deterministic model is not the minimum amongst all the operating cost threshold levels listed in Panels D and E of Table 10, so the deterministic solution cannot be interpreted as a floor benchmark level. Adopting the deterministic model as a yardstick is sub-optimal for a positive volatility, but it is impossible to tell from the deterministic solution whether it represents an under- or an over-estimate of the true figure.

The effect of variations in the correlation ρ between the levels for the operating cost and salvage value on the timing boundary is illustrated in Panel F of Table 10. Normally, poorly performing assets can only command low salvage values, so we would expect ρ to be negative since random disturbances that produce a rise in the operating cost are most likely to be reflected in a fall in the salvage value. For the reason given above, variations in ρ have no impact on the timing boundary for a zero salvage value threshold. Panel F reveals that for salvage value thresholds exceeding zero, the operating cost threshold declines as the correlation coefficient declines and becomes increasingly more negative and that this is most severe for assets having a high salvage value or for older assets. This effect, which is similar to the response of the operating cost threshold due to an increase in the salvage value volatility, is again counter-intuitive since if the correlation coefficient is negative, a random disturbance that causes the operating cost to rise would also produce a fall in the salvage value, which leads to a rise in the net re-investment cost. However, such a disturbance is unfortunate on two counts, because it raises the operating cost and lowers the salvage value obtainable if the asset is replaced, and these two outcomes are not mutually compensatory. The decrease in operating cost threshold arising from a reduction in the correlation coefficient signals that the consequences for the operating cost and salvage value levels do not compensate for each other. This suggests that the fall in the operating cost threshold observed for a drop in the correlation coefficient is acting as a kind of protection against suffering from both types of loss. Alternatively, if the correlation coefficient is one, the resulting higher operating cost threshold means that it is economically justified to prolong the asset use, since any current value decrease due to a higher operating cost is redeemed by a future value gain in the salvage value.

Panel G of Table 10 illustrates the effects of variations in the salvage value drift rate on the operating cost threshold. As before, when the salvage value threshold is zero, changes in the salvage value drift rate have no impact on the boundary, since the salvage value is absent from the value matching relationship. When the salvage value threshold exceeds zero, the response of the replacement policy to a negative increase in the drift rate is to lower the operating cost threshold. This can be explained in the following way. It can be inferred from the entries in Panel G that, while keeping the operating cost threshold constant, a fall in the salvage value drift rate produces a rise in the salvage value threshold. If an asset has a salvage value that deteriorates more intensely, then this asset would have to be replaced earlier, at a higher salvage value, than other assets.

An increase in the declining balance rate advances the depreciation tax shield, and this should lead to more a liberal replacement policy. Panel H of Table 10 illustrates the effects of changes in the declining balance rate on the operating cost threshold. The entries reveal a mixed picture. An increase in the declining balance rate does yield a fall in the operating cost threshold for older assets $\left(\hat{T}_4 \geq 20\right)$ and for newly installed assets, but for assets having an in-between age, the operating cost threshold rises. This contrasting behavior can be explained through considering the value matching relationship (27). If the threshold for the depreciation charge is close to its reversionary level and the asset is young, then the tax shield for the incumbent neutralizes the tax shield for the replica, so the re-investment cost is reduced by an amount equaling the residual depreciation tax shield. Similarly, if the depreciation threshold is close to zero and the asset is old, the incumbent tax shield and the residual tax shield neutralize each other, so the re-investment cost is reduced by an amount equaling the replica depreciation shield. This effect becomes less intense for assets of in-between years, and consequently, the reduction in the re-investment cost is less, and this is reflected in a lower operating cost threshold.

The effect of a tax rate change on the replacement policy is illustrated in Panel I of Table 10. We would normally expect a tax rate increase to make the asset less attractive financially, and consequently, the operating cost threshold would rise to compensate the tax rate increase. This occurrence is recorded only for older assets, assets having an age threshold greater than 2.5 years, and the effect intensifies with age. In contrast, for newly installed and very young assets, the opposite occurrence happens and the operating cost threshold falls in response to a tax rate increase. In general, we obtain the expected result that a tax rate rise leads to an increase in the operating cost threshold, with the exception for younger assets.

Changing the tax rate to zero makes the depreciation charge irrelevant as far as the replacement decision is concerned. The effect of $\tau = 0$ on the value matching relationship is to eliminate the depreciation variables, so (27) becomes:

$$A_{41}\hat{C}_4^{\eta_{41}}\hat{S}_4^{\gamma_{41}} - \frac{\hat{C}_4}{r-\theta_C} = A_{41}C_I^{\eta_{41}}S_I^{\gamma_{41}}D_I^{\lambda_{41}} - \frac{C_I}{r-\theta_C} + \hat{S}_4 - K\,, \tag{37}$$

which is similar in form to the value matching relationship for the replacement-salvage model, (7). If the capital expenditure could be fully expensed for tax purposes, then (7) would be expressed as:

$$A_{11}\hat{C}_1^{\eta_{11}}\hat{S}_1^{\gamma_{11}} - \frac{\hat{C}_1(1-\tau)}{r-\theta_C} = A_{11}C_I^{\eta_{11}}S_I^{\gamma_{11}} - \frac{C_I(1-\tau)}{r-\theta_C} + (1-\tau)\hat{S}_1 - (1-\tau)K\,. \tag{38}$$

It is easily demonstrated that the solutions for the two revised expressions, (37) and (38), are identical. This means that the timing boundary is invariant to the tax rate when the re-investment cost is fully expensed for tax purposes, so the replacement policy becomes immune to tax rate changes and is not distorted by the tax rate level. This finding endorses the conclusion of Smith (1963), which is based on a deterministic NPV evaluation. The timing boundary for the replacement-salvage model is illustrated in Panel J of Table 5. By comparing this boundary with that for the replacement-salvage-depreciation model with various declining balance rates, Panel H of Table 10, we observe that the former is always less than the latter for assets having a lifetime greater than the expected lifetime. In contrast, if the asset is recently installed, the latter is less than the former, and the source of the difference lies in their disparate forms of the incumbent and replica option value.

The effects of the operating cost drift rate and the risk-free rate on the timing boundary are unsurprising. Panels J and K of Table 10 record respectively the response of the operating cost threshold due to the separate variations in the operating cost drift rate and the risk-free rate. An increase in either of these two parameters produces a rise in the operating cost threshold. The magnitude of the positive change produced by a rise in the risk-free rate is greatest for the oldest assets, because of the time value of money, and for assets with the lowest salvage value, since a low salvage value is associated with older assets. The magnitude of the positive change produced by a rise in the operating cost drift rate is much less intense, but mixed. The magnitude is lowest for assets with an average age and an average salvage value, but increases slightly for younger and older assets, and for assets having a lower and higher salvage value.

Conclusion

We have applied the quasi-analytical method to the problem of replacing an asset, whose operating cost and salvage value deteriorate stochastically with age. The after-tax timing boundary is determined as the solution to a set of simultaneous equations. The approach has the advantage of analytical transparency. The standard finding for the one-factor real option model that for an investment to be economically justified, the value of an opportunity has to exceed the investment expenditure is shown to also apply to two- and three-factor replacement models. We prove analytically that for a replacement to be economically viable, the incremental value that it renders has to be greater than the re-investment cost less any after-tax cash flows from disposal. Not only does the analytical approach cope effectively in the absence of dimension reducing transformation, without recourse to onerous numerical methods, but it also has the versatility for dealing with models with more than two factors. We show how this method is extendable for a three-factor model, when depreciation is included as an additional factor. Finally, since this characterization of the replacement model represents the salvage value explicitly in the formulation, the infinite chain can be terminated when the incumbent value equals the salvage value. From this identity, the minimum reversionary revenue level that can sustain an economically justified replacement can be found.

The timing boundaries for a variety of replacement models are investigated. The effect of including a variable representing the salvage value or the depreciation in the formulation is to lower the operating cost threshold. Additionally, we obtain distinct replacement policies if a variable is treated as deterministic instead of stochastic. All this points to the view that the appropriate viable conditions justifying a replacement depends on the context of the decision situation, and that full knowledge on the asset characteristics as well as the circumstances surrounding its operations is essential before an optimal decision on replacement can be formulated.

There is a fall in the operating cost threshold due to including the salvage value, depreciation charge, or both, because of the injection of a positive after-tax cash flow obtainable from the disposal. This effect is most intense for younger assets because of the residual depreciation tax shield, and for assets having a higher salvage value. The replacement policy also depends on the number of remaining replacement opportunities. The operating cost threshold is significantly lower for multiple than for a single opportunity since in the latter case, the net re-investment cost can only be recouped during a single occasion. This implies that the trajectory of prevailing values starting from their respective reversionary levels always hits the multiple boundary before hitting the single boundary. Unless management decrees a single replacement policy, a multiple replacement decision is always enacted and the replacement chain cannot be terminated. Alternatively, we can determine the reversionary revenue level that is conducive to terminating the chain, which occurs when the incumbent value is exactly matched by the after-tax disposal value. We find that this implied reversionary level is almost invariant, so it provides practitioners with a simple decision rule for ending the chain.

Variations in the operating cost volatility have an expected, negative impact on the operating cost threshold. However, the response to changes in the salvage value volatility and the correlation coefficient between the operating cost and salvage value are surprising. There is a decrease in the operating cost threshold due to either a rise in the salvage value volatility, although the effect is quite modest, or to a fall in the correlation coefficient. One possible interpretation of these findings is the workings of some form of protection. There exists a double misfortune if a rise in the operating cost accompanies a fall in the salvage value, and this misfortune is amplified for higher salvage value volatilities. As a protection against this double misfortune, replacing the asset at a lower operating cost threshold may be economically desirable, since it avoids experiencing a salvage value fall and a rise in the net re-investment cost. Certainly, we find that the response of the boundary to either an increase in salvage value volatility or a decrease in the correlation coefficient is greatest for the highest salvage value thresholds, which is when the extent of the possible loss is at its greatest.

The responses of the timing boundary to changes in the declining balance rate and the tax rate are mixed. Clearly, a higher declining balance rate and a lower tax rate are expected to lower the operating cost threshold because of the respective benefits in advancing and increasing the cash flow. However, neither an increased declining balance rate nor a reduced tax rate lowers universally the timing boundary, and the expected result is only obtainable for specific cases. If the capital expenditure is allowed to be fully expense for tax purposes, the replacement policy is invariant to the tax rate and the resulting operating cost threshold is lower than that for any of the declining balance rates provided that the asset is at least older than its expected lifetime. A fully expensed re-investment cost for tax purposes has the merits that the optimal replacement policy is interpretable as a simple rule, replacement is accelerated because of a mainly lower operating cost threshold, and the authorities receive the tax receipts earlier instead of being distributed over the asset lifetime.

The significant response of the timing boundary to the inclusion of the salvage value and the depreciation charge testifies to the limitations inherent in previous stochastic replacement models. Restricted to one-factor formulations because of the curse of dimensionality, more representative replacement models have not been devised and questions on the effect of the additional factors and their interactions on the solution have been evaded. Through our analysis, we have demonstrated that the quasi-analytical method is a key for unlocking these questions, and possibly, its use can be extended for investigating other kinds of investment opportunities than replacement.

Appendix: Deterministic Model

In this Appendix, we find the optimal cycle time $\tilde{T}$ for the deterministic replacement model with salvage and depreciation, for constant revenues, and varying operating costs, depreciation charge and salvage price. We then show that identical solutions are obtained for the deterministic model and the stochastic model with zero volatilities.

Following Lutz and Lutz (1951) and Howe and McCabe (1983), the present value V_T for the asset with lifetime T is the discounted stream of after-tax net cash flows:

$$\begin{aligned} V_T &= \int_0^T \left\{ (1-\tau) P_I - (1-\tau) C_t + \tau D_t \right\} \mathrm{e}^{-\mu t} \mathrm{d}t \\ &= \int_0^T \left\{ (1-\tau) P_I - (1-\tau) C_I \mathrm{e}^{\alpha_C t} + \tau D_I \mathrm{e}^{-\theta_D t} \right\} \mathrm{e}^{-\mu t} \mathrm{d}t \\ &= \frac{(1-\tau) P_I \left(1-\mathrm{e}^{-\mu T}\right)}{\mu} - \frac{(1-\tau) C_I \left(1-\mathrm{e}^{-(\mu-\alpha_C)T}\right)}{\mu-\alpha_C} + \frac{\tau D_I \left(1-\mathrm{e}^{-(\mu+\theta_D)T}\right)}{\mu+\theta_D}, \end{aligned} \tag{39}$$

where μ denotes the appropriate risk-adjusted discount rate. The asset is financially viable for some definite lifetime so $\mu > \alpha_C$. By adapting the result by Lutz and Lutz (1951), the optimal cycle time for the asset is found from maximizing the value of the infinite chain W_T, where:

$$W_T = V_T + \left(W_T + (1-\tau) S_T + \tau D_T / \theta_D - K \right) \mathrm{e}^{-\mu T} \tag{40}$$

where D_T / θ_D denotes the residual depreciation of the incumbent at replacement and $S_T = S_0 \mathrm{e}^{\alpha_S T}$ its salvage value at replacement. From (40), the first order condition for an optimal cycle time is:

$$\left[\frac{\mathrm{d}V_T}{\mathrm{d}T} - \mu\left(W_T + (1-\tau) S_T + \tau D_T / \theta_D - K \right) \mathrm{e}^{-\mu T} + \mu\left((1-\tau) \frac{\mathrm{d}S_T}{\mathrm{d}T} + \frac{\tau}{\theta_D} \frac{\mathrm{d}D_T}{\mathrm{d}T} \right) \mathrm{e}^{-\mu T} \right]\Bigg|_{T=\tilde{T}} = 0\,. \tag{41}$$

From (39) and (40), (41) simplifies to:

$$\begin{aligned} &\frac{(1-\tau) C_{\tilde{T}}}{\mu} \left\{ 1 + \frac{\alpha_C \mathrm{e}^{-\mu\tilde{T}}}{\mu-\alpha_C} \right\} + \tau D_{\tilde{T}} \left\{ \frac{1}{\theta_D} - \frac{\mathrm{e}^{-\mu\tilde{T}}}{\mu+\theta_D} \right\} + (1-\tau) S_{\tilde{T}} \left\{ 1 - \alpha_S \frac{\left(1-\mathrm{e}^{-\mu\tilde{T}}\right)}{\mu} \right\} \\ &\qquad = \frac{(1-\tau) C_I}{\mu-\alpha_C} - \frac{\tau D_I}{\mu+\theta_D} + K. \end{aligned} \tag{42}$$

By noting that:

$$C_{\tilde{T}} = C_I \mathrm{e}^{\alpha_C \tilde{T}}, D_{\tilde{T}} = D_I \mathrm{e}^{-\theta_D \tilde{T}}, S_{\tilde{T}} = S_I \mathrm{e}^{\alpha_S \tilde{T}},$$

the solution to $\tilde{T}$ is obtainable by solving (42) numerically.

The optimal cycle time for the deterministic replacement model with salvage is obtainable from (42) by setting $D_I = 0$, which implies $D_{\tilde{T}} = 0$.

We now show that the deterministic solution (42) is obtainable from the solution to the stochastic replacement model with operating costs, salvage value and depreciation (30). The stochastic solution is recast in a dynamic programming framework by setting $\alpha_C = \theta_C$, $\alpha_S = \theta_S$ and $\mu = r$. For $\sigma_C = \sigma_S = 0$, then from (24):

$$\theta_C \eta_{41} + \theta_S \gamma_{41} - \theta_D \lambda_{41} - r = 0, \text{ so } \frac{C_I^{\eta_{41}} S_I^{\gamma_{41}} D_I^{\lambda_{41}}}{\hat{C}_4^{\eta_{41}} \hat{S}_4^{\gamma_{41}} \hat{D}_4^{\lambda_{41}}} = \mathrm{e}^{-r\hat{T}}.$$

By making these substitutions in (30), then it is straightforward to show using the smooth pasting conditions (28), that the result is identical to the deterministic solution (42).

Since the solutions to the stochastic and deterministic replacement models with operating costs, salvage value and depreciation are identical for zero volatilities, clearly, this finding carries over for the model with operating costs and salvage value.

Table 1
Constituent Equations for Replacement Model with Salvage

Model	**Equations**
Multiple Replacement Opportunity	Reduced Form Value Matching Relationship
	$\frac{\hat{C}_1(1-\tau)}{\eta_{11}(r-\theta_C)}\left[\eta_{11}+\gamma_{11}-1+\frac{C_I^{\eta_{11}}S_I^{\gamma_{11}}}{\hat{C}_1^{\eta_{11}}\hat{S}_1^{\gamma_{11}}}\right]-\frac{C_I(1-\tau)}{r-\theta_C}-K=0$
	Reduced Form Smooth Pasting Condition
	$\frac{\gamma_{11}\hat{C}_1(1-\tau)}{(r-\theta_C)}-\eta_{11}\hat{S}_1(1-\tau)=0$
	Characteristic Root Equation
	$\tfrac{1}{2}\sigma_C^2\eta_{11}(\eta_{11}-1)+\rho\sigma_C\sigma_S\eta_{11}\gamma_{11}+\tfrac{1}{2}\sigma_S^2\gamma_{11}(\gamma_{11}-1)+\theta_C\eta_{11}+\theta_S\gamma_{11}-r=0$
Single Replacement Opportunity	Reduced Form Value Matching Relationship
	$\frac{\hat{C}_{1s}(1-\tau)}{\eta_{11s}(r-\theta_C)}[\eta_{11s}+\gamma_{11s}-1]-\frac{C_I(1-\tau)}{r-\theta_C}-K=0$
	Reduced Form Smooth Pasting Condition
	$\frac{\gamma_{11s}\hat{C}_{1s}(1-\tau)}{(r-\theta_C)}-\eta_{11s}\hat{S}_{1s}(1-\tau)=0$
	Characteristic Root Equation
	$\tfrac{1}{2}\sigma_C^2\eta_{11s}(\eta_{11s}-1)+\rho\sigma_C\sigma_S\eta_{11s}\gamma_{11s}+\tfrac{1}{2}\sigma_S^2\gamma_{11s}(\gamma_{11s}-1)+\theta_C\eta_{11s}+\theta_S\gamma_{11s}-r=$

Zero Salvage Value

Reduced Form Value Matching Relationship

$$\frac{\hat{C}_{21}\left(1-\tau\right)}{\eta_{21}\left(r-\theta_C\right)}\left[\eta_{21}-1+\frac{C_I^{\eta_{21}}}{\hat{C}_1^{\eta_{21}}}\right]-\frac{C_I\left(1-\tau\right)}{r-\theta_C}-K=0$$

Characteristic Root Equation

$$\tfrac{1}{2}\sigma_C^2\eta_{21}\left(\eta_{21}-1\right)+\theta_C\eta_{21}-r=0$$

Table 2

Parametric Data for Replacement Model with Salvage

Replacement re-investment cost	K	100
Initial operating cost for replica	C_I	10
Risk neutral operating cost drift rate	θ_C	4%
Operating cost volatility	σ_C	25%
Initial salvage value for replica	S_I	60
Risk neutral salvage value drift rate	θ_S	-5%
Salvage value volatility	σ_S	25%
Operating cost salvage value correlation coefficient	ρ	0%
Risk-free interest rate	r	7%
Tax rate	τ	30%

Table 3

Representative Values along the Multiple Replacement with Salvage Boundary

$\hat{S}_1$	$\hat{C}_1$	η_{11}	γ_{11}	$\left(S_I / \hat{S}_1\right)^{\gamma_{11}}$	$\hat{P}_{3I}$
60.0	25.812	1.4447	0.10075	1.0000	22.740
50.0	27.223	1.4278	0.07867	1.0144	22.532
40.0	28.755	1.4122	0.05893	1.0242	22.384
30.0	30.409	1.3980	0.04138	1.0291	22.301
20.0	32.193	1.3851	0.02582	1.0288	22.286
10.0	34.132	1.3736	0.01207	1.0219	22.359
0.0	36.397	1.3632	0.00000	1.0000	22.627

Table 4

Representative Values along the Single Replacement with Salvage Boundary

$\hat{S}_{1s}$	$\hat{C}_{1s}$	η_{11s}	γ_{11s}
60.0	42.743	1.4127	0.05949
50.0	44.528	1.4028	0.04726
40.0	46.326	1.3937	0.03610
30.0	48.135	1.3852	0.02590
20.0	49.954	1.3773	0.01654
10.0	51.782	1.3700	0.00794
0.0	53.619	1.3632	0.00000

Table 5

Effects of Parametric Data Changes on Multiple Replacement with Salvage Boundary

$\hat{S}_1$	0	10	20	30	40	50	60
Panel A: The effect of re-investment cost changes on the boundary							
$K = 80$	32.446	30.048	28.033	26.205	24.538	23.029	21.679
$K = 100$	36.397	34.132	32.193	30.409	28.755	27.223	25.812
$K = 120$	40.229	38.052	36.156	34.393	32.738	31.182	29.723
Panel B: The effect of reversionary operating cost changes on the boundary							
$C_I = 8$	32.938	30.773	28.880	27.122	25.481	23.951	22.534
$C_I = 10$	36.397	34.132	32.193	30.409	28.755	27.223	25.812
$C_I = 12$	39.738	37.376	35.390	33.579	31.912	30.376	28.969
Panel C: The effect of operating cost drift rate changes on the boundary							
$\theta_C = 2\%$	36.056	33.912	32.001	30.211	28.525	26.941	25.457
$\theta_C = 4\%$	36.397	34.132	32.193	30.409	28.755	27.223	25.812
Panel D: The effect of operating cost volatility changes on the boundary							
$\sigma_C = 0\%$	27.004	25.363	23.950	22.659	21.470	20.378	19.379
$\sigma_C = 10\%$	28.928	27.178	25.679	24.310	23.051	21.894	20.835
$\sigma_C = 20\%$	33.498	31.440	29.681	28.068	26.578	25.202	23.938
$\sigma_C = 25\%$	36.397	34.132	32.193	30.409	28.755	27.223	25.812
$\sigma_C = 30\%$	39.635	37.137	34.990	33.009	31.165	29.451	27.865
Panel E: The effect of reversionary salvage value changes on the boundary							
$S_I = 50$	36.397	34.180	32.297	30.577	28.991	27.530	26.189
$S_I = 60$	36.397	34.132	32.193	30.409	28.755	27.223	25.812
$S_I = 70$	36.397	34.091	32.104	30.264	28.549	26.952	25.473
Panel F: The effect of salvage value drift rate changes on the boundary							

$\theta_S = -4\%$	36.397	34.186	32.296	30.557	28.942	27.444	26.060
$\theta_S = -5\%$	36.397	34.132	32.193	30.409	28.755	27.223	25.812
$\theta_S = -6\%$	36.397	34.079	32.090	30.263	28.570	27.006	25.567
Panel G: The effect of salvage value volatility changes on the boundary							
$\sigma_S = 0\%$	36.397	34.298	32.509	30.856	29.312	27.870	26.527
$\sigma_S = 10\%$	36.397	34.272	32.458	30.783	29.222	27.764	26.409
$\sigma_S = 20\%$	36.397	34.192	32.306	30.569	28.953	27.453	26.064
$\sigma_S = 25\%$	36.397	34.132	32.193	30.409	28.755	27.223	25.812
$\sigma_S = 30\%$	36.397	34.059	32.056	30.216	28.517	26.949	25.511
Panel H: The effect of operating cost salvage value correlation changes on the boundary							
$\rho = -1$	36.397	33.672	31.304	29.129	27.124	25.285	23.608
$\rho = -0.5$	36.397	33.902	31.748	29.768	27.937	26.251	24.705
$\rho = 0$	36.397	34.132	32.193	30.409	28.755	27.223	25.812
$\rho = 0.5$	36.397	34.363	32.639	31.054	29.579	28.207	26.933
$\rho = 1$	36.397	34.593	33.086	31.701	30.410	29.203	28.075
Panel I: The effect of risk-free rate changes on the boundary							
$r = 6\%$	34.876	32.683	30.853	29.191	27.669	26.275	25.007
$r = 7\%$	36.397	34.132	32.193	30.409	28.755	27.223	25.812
$r = 8\%$	37.898	35.560	33.514	31.611	29.830	28.166	26.617
Panel J: The effect of tax rate changes on the boundary							
$\tau = 15\%$	32.919	30.539	28.537	26.716	25.052	23.540	22.182
$\tau = 30\%$	36.397	34.132	32.193	30.409	28.755	27.223	25.812
$\tau = 45\%$	41.600	39.447	37.563	35.804	34.147	32.583	31.110

Table 6
Constituent Equations for Replacement Model with Salvage and Depreciation

Model	Equations	Source
Multiple Replacement Opportunity	Reduced Form Value Matching Relationship	
	$\frac{\hat{C}_4(1-\tau)}{\eta_{41}(r-\theta_C)}\left[\eta_{41}+\gamma_{41}+\lambda_{41}-1+\frac{C_I^{\eta_{41}}S_I^{\gamma_{41}}D_I^{\lambda_{41}}}{\hat{C}_4^{\eta_{41}}\hat{S}_4^{\gamma_{41}}\hat{D}_4^{\lambda_{41}}}\right]-\frac{C_I(1-\tau)}{r-\theta_C}+\frac{D_I\tau}{r+\theta_D}-K=0$	(30)
	Two Reduced Form Smooth Pasting Conditions	
	$\frac{\gamma_{41}\hat{C}_4(1-\tau)}{(r-\theta_C)}-\eta_{41}\hat{S}_4(1-\tau)=0$	
	$\frac{\lambda_{41}\hat{C}_4(1-\tau)}{(r-\theta_C)}-\frac{\eta_{41}\hat{D}_4\tau r}{\theta_D(r+\theta_D)}=0$	(28)
	Characteristic Root Equation	
	$\frac{1}{2}\sigma_C^2\eta_{41}(\eta_{41}-1)+\rho\sigma_C\sigma_S\eta_{41}\gamma_{41}+\frac{1}{2}\sigma_S^2\gamma_{41}(\gamma_{41}-1)+\theta_C\eta_{41}+\theta_S\gamma_{41}-\theta_D\lambda_{41}-r=0.$	(28)

(24)

Single Replacement Opportunity

Reduced Form Value Matching Relationship

$$\frac{\hat{C}_{4s}(1-\tau)}{\eta_{41s}(r-\theta_C)}\left[\eta_{41s}+\gamma_{41s}+\lambda_{41s}-1\right]-\frac{C_I(1-\tau)}{r-\theta_C}+\frac{D_I\tau}{r+\theta_D}-K=0$$

(32)

Two Reduced Form Smooth Pasting Conditions

$$\frac{\gamma_{41s}\hat{C}_{4s}(1-\tau)}{(r-\theta_C)}-\eta_{41s}\hat{S}_{4s}(1-\tau)=0 \qquad (28)$$

$$\frac{\lambda_{41s}\hat{C}_{4s}(1-\tau)}{(r-\theta_C)}-\frac{\eta_{41s}\hat{D}_{4s}\tau r}{\theta_D(r+\theta_D)}=0$$

Characteristic Root Equation (28)

$$\tfrac{1}{2}\sigma_C^2\eta_{11s}(\eta_{11s}-1)+\rho\sigma_C\sigma_S\eta_{11s}\gamma_{11s}+\tfrac{1}{2}\sigma_S^2\gamma_{11s}(\gamma_{11s}-1)+\theta_C\eta_{11s}+\theta_S\gamma_{11s}-r=0$$

(24)

Table 7

Supplementary Parametric Data for Replacement Model

with Salvage and Depreciation

Initial depreciation value for replica	D_I	10
Depreciation rate	θ_D	10%

Table 8

Representative Values along the Boundary

for the Multiple Replacement Model with Salvage and Depreciation

T^	$\hat{T}=0$	$\hat{T}=2.5$	$\hat{T}=5$	$\hat{T}=10$	$\hat{T}=20$	$\hat{T}=40$	$\hat{T}=\infty$
Discriminatory boundary values for $\hat{S}=0$							
$\hat{C}$	29.540	30.176	30.700	31.478	32.322	32.818	32.919
η	1.3895	1.3832	1.3785	1.3722	1.3664	1.3636	1.3632
γ	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
λ	0.02490	0.01890	0.01442	0.00849	0.00303	0.00041	0.00000
$\hat{P}_{5I}$	20.555	20.468	20.417	20.377	20.391	20.443	20.464
Discriminatory boundary values for $\hat{S}=10$							
$\hat{C}$	27.099	27.730	28.256	29.044	29.912	30.432	30.539
η	1.4052	1.3979	1.3925	1.3852	1.3785	1.3753	1.3748
γ	0.01556	0.01512	0.01478	0.01431	0.01383	0.01356	0.01350
λ	0.02745	0.02079	0.01582	0.00929	0.00330	0.00045	0.00000
$\hat{P}_{5I}$	20.169	20.080	20.032	20.001	20.032	20.100	20.126
Discriminatory boundary values for $\hat{S}=20$							
$\hat{C}$	25.080	25.699	26.219	27.008	27.889	28.424	28.537
η	1.4230	1.4147	1.4084	1.4001	1.3923	1.3886	1.3879
γ	0.03404	0.03303	0.03223	0.03110	0.02995	0.02931	0.02918
λ	0.03004	0.02270	0.01725	0.01010	0.00358	0.00048	0.00000
$\hat{P}_{5I}$	20.032	19.939	19.890	19.862	19.904	19.982	20.011
Discriminatory boundary values for $\hat{S}=30$							

$\hat{C}$	23.271	23.872	24.382	25.166	26.052	26.599	26.716
η	1.4432	1.4337	1.4265	1.4169	1.4079	1.4035	1.4028
γ	0.05582	0.05405	0.05265	0.05067	0.04864	0.04749	0.04726
λ	0.03283	0.02476	0.01879	0.01097	0.00387	0.00052	0.00000
$\hat{P}_{5I}$	20.012	19.911	19.858	19.831	19.880	19.967	19.999

Discriminatory boundary values for $\hat{S} = 40$

$\hat{C}$	21.648	22.225	22.722	23.493	24.378	24.932	25.052
η	1.4659	1.4550	1.4468	1.4358	1.4254	1.4203	1.4194
γ	0.08126	0.07856	0.07641	0.07334	0.07017	0.06836	0.06799
λ	0.03585	0.02699	0.02045	0.01190	0.00419	0.00056	0.00000
$\hat{P}_{5I}$	20.090	19.979	19.921	19.891	19.944	20.038	20.072

Discriminatory boundary values for $\hat{S} = 50$

$\hat{C}$	20.211	20.759	21.237	21.988	22.863	23.419	23.540
η	1.4910	1.4786	1.4693	1.4567	1.4448	1.4388	1.4378
γ	0.11065	0.10684	0.10378	0.09937	0.09479	0.09216	0.09162
λ	0.03905	0.02937	0.02222	0.01290	0.00453	0.00061	0.00000
$\hat{P}_{5I}$	20.264	20.141	20.076	20.039	20.091	20.189	20.225

Discriminatory boundary values for $\hat{S} = 60$

$\hat{C}$	18.963	19.478	19.932	20.654	21.509	22.060	22.182
η	1.5180	1.5043	1.4938	1.4795	1.4660	1.4591	1.4579
γ	0.14410	0.13901	0.13490	0.12894	0.12268	0.11906	0.11830
λ	0.04238	0.03184	0.02407	0.01395	0.00489	0.00066	0.00000
$\hat{P}_{5I}$	20.535	20.397	20.322	20.274	20.320	20.418	20.456

Table 9

Representative Values along the Boundary

for the Single Replacement Model with Salvage and Depreciation

T^	$\hat{T}=0$	$\hat{T}=2.5$	$\hat{T}=5$	$\hat{T}=10$	$\hat{T}=20$	$\hat{T}=40$	$\hat{T}=\infty$
Discriminatory boundary values for $\hat{S}=0$							
$\hat{C}$	47.250	48.029	48.637	49.480	50.301	50.714	50.780
η	1.3796	1.3757	1.3728	1.3689	1.3653	1.3635	1.3632
γ	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
λ	0.01546	0.01181	0.00906	0.00539	0.00195	0.00027	0.00000
Discriminatory boundary values for $\hat{S}=10$							
$\hat{C}$	45.423	46.200	46.806	47.646	48.466	48.878	48.944
η	1.3881	1.3839	1.3808	1.3766	1.3727	1.3707	1.3704
γ	0.00917	0.00899	0.00885	0.00867	0.00850	0.00841	0.00840
λ	0.01618	0.01235	0.00947	0.00563	0.00203	0.00028	0.00000
Discriminatory boundary values for $\hat{S}=20$							
$\hat{C}$	43.605	44.380	44.984	45.822	46.640	47.050	47.116
η	1.3974	1.3928	1.3894	1.3849	1.3806	1.3785	1.3782
γ	0.01923	0.01883	0.01853	0.01813	0.01776	0.01758	0.01755
λ	0.01697	0.01294	0.00992	0.00589	0.00212	0.00029	0.00000
Discriminatory boundary values for $\hat{S}=30$							
$\hat{C}$	41.799	42.571	43.173	44.008	44.823	45.233	45.298
η	1.4074	1.4025	1.3988	1.3938	1.3892	1.3869	1.3866
γ	0.03030	0.02965	0.02916	0.02850	0.02789	0.02760	0.02755

λ	0.01783	0.01358	0.01040	0.00617	0.00222	0.00030	0.00000

Discriminatory boundary values for $\hat{S} = 40$

$\hat{C}$	40.006	40.774	41.374	42.206	43.018	43.426	43.491
η	1.4184	1.4130	1.4089	1.4035	1.3985	1.3960	1.3957
γ	0.04254	0.04158	0.04086	0.03991	0.03901	0.03858	0.03851
λ	0.01877	0.01429	0.01094	0.00648	0.00233	0.00032	0.00000

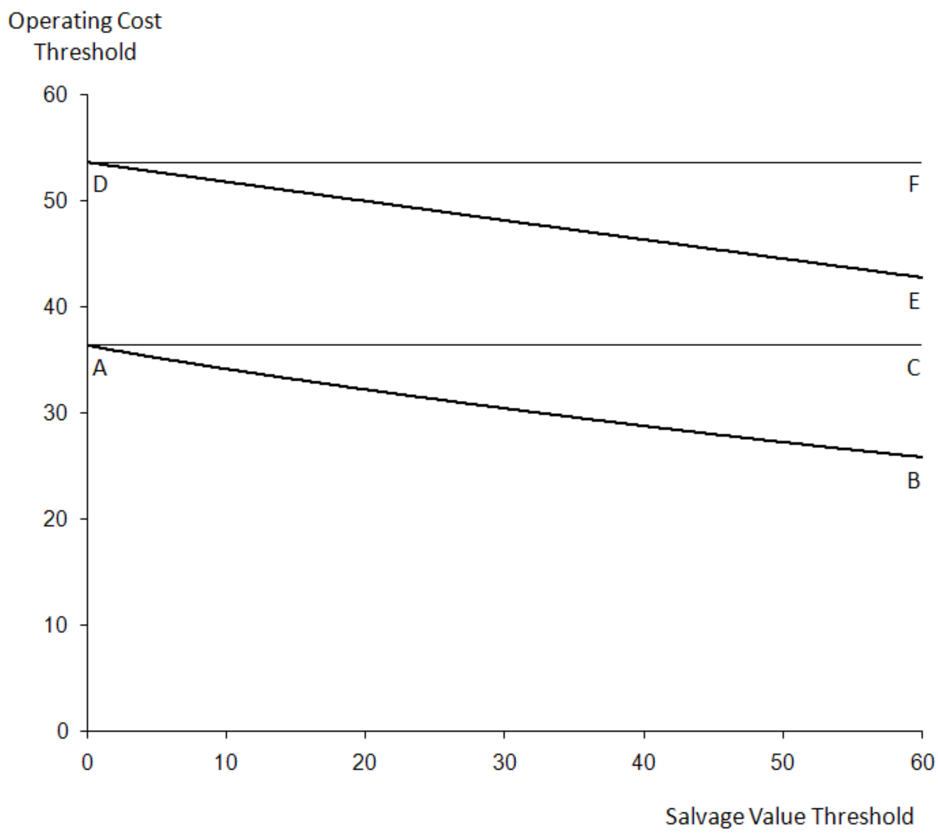
Discriminatory boundary values for $\hat{S} = 50$

$\hat{C}$	38.227	38.992	39.588	40.417	41.225	41.631	41.697
η	1.4303	1.4244	1.4200	1.4141	1.4086	1.4059	1.4055
γ	0.05612	0.05479	0.05380	0.05248	0.05125	0.05066	0.05056
λ	0.01981	0.01506	0.01152	0.00681	0.00245	0.00033	0.00000

Discriminatory boundary values for $\hat{S} = 60$

$\hat{C}$	36.466	37.226	37.819	38.642	39.447	39.851	39.916
η	1.4433	1.4368	1.4320	1.4255	1.4195	1.4166	1.4161
γ	0.07124	0.06948	0.06815	0.06640	0.06477	0.06399	0.06386
λ	0.02095	0.01591	0.01216	0.00719	0.00258	0.00035	0.00000

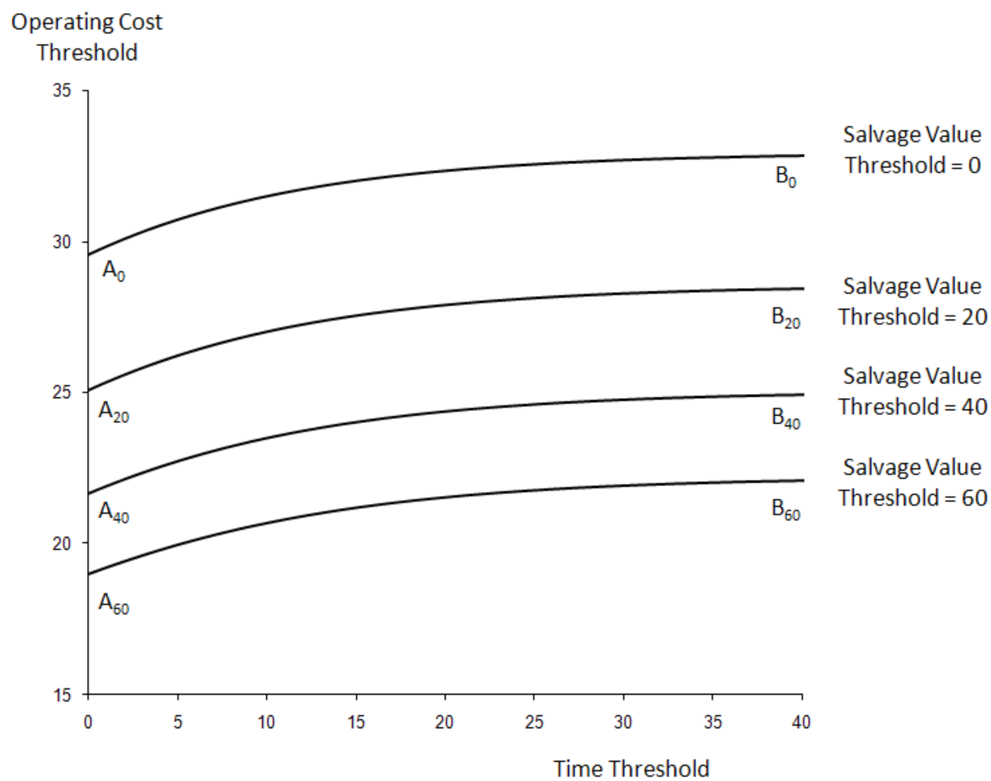
Figure 1



Boundaries for the Replacement with Salvage Model

AB and DE depict the boundaries for the multiple and single replacement model variants based on the data exhibited in Table 1, while AC and DF depict the same boundaries for a zero salvage value. The respective continuance and replacement regions lie below and above each boundary. Representative values along the boundaries AB and DE are presented in Tables 2 and 3 respectively. For a zero salvage value, its threshold $\hat{S}$ and parameter value γ_1 are both zero, and the operating cost thresholds are 36.397 and 53.619 for the multiple and single replacement models respectively.

Figure 2
Optimal Timing Boundary for Multiple Replacement Model
with Salvage Value and Depreciation



The various optimal timing boundaries are represented by the generic lines AB, where the subscript refers to the value of the salvage value threshold, based on the parametric values presented in Tables 1 and 5. The respective continuance and replacement decision regions lie below and above each boundary. Representative values along each of the boundaries are shown in Table 6.

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