Expected Time to Invest in a New Location

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Abstract

In this paper we study the expected time to invest in a new location. In particular we derive expected values of the optimal timing regarding the decision of relocation of a company. We address two classes of scenarios. In the first one we assume that new (and potentially more efficient) spots become available according to a non-homogeneous Poisson process, whereas in the second one we assume a conditional Poisson process. For both scenarios we derive mathematical expressions for the expected value of the firm in specific situations, where the intensity function is some particular function. We end up the paper presenting numerical illustrations of the derived results.

Keywords: Relocation, non-homogeneous Poisson process, conditional Poisson process, hitting time, expected value, gamma function.

1 Motivation

In this paper we study the expected time to invest in a new location, which is a key question when it comes to the relocation problem that most companies face nowadays. For that reason we start this paper by presenting the problem in a pure economical setting, before explaining the mathematical formulation.

The analysis of the problem of production units relocation for third countries, supposedly more competitive at the level of production factors and/or incentives to investment, has assumed an increasing significance in western economies, when confronted with a full array of problems induced by the fact of a growing number of companies resorting to this kind of solutions, in order to improve their competitive edge in the market. In fact, the overall movement in the sense of the growing abolition of protectionist barriers of technical and political nature has led to the creation of a more accentuated competitive environment, where a great part of economies (mainly the most industrialized) are subject to geo-economic insertion movements in widen spot. Scenarios like this, marked by the emergence of new markets, by the free circulation of capital and investments at a world scale,and by production conditions' homogeneity calls forth a scenario favourable to the production units' relocation.

The overall conclusion in several studies about production units relocation is that it only reveals to be attractive when the existent comparative advantages (different conditions) between the destination and origin country overcompensate, significantly, the costs needed to support the implementation of this relocation process.

The papers which emphasize the minimization of production costs have focused, mostly, on the analysis of potential explicative factors regarding the relocation of production units' phenomenon. In some cases, the emphasis relies on factors external to companies and/or its markets (i.e. the need of processes' rationalization, deficiencies on the origin markets), while in some other cases the emphasis is put upon factors of external or institutional nature (taxes, incentive policies, market venture, etc.). In the first case, it's worth of notice works like Kogut and Kulatilaka (1994), Sleuwaegen and Pennings (2002) and Siegfried and Evans (1994). In the second, outstand the papers of Black and Hoyt (1989), Belderbos and Sleuwaegen (1996), Haaparanta (1996), G. Bekaert and Viskanda (1998), Devereux and Griffith (1998), Dunning (1998), C. Head and Swenson (1999), Haufler and Wooton (1999), Chakrabarti (2001), Pantzalis (2001), and Berry (2002).

Regarding the potential explicative internal factors two are worth of notice: the existence of comparative advantages on the destination locations, and the possibility of increasing the operational flexibility by means of creating a network of subsidiaries spread in several locations. Siegfried and Evans (1994) have found that phenomena like low rate of return, scarce demand, low capital intensity and small dimension companies in a given location have a significant impact on the propensity of abandoning a certain location. Under these circumstances, companies are naturally tempted to remove their production units to locations which offer them comparative advantages (vidé Hymer (1960), quoted by Kogut and Kulatilaka (1994)). Also companies in a restructuring process are more opened to relocation solutions (vidé Sleuwaegen and Pennings (2002)).

The group of potential external explicative factors is wide: taxes, direct incentives to foreign investment, level of development and wealth of each national location, level of venture of each market, and nature of local culture and institutions. The level of economic development of each country or region is also a potential factor of influence on the decision of relocating a production unit (v.g., studies of Belderbos and Sleuwaegen (1996); Dunning (1998); and Pantzalis (2001)). In general, the most developed countries offer the access to consumers with higher incomes and higher levels of education, less risk, easiness on apprehending know-how and institutional protection for investments. On the contrary, less developed countries offer great rate of return, low costs and abundant resources (land, work and capital). In specific cases, like China and India, there is still the access to big markets and elevated population growth rates. Complementary, Haufler and Wooton (1999) show that whenever the transaction costs and productivity incomes are taken into account, the country's dimension is also a factor to consider on the attractiveness of multinational companies.

An alternative approach has occurred from the need of getting the value of flexibility over uncertainty. These works present as a common characterizing element the fact of using stochastic models tendentious to value incrusted options. De-Meza and van der Ploeg (1987) present a model with one period of production relocation. Subsequently, Kogut (1983) and Kogut (1985) present and describe quantitatively multiperiod stochastic models which incorporate explicitly the option's valuation. Such model was formally analysed by Kogut and Kulatilaka (1988). Kogut and Kulatilaka (1994) have used stochastic dynamic programming models for the treatment of flexibility afforded by a portfolio of this kind of options. Their results suggest that location will be decided taking into consideration the relative price of local inputs, and the value of the relocation option may be affected by several sources of uncertainty, such as social conflicts, governmental policies, local suppliers, interest rates, etc.

Partisans of Real Options Analysis believe that companies may act pro-actively over uncertainties, in a way to obtain advantages (v.g., Kogut (1983), Kogut (1989), D. Hurry and Bowman (1992), Bowman and Hurry (1993), and Sanchez (1993); and McGrath (1997)). Reuer and Leiblein (2000) have studied, in this context, the risk implications on the loss of international investment, in a management flexibility context in multinationals and joint ventures. In their perspectives these investments may enable the creation of real options which enable these companies to avoid income losses, through the substitution of activity between countries, according to their own contexts. Also Buckley and Casson (1998) refer that the value of flexibility may encourage companies to produce the same product in numerous locations, in order to allow the change of production among them, according to circumstances. In this sense, Miller and Reuer (1998c) and Miller and Reuer (1998a) empirical results suggest that international strategic options about production location reduce the volatility of the stockholders' rate of return.

The optimal timing regarding decision making is absolutely crucial in a volatile context. The right decision may have quite little significance if taken in the wrong moment, Rivoli and Salorio (1996). Thus, Campa (1994) refers that the more volatile is the context the most likely is the change and the bigger will be the advantage of waiting until all imminent changes occur. This strategy requires that the company defines in advance a set of possible locations, as referred in Buckley and Casson (1998).

We end up this section presenting the remainder organization of the paper. In Section (2) we present the mathematical formulation of the paper; we skip some technical details, as they are not the key point in this paper. Next, in Section (3), we provide the first set of results concerning the expected time to invest in a new location when information concerning new (and more efficient) locations arrive according to a non-homogeneous Poisson process, whereas in Section (4) we tackle the problem when the process is a conditional Poisson process. In Section (5) we show some numerical results, in the form of 2D and 3D plots, and finally in Section (6) we present conclusions concerning this work.

Finally, a word about notation used in this paper and notably concerning random variables. If X is a random variable, we denote its distribution function by $F_X(.)$ and its density function by $f_X(.)$. Moreover, $\mathbb{E}[X] = \int u dF_X(u)$ denotes its expected value. We use the symbol $\Box$ to denote end of a proof of a lemma or a theorem, and i.i.d. means *independent and identically distributed*.

We also note that in order to improve the readability of the paper, we present all the mathematical proofs in the appendix (see Section (7)).

2 Mathematical formulation

In this section we introduce the main concepts and notation that we use to tackle the problem mathematically. We note, in addition, that although the main objective is the characterization of the expected time to invest in a new location, the mathematical results that we derive are general enough, and therefore can be applied to other frameworks.

As stated in the previous section, the efficiency of a company changes along its time life; let us denote the efficiency of the company at time t by $\theta(t)$. Thus $\Theta = \{\theta(t), t \geq 0\}$ is a stochastic process (in continuous time), and denotes the efficiency process. For example, $\theta(0)$ is simply the initial efficiency. In addition, as the efficiency varies over time, we denote by $\{T_i, i \in \mathbb{N}\}$ the sequence of times when changes in the efficiency occur. For instance, T_1 is the first time that there is a change in the efficiency. We note that in this paper, as we are interested only in the relocation issue, we assume that effiency can only change because a new spot has become available.

Furthermore, everytime that there is a change in the efficiency, its increment is denoted by U_i. Therefore $\{U_i, i \in \mathbb{N}\}$ is simply the sequence of jumps in the efficiency process, whereas $\{T_i, i \in \mathbb{N}\}$ is the sequence of jump times, such that

$$\theta(T_i) = \theta(T_i^-) + U_i.$$

Moreover, we assume that $\{T_i, i \in \mathbb{N}\}$ is also the sequence of events of a point process, that we denote by $N = \{N(t), t \geq 0\}$. Thus, in the setting of relocation problems, the process N is simply the process of arrival of information concerning new (and available) locations where the company can produce in a more efficient way.

With the previous description, it follows that for each $t \geq 0$, $\theta(t)$ can be written as follows:

$$\theta(t) = \theta(0) + \sum_{i=1}^{N(t)} U_i.$$

For example, if N is a Poisson process, then it follows that Θ is a compound Poisson process, Ross (1996).

Now, for a given value a, we denote by $T(a)$ the following variable:

$$T(a) = \inf\{t \geq 0 : \theta(t) \geq a\}. \tag{1}$$

Note that in fact $T(a)$ is an hitting time, in the sense that it is the first time that the efficiency process Θ hits the value a, where this a can be an arbitrary quantity.

This is the general definition of an hitting time. But in terms of the relocation problem we can be more precise, as we explain next. We know from Dixit and Pindyck (1994) that for the relocation problem there is a value, usually denoted by $\theta^\star$, that triggers a relocation, so that if $\theta(t) > \theta^\star$ the firm decides to invest in this new location, whereas if $\theta(t) \leq \theta^\star$ the optimal decision is to stay in its current site and wait for other locations to become available. In fact, the sequence $\{T_i, i \in I\!N\}$ is a sequence of decision times, as at each time T_i the firm has to decide between *continuing* in the present location or *stop*, and move to a new location. As we stated previously, this decision strongly depends on the relationship between the current efficiency of the firm and the efficiency that it will achieve in the new location. In order to justify a change in location, the corresponding efficiency gains need to overcompensate the resultant relocation costs.

Thus, if the firm acts always optimally, the variable $T(\theta^\star)$, as defined in Equation (1), is simply the optimal time of relocation. Following Dixit and Pindyck (1994), we call the value $\theta^\star$ the *optimal switching level*.

In order to contemplate more general situations (where a is not necessarily the optimal switching level but an arbitrary level of efficiency), for the time being we let a be an arbitrary non-negative value (we assume, without loss of generality, that the efficiency process takes only non-negative values).

We note trivially that $T(a)$ is a non-negative random variable, and therefore it follows that $I\!E[T(a)] = \int_0^\infty u F_{T(a)}(du) = \int_0^\infty (1 - P(T(a) \leq t))\, du$, Ross (1996), where (according to our notation) $F_{T(a)}$ denotes the distribution function of $T(a)$. Therefore it follows that:

$$
\begin{aligned}
I\!E[T(a)] &= \int_0^\infty (1 - P(T(a) \leq t))\, dt = \int_0^\infty P\left(\theta(t) < a\right) dt \\
&= \int_0^\infty \sum_{k=0}^{\infty} P\left(\sum_{n=1}^{k} U_n < a - \theta(0)\right) P(N(t) = k) dt \\
&= \frac{1}{\lambda} + \int_0^\infty \left[\sum_{k=1}^{\infty} \left(\int_0^{a-\theta(a)} f_{\sum_{n=1}^k U_n}(x) dx\right) \frac{e^{-\lambda t}(\lambda t)^k}{k!}\right] dt \qquad (2)
\end{aligned}
$$

where $f_{\sum_{n=1}^k U_n}(.)$ denotes the density function of the sum of i.i.d. random variables $U_1, U_2, \ldots, U_k$.

The aim of this paper is to present results concerning properties of the hitting time $T(a)$ for different characterizations of the counting process N. In particular we consider the following cases:

- N is a non-homogeneous Poisson process (NHPP);
- N is a conditional Poisson process.

Note, trivially, that if the increments $\{U_i\}$ are deterministic, and equal to a certain value U, then the number of jumps in the counting process until the process $\{\theta(t), t \geq 0\}$ hits level a, $n(a)$, is given by:

$$n(a) = \lfloor \frac{a - \theta(0)}{U} \rfloor + 1.$$

Note that in this case $n(a)$ is deterministic, although $T(a)$ is random, as the arrival process is regulated by a stochastic mechanism. Furthermore, the conditional distribution of $T(a)$, given $n(a)$, is such that

$$P(T(a) \leq t | n(a)) = \sum_{n=n(a)}^{\infty} P(N(t) = n | n(a)).$$

As this is not the main objective of the paper, we leave to the reader some important references, where one can find a complete description of the mathematical formulation of the relocation problem. The economic framework that we follow in this paper follows closely the one considered by Farzin et al. (1998); Huisman (2000); Couto (2006). For instance, Huisman (2000) proved that if the counting process $N = \{N(t), t \geq 0\}$ is a Poisson process with (constant) arrival rate λ, then:

$$E[T(\theta^\star) \ | n(\theta^\star)] = \frac{n(\theta^\star)}{\lambda}. \tag{3}$$

See also Couto (2006) for a complete description of the associated decision problem in the relocation question, and the balance between inherent costs/losses in keeping the current location or changing to a new one.

In the following sections we derive the expected value of the hitting times and, as we will see, the main results involving gamma functions. For this reason we specify here the notation that we use concerning these functions, namely:

$$\Gamma(z) = \int_0^{+\infty} t^{(z-1)} e^{-t} \ dt, \quad \Gamma(a, z) = \int_z^{+\infty} t^{(a-1)} e^{-t} \ dt, \quad \gamma(a, z) = \int_0^{z} t^{(a-1)} e^{-t} \ dt$$

denote the gamma, upper gamma and lower gamma functions, respectively. In the appendix (at the end of the paper) we recall the main properties of these functions and also an auxiliary result that we have derived, that up to our knowledge is new (see Lemma (7.1)).

Finally in order to keep notation simply, instead of $n(a)$ and $T(a)$ we use simply n and T. Clearly that the main interest is in the relocation time, $T(\theta^\star)$, but in order to keep the results as general as possible we let a denote a (non-negative) arbitrary level.

3 Non-Homogeneous Poisson Process

In the majoraty of the papers where the relocation problem is addressed, the Poisson process (PP) is the usual choice. But in more realistic situations this is not the natural choice, as, like in production units relocation, the intensity rate depend on striking time events. For modelling such phenomenon the non-homogeneous Poisson process (NHPP) is the most appropriate. The choice of the non-homogeneous Poisson process to model the shocks' time events was first proposed by J. Esary and Walkup (1967)

Let $\Lambda(t) = \int_0^t \lambda(s)ds$, so that

$$E[T \mid n] = \int_0^{+\infty} P(T \geq t \mid n)\ dt = \sum_{m=0}^{n-1} \int_0^{+\infty} e^{-\Lambda(t)}\ \frac{(\Lambda(t))^m}{m!}\ dt. \tag{4}$$

In order to obtain formal expressions for the expected value of T, we assume particular forms of the intensity function $\lambda(.)$, namely: linear function, polynomial type, and ladder type.

- **Case 1**: $N \sim \text{NHPP}(\lambda(t) = at)$.

In this case we assume that the intensity function is a linear (and continuous) function of time, and a denotes the slop. Note, that the higher the value of parameter a, the more pronounced is the increase in the intensity of the process.

As an example of this phenomenon we have multiple relocation cases on the contracting sector, from the North and East of England to Poland, Czech Republic and Slovakia in a short term. Similarly, although relocation has been operated in different destinations, we may present as an example the case of the textile sector of the North and East of England which was virtually eliminated on the last five years, face to the relocation of the manufacturing units for outer Europe (namely to the North Africa and Middle-East), in response to the redirecting of demand from big clients like Marks & Spencer, and to the disinvestment of major multinationals in order to open new production units in new locations within European Union. At the same time, in the USA, the McKinsey Global Institute foresees a growth of 30 to 40% on relocations, until 2009, and the Forrester Research points out the loss of 3,3 million jobs until 2015, in result of this kind of processes Drezner (2004).

In the following theorem we derive the expected value of T.

Theorem 3.1 *If $N \sim NHPP(\lambda(t) = at)$ then*

$$E[T \mid n] = \left(\frac{2}{a}\right)^{\frac{1}{2}} \frac{\Gamma(n + \frac{1}{2})}{\Gamma(n)} \tag{5}$$

Note that $E[T|n]$ is a decreasing function of a, as it would be expected.

- **Case 2**: $N \sim\ PPNH(\lambda(t) = at^{b-1})$, with $b \neq 0$.

This case generalizes the previous one, setting $b = 2$; note, in addition, that if we set $b = 1$ then we get the Poisson process. It follows trivially that:

$$\Lambda(t) = \int_0^t ax^{b-1}dx = \frac{a\ t^b}{b}. \tag{6}$$

In this case the expected value is as follows:

Theorem 3.2 *If* $N \sim PPNH(\lambda(t) = at^{b-1})$ *then*

$$E[T\ |n] = \left(\frac{b}{a}\right)^{\frac{1}{b}} \frac{\Gamma(n+\frac{1}{b})}{\Gamma(n)} \tag{7}$$

We note, once again, that $E[T\ |n]$ is a decreasing function of a, with fixed b, and decreasing in b, with fixed a. In addition, if we set $b = 1$ we end up with Equation (3), whereas if $b = 2$ we get Equation (5).

Corollary 3.3

$$E[T\ |\lambda(t) = at^{b-1}, n] = \frac{E[T\ |\lambda(t) = t^{b-1}, n]}{a^{\frac{1}{b}}}. \tag{8}$$

The result expressed in this last corollary describes the relation between the expected value of the hitting time T, which intensity rate is a monomer, and the expected value of the hitting time of a process which intensity rate is a monic monomial of the same degree. We skip the proof, as it follows straightforward from Equation (7).

- **Case 3**: $N \sim PPNH(\lambda(t))$, with $\lambda(.)$ given by:

$$\lambda(t) = \begin{cases} a, & \text{if } t \in \bigcup_{s=0}^{+\infty}[2s, 2s+1[\\ b, & \text{if } t \in \bigcup_{s=0}^{+\infty}[2s+1, 2s+2[\end{cases}. \tag{9}$$

Thus, in this case the intensity rate oscillates between two different values (a and b), changing every time unit interval. This case is also a generalization of the Poisson process (setting $a = b$).

We note that this alternate behaviour of the intensity rate can be adequate to model the covering of financial risk on the exposure to exchange risk, studied by Simkins and Laux (1996), and Allayannis and Ofek (2001). Allayannis and Ofek (2001) have examined the combined influence of the covering of financial and operational risk regarding the exchange risk, such as D. Carter and Simkins (2003). The covering of operational risk functions as a real option, since that certain companies are less exposed to downside risk and more exposed to upside risk. It is usually assumed in the literature that research on the exchange rate effect on this domain has started with the work of Jorion (1990), followed by Bodnar and Gentry (1993), Bartov and Bodnar (1994), E. Bartov and Kaul (1996), E. Chow and Solt (1997a,b) and A. Martin and Akhigbe (1999). Are also included here the works of Kogut and Kulatilaka (1994), Miller and Reuer (1998c,b) and S. Martzoukos and Trigeorgis (2002). Rangan (1998) refers to the management of operational flexibility in response to changes on the currency parity, where is set the methodology of this model.

Let us start by assuming that $b = 0$, so that arrivals can only take place in unit intervals with origin in even times. This case corresponds roughly to an on-off process, where the intervals on-off are deterministically determined.

In this case [1]

$$\Lambda(t) = \begin{cases} a(t-s), & se\ t \in \bigcup_{s=0}^{+\infty} [2s, 2s+1[\\ a(s+1), & se\ t \in \bigcup_{s=0}^{+\infty} [2s+1, 2s+2[\end{cases}. \tag{10}$$

Note that it follows from Equation (10) that:

$$\sum_{n=0}^{n-1} \frac{\Lambda(t)^n}{n!} = \begin{cases} e^{a(t-s)}\ \frac{\Gamma(n,\ a(t-s))}{\Gamma(n)}, & \text{if } t \in \bigcup_{s=0}^{+\infty} [2s, 2s+1[\\ e^{a(s+1)}\ \frac{\Gamma(n,\ a(s+1))}{\Gamma(n)}, & \text{if } t \in \bigcup_{s=0}^{+\infty} [2s+1, 2s+2[\end{cases}. \tag{11}$$

Theorem 3.4

$$E[T\ |n] = \frac{n}{a} + \sum_{s=1}^{+\infty} \frac{\Gamma(n, as)}{\Gamma(n)}. \tag{12}$$

Remark that the previous result can be extended to the case where a is not constant, i.e., suppose that

$$\lambda(t) = \sum_{s=0}^{\infty} a_s 1_{\{t \in [2s, 2s+1[\}} \tag{13}$$

where $\{a_s, s \in I\!N\}$ is a sequence of real values. We note that this is a huge generalization when compared to the cases that are usually studied, as this can accommodate many of the interesting and real situations that we can find.

For this case we can also derive the expected value of T as follows:

Theorem 3.5 *If $N \sim NHPP(\lambda(t))$, with $\lambda(.)$ given by Equation (13), then*

$$\begin{aligned} E[T\ |n] = &\frac{n}{a_0} + \sum_{s=1}^{\infty} \left[\left(\frac{a_s - a_{s-1}}{a_s a_{s-1}} \right) \left(\frac{\sum_{i=0}^{s-1} a_i\ \Gamma(n, \sum_{i=0}^{s-1} a_i) + \Gamma(n+1, \sum_{i=0}^{s-1} a_i)}{\Gamma(n)} \right) \right] + \\ &+ \sum_{s=0}^{+\infty} \frac{\Gamma(n, \sum_{i=0}^{s} a_i)}{\Gamma(n)}. \end{aligned} \tag{14}$$

Let us now consider the initial case (9), with b being an arbitrary value. In this case we have the following result:

[1] Note that the symmetrical case (i.e., arrivals can only take place in unit intervals with origin in odd times) can also be analysed in a similar way.

Theorem 3.6

$$E[T\ |n] = \frac{n}{b} + \left(\frac{b-a}{ab}\right)\sum_{s=0}^{+\infty}\Bigg[\left(\frac{(b+a)(s+1)\ \Gamma(n,\ (b+a)(s+1)) - \Gamma(n+1,\ (b+a)(s+1))}{\Gamma(n)}\right) -$$
$$-\left(\frac{(b+(b+a)s)\ \Gamma(n,\ (b+(b+a)s)) - \Gamma(n+1,\ b+(b+a)s)}{\Gamma(n)}\right)\Bigg]. \tag{15}$$

4 Conditional Poisson Process

In this section we assume that the process N is a conditional Poisson process, which intensity is a function of a random variable, that we denote by X, as follows:

$$N|X = x\ \sim PPHN(\lambda(t|x) = f(x)),$$

where f is a positive function.

In this section we assume the same intensity functions as we assumed in the previous section, but for each case we derive results concerning different distributions for the random variable X.

- **Case 1**: $N|X = x \sim\ PPNH(\lambda(t|x) = x\ t^{b-1})$

In the next theorem we prove that the expected value of T depends on the distribution of the random variable X only through the expected value of a particular function of X.

Theorem 4.1 *If $E[|T|\ |n] < \infty$, then $E[T\ |n]$ depends on X through $E[X^{-\ \frac{1}{b}}]$.*

In order to incorporate the information on the intensity rate, we use next the notation $E[T|\lambda(t|X = x), n]$ to denote the expected value of T, given n, when the intensity rate is $\{\lambda(t|X = x), t \geq 0\}$.

In view of Theorem (4.1), we have the following corollary:

Corollary 4.2

$$E[T\ |\lambda(t|X = x) = x\ t^{b-1},\ n] = E[T\ |\lambda(t) = t^{b-1},\ n]\ E[X^{-\ \frac{1}{b}}]. \tag{16}$$

Next we consider several instances of distributions for X. In view of Equation (16), in order to compute the conditional expected value we have only to derive $E[X^{-\ \frac{1}{b}}]$.

- Let X be a discrete r.v., with probability function

$$P(X = x) = \begin{cases} p, & se\ x = a \\ 1-p, & se\ x = c \\ 0, & c.c. \end{cases} \tag{17}$$

where $a, c \neq 0$. Then

$$E[T\ |n] = b^{\frac{1}{b}}\ \frac{\Gamma(n+\frac{1}{b})}{\Gamma(n)}\left(p\left(\frac{1}{a}\right)^{\frac{1}{b}}+(1-p)\left(\frac{1}{c}\right)^{\frac{1}{b}}\right). \tag{18}$$

using Equation (16).

- Let $X \sim Unif(a,c)$ $(0 < a < c)$. If $b = 1$:

$$E[T\ |n] = n\frac{log\left(\frac{c}{a}\right)}{c-a} \tag{19}$$

whereas if $b > 1$:

$$E[T\ |n] = \frac{\Gamma(n+\frac{1}{b})}{\Gamma(n)}\left(\frac{b}{b-1}\right)\left[\frac{c\left(\frac{b}{c}\right)^{\frac{1}{b}}-a\left(\frac{b}{a}\right)^{\frac{1}{b}}}{c-a}\right]. \tag{20}$$

- If $X \sim Gama(m, \alpha)$ $(m > \frac{1}{b})$ then

$$E[T\ |n] = \left(\alpha b\right)^{\frac{1}{b}}\left(\frac{\Gamma(n+\frac{1}{b})\ \Gamma(m-\frac{1}{b})}{\Gamma(n)\Gamma(m)}\right). \tag{21}$$

We note that if $X \sim Exponencial(1)$, then $E[X^{-1}]$ does not exist, and therefore $E[T\ |n]$ does not exist either.

- **Case 2**: $N|X = x \sim\ PPNH(\lambda(t|x))$, where

$$\lambda(t|x) = \begin{cases} x, & se\ t \in \bigcup_{s=0}^{+\infty}[2s, 2s+1[\\ 0, & se\ t \in \bigcup_{s=0}^{+\infty}[2s+1, 2s+2[\end{cases} \tag{22}$$

As before, we consider several situations for the r.v. X.

- If X has probability function given by Equation (17), then:

$$E[T\ |n] = p\ \left(\frac{n}{a}+\sum_{s=1}^{+\infty}\frac{\Gamma(n,as)}{\Gamma(n)}\right)+(1-p)\ \left(\frac{n}{c}+\sum_{s=1}^{+\infty}\frac{\Gamma(n,cs)}{\Gamma(n)}\right). \tag{23}$$

- If $X \sim Unif(a,c)$ $(0 < a < c)$, then

$$\begin{aligned} E[T\ |n] = {} & n\frac{log\left(\frac{c}{a}\right)}{c-a}+ \\ & +\sum_{s=1}^{+\infty}\frac{1}{s(c-a)}\left[\frac{cs\ \Gamma(n,cs)-\Gamma(n+1,cs)-(as\ \Gamma(n,as)-\Gamma(n+1,as))}{\Gamma(n)}\right]. \end{aligned} \tag{24}$$

5 Numerical Illustration

In this section we illustrate numerically the results that we derived in the previous two sections.

Assume that the arrival process is modelled by a NHPP, with linear arrival rate $\lambda(t) = at$. In Figure (1) we present the 3D plot of the conditional expected value of the relocation time T, according to Equation (5), as a function of a and n.

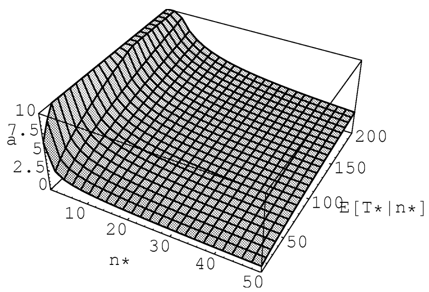

Figure 1: Plot of $E[T|n]$ for an NHPP with linear intensity function, as a function of a and n.

For example, if $a = 2$, then the plot of $E[T|n]$ is the following:

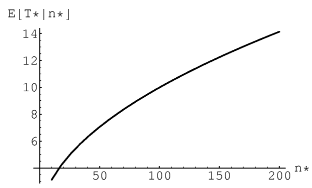

Figure 2: Plot of $E[T|n]$ for an NHPP with linear intensity function, as a function of n, with $a = 2$.

Using once again the value $a = 2$ as a reference value, we plot in Figure (3) the first derivative of $E[T|n]$ as a function of n. We conclude from the observation of the plot that the increase of $E[T|n]$ is larger with the increase of small values of n than with the increase of large values of n. Therefore there is a larger increase in the expected relocation time for small values of n than for larger values of n.

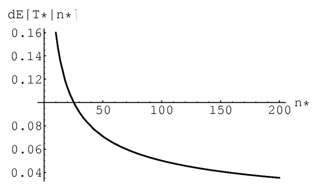

Figure 3: Plot of $dE[T|n]$ as a function of n, with $a = 2$.

Suppose now that the intensity function is of monomial type, as in Equation (6). In this case, setting $a = 2$, we get the following 3D picture (see Figure (4)).

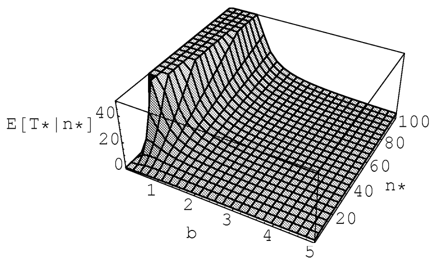

Figure 4: Plot of $E[T|n]$ for an NHPP with monomial intensity function, as a function of b and n.

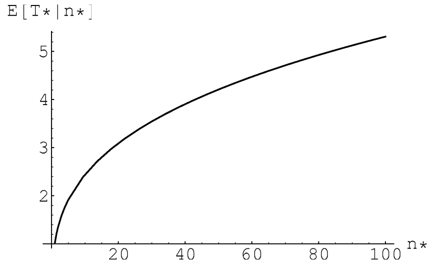

Figure 5: Plot of $E[T|n]$ for an NHPP with monomial intensity function, as a function of n.

For example, using the reference values $a = 2$ and $b = 3$ (quadratic type), we get the numerical results illustrated in Figure (5).

If we consider the first derivative of $E[T|n]$ in order to n, we conclude that the increase in $E[T|n]$ is larger for an increase in small values of n than in large values of n.

Next we consider a conditional Poisson process, also with monomial intensity rate but now depending on a random variable, that we denote by X. If X has probability function given by Equation (17), with $p = 0.5$, $a = 1$ and $c = 2$, then the plot corresponding to $E[T|n]$ is as follows (see Figure (6)):

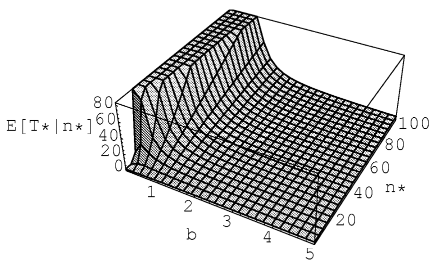

Figure 6: Plot of $E[T|n]$ for a conditional PP with monomial intensity function, as a function of b and n.

6 Concluding Remarks

Nowadays, the problem of relocation production is especially relevant because we are facing a period of globalization and market integration with multiple socio-economic implications. In this work we focus on the perceived increase in efficiency needed to justify the decision to relocate production units from one place to another, in a real options context.

In particular we considered the problem of the characterization of the time until (optimal) relocation for different scenarios, with special emphasis in the non-homogeneous and the conditional Poisson processes. For particular cases of rate functions we have been able to derive closed mathematical expressions for the expected relocation times.

We note that the results that we have derived are in occurence with the economic rationale; in particular the numerical illustrations show that there is a larger increase in the expected time until relocation with small values of $n(\theta^\star)$ (recall that $n(\theta^\star)$ denotes the number of localizations that become available until the optimal one, the one that triggers a relocation, becomes available) than with larger values of $n(\theta^\star)$.

The cases that we have dealt with in this paper (the particular instances of intensity functions for the considered Poisson processes) can be extended to other situations. Therefore, other intensity rates can be analysed in a similar way, but with different mathematical manipulations. In addition, more moments of T can be derived (namely, the variance), following the same kind of analysis (but certainly with more tedious calculus).

Naturally, a simple framework as the one proposed here has some limits that could and should be overcomed in another works.

References

A. Martin, J. M. and A. Akhigbe (1999). Economic exchange rate exposure of u.s.-based mnc's operating in europe. *The Financial Review 34*, 21–36.

Allayannis, G. and E. Ofek (2001). Exchange rate exposure, hedging, and the use of foreign currency derivatives. *Journal of International Money and Finance 20*, 273–296.

Bartov, E. and G. Bodnar (1994). Firm valuation, earnings expectations, and the exchange rate exposure effect. *The Journal of Finance 44*, 1755–1785.

Belderbos, R. and L. Sleuwaegen (1996). Japanese firms and the decision to invest abroad: Business groups and regional core networks. *Review of Economic and Statistics 78*, 214–220.

Berry, H. (2002). *The Influence of Location and Multinational Network Effects on Firm Value: Evidence from US Manufacturing Firms, 1981-2000.* Working Paper, Center for International Business Education and Research at the UCLA Anderson Graduate School of Management.

Black, D. and W. Hoyt (1989). Bidding for firms. *American Economic Review 79*, 1249–1256.

Bodnar, G. and W. Gentry (1993). Exchange rate exposure and industry characteristics: Evidence from canada, japan and u.s. *Journal of International Money and Finance 12*, 29–45.

Bowman, E. and D. Hurry (1993). Strategy through the options lens: An integrated view of resource investments and the incremental-choice process. *Academy of Management Review 18*, 760–782.

Buckley, P. and M. Casson (1998). Models of the multinational enterprise. *Journal of International Business Studies 29*(1), 21–44.

C. Head, J. R. and D. Swenson (1999). Attracting foreign manufacturing: Investment promotion and agglomeration. *Regional Science and Urban Economics 29*, 197–218.

Campa, J. (1994). Multinational investment under uncertainty in the chemical processing industries. *Journal of International Business Studies 25*(3), 557–578.

Chakrabarti, A. (2001). The determinants of foreign direct investment: Sensitivity analyses of cross-country regressions. *Kyklos 54*, 89–114.

Couto, G. (2006). *Opções Reais e Decisão sob Incerteza no Processo de Relocalização.* Ph. D. thesis, Instituto Superior de Economia e Gestão, Technical University of Lisbon, Portugal.

D. Carter, C. P. and B. Simkins (2003). *Asymmetric Exposure to Foreign-Exchange Risk : Financial and Real Option Hedges Implemented by U.S. Multinational Corporations.* Working Paper, Department of Finance, College of Business Administration, Oklahoma State University, 14 February.

D. Hurry, A. M. and E. Bowman (1992). Calls on high-technology: Japanese exploration of venture capital investments in the united states. *Strategic Management Journal 13*, 85–101.

De-Meza, D. and F. van der Ploeg (1987). Production flexibility as a motive for multinationality. *Journal of Industrial Economy 35*, 343–352.

Devereux, M. and R. Griffith (1998). Taxes and the location of production: Evidence from a panel of us multinationals. *Journal of Public Economics 68*, 335–367.

Dixit, A. and R. Pindyck (1994). *Investment Under Uncertainty.* Princeton University Press.

Drezner, D. (2004). *The Outsourcing Bogeyman.* Working Paper, Foreign Affairs, May/June. Published by Council on Foreign Affairs.

Dunning, J. (1998). Location and the multinational enterprise: A neglected factor? *Journal of International Business Studies 29*(1), 45–66.

E. Bartov, G. B. and A. Kaul (1996). Exchange rate variability and the riskiness of u.s. multinational firms: Evidence from the breakdown of the bretton woods system. *Journal of Financial Economics 42*, 105–132.

E. Chow, W. L. and M. Solt (1997a). The economic exposure of u.s. multinational firms. *The Journal of Financial Research 20*, 1991–210.

E. Chow, W. L. and M. Solt (1997b). The exchange-rate risk exposure of asset returns. *Journal of Business 70*, 105–123.

Farzin, Y., K. Huisman, and P. Kort (1998). Optimal timing of technology adoption. *Journal of Economic Dynamics and Control 22*, 779–799.

G. Bekaert, C. Erb, C. H. and T. Viskanda (1998). Distributional characteristics of emerging markets returns and asset allocation. *Journal of Portfolio Management 21*.

Haaparanta, P. (1996). Competition for foreign direct investments. *Journal of Public Economics 63*, 141–153.

Haufler, A. and I. Wooton (1999). Country size and tax competition for foreign direct investment. *Journal of Public Economics 71*, 121–139.

Huisman, K. (2000). *Technology Investment: a Game Theoretic Real Options*. Ph. D. thesis, Tilburg University, Department of Econometrics, CentER Dissertation Series Centre for Quantitative Methods in Eindhoven, Tilburg, The Netherlands.

Hymer, S. (1960). *The International Operations of National Firms*. Ph. D. thesis, MIT, Cambridge, MA.

J. Esary, F. P. and D. Walkup (1967). Association of random variables with applications. *Ann. Math. Statist. 38*, 1466–1474.

Jorion, P. (1990). The exchange rate exposure of u.s. multinationals. *Journal of Business 63*, 331–345.

Kogut, B. (1983). *Foreign Direct Investment as a Sequential Process*. In C.P. Kindleberger e D.B. Audretsch (Ed.), The Multinational Corporation in the 1980's. MIT Press, Cambridge, MA, 38-56.

Kogut, B. (1985). Designing global strategies: Profiting from operating flexibility. *Sloan Management Review 26*, 27–38.

Kogut, B. (1989). A note on global strategies. *Strategic Management Journal 10*, 383–389.

Kogut, B. and N. Kulatilaka (1988). *Multinational Flexibility and the Theory of Foreign Direct Investment*. Working Paper, 88-10, Reginald H. Jones Center for Management Policy, Strategy and Organization, University of Pensylvania, July.

Kogut, B. and N. Kulatilaka (1994). Options thinking and platform investments: Investing in opportunity. *California Management Review 36*(2), 52–71.

McGrath, R. (1997). A real options logic for initiating technology positioning investments. *Academy of Management Review 22*(4), 974–996.

Miller, K. and J. Reuer (1998a). Asymmetric corporate exposures to foreign exchange rate changes. *Strategic Management Journal 19*, 1183–1191.

Miller, K. and J. Reuer (1998b). Asymmetric corporate exposures to foreign exchange rate changes. *Strategic Management Journal 19*, 1183–1191.

Miller, K. and J. Reuer (1998c). Firm strategy and economic exposure to foreign exchange rate movements. *Journal of International Business Studies 29*(3), 493–514.

Pantzalis, C. (2001). Does location matter? an empirical analysis of geographic scope and mnc market valuation. *Journal of International Business Studies 32*(1), 133–155.

Rangan, S. (1998). Do multinationals operate flexibly? theory and evidence. *Journal of International Business Studies 29*(2), 217–237.

Reuer, J. and M. Leiblein (2000). Downside risk implications of multinationality and international joint ventures. *Academy of Management Journal 43*(2), 203–214.

Rivoli, P. and E. Salorio (1996). Foreign direct investment under uncertainty. *Journal of International Business Studies 27*(2), 335–354.

Ross, S. (1996). *Stochastic Processes.* New York: Wiley.

S. Martzoukos, N. P. and L. Trigeorgis (2002). *Capital Investment Decision Networks with Partial Reversibility and Stochastic (Utilization-Dependent) Switching Costs.* Working Paper, September, University of Cyprus.

Sanchez, R. (1993). *Strategic Flexibility, Firm Organization, and Managerial Work in Dynamic Markets: A Strategic-Options Perspective.* In P. Shrivastava, A. Huff, J. Dutton (Eds.). Advances in Strategic Management, 9, 251-291. Greenwich, CT: JAI Press.

Siegfried, J. and L. Evans (1994). Empirical studies of entry and exit: A survey of the evidence. *Review of Industrial Organization 9*, 121–155.

Simkins, B. and P. Laux (1996). *Derivative use and the Exchange Rate Risk of Large U.S. Corporations.* Conference Proceedings: The Second International Finance Conference, Georgia Tech University.

Sleuwaegen, L. and E. Pennings (2002). *New Empirical Evidence on the International Relocation of Production.* Working Paper, Erasmus University Rotterdam.

7 Appendix

Before we present the proofs of the theorems that we presented in the above sections, we make a small interlude concerning main properties of the gamma functions, as many results that we have derived use the gamma functions and respective properties.

We recall that such functions have the following known properties: for $a \in \mathbb{Z}$:

$$\Gamma(a) = (a-1), \quad \Gamma(a) = \Gamma(a,z) + \gamma(a,z) \tag{25}$$

$$\int \Gamma(a,z)\ dz = z\Gamma(a,z) - \Gamma(a+1,z), \quad \frac{\partial\ \gamma(a,x)}{\partial x} = -\frac{\partial\ \Gamma(a,x)}{\partial x} = x^{a-1}e^{-x}. \tag{26}$$

Furthermore:

$$\Gamma(a,z) = (a-1)!\ e^{-z} \sum_{k=0}^{a-1} \frac{z^k}{k!}. \tag{27}$$

and if $\mathcal{R}(a+b) > 0$ and $\mathcal{R}(a) > 0$:

$$\int_0^{+\infty} t^{a-1}\ \Gamma(b,\ t)\ dt = \frac{\Gamma(a+b)}{a}. \tag{28}$$

Next we present two results concerning the gamma function, that will be useful in the sequel. The proof can be found in the appendix.

Lemma 7.1 *For $n \in \mathbb{N}$ and $a \in \mathbb{R}$:*

$$\sum_{k=1}^{n} \frac{\Gamma(k,a)}{\Gamma(k)} = \frac{\Gamma(n+1,a) - a\ \Gamma(n,a)}{\Gamma(n)} \tag{29}$$

$$\sum_{k=0}^{n-1} \frac{\Gamma(k+a)}{\Gamma(k+1)} = \frac{\Gamma(n+a)}{a\ \Gamma(n)}. \tag{30}$$

Proof of Lemma (7.1):

From the definition of the gamma and upper gamma functions, and from Equation (27), it follows that

$$\frac{\Gamma(n,a)}{\Gamma(n)} = \sum_{k=0}^{n-1} \frac{e^{-a}\ a^k}{k!}. \tag{31}$$

Thus

$$\frac{\Gamma(n,a)}{\Gamma(n)} = \sum_{k=0}^{n-1} -\left[\frac{d}{da}\left(\frac{\Gamma(k+1,a)}{\Gamma(k+1)}\right)\right] = \frac{d}{da}\left[-\sum_{k=0}^{n-1}\left(\frac{\Gamma(k+1,a)}{\Gamma(k+1)}\right)\right] \tag{32}$$

and therefore

$$\int \frac{\Gamma(n,a)}{\Gamma(n)}\ da = -\sum_{k=0}^{n-1}\left(\frac{\Gamma(k+1,a)}{\Gamma(k+1)}\right) \tag{33}$$

Now Equation (29) follows, in view of Equations (33) and (26).

In order to prove Equation (30) we note that in view of the definition of the gamma function, it follows that:

$$\begin{aligned}
\sum_{k=0}^{n-1} \frac{\Gamma(k+a)}{\Gamma(k+1)} &= \sum_{k=0}^{n-1} \frac{1}{\Gamma(k+1)} \left(\int_0^{+\infty} t^{k+a-1}\ e^{-t}\ dt \right) \\
&= \int_0^{+\infty} e^{-t}\ t^{a-1} \left(\sum_{k=0}^{n-1} \frac{t^k}{k!} \right)\ dt \\
&= \int_0^{+\infty} e^{-t}\ t^{a-1} \left(\frac{\Gamma(n,t)}{e^{-t}\ \Gamma(n)} \right)\ dt \\
&= \frac{1}{\Gamma(n)} \left(\int_0^{+\infty} t^{a-1}\ \Gamma(n,t)\ dt \right) \qquad (34)
\end{aligned}$$

where in Equation (34) we used Equation (27). Now the result follows, in view of the definition of the gamma function.

□

Proof of Theorem (3.1):

It follows from Equation (4) that:

$$\begin{aligned}
E[T\ |n] &= \sum_{n=0}^{n-1} \frac{1}{n!} \int_0^{+\infty} e^{-\frac{at^2}{2}} \left(\frac{a}{2}\right)^n t^{2n}\ dt \\
&= \sum_{n=0}^{n-1} \frac{1}{n!} \left(\frac{a}{2}\right)^n \int_0^{+\infty} e^{-\frac{az}{2}}\ z^n \left(\frac{1}{2} z^{-\frac{1}{2}}\right)\ dz \\
&= \sum_{n=0}^{n-1} \frac{1}{n!} \frac{a^n}{2^{n+1}} \int_0^{+\infty} e^{-\frac{az}{2}}\ z^{(n-\frac{1}{2})}\ dz \qquad (35) \\
&= \sum_{n=0}^{n-1} \frac{1}{n!} \frac{a^n}{2^{(n+1)}} \frac{\Gamma(n+\frac{1}{2})}{\left(\frac{a}{2}\right)^{(n+\frac{1}{2})}} \\
&= \sum_{n=0}^{n-1} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{2a}\ \Gamma(n+1)}.
\end{aligned}$$

Using Equation (30):

$$\begin{aligned}
E[T\ |n] &= \left(\frac{1}{\sqrt{2a}}\right) \frac{2\Gamma(n+\frac{1}{2})}{\Gamma(n)} \\
&= \left(\frac{2}{a}\right)^{\frac{1}{2}} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n)}
\end{aligned}$$

which proves Equation (5).

□

Proof of Theorem (3.2):

From Equation (4) it follows that:

$$E[T\ |n] = \sum_{n=0}^{n-1}\left(\int_0^{+\infty} \frac{e^{-\frac{a\ t^b}{b}}(\frac{a\ t^b}{b})^n}{n!}\ dt\right).$$

Performing a change of variable, $z = \frac{a\ t^b}{b}$, and using the gamma definition and the relation (30), we end up with:

$$\begin{aligned} E[T\ |n] &= \sum_{n=0}^{n-1}\left(\int_0^{+\infty} e^{-z}\frac{z^n}{n!}(b\ z)^{\frac{1-b}{b}}a^{-\frac{1}{b}}\ dz\right) \\ &= \sum_{n=0}^{n-1}\left(\left(\frac{b}{a}\right)^{\frac{1}{b}}\frac{1}{b\ n!}\int_0^{+\infty} e^{-z}z^{n+\frac{1}{b}-1}\ dz\right) \\ &= \left(\frac{b}{a}\right)^{\frac{1}{b}}\sum_{n=0}^{n-1}\left(\frac{\Gamma(n+\frac{1}{b})}{b\ \Gamma(n+1)}\right). \end{aligned}$$

□

Proof of Theorem (3.4):

It follows from Equation (4) that:

$$\begin{aligned} E[T\ |n] &= \frac{1}{(n-1)!}\sum_{s=0}^{+\infty}\left(\int_{2s}^{2s+1}\Gamma(n,a(t-s))\ dt\ + \int_{2s+1}^{2s+2}\Gamma(n,a(s+1))\ dt\right) \\ &= \frac{1}{(n-1)!}\sum_{s=0}^{+\infty}\left(\int_{2s}^{2s+1}\Gamma(n,a(t-s))\ dt\ + \Gamma(n,a(s+1))\right). \end{aligned} \tag{36}$$

In view of the first relation of Equation (26), we have the following equality:

$$\int_{2s}^{2s+1}\Gamma(n,a(t-s))\ dt = \frac{1}{a}\Big(-as\ \Gamma(n,as)+a(s+1)\ \Gamma(n,a(s+1))+\Gamma(n+1,as)-\Gamma(n+1,a(s+1))\Big)$$

and therefore

$$
\begin{aligned}
E[T\ |n] &= \frac{1}{\Gamma(n)} \sum_{s=0}^{+\infty} \left[\frac{1}{a} \Big(-as\Gamma(n, as) + a(s+1)\ \Gamma(n, a(s+1)) + \Gamma(n+1, as) - \right. \\
&\left. - \Gamma(n+1, a(s+1)) \Big) + \Gamma(n, a(s+1)) \ \right] \\
&= \frac{\Gamma(n+1)}{a\ \Gamma(n)} + \sum_{s=0}^{+\infty} \frac{\Gamma(n, a(s+1))}{\Gamma(n)} \\
&= \frac{n}{a} + \sum_{s=1}^{+\infty} \frac{\Gamma(n, as)}{\Gamma(n)}.
\end{aligned}
\tag{37}
$$

□

Proof of Theorem (3.5):

From Equation (4) it follows that:

$$
\begin{aligned}
E[T\ |n] &= \frac{1}{\Gamma(n)} \sum_{s=0}^{+\infty} \left(\int_{2s}^{2s+1} \Gamma(n, \sum_{i=0}^{s-1} a_i + a_s(t-2s))\ dt\ + \int_{2s+1}^{2s+2} \Gamma(n,\ \sum_{i=0}^{s} a_i)\ dt \ \right) \\
&= \frac{1}{\Gamma(n)} \sum_{s=0}^{+\infty} \left(\int_{2s}^{2s+1} \Gamma(n, \sum_{i=0}^{s-1} a_i + a_s(t-2s))\ dt\ + \Gamma(n,\ \sum_{i=0}^{s} a_i) \right).
\end{aligned}
\tag{38}
$$

Now, using the integral property of the gamma function (26):

$$
\begin{aligned}
\int_{2s}^{2s+1} \Gamma(n, \sum_{i=0}^{s-1} a_i + a_s(t-2s))\ dt = \frac{1}{a_s} \Bigg[& \sum_{i=0}^{s} a_i\ \Gamma(n,\ \sum_{j=0}^{s} a_j) - \sum_{i=0}^{s-1} a_i\ \Gamma(n,\ \sum_{j=0}^{s-1} a_j) + \\
& + \Gamma(n+1,\ \sum_{i=0}^{s-1} a_i) - \Gamma(n+1,\ \sum_{i=0}^{s} a_i) \Bigg].
\end{aligned}
\tag{39}
$$

Plugging this last result in Equation(38), we get Equation(14), as:

$$
\begin{aligned}
E[T\ |n] &= \frac{1}{\Gamma(n)} \sum_{s=0}^{+\infty} \left(\frac{1}{a_s} \left[\sum_{i=0}^{s} a_i\ \Gamma(n,\ \sum_{j=0}^{s} a_j) - \sum_{i=0}^{s-1} a_i\ \Gamma(n,\ \sum_{j=0}^{s-1} a_j) + \Gamma(n+1,\ \sum_{i=0}^{s-1} a_i) - \right. \right. \\
&\left. \left. - \Gamma(n+1,\ \sum_{i=0}^{s} a_i) \right] + \Gamma(n,\ \sum_{i=0}^{s} a_i) \right) \\
&= \frac{n}{a_0} + \sum_{s=1}^{\infty} \left[\left(\frac{a_s - a_{s-1}}{a_s a_{s-1}} \right) \left(\frac{\sum_{i=0}^{s-1} a_i\ \Gamma(n, \sum_{j=0}^{s-1} a_j) + \Gamma(n+1, \sum_{j=0}^{s-1} a_j)}{\Gamma(n)} \right) \right] + \\
&+ \sum_{s=0}^{+\infty} \frac{\Gamma(n, \sum_{i=0}^{s} a_i)}{\Gamma(n)}.
\end{aligned}
$$

$\square$

Proof of Theorem (3.6):

Using (4), we conclude that:

$$
\begin{aligned}
E[T\ |n] &= \frac{1}{(n-1)!}\sum_{s=0}^{+\infty}\left(\int_{2s}^{2s+1}\Gamma(n,\ b(t-s)+as)\ dt\ +\right.\\
&\left. +\int_{2s+1}^{2s+2}\Gamma(n,\ b(s+1)+a(t-s-1))\ dt\ \right)\\
&= \frac{1}{(n-1)!}\sum_{s=0}^{+\infty}\left(A(s)+B(s)\right).
\end{aligned}
$$

where $A(s) = \displaystyle\int_{2s}^{2s+1}\Gamma(n,\ b(t-s)+as)\ dt$ and $B(s) = \displaystyle\int_{2s+1}^{2s+2}\Gamma(n,\ b(s+1)+a(t-s-1))\ dt$. Note that

$$
\begin{aligned}
A(s) &= \int_{2s}^{2s+1}(n-1)!\ e^{-(b(t-s)+as)}\sum_{k=0}^{n-1}\frac{(b(t-s)+as)^k}{k!}\ dt\\
&= (n-1)!\ \sum_{k=0}^{n-1}\frac{1}{k!}\int_{2s}^{2s+1} e^{-(b(t-s)+as)}(b(t-s)+as)^k\ dt\\
&= (n-1)!\ \sum_{k=0}^{n-1}\frac{1}{k!}\int_{(b+a)s}^{b(s+1)+as}\frac{1}{b}\ e^{-z}z^k\ dz\\
&= \frac{(n-1)!}{b}\ \sum_{k=0}^{n-1}\frac{1}{k!}\big[\gamma(k+1,(b+a)s+b)-\gamma(k+1,(b+a)s)\big]\\
&= \frac{(n-1)!}{b}\ \sum_{k=0}^{n-1}\frac{1}{k!}\big[\Gamma(k+1,(b+a)s)-\Gamma(k+1,(b+a)s+b)\big].
\end{aligned}
$$

and, similarly,

$$
B(s) = \frac{(n-1)!}{a}\ \sum_{k=0}^{n-1}\frac{1}{k!}\big[\Gamma(k+1,(b+a)s+b)-\Gamma(k+1,(b+a)(s+1))\big].
$$

Therefore:

$$E[T\ |n] = \frac{1}{(n-1)!}\sum_{s=0}^{+\infty}\Bigg[\Bigg(\frac{(n-1)!}{b}\ \sum_{k=0}^{n-1}\frac{1}{k!}\big[\Gamma(k+1,(b+a)s)-\Gamma(k+1,(b+a)s+b)\big]\Bigg)+$$

$$+\frac{(n-1)!}{a}\ \sum_{k=0}^{n-1}\frac{1}{k!}\big[\Gamma(k+1,(b+a)s+b)-\Gamma(k+1,(b+a)(s+1))\big]\Bigg)\Bigg]$$

$$=\sum_{k=0}^{n-1}\frac{1}{k!}\sum_{s=0}^{+\infty}\Bigg[\frac{1}{b}\Bigg(\Gamma(k+1,(b+a)s)-\Gamma(k+1,(b+a)s+b)\Bigg)+$$

$$+\frac{1}{a}\Bigg(\Gamma(k+1,(b+a)s+b)-\Gamma(k+1,(b+a)(s+1))\Bigg)\Bigg]$$

$$=\sum_{k=0}^{n-1}\frac{1}{k!}\Bigg(\frac{1}{b}\Gamma(k+1)\ -\sum_{s=0}^{+\infty}\Bigg(\frac{b-a}{ab}\Bigg)\Bigg[\Gamma(k+1,(b+a)(s+1))-\Gamma(k+1,(b+a)s+b)\Bigg]\Bigg)$$

$$=\sum_{k=0}^{n-1}\frac{1}{k!}\Bigg(\frac{k!}{b}+\ \sum_{s=0}^{+\infty}\Bigg(\frac{b-a}{ab}\Bigg)\Bigg[\Gamma(k+1,(b+a)s+b)-\Gamma(k+1,(b+a)(s+1))\Bigg]\Bigg)$$

$$=\frac{n}{b}+\Bigg(\frac{b-a}{ab}\Bigg)\sum_{s=0}^{+\infty}\sum_{k=0}^{n-1}\Bigg[\frac{\Gamma(k+1,\ (b+a)s+b)-\Gamma(k+1,\ (b+a)(s+1))}{\Gamma(k+1)}\Bigg].$$

Alternatively, from Equation (4) we conclude that:

$$A(s)=\frac{1}{b}\Bigg[(b+(b+a)s)\ \Gamma(n,b+(b+a)s)-(b+a)s\ \Gamma(n,(b+a)s)+\Gamma(n+1,(b+a)s)-$$

$$-\Gamma(n+1,b+(b+a)s)\Bigg].$$

and

$$B(s)=\frac{1}{a}\Bigg[((b+a)(s+1))\ \Gamma(n,(b+a)(s+1))-(b+(b+a)s)\ \Gamma(n,b+(b+a)s)-$$

$$-\Gamma(n+1,(b+a)(s+1))+\Gamma(n+1,b+(b+a)s)\Bigg]$$

and thus the result follows.

□

Proof of Theorem (4.1):

As

$$E[T\ |n]=\int_{x\in D}\ E[T\ |X=x,\ n]\ dF(x).$$

it follows that

$$\begin{aligned} E[T \mid n] &= \int_{x \in D} \left(\frac{b}{x}\right)^{\frac{1}{b}} \frac{\Gamma(n+\frac{1}{b})}{\Gamma(n)}\, dF(x) \\ &= b^{\frac{1}{b}} \frac{\Gamma(n+\frac{1}{b})}{\Gamma(n)} \int_{x \in D} x^{-\frac{1}{b}}\, dF(x) \\ &= b^{\frac{1}{b}} \frac{\Gamma(n+\frac{1}{b})}{\Gamma(n)}\, E[X^{-\frac{1}{b}}]. \end{aligned}$$

□